3 Constructible Figures and Numbers

3.1 Constructions Using a Compass and Straightedge

In this section, we will do constructions that follow some *rules of construction*. This is closer to how the ancient Greeks did geometry.

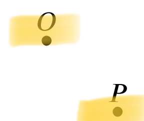
We start with two points O,P and construct new points using these rules. The rules allow us **only** to draw lines (rule 1, using a straightedge or unmarked ruler), and circles (rule 2, using a compass) **using points that were already constructed**. New points can only be obtained as intersections of these lines and circles, and can then be used to draw new ones.

How and Why

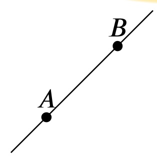
The focus here is not only in proofs but in constructing. However, you still have to prove that your construction works! And for this we will use many of the results seen so far. We will refer to this as “how and why", and both parts will usually be required.

Definition 3.1 (The Rules for constructions)**.**

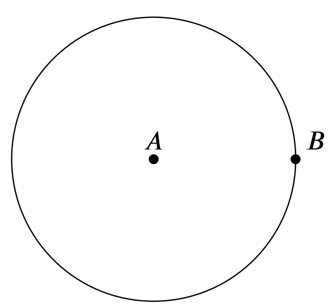
0 We start with two base constructed points.



1 We can draw the line through any two **constructed** points A and B



2 We can draw the circle $\mathscr{C}_A(B)$ with a **constructed center** A through a **constructed point** B



$\mathscr{C}_A(B)$

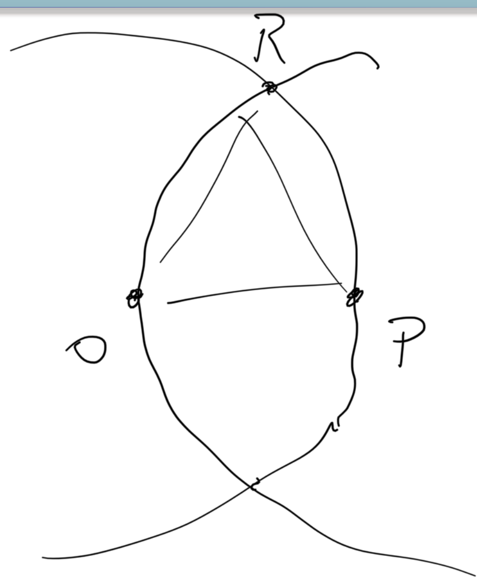
3 The intersections of constructed lines and circles become constructed points.

Definition 3.2. We say that a point is *constructible* if it is possible to construct it from O and P by applying rules 1-3 a finite number of times.

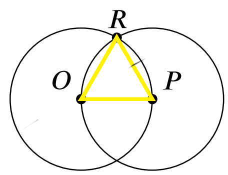
We will now do several basic constructions, showing in each case *how and why*. When we say *We can...* we mean that we can do that using the rules of construction.

Proposition 3.3. *We can construct an equilateral triangle.*

How can we do this?



Proof. *How*: We construct $\mathscr{C}_O(P)$ and $\mathscr{C}_P(O)$, we choose a point in their intersection that we denote R. (this is how R is **constructed** using rules 2 and 3).



Why: We show that $\triangle$ OPR is equilateral: since R is in $\mathscr{C}_O(P)$ and $\mathscr{C}_P(O)$, then $|RO| = |PO| = |PR|$.

□

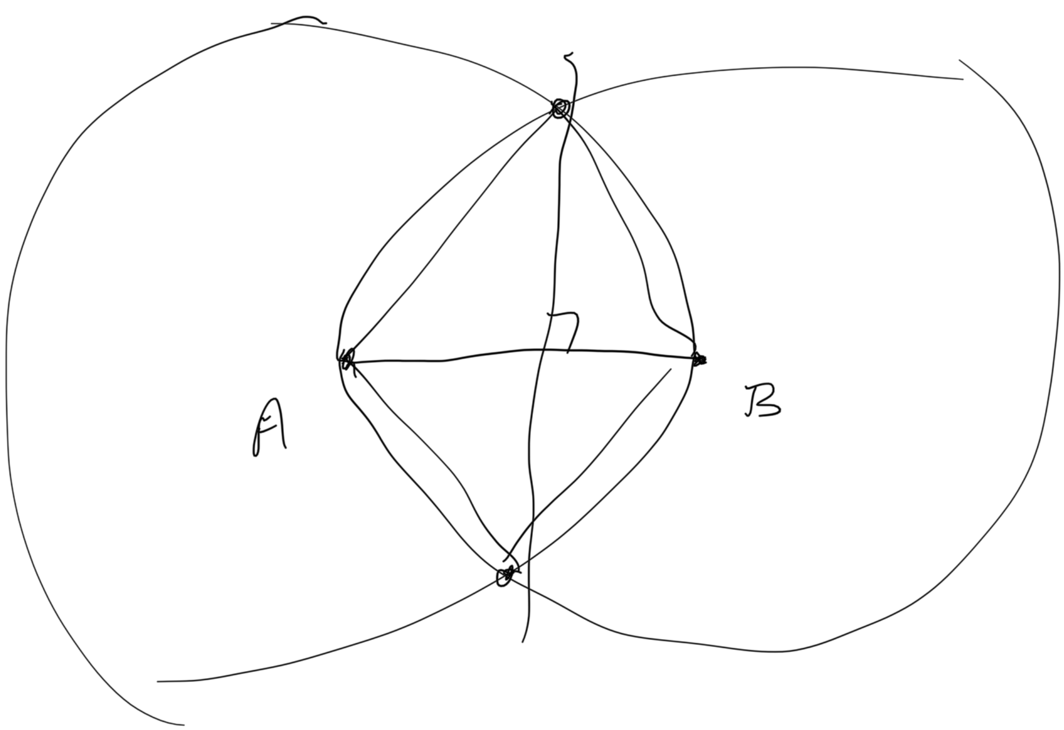
Proposition 3.4. *We can construct perpendicular bisectors.*

How can we do this?

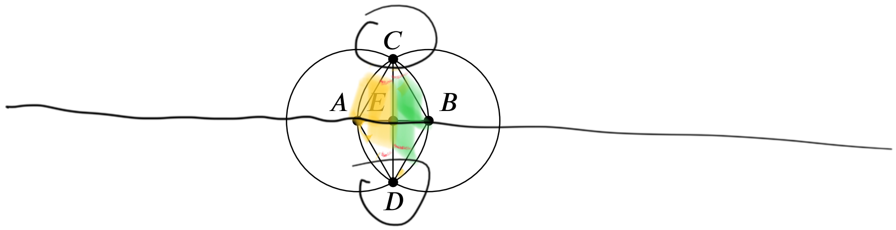
$\overline{AB}$

A, B

(Of any constructed segment, with constructed endpoints)



Proof. *How*: Let A, B be constructed points. $\mathscr{C}_A(B)$ and $\mathscr{C}_B(A)$ meet at C,D. Then the line through C and D (constructed using rule 1) is the perpendicular bisector of AB.



Why: As in Proposition 3.3, we get five congruent segments, and six 60° angles. By SSS, $\triangle ACD \cong \triangle BCD$, then the 4 angles ($\measuredangle$ ACE, $\measuredangle$ BCE, $\measuredangle$ ADE, $\measuredangle$ BDE) are congruent, then they are 30° each and AB $\perp$ CD. By SAS, $\triangle$ CAE $\cong$ $\triangle$ CBE and then |AE| = |BE|... (or just note that both AE and BE are heights of congruent triangles $\triangle$ ACD and $\triangle$ BCD.) ☐

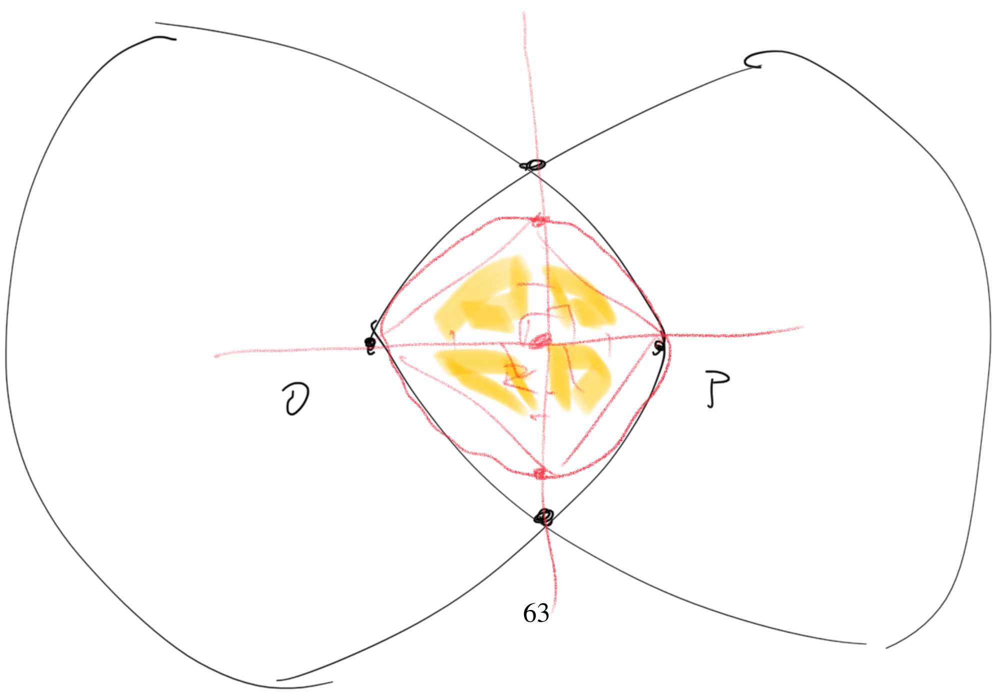
We get two *extra things for free* from this construction:

Remark 3.5. Note that in particular, we can construct midpoints of segments.

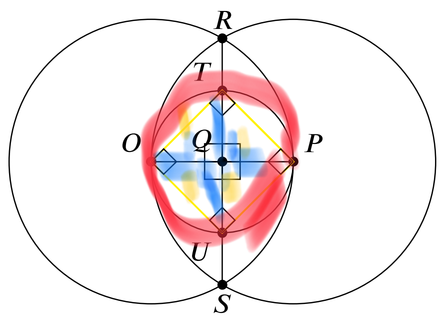
Remark 3.6. Note that we have shown that $AB \perp CD$, and $|CE| = |ED|$, so D is the reflection of C with respect to the line extending AB.

Proposition 3.7. *We can construct a square with OP as its diagonal.*

How can we do this?



Proof. *How*: We first construct the perpendicular bisector RS of OP as in Proposition 3.4. $\mathscr{C}_O(P)$, $\mathscr{C}_P(O)$ meet at two points R, S, we draw the line through them and we denote Q its intersection with OP. The circle $\mathscr{C}_Q(P) = \mathscr{C}_Q(O)$ meets RS at the points T and U. Then OTPU is a square.

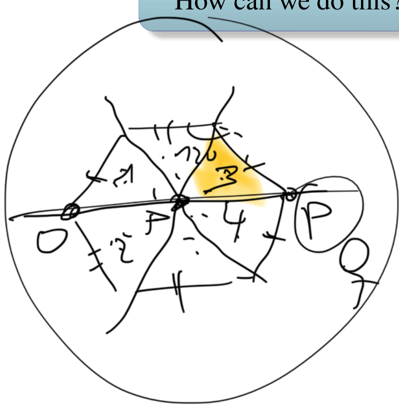


Why: Since O, P, T, and U are in $\mathscr{C}_Q(P)$, then $|OP| = |TQ| = |PQ| = |UP|$. Since RS is the perpendicular bisector of OP, then the 4 angles (∡OQT, ∡TQP, ∡PQU, ∡UQO) are right angles.

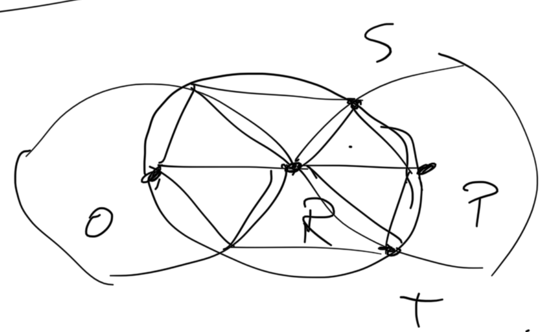
Then by SAS (or by the Pythagorean Theorem), we get that the 4 triangles forming OTPU are congruent, then so are the segments OT, TP, PU, UO. The 4 angles of OTPU are each 90° by Star Trek for diameters (Remark 2.5), or because each is split into two 45° angles. □

Proposition 3.8. *We can construct a regular hexagon.*

How can we do this?



How 1

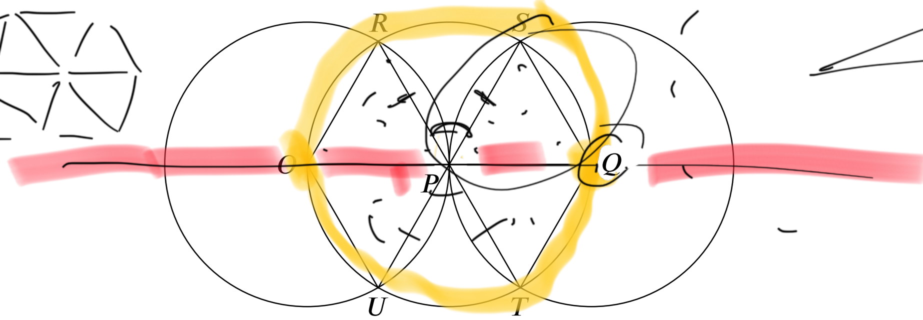


How 2

Proof. *How:* We draw the line through O and P and the circle $\mathscr{C}_P(O)$, they meet at Q. We then draw $\mathscr{C}_O(P)$ and $\mathscr{C}_Q(P)$, and denote the intersections of these two circles with $\mathscr{C}_P(O)$ by R,S,T, and U. We can connect RS and UT by rule 1 of constructions, and then ORSQTU is a regular hexagon.

a new point

n-gon

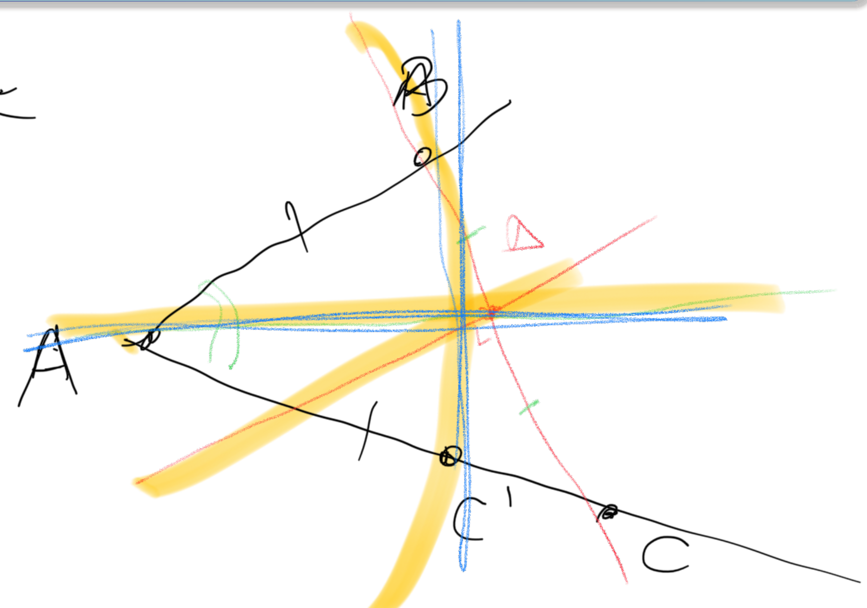


Why: Since we did the construction in Proposition 3.3 4 times, we have 4 equilateral triangles. The remaining two triangles ($\triangle PRS$ and $\triangle PUT$) are equilateral since they have two congruent sides and a 60° angle between them. We have then a regular hexagon formed by 6 equilateral triangles. ☐

Proposition 3.9. *We can bisect any angle*

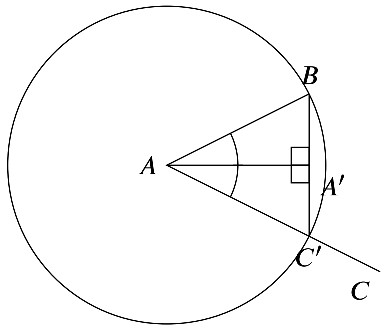
How can we do this?

$\measuredangle BAC$



H.W

Proof. <u>*How*</u>: Given $\measuredangle BAC$, first we draw $\mathscr{C}_A(B)$, which meets the ray $\overrightarrow{AC}$ at C'. The perpendicular bisector of BC' (that we can construct by Proposition 3.4) is the angle bisector of $\measuredangle BAC$.



<u>*Why*</u>: Since C' is in $\mathscr{C}_A(B)$, $|AB| = |AC'|$. So $\triangle ABC'$ is isosceles, and in exercise 2a) of Assignment 3 you have shown that then the median AA' is the perpendicular bisector of BC'. By SSS we get then $\triangle BAA'$ and $\triangle C'AA'$, so $\measuredangle BAA' = \measuredangle C'AA'$ as desired. □

Can we cut any angle into 4 equal parts? Yes, we just bisect the angle and then bisect the bisected angles. Can we cut any angle into 3 equal parts? This question is *way deeper* than it may look at first sight...

We will show now that the constructions we did when we showed how to construct the parallel line through a point (as in Playfair's version of the 5th axiom) can be done with a straightedge and a compass.

Proposition 3.10. *We can construct the reflection of any constructed point with respect to any constructed line.*

Proof. <u>*How*</u>: Given constructed points P, A, and B, we show how to construct the reflection of P with respect to the line ℓ through A. and B. Draw $\mathscr{C}_P(A)$ which meets ℓ at another point C. Then do the same construction we did for constructing the perpendicular bisector of $\overline{AC}$ in Proposition 3.9 (drawing the circles $\mathscr{C}_A(C)$ and $\mathscr{C}_C(A)$, that meet at P and Q). Then Q is the reflection of P with respect to ℓ.