The Grothendieck Construction for Double Categories

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Abstract

We develop a Grothendieck construction of a double category, for a double functor with domain an arbitrary double category, taking its values in double categories. We show two basic properties of this construction, that correspond respectively to the facts that the classical Grothendieck construction for categories is a lax colimit, and that it is the value on objects of a full and faithful 2-functor. We relate our construction to several others in the literature, and as an application we compute 2-colimits of categories.

1 Introduction

The Grothendieck construction, associating a cofibration $\mathbf{Gr}F$ to each pseudofunctor $F : \mathbf{C} \to \mathcal{C}\mathrm{at}$ into the 2-category of categories, is a basic tool in category theory for which standard references are [2, 14, 15, 17, 19]. Note that for convenience we consider here the case in which F is covariant, formally dual to the original construction. Two of its basic properties are:

A. $\mathbf{Gr}F$ is the lax colimit of F in $\mathcal{C}\mathrm{at}$.

B. $\mathbf{Gr}F$ is the value on objects of a 2-functor, which is an equivalence of 2-categories

$$\mathbf{Gr} \colon \mathbf{Hom}_p(\mathbf{C}, \mathcal{C}\mathrm{at}) \longrightarrow \mathsf{coFib}(\mathbf{C})$$

where the 2-category on the left has pseudonatural transformations as arrows, and the one on the right is the 2-category of cofibrations over $\mathbf{C}$.

A detailed textbook exposition can also be found in the more recent [18, §10], where §10.2 is devoted to showing the result in **A**, and §§10.3-10-6 to **B**. Note that the result **A** is independent of the notion of cofibration. Furthermore, the result **B** can be naturally split in two parts, one of which is also independent of this notion:

B.1. $\mathbf{Gr}F$ is the value on objects of a 2-functor

$$\mathbf{Gr} \colon \mathbf{Hom}_\ell(\mathbf{C}, \mathcal{C}\mathrm{at}) \longrightarrow \mathcal{C}\mathrm{at}/\mathbf{C}$$

where the 2-category on the left has lax transformations as arrows and the one on the right is a slice 2-category, which is full and faithful.

B.2. The assignation $F \mapsto \mathbf{Gr}F$ into $\mathsf{coFib}(\mathbf{C})$ is essentially surjective[1].

In this article we perform a Grothendieck construction for double categories, and we show some of its basic properties corresponding to **A** and **B.1** above. The fibration aspect of this construction will be discussed in a sequel.

Double categories, that can be defined as internal categories in $\mathcal{C}\mathrm{at}$, are a basic 2-dimensional categorical structure that go back to [8]. We recall in Section 2 the main definitions and constructions of the theory. A distinguishing feature of such a structure is that there are two different types of arrows between its objects, usually referred to as horizontal and vertical, and double cells relating them. There are double functors between double categories, defining a category DblCat, and transformations between them that come in two *flavors*, horizontal and vertical, defining two

[1] One can then deduce **B** by verifying that the bijection $\mathbf{Hom}_\ell(\mathbf{C}, \mathcal{C}\mathrm{at})(F, G) \longrightarrow \mathcal{C}\mathrm{at}/\mathbf{C}(\mathbf{Gr}F, \mathbf{Gr}G)$ in **B.1** restricts to a bijection $\mathbf{Hom}_p(\mathbf{C}, \mathcal{C}\mathrm{at}) \longrightarrow \mathsf{coFib}(\mathbf{C})(\mathbf{Gr}F, \mathbf{Gr}G)$, so that the 2-functor in **B** is also fully faithful.

2-categories $\mathcal{D}\mathbf{blCat}_h$ and $\mathcal{D}\mathbf{blCat}_v$. Furthermore, there is a DblCat-enriched category, $\mathcal{D}\mathbf{blCat}$, whose horizontal (resp. vertical) underlying 2-category is $\mathcal{D}\mathbf{blCat}_h$ (resp. $\mathcal{D}\mathbf{blCat}_v$). The double cells in the hom double categories of $\mathcal{D}\mathbf{blCat}$ are the modifications (see for example [11, 1.6]), that relate the horizontal and vertical transformations.

In our construction the category $\mathbf{C}$ above will be replaced by an arbitrary double category $\mathbb{D}$. Noting that there is no natural *double category of double categories*, the question arises of which structure should play the role of the 2-category $\mathcal{C}$at. Containing both horizontal and vertical aspects, $\mathcal{D}\mathbf{blCat}$ is a more complete structure, so we are led to consider it as the codomain of a morphism from a double category. We consider what a double functor F mapping the objects of $\mathbb{D}$ to double categories could be. Note that, since arrows in $\mathcal{D}\mathbf{blCat}$ have only one *type*, namely double functors, such an F should map both horizontal and vertical arrows of $\mathbb{D}$ to double functors. At the next level one is forced to make a choice, since double cells of $\mathbb{D}$ could be sent to either vertical or horizontal transformations. We choose the definition with vertical tranformations in this paper and the reader can easily switch the appropriate occurrences of 'horizontal' and 'vertical' in this paper to obtain the theory and results for double indexing functors in terms of horizontal transformations. We call such a morphism a vertical double functor $F\colon \mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$ (note that it amounts to a double functor $F\colon \mathbb{D} \to \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ into a quintet double category). For such an F, in Section 3 we construct a double category $\mathbb{G}\mathrm{r}F$. We show in Example 3.12 how several constructions in the literature can be seen as instances of ours. The following results, whose content we explain below, correspond respectively to the results **A** and **B.1** above:

Theorem 5.1. Given a vertical double functor $F\colon \mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$, the double Grothendieck construction $\mathbb{G}\mathrm{r}F$ is the *doubly lax colimit* of F in $\mathcal{D}\mathbf{blCat}$.

Theorem 6.5. The double Grothendieck construction is the value on objects of a DblCat-functor

$$\mathbb{G}r\colon \mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{D}\mathbf{blCat})_{d\ell} \longrightarrow \mathcal{D}\mathbf{blCat}/\mathbb{D},$$

which is locally an isomorphism of double categories $\mathbb{H}\mathsf{om}_{d\ell}(F,G) \longrightarrow (\mathcal{D}\mathbf{blCat}/\mathbb{D})(\mathbb{G}rF, \mathbb{G}rG)$.

The notion of doubly lax colimit appearing above is introduced in Section 4. It is at the same time more general and more convenient for computations to consider here an arbitrary DblCat-category $\mathcal{C}$ to play the role of $\mathcal{D}\mathbf{blCat}$ above. In Section 4.1 we define a double category $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$, whose double cells are the (vertical) double cylinders. Double cylinders are given by six 2-cells forming a cube, and their horizontal and vertical composition is given by their *pasting* in a shared face. The basic idea of fitting arrows and 2-cells in the form of cylinders, and of composing them by pasting, as far as we know first appeared in [1], where a bicategory of cylinders is constructed for a bicategory $\mathcal{B}$, which is a strict 2-category when so is $\mathcal{B}$. The cylinders in the present paper extend these *in a third direction*, and this is from where they get their name. We present the structure in 3-dimensional pictures to visualize the computations (see for example (4.3)).

Vertical double functors can be made to fit in the *top* and *bottom* faces of the double cylinders, and so (analogously to the case of bicategories and their morphisms [1]), double cylinders lead to the definition of a *doubly lax transformation* between vertical double functors (see Definition 4.11 for details). Unlike previously considered lax transformations, we get *laxness* in both the horizontal and vertical directions. A doubly lax transformation $F \Longrightarrow G$ will be given by a double functor $\mathbb{D} \longrightarrow \mathbb{C}\mathsf{yl}_v(\mathcal{C})$, and in this way the double category $\mathbb{H}\mathsf{om}_{d\ell}(F,G)$ appearing in Theorem 6.5 above can be defined, with doubly lax transformations, interpreted as double functors, as objects, horizontal transformations as horizontal arrows, vertical transformations as vertical arrows, and modifications as double cells. This, in turn, leads to the definition of doubly lax (co)limit, when either F or G are taken to be constant double functors. The universal property of such a limit is given by an isomorphism of double categories (like lax limits in 2-categories involve an isomorphism of categories, see for example [20, §5]).

In Section 5 (and in Appendix A) we show that this is the universal property characterizing the double category $\mathbb{G}rF$, that is, we show Theorem 5.1 above. Note that, even though the vertical double functor $F\colon \mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$ can also be seen as a double functor $F\colon \mathbb{D} \longrightarrow \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$, the full structure of $\mathcal{D}\mathbf{blCat}$, including the horizontal transformations, appears in this universal property. As an application of Theorem 5.1, in Section 5.1 we compute 2-colimits of categories, by mimicking in the context of double categories the way in which the lax colimit result **A** above is used to compute the colimit of a set-valued functor as the connected components of its category of elements.

In Section 6, we show first that the double category $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$ is in fact part of a *richer* structure formed by the double cylinders, namely that of a category object (internal category) $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$ in the category DblCat. The composition in this internal category is given by pasting the double cylinders in the third dimension, which may also be viewed as an extension of the original construction in [1]. A double category $\mathbb{D}$ can be thought of as a discrete category object $\mathbb{D}_{\mathrm{disc}}$ in DblCat, vertical double functors $\mathbb{D} \xrightarrow{v} \mathcal{C}$ amount to internal functors from $\mathbb{D}_{\mathrm{disc}}$ to $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$, and doubly lax transformations amount to internal natural transformations. In this way the DblCat-enriched category $\mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{C})_{d\ell}$ appearing in Theorem 6.5 above is defined, in which objects are vertical double functors $\mathbb{D} \xrightarrow{v} \mathcal{C}$ and $\mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{C})_{d\ell}(F, G) = \mathbb{H}\mathsf{om}_{d\ell}(F, G)$.

After giving these definitions, we show how the universal property of $\mathbb{G}rF$ given in Theorem 5.1 can be used to prove Theorem 6.5 avoiding most of the computations needed for a direct proof. This technique that we discovered, of applying the lax colimit property **A** to show the higher isomorphism property **B.1** of the Grothendieck construction, does not seem to have appeared before in the literature on the subject.

2 Preliminaries on Double Categories

2.1 Notation

We use various types of higher categorical structures and will as much as possible distinguish between them by using different types of type-faces. 2-Categories will be denoted by boldface upper case letters: **A**, **B**, **C**, etc. Double categories will be denoted by blackboard bold upper case letters: $\mathbb{B}$, $\mathbb{C}$, $\mathbb{D}$, etc. We will be considering two types of 3-dimensional structures: (weak) 3-categories and categories enriched in double categories. We will denote both by calligraphic upper case letters: $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$, etc.

We will frequently use the following functors between the category DblCat of double categories and the category of 2-categories. The functor $\mathbf{V}$ assigns to each double category $\mathbb{D}$ the 2-category $\mathbf{V}\mathbb{D}$ of objects, vertical arrows, and vertically globular cells of $\mathbb{D}$, also denoted as the *vertical 2-category of* $\mathbb{D}$. Similarly, $\mathbf{H}\mathbb{D}$ is the horizontal 2-category of $\mathbb{D}$. In the opposite direction, a 2-category gives rise to three double categories that will play a role in this work: $\mathbb{V}\mathbf{A}$, the double category with the same objects as $\mathbf{A}$, only identity horizontal arrows, the arrows of $\mathbf{A}$ as vertical arrows, and the cells of $\mathbf{A}$ as vertically globular double cells. This is also called the *vertical double category of a 2-category*. Dually, $\mathbb{H}\mathbf{A}$ is the horizontal double category of a 2-category $\mathbf{A}$. Whenever we apply the functors $\mathbb{Q}$, $\mathbb{H}$ or $\mathbb{V}$ to a mere category, rather than a 2-category, we will consider the category as a 2-category with only identity 2-cells, as usual. In particular, when $\mathbf{A}$ is a category, $\mathbb{Q}\mathbf{A}$ is the double category with commutative squares in $\mathbf{A}$ as double cells.

We denote the horizontal composition in a double category by $\circ$, and the vertical composition by $\bullet$. We decorate the vertical arrows of a double category with a $\bullet$ symbol. We denote horizontal identity arrows by $1_A \colon A \longrightarrow A$ and vertical identity arrows by $1^\bullet_A \colon A \longrightarrow\!\!\!\!\!\!\bullet\;\; A$. Given a horizontal arrow f, resp. vertical arrow v we denote its identity double cell by $1^\bullet_f$, resp. 1_v. We denote the double cell $1_{1^\bullet_A} = 1^\bullet_{1_A}$ by $1^\square_A$.

Since DblCat-enriched categories play a central role in this paper we will explicitly introduce the ways in which we will refer to the arrows and cells in these categories. Let $\mathcal{C}$ be a DblCat-enriched category. So for each pair of objects $A, B \in \mathcal{C}$, $\mathcal{C}(A, B)$ is a double category. The objects of these double categories will be viewed as the *arrows* of $\mathcal{C}$, the horizontal and vertical arrows in these hom double categories will be viewed as the *horizontal* and *vertical* (respectively) *2-cells* of $\mathcal{C}$ and the double cells of the hom double categories will be viewed as the *triple cells* of $\mathcal{C}$.

2.2 Transformations for Double Functors

The notion of double functor is straightforward to define. However, for transformations between double functors we have two options.

Definition 2.1. Let $F, F' : \mathbb{A} \to \mathbb{B}$ be double functors between double categories $\mathbb{A}$ and $\mathbb{B}$. A *vertical transformation* $F \overset{U}{\bullet\!\!\Longrightarrow} F'$ assigns to each object $A \in \mathbb{A}$ a vertical arrow $FA \overset{U_A}{\bullet\!\!\longrightarrow} F'A$, and to each horizontal arrow $A \xrightarrow{f} B$ in $\mathbb{A}$ a double cell $U_f : \left(U_A \ \substack{Ff \\ F'f}\ U_B\right)$ in $\mathbb{B}$, such that

(v.1) for each object A in $\mathbb{A}$ we have $U_{1_A} = 1_{U_A}$,

(v.2) for horizontal arrows $A \xrightarrow{f} B \xrightarrow{g} C$ we have $U_g \circ U_f = U_{g \circ f}$,

(v.3) for each double cell $\Phi : \left(u \ \substack{f \\ f'}\ v\right)$, we have $\frac{F\Phi}{U_{f'}} = \frac{U_f}{F'\Phi}$.

The dual notion of *horizontal transformation* $F \overset{H}{\Longrightarrow} G$ is given by $FA \xrightarrow{H_A} GA$ for each A, $H_u : \left(Fu \ \substack{HA \\ HA'}\ Gu\right)$ for each $A \overset{u}{\bullet\!\!\longrightarrow} A'$, satisfying dual axioms (h.1)-(h.3). The details are spelled out in [11, 1.4].

Remark 2.2. It follows from these axioms that vertical transformations are *natural* in the vertical direction, see for example [11, Remark 1.5]. In particular, we have:

(v.*) for each vertical arrow $A \overset{u}{\bullet\!\!\longrightarrow} A'$, $\alpha_{A'} \bullet Fu = Gu \bullet \alpha_A$

We have thus a 2-category $\mathcal{D}\mathbf{blCat}_v$ of double categories, double functors and vertical transformations, and respectively a 2-category $\mathcal{D}\mathbf{blCat}_h$ with horizontal transformations instead. They can be combined in a 3-dimensional structure as follows.

For $\mathbb{A}, \mathbb{B}$ double categories, there is an exponential double category $\mathbb{B}^{\mathbb{A}}$ (see for example [11, 1.6]) consisting of double functors, horizontal and vertical transformations, and modifications

$$\begin{tikzcd} F \arrow[r, Rightarrow, "H"] \arrow[d, Rightarrow, "U"'] \arrow[dr, phantom, "M"] & G \arrow[d, Rightarrow, "V"] \\ F' \arrow[r, Rightarrow, "H'"'] & G' \end{tikzcd}$$

as double cells, which are given by double cells $M_A : \left(U_A \ \substack{H_A \\ H'_A}\ V_A\right)$ of $\mathbb{B}$, one for each object A of $\mathbb{D}$, satisfying the axioms

(m.1) $(M_A \mid V_f) = (U_f \mid M_B)$ for each $A \xrightarrow{f} B$,

(m.2) a symmetric condition for each $A \overset{u}{\bullet\!\!\longrightarrow} A'$.

In this way, double categories form a category enriched in DblCat (i.e. DblCat-category), which we denote by $\mathcal{D}\mathbf{blCat}$, $\mathcal{D}\mathbf{blCat}(\mathbb{A}, \mathbb{B}) = \mathbb{B}^{\mathbb{A}}$. Note that this is now a 3-dimensional structure in which we have only one type of arrow, horizontal and vertical 2-cells between them, and triple cells corresponding to modifications filling the local squares. We consider all compositions to be strict, since this is enough for our purposes in the present article, but we note that a *pseudo* case of this structure, with which we will not deal here, is defined in [9] and called a *locally cubical bicategory*. We denote the composition $\mathcal{D}\mathbf{blCat}(\mathbb{B}, \mathbb{C}) \times \mathcal{D}\mathbf{blCat}(\mathbb{A}, \mathbb{B}) \longrightarrow \mathcal{D}\mathbf{blCat}(\mathbb{A}, \mathbb{C})$ by juxtaposition.

As mentioned in the introduction, $\mathcal{D}\mathbf{blCat}$ is our example *to have in mind* as we will work with an arbitrary DblCat-category $\mathcal{C}$.

Notation 2.3. For objects A, B, C in any DblCat-enriched category $\mathcal{C}$, the composition $\mathcal{C}(B, C) \times \mathcal{C}(A, B) \longrightarrow \mathcal{C}(A, C)$ induces a 2-functor $\mathbf{V}\mathcal{C}(B, C) \times \mathbf{V}\mathcal{C}(A, B) \longrightarrow \mathbf{V}\mathcal{C}(A, C)$. In this way a 3-category $\mathcal{C}_v$ of objects, vertical arrows, vertical globular cells and vertical globular modifications is defined: the *vertical part* of $\mathcal{C}$. We will denote its underlying 2-category also by $\mathcal{C}_v$. The 2-category $\mathcal{D}\mathbf{blCat}_v$ is recovered in this way from $\mathcal{D}\mathbf{blCat}$, and dually for $\mathcal{D}\mathbf{blCat}_h$.

Let D be an object in a DblCat-enriched category $\mathcal{C}$ (we will only need the case $\mathcal{C} = \mathcal{D}\mathbf{blCat}$ in this paper). We can easily generalize the notion of “slice category” or “over category” to this context as follows:

Definition 2.4. For a double category $\mathbb{D}$, we have a DblCat-enriched category $\mathcal{D}\mathbf{blCat}/\mathbb{D}$, whose objects are double functors $\mathbb{E} \xrightarrow{F} \mathbb{D}$, and whose Hom double categories are defined as follows: for another double functor $\mathbb{E}' \xrightarrow{G} \mathbb{D}$, $(\mathcal{D}\mathbf{blCat}/\mathbb{D})(\mathbb{E} \xrightarrow{F} \mathbb{D}, \mathbb{E}' \xrightarrow{G} \mathbb{D})$ is the sub-double category of $\mathcal{D}\mathbf{blCat}(\mathbb{E}, \mathbb{E}')$ whose

1. Objects are given by commutative triangles, that is double functors $\mathbb{E} \xrightarrow{\eta} \mathbb{E}'$ such that $G\eta = F$.
2. Vertical arrows $\eta \overset{R}{\bullet\!\!\!\longrightarrow} \eta'$ are given by vertical transformations $\eta \overset{R}{\bullet\!\!\!\Longrightarrow} \eta'$ such that $GR = 1^{\bullet}_F$
3. Dually, horizontal arrows $\eta \xrightarrow{K} \zeta$ are given by vertical transformations $\eta \overset{K}{\Longrightarrow} \zeta$ such that $GK = 1_F$
4. Double cells N are given by modifications N such that $GN = 1^{\square}_F$

3 The Grothendieck Construction for Double Categories

We consider now, as in the introduction of the paper, a double functor $F \colon \mathbb{D} \longrightarrow \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ and construct a double category that we will refer to as the Grothendieck Construction of F.

Definition 3.1. Let $\mathbb{D}$ be a double category, and $\mathbb{D} \xrightarrow{F} \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ be a double functor. We construct the double category $\mathbb{G}rF = \int_{\mathbb{D}} F$ as follows:

- Objects are pairs (A, x) where A is an object in $\mathbb{D}$ and x is an object in FA,
- Vertical arrows are $(A, x) \overset{(u,\rho)}{\bullet\!\!\!\longrightarrow} (A', x')$, where $A \overset{u}{\bullet\!\!\!\longrightarrow} A'$ is a vertical arrow in $\mathbb{D}$, and $Fux \overset{\rho}{\bullet\!\!\!\longrightarrow} x'$ is a vertical arrow in FA'.
- Horizontal arrows are $(A, x) \xrightarrow{(f,\varphi)} (B, y)$, where $A \xrightarrow{f} B$ is a horizontal arrow in $\mathbb{D}$, and $Ffx \xrightarrow{\varphi} y$ is a horizontal arrow in FA.
- Double cells are
$$\begin{tikzcd}
(A,x) \arrow[r, "{(f,\varphi)}"] \arrow[d, "{(u,\rho)}"', "\bullet" marking] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (B,y) \arrow[d, "{(v,\lambda)}", "\bullet" marking] \\
(A',x') \arrow[r, "{(f',\varphi')}"'] & (B',y')
\end{tikzcd}$$
, where $\alpha : (u \;{}^{f}_{f'}\; v)$ is a double cell in $\mathbb{D}$ and Φ is a double cell in FB':
$$\begin{tikzcd}
FvFfx \arrow[r, "Fv\varphi"] \arrow[d, "(F\alpha)_x"', "\bullet" marking] & Fvy \arrow[dd, "\lambda", "\bullet" marking] \\
Ff'Fux \arrow[d, "Ff'\rho"', "\bullet" marking] \arrow[r, phantom, "\Phi"] & {} \\
Ff'x' \arrow[r, "\varphi'"'] & y'
\end{tikzcd} \tag{3.2}$$

-Vertical composition: Given vertical arrows $(A, x) \overset{(u,\rho)}{\bullet\!\!\!\longrightarrow} (A', x') \overset{(u',\rho')}{\bullet\!\!\!\longrightarrow} (A'', x'')$, their composition is $(A, x) \overset{(u' \bullet u, \rho' \bullet Fu'\rho)}{\bullet\!\!\!\longrightarrow} (A'', x'')$. We define the vertical composition of double cells as follows:

$$
\begin{tikzcd}
(A,x) \arrow[r, "(f,\varphi)"] \arrow[d, "(u,\rho)"', "\bullet" marking] \arrow[dr, phantom, "(\alpha,\Phi)"] & (B,y) \arrow[d, "(v,\lambda)", "\bullet" marking] \\
(A',x') \arrow[r, "(f',\varphi')"] \arrow[d, "(u',\rho')"', "\bullet" marking] \arrow[dr, phantom, "(\alpha',\Phi')"] & (B',y') \arrow[d, "(v',\lambda')", "\bullet" marking] \\
(A'',x'') \arrow[r, "(f'',\varphi'')"'] & (B'',y'')
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
(A,x) \arrow[rr, "(f,\varphi)"] \arrow[d, "(u'\bullet u,\rho'\bullet Fu'\rho)"', "\bullet" marking] & {} \arrow[d, phantom, "(\alpha'\bullet\alpha,\Phi'\bullet Fv'\Phi)"] & (B,y) \arrow[d, "(v'\bullet v,\lambda'\bullet Fv'\lambda)", "\bullet" marking] \\
(A'',x'') \arrow[rr, "(f'',\varphi'')"'] & {} & (B'',y''),
\end{tikzcd}
$$

where the double cell $\Phi' \bullet Fv'\Phi$ is explicitly computed as the pasting

$$
\begin{tikzcd}
Fv'FvFfx \arrow[r, "Fv'Fv\varphi"] \arrow[d, "Fv'(F\alpha)_x"', "\bullet" marking] & Fv'Fvy \arrow[dd, "Fv'\lambda", "\bullet" marking] \\
Fv'Ff'Fux \arrow[d, "Fv'Ff'\rho"', "\bullet" marking] \arrow[r, phantom, "Fv'\Phi"] & {} \\
Fv'Ff'x' \arrow[r, "Fv'\varphi'"] \arrow[d, "(F\alpha')_{x'}"', "\bullet" marking] & Fv'y' \arrow[dd, "\lambda'", "\bullet" marking] \\
Ff''Fu'x' \arrow[d, "Ff''\rho'"', "\bullet" marking] \arrow[r, phantom, "\Phi'"] & {} \\
Ff''x'' \arrow[r, "\varphi''"'] & y''
\end{tikzcd}
$$

Note that, by the definition of the double cells of $\mathbb{G}rF$, as given in (3.2), the vertical arrow on the left of this double cell should equal the composition

$$
Fv'FvFfx \xrightarrow{Fv'(F\alpha)_x} Fv'Ff'Fux \xrightarrow{(F\alpha')_{Fux}} Ff''Fu'Fux \xrightarrow{Ff''Fu'\rho} Ff''Fu'x' \xrightarrow{Ff''\rho'} Ff''x''
$$

This is so by (v.*) in Remark 2.2, applied to the vertical natural transformation $F\alpha'$ and the vertical arrow ρ.

- Horizontal composition: Given horizontal arrows $(A,x) \xrightarrow{(f,\varphi)} (B,y) \xrightarrow{(g,\psi)} (C,z)$, their composition is $(A,x) \xrightarrow{(g\circ f,\psi\circ Fg\varphi)} (C,z)$. We define the horizontal composition of double cells as follows:

$$
\begin{tikzcd}
(A,x) \arrow[r, "(f,\varphi)"] \arrow[d, "(u,\rho)"', "\bullet" marking] \arrow[dr, phantom, "(\alpha,\Phi)"] & (B,y) \arrow[r, "(g,\psi)"] \arrow[d, "(v,\lambda)", "\bullet" marking] \arrow[dr, phantom, "(\beta,\Psi)"] & (C,z) \arrow[d, "(w,\mu)", "\bullet" marking] \\
(A',x') \arrow[r, "(f',\varphi')"'] & (B',y') \arrow[r, "(g',\psi')"'] & (C',z')
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
(A,x) \arrow[rr, "(g\circ f,\psi\circ Fg\varphi)"] \arrow[d, "(u,\rho)"', "\bullet" marking] & {} \arrow[d, phantom, "(\beta\circ\alpha,\Lambda)"] & (C,z) \arrow[d, "(w,\mu)", "\bullet" marking] \\
(A',x') \arrow[rr, "(g'\circ f',\psi'\circ Fg'\varphi')"'] & {} & (C',z')
\end{tikzcd}
$$

where Λ is the double cell $\left(\frac{(F\beta)_\varphi}{Fg'\Phi} \,\middle|\, \Psi\right)$ in FC', written explicitly as

$$
\begin{tikzcd}
FwFgFfx \arrow[r, "FwFg\varphi"] \arrow[d, "(F\beta)_{Ffx}"'] \arrow[r, phantom, shift right=8ex, "(F\beta)_\varphi"] & FwFgy \arrow[r, "Fw\psi"] \arrow[d, "(F\beta)_y"] & Fwz \arrow[ddd, "\mu"] \\
Fg'FvFfx \arrow[r, "Fg'Fv\varphi"] \arrow[d, "Fg'(F\alpha)_x"'] & Fg'Fvy \arrow[dd, "Fg'\lambda"] \arrow[r, phantom, "\Psi"] & \\
Fg'Ff'Fux \arrow[d, "Fg'Ff'\rho"'] \arrow[r, phantom, "Fg'\Phi"] & & \\
Fg'Ff'x' \arrow[r, "Fg'\varphi'"'] & Fg'y' \arrow[r, "\psi'"'] & z'
\end{tikzcd}
\tag{3.3}
$$

It can be checked that both compositions are associative and have units given by those of $\mathbb{D}$. To show middle four interchange, consider double cells fitting as:

$$
\begin{tikzcd}
(A,x) \arrow[r, "{(f,\varphi)}"] \arrow[d, "{(u,\rho)}"'] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (B,y) \arrow[r, "{(g,\psi)}"] \arrow[d, "{(v,\lambda)}"] \arrow[dr, phantom, "{(\beta,\Psi)}"] & (C,z) \arrow[d, "{(w,\mu)}"] \\
(A',x') \arrow[r, "{(f',\varphi')}"'] \arrow[d, "{(u',\rho')}"'] \arrow[dr, phantom, "{(\alpha',\Phi')}"] & (B',y') \arrow[r, "{(g',\psi')}"'] \arrow[d, "{(v',\lambda')}"] \arrow[dr, phantom, "{(\beta',\Psi')}"] & (C',z') \arrow[d, "{(w',\mu')}"] \\
(A'',x'') \arrow[r, "{(f'',\varphi'')}"'] & (B'',y'') \arrow[r, "{(g'',\psi'')}"'] & (C'',z'')
\end{tikzcd}
$$

We compute:

$$((\beta',\Psi')\bullet(\beta,\Psi))\circ((\alpha',\Phi')\bullet(\alpha,\Phi)) = (\beta'\bullet\beta, \Psi'\bullet Fw'\Psi)\circ(\alpha'\bullet\alpha, \Phi'\bullet Fv'\Phi) = ((\beta'\bullet\beta)\circ(\alpha'\bullet\alpha), \text{A}),$$

$$
\begin{aligned}
((\beta',\Psi')\circ(\alpha',\Phi'))\bullet((\beta,\Psi)\circ(\alpha,\Phi)) &= ((\beta'\circ\alpha', \big(\frac{F(\beta')_{\varphi'}}{F(g'')(\Phi')}\ \Big|\ \Psi'\big))\bullet((\beta\circ\alpha, \big(\frac{F(\beta)_{\varphi}}{F(g')(\Phi)}\ \Big|\ \Psi\big))) \\
&= ((\beta'\circ\alpha')\bullet(\beta\circ\alpha), \text{B}),
\end{aligned}
$$

where A and B are the double cells (for readability we omit the labels on objects and arrows):

$$
\text{A}:\
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(w')F(\beta)_\varphi"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(\beta')_{F(v)(\varphi)}"] & \cdot \arrow[d] \arrow[r, phantom, "F(w')(\Psi)"] & {} \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(g'')F(v')(\Phi)"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(g'')(\Phi')"] & \cdot \arrow[d] \arrow[r, phantom, "\Psi'"] & {} \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
\quad , \quad
\text{B}:\
\begin{tikzcd}
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(w')F(\beta)_\varphi"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(w')F(g')(\Phi)"] & \cdot \arrow[d] \arrow[r, phantom, "F(w')(\Psi)"] & {} \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(\beta')_{\varphi'}"] & \cdot \arrow[r] \arrow[d] & \cdot \arrow[dd] \\
\cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "F(g'')(\Phi')"] & \cdot \arrow[d] \arrow[r, phantom, "\Psi'"] & {} \\
\cdot \arrow[r] & \cdot \arrow[r] & \cdot
\end{tikzcd}
$$

Clearly, it suffices to show that the shaded regions are equal, which is precisely the content of Axiom (v.3) in Definition 2.1, applied to the vertical transformation $F(\beta)$ and the double cell Φ.

Example 3.4. We have the double functor $\Delta\mathbb{E}\colon \mathbb{D} \longrightarrow \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ constant at $\mathbb{E}$, mapping all the objects of $\mathbb{D}$ to $\mathbb{E}$ and all the arrows and double cells of $\mathbb{D}$ to identities. Then it's immediate to check that $\mathbb{G}r(\Delta\mathbb{E}) = \mathbb{D} \times \mathbb{E}$, that is the Cartesian product of double categories.

Remark 3.5. There is a projection double functor $\int_{\mathbb{D}} F \longrightarrow \mathbb{D}$ and the double categories FA can be recovered, as for fibrations of categories, as the preimages of A, 1_A, $1^{\bullet}_A$, and $1^{\square}_A$.

Remark 3.6. Any horizontal arrow (f, φ) as above can be factored as the composition

$$(A, x) \xrightarrow{(f,1_{Ffx})} (B, Ffx) \xrightarrow{(1_B,\varphi)} (B, y). \tag{3.7}$$

Dually, any vertical arrow (u, ρ) can be factored as

$$(A, x) \xrightarrow{(u,1^{\bullet}_{Fux})} (A', Fux) \xrightarrow{(1^{\bullet}_{A'},\rho)} (A', x') \tag{3.8}$$

And any double cell (α, Φ) as above can be factored as the composition

$$\begin{tikzcd}
(A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[dd, "{(u,1^{\bullet}_{Fux})}"'] & (B,Ffx) \arrow[r, "{(1_B,\varphi)}"] \arrow[d, "{(v,1^{\bullet}_{F(vf)x})}"'] \arrow[dr, phantom, "{(1_v,1^{\bullet}_{Fv\varphi})}"] & (B,y) \arrow[d, "{(v,1^{\bullet}_{Fvy})}"] \\
\arrow[r, phantom, "{(\alpha,1_{(F\alpha)_x})}"] & (B',FvFfx) \arrow[r, "{(1_{B'},Fv\varphi)}"'] \arrow[d, "{(1^{\bullet}_{B'},(F\alpha)_x)}"] & (B',Fvy) \arrow[dd, "{(1^{\bullet}_{B'},\lambda)}"] \\
(A',Fux) \arrow[r, "{(f',1_{F(f'u)x})}"] \arrow[d, "{(1^{\bullet}_{A'},\rho)}"'] \arrow[dr, phantom, "{(1^{\bullet}_{f'},1_{Ff'\rho})}"] & (B',Ff'Fux) \arrow[d, "{(1^{\bullet}_{B'},Ff'\rho)}"] \arrow[r, phantom, "{(1^{\square}_{B'},\Phi)}"] & \\
(A',x') \arrow[r, "{(f',1_{Ff'x'})}"'] & (B',Ff'x') \arrow[r, "{(1_{B'},\varphi')}"'] & (B',y')
\end{tikzcd} \tag{3.9}$$

We will use this factorization in the proof of Theorem 5.1 below, see (5.4). We note that this is the double categorical version of the usual Cartesian-vertical factorization for the theory of fibred categories.

Notation 3.10. We will refer to the double functors $\mathbb{D} \longrightarrow \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ as *vertical double functors* $\mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$ (see also Definition 4.8). Note that it looks as if at this point we have lost the horizontal data of $\mathcal{D}\mathbf{blCat}$. We will regain use of the horizontal structure in the notion of *doubly lax transformation* between such vertical double functors (see Definition 4.11). This notion of lax transformation leads in turn to notions of lax cone and cocone, when one of the two vertical functors is constant, and we will show in Section 5 that $\int_{\mathbb{D}} F$ is the nadir of a universal lax cocone of the diagram given by F, that is, its doubly lax colimit. This result involves the whole structure of $\mathcal{D}\mathbf{blCat}$, including the horizontal transformations and the modifications.

There are several constructions in the literature of 2-categories and double categories, that are associated to indexed (2-)categories, and that are related to particular cases of the construction above. In Example 3.12 we give five examples of such constructions; the original constructions can be found in [4], [12], and [22] (for a summary of the results, see equations (3.13) to (3.17)). Before we start we recall one more piece of notation.

Notation 3.11. Recall (see for example [21, p.297]) that, given a 2-category $\mathcal{X}$ and a specified family Σ of arrows, closed under composition and identities, we write $\mathbb{Q}^{\Sigma}\mathcal{X}$ for the vertically full sub-double category of $\mathbb{Q}\mathcal{X}$ in which the horizontal arrows are required to be in Σ.

Example 3.12. Recall that given a 2-functor $F\colon \mathbf{A} \longrightarrow \mathbf{2\text{-}Cat}$ into the 2-category of 2-categories, 2-functors, and 2-natural transformations, [4, 2.2.1] introduces a 2-categorical Grothendieck construction that we will denote by $\int_{\mathbf{A}} F$, see also [6]. We obtain several double categorical extensions of this 2-category in the following ways:

1. If we compose F with the 2-functor $\mathbf{2\text{-}Cat} \xrightarrow{\mathbb{V}} \mathcal{D}\mathbf{blCat}_v$ we get $\mathbf{A} \xrightarrow{\mathbb{V}\circ F} \mathcal{D}\mathbf{blCat}_v$, and then applying $\mathbb{Q}$ we obtain a double functor $\mathbb{Q}(\mathbb{V}\circ F)\colon \mathbb{Q}\mathbf{A} \to \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$. The construction in Definition 3.1 can then be applied to this double functor, and it yields a double category $\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ that we want to compare with $\int_{\mathbf{A}} F$. It is immediate to check that $\mathbf{V}\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F) = \int_{\mathbf{A}} F$ and that the horizontal arrows of $\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ are those of $\int_{\mathbf{A}} F$ of the form (f, id) (that is, the family *Cart* of chosen coCartesian arrows of the cofibration corresponding to F). Double cells of $\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ correspond to 2-cells of $\int_{\mathbf{A}} F$ between the appropriate compositions of 1-cells, so we conclude that

$$\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F) = \mathbb{Q}^{Cart}\int_{\mathbf{A}} F, \tag{3.13}$$

   Without getting into the details, let us mention that for the case in which F is a *weight*; that is, a 2-functor $\mathbf{A} \xrightarrow{W} \mathbf{Cat}$, the family *Cart* is relevant regarding W-weighted 2-dimensional limits [7].

2. If we want to recover $\int_{\mathbf{A}} F$ without specifying its Cartesian arrows, we can restrict the double functor $\mathbb{Q}(\mathbb{V}\circ F)$ from (1) to $\mathbb{V}\mathbf{A}$, as in (3.14) below. In this case the horizontal arrows of $\int_{\mathbb{V}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F)$ are only identities and so this double category equals $\mathbb{V}\int_{\mathbf{A}} F$. Noting that the diagram of double functors on the left below is commutative, where the two arrows going downwards are inclusions, we conclude the equalities on the right:

$$\begin{tikzcd} \mathbb{V}\mathbf{A} \arrow[r, "\mathbb{V}(\mathbb{V}\circ F)"] \arrow[d] & \mathbb{V}\mathcal{D}\mathbf{blCat}_v \arrow[d] \\ \mathbb{Q}\mathbf{A} \arrow[r, "\mathbb{Q}(\mathbb{V}\circ F)"] & \mathbb{Q}\mathcal{D}\mathbf{blCat}_v \end{tikzcd} \qquad \int_{\mathbb{V}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F) = \int_{\mathbb{V}\mathbf{A}} \mathbb{V}(\mathbb{V}\circ F) = \mathbb{V}\int_{\mathbf{A}} F. \tag{3.14}$$

3. If instead we restrict $\mathbb{Q}(\mathbb{V}\circ F)$ to $\mathbb{H}\mathbf{A}$, our construction yields a double category which is also a relevant construction for F: in this case objects are pairs (C, x) as usual, vertical arrows are of the form (id, ρ), horizontal arrows are of the form (f, id) and double cells are of the form

$$\begin{tikzcd} (C,x) \arrow[r, "{(f,id)}"] \arrow[d, "{(id,\rho)}"'] \arrow[dr, phantom, "{(\alpha,\Phi)}"] & (D,y) \arrow[d, "{(id,\lambda)}"] \\ (C,x') \arrow[r, "{(f',id)}"'] & (D,y') \end{tikzcd}$$

   where $f \overset{\alpha}{\Longrightarrow} f'$ is a 2-cell in $\mathbf{A}$ and Φ is a 2-cell in FD (note that $Ffx = y$, $Ff'x' = y'$):

$$\begin{tikzcd} Ffx \arrow[r, "id"] \arrow[d, "(F\alpha)_x"'] \arrow[dr, phantom, "\Phi"] & Ffx \arrow[d, "\lambda"] \\ Ff'x \arrow[r, "Ff'\rho"'] & Ff'x' \end{tikzcd} .$$

   The existence of this double category (for the case in which F is a weight W as above) goes back to at least [24], see also [12, 1.2] for its relation with 2-limits. This double category is called a "double category of elements" and denoted by $\mathbb{E}\mathsf{l}(F)$. We have thus:

$$\int_{\mathbb{H}\mathbf{A}} \mathbb{Q}(\mathbb{V}\circ F) = \mathbb{E}\mathsf{l}(F). \tag{3.15}$$

4. Note that we can also compose $\mathbf{A} \xrightarrow{F} \mathbf{2\text{-}Cat}$ with the 2-functor $\mathbf{2\text{-}Cat} \xrightarrow{\mathbb{Q}} \mathcal{D}\mathbf{blCat}_v$ instead of $\mathbb{V}$. If we consider the double functor $\mathbb{Q}\mathbf{A} \xrightarrow{\mathbb{Q}(\mathbb{Q}\circ F)} \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$ that arises in this way, it can be checked just as in Example 3.12.1 that:
$$\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}(\mathbb{Q}\circ F) = \mathbb{Q}\int_{\mathbf{A}} F \tag{3.16}$$
The extra double categorical structure present here allows for the following construction:

5. Note that there is a 2-functor *horizontal flip* $\mathcal{D}\mathbf{blCat}_v \longrightarrow \mathcal{D}\mathbf{blCat}_v$, mapping each double category $\mathbb{A}$ to the double category $\mathbb{A}^\wedge$ in which the direction of the horizontal arrows is reversed. If (before applying $\mathbb{Q}$ to the composition) we compose $\mathbb{Q}\circ F$ with this 2-functor, then we get a different double functor $\mathbb{Q}\mathbf{A} \xrightarrow{\mathbb{Q}((\mathbb{Q}\circ F)^\wedge)} \mathbb{Q}\mathcal{D}\mathbf{blCat}_v$, which maps each object A to $(\mathbb{Q}FA)^\wedge$. We will now give a detailed description of the result of the construction in Definition 3.1 when applied to this double functor.

   Objects and vertical arrows are the usual ones, that is, the same as in $\int_{\mathbf{A}} F$, and the same as in Definition 3.1. But for horizontal arrows the direction of the arrow in the second coordinate is *reversed*: a horizontal arrow $(C,x) \xrightarrow{(f,\varphi)} (D,y)$ is given by an arrow $C \xrightarrow{f} D$ of $\mathbf{A}$, and an arrow $y \xrightarrow{\varphi} Ffx$ in FD. We observe that this double category combines the 2-categorical Grothendieck construction of F with its *fiberwise opposite*. The construction of this double category was recently been considered, for the case of indexed 1-categories, in [22], where it is referred to as a "Grothendieck double construction" and denoted $F \wr F^{op}$. We refer the interested reader to that article, and to the references therein, for the link of the *fiberwise opposite* of a fibration with the theory of *lenses*, as well as the prospective applications of this construction to dynamical systems. Note that the vertical morphisms are the ones that get reversed in op. cit., which corresponds to choosing to work with horizontal natural transformations instead of vertical ones in Definition 3.1. We leave it to the reader to check that the double cell in (3.2), with Φ an identity, matches diagram (3) in [22, Def. 3.8]. We conclude:
$$\int_{\mathbb{Q}\mathbf{A}} \mathbb{Q}((\mathbb{Q}\circ F)^\wedge) = F \wr F^{op}. \tag{3.17}$$

4 Double cylinders and doubly lax limits

We fix throughout this section a DblCat-category $\mathcal{C}$.

Remark 4.1. Before we start the explicit computations, we believe a warning on notation is appropriate. Recall that for each pair of objects A, B we denote the horizontal composition in $\mathcal{C}(A,B)$ by $\circ$, the vertical composition by $\bullet$, and that we denote the composition $\mathcal{C}(B,C) \times \mathcal{C}(A,B) \longrightarrow \mathcal{C}(A,C)$ by juxtaposition.

We allow ourselves to represent identities with the same letter of the involved object, and we do this "for all dimensions". This simplifies the notation a great deal, but the reader should be warned, and remember that anytime that *dimensions don't seem to match* it is probably due to this. For example, in diagram (4.3) below, each of the six 2-cells is a composition of an actual 2-cell with a 2-cell that is an identity coming from an arrow (note that four of them are vertical identities and two are horizontal; it is the *type* of the actual 2-cell that determines the *type* of the identity). For a more explicit example, the following diagram in $\mathcal{C}$:
$$\begin{tikzcd} & & \cdot \arrow[dr, "v'"] \arrow[dd, phantom, "\Downarrow\beta"] & \\ \cdot \arrow[r, "f"] & \cdot \arrow[ur, "k"] \arrow[dr, "v"'] & & \cdot \\ & & \cdot \arrow[ur, "\overline{k}"'] & \end{tikzcd}$$
denotes a vertical 2-cell $v'kf \overset{\beta f}{\Longrightarrow} \overline{k}vf$, which is the composition in $\mathcal{C}$ of β and the vertical identity on f; that

is, the value of the "composition" double functor at the vertical arrow $(\beta, 1_f^\bullet)$, note that this is a vertical arrow of a double category of the form $\mathcal{C}(B,C) \times \mathcal{C}(A,B)$. These types of compositions are usually called *whiskering* with an arrow, but we will also use this notation for compositions that are not instances of whiskering, for example: a configuration in $\mathcal{C}$ of the form

$$\begin{tikzcd} \cdot \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \cdot \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & \cdot \end{tikzcd}$$

gives rise to a triple cell

$$\begin{tikzcd} gf \arrow[r, Rightarrow, "g\alpha"] \arrow[d, Rightarrow, "\beta f"'] \arrow[dr, phantom, "\beta\alpha"] & gf' \arrow[d, Rightarrow, "\beta f'"] \\ g'f \arrow[r, Rightarrow, "g'\alpha"'] & g'f' \end{tikzcd}$$

in $\mathcal{C}$, which is in fact the value of the composition double functor at the double cell $(1_\beta, 1_\alpha^\bullet)$. A similar, dual triple cell exists if α is horizontal and β vertical. We will sometimes refer to triple cells of this kind as *thin* triple cells. Just as whiskering is used in pasting diagrams in 2-categories the thin triple cells will be used to form pasting double cells. For an example, see diagram (4.6) below.

4.1 Double cylinders

Definition 4.2. We define a double category $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$ of (vertical[2]) double cylinders.

- An *object* f is given by an arrow of $\mathcal{C}$, that we write vertically as $f\big\downarrow$

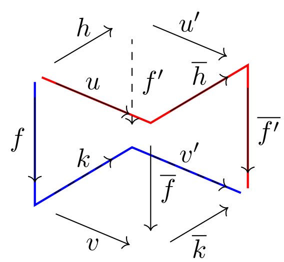
- A *vertical arrow* $f \xrightarrow{(u,\mu,v)} \overline{f}$ is given by a vertical 2-cell that we write as
$$\begin{tikzcd} \cdot \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\mu"] & \cdot \arrow[d, "\overline{f}"] \\ \cdot \arrow[r, "v"'] & \cdot \end{tikzcd}$$

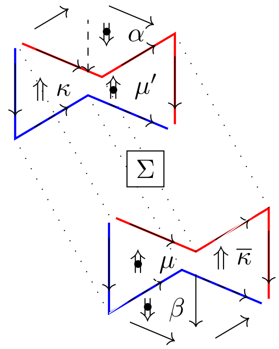
- A *horizontal arrow* $f \xrightarrow{(h,\kappa,k)} f'$ is given by a horizontal 2-cell that we write as
$$\begin{tikzcd} \cdot \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, phantom, "\Uparrow\kappa"] & \cdot \arrow[d, "f'"] \\ \cdot \arrow[r, "k"'] & \cdot \end{tikzcd}$$

- A *double cell*, that we call a *double cylinder*,
$$\begin{tikzcd} f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"] \\ \overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \end{tikzcd}$$
is given by two vertical 2-cells that we write as
$$\begin{tikzcd} & \cdot \arrow[dr, "u'"] & \\ \cdot \arrow[ur, "h"] \arrow[dr, "u"'] & \Downarrow\alpha & \cdot \\ & \cdot \arrow[ur, "\overline{h}"'] & \end{tikzcd} , \quad \begin{tikzcd} & \cdot \arrow[dr, "v'"] & \\ \cdot \arrow[ur, "k"] \arrow[dr, "v"'] & \Downarrow\beta & \cdot \\ & \cdot \arrow[ur, "\overline{k}"'] & \end{tikzcd}$$
and a triple cell Σ as in the left below, that we represent as a *filling* for the cube (or cylinder) formed by the six 2-cells on the right in the following diagram (recall Remark 4.1):

[2] There is a dual notion of horizontal double cylinder that we won't describe explicitly, as we won't be using it.

$$
\begin{tikzcd}
{\color{blue}v'kf} \arrow[r, Rightarrow, "v'\kappa"] \arrow[d, Rightarrow, "\beta f"] & v'f'h \arrow[d, Rightarrow, "\mu' h"] \\
\overline{k}vf \arrow[r, phantom, "\Sigma"] \arrow[d, Rightarrow, "\overline{k}\mu"] & \overline{f'}u'h \arrow[d, Rightarrow, "\overline{f'}\alpha"] \\
\overline{k}\,\overline{f}u \arrow[r, Rightarrow, "\overline{\kappa}u"] & {\color{red}\overline{f'}\,\overline{h}u}
\end{tikzcd}
$$





(4.3)

Note that the 2-cells fitting into such a cube are double cells in quintet double categories, more precisely α, μ, β, ν are double cells in $\mathbb{Q}\mathcal{C}_v$ (as in Notation 2.3) that we write $\begin{array}{ccc} & \xrightarrow{h} & \\ u\downarrow & \Downarrow\ \alpha & \downarrow u' \\ & \xrightarrow[\overline{h}]{} & \end{array}$, $\begin{array}{ccc} & \xrightarrow{f} & \\ u\downarrow & \Downarrow\ \mu & \downarrow v \\ & \xrightarrow[\overline{f}]{} & \end{array}$, etc. We write κ, $\overline{\kappa}$ as $\begin{array}{ccc} & \xrightarrow{h} & \\ f\downarrow & \Uparrow\kappa & \downarrow f' \\ & \xrightarrow[k]{} & \end{array}$, etc, that is as double cells in the transpose[3] of the double category $\mathbb{Q}\mathcal{C}_h$. The composition of double cylinders extends the composition of double cells in these double categories.

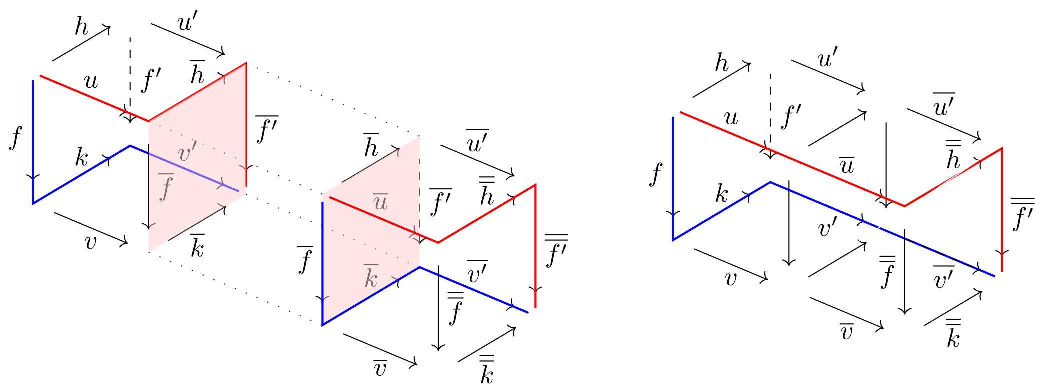
- **Vertical composition.** The composition of two vertical arrows $f \xrightarrow{(u,\mu,v)} \overline{f} \xrightarrow{(\overline{u},\overline{\mu},\overline{v})} \overline{\overline{f}}$ is the vertical arrow $f \xrightarrow{(\overline{u}u,\overline{\mu}\bullet\mu,\overline{v}v)} \overline{\overline{f}}$, where the vertical 2-cell $\begin{array}{cccc} f\downarrow & \overset{u}{\searrow} & & \\ & & \overset{\overline{u}}{\searrow} & \\ & \Uparrow\ \overline{\mu}\bullet\mu & & \\ & \underset{v}{\searrow} & & \downarrow\overline{\overline{f}} \\ & & \underset{\overline{v}}{\searrow} & \end{array}$ is defined as the vertical pasting of μ and $\overline{\mu}$, sharing $\overline{f}$, in the double category $\mathbb{Q}\mathcal{C}_v$. Equivalently, $\overline{\mu} \bullet \mu$ is the vertical pasting in the double category $\mathcal{C}(a,b)$, where $a = \mathrm{dom}(f)$, $b = \mathrm{cod}(\overline{\overline{f}})$, of the vertical 2-cells $\overline{v}\mu$ and $\overline{\mu}u$ (which are vertical arrows of this double category), that is $\overline{\mu} \bullet \mu = (\overline{\mu}u) \bullet (\overline{v}\mu)$.

The vertical composition of double cylinders is an operation

$$
\begin{tikzcd}
f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"] \\
\overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] \arrow[d, "{(\overline{u},\overline{\mu},\overline{v})}"'] \arrow[dr, phantom, "{(\overline{\alpha},\overline{\Sigma},\overline{\beta})}"] & \overline{f'} \arrow[d, "{(\overline{u'},\overline{\mu'},\overline{v'})}"] \\
\overline{\overline{f}} \arrow[r, "{(\overline{\overline{h}},\overline{\overline{\kappa}},\overline{\overline{k}})}"'] & \overline{\overline{f'}}
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(\overline{u}u,\overline{\mu}\bullet\mu,\overline{v}v)}"'] \arrow[dr, phantom, "{(\overline{\alpha}\bullet\alpha,\overline{\Sigma}\bullet\Sigma,\overline{\beta}\bullet\beta)}"] & f' \arrow[d, "{(\overline{u'}u',\overline{\mu'}\bullet\mu',\overline{v'}v')}"] \\
\overline{\overline{f}} \arrow[r, "{(\overline{\overline{h}},\overline{\overline{\kappa}},\overline{\overline{k}})}"'] & \overline{\overline{f'}}
\end{tikzcd}
$$

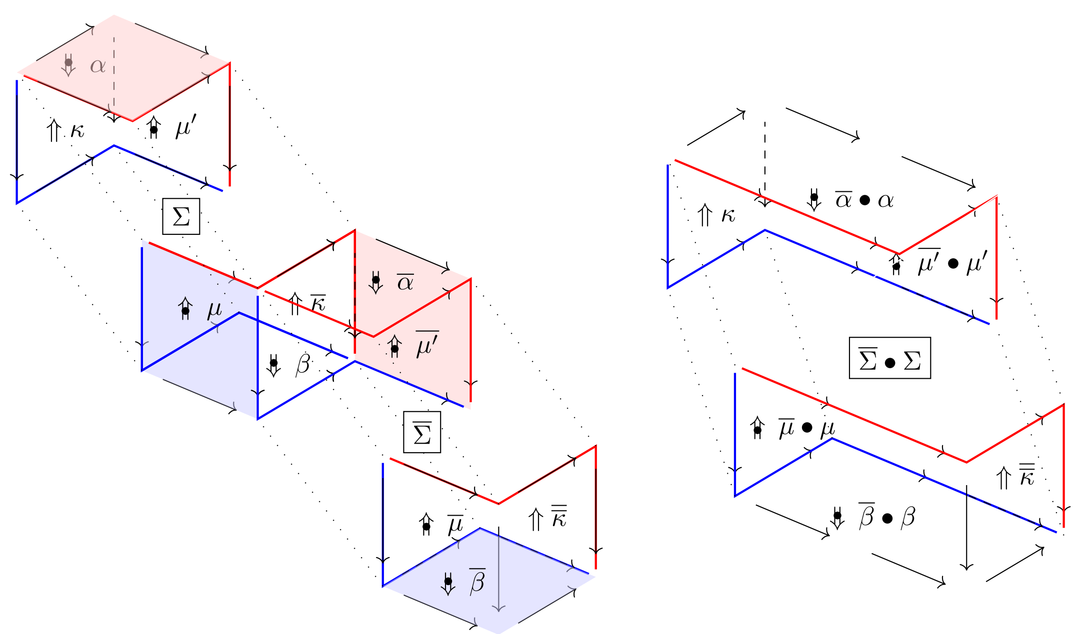
that is given by their *pasting* in the shared face $\overline{\kappa}$. At the level of 1-cells we picture this as follows

[3] this is done merely for convenience in the notation, so that the arrows h, k are always horizontal.



(4.4)

At the level of 2- and triple cells, we picture this as



The vertical 2-cells $\overline{\alpha} \bullet \alpha$, $\overline{\beta} \bullet \beta$ are defined just like $\overline{\mu} \bullet \mu$ is. In an analogous fashion, we define the triple cell $\overline{\Sigma} \bullet \Sigma$ as the vertical composition of the triple cells $\overline{v'}\Sigma$ and $\overline{\Sigma}u$ in the double category $\mathcal{C}(a, b)$, where $a = \mathrm{dom}(f), b = \mathrm{cod}(\overline{\overline{f'}})$ as in the following diagram. For the first equality, we are using that the composition in $\mathcal{C}$ commutes with the vertical composition for the pairs of 2-cells μ, $\overline{\beta}$, marked in blue and α, $\overline{\mu'}$ marked in red.

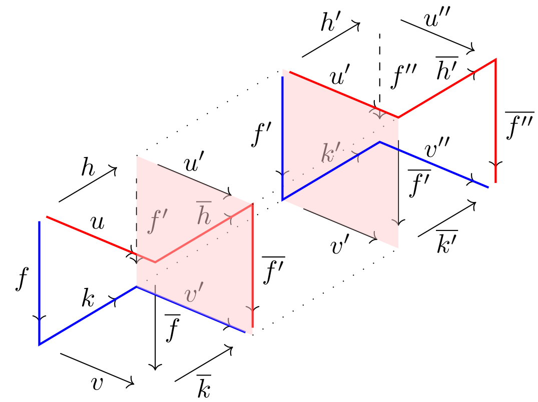
$$
\begin{tikzcd}[column sep=small]
\overline{v'}v'kf \arrow[rr, Rightarrow, "\overline{v'}v'\kappa"] \arrow[d, Rightarrow, "\overline{v'}\beta f"] & & \overline{v'}v'f'h \arrow[d, Rightarrow, "\overline{v'}\mu' h"] \\
\overline{v'}\,\overline{k}vf \arrow[d, Rightarrow, "\overline{v'}\,\overline{k}\mu"] & \overline{v'}\Sigma & \overline{v'}\,\overline{f'}u'h \arrow[d, Rightarrow, "\overline{v'}\,\overline{f'}\alpha"] \\
\overline{v'}\,\overline{k}\,\overline{f}u \arrow[rr, Rightarrow, "\overline{v'}\,\overline{\kappa}u"] \arrow[d, Rightarrow, "\overline{\beta}\,\overline{f}u"] & & \overline{v'}\,\overline{f'}\,\overline{h}u \arrow[d, Rightarrow, "\overline{\mu'}\,\overline{h}u"] \\
\overline{\overline{k}}\,\overline{v}\,\overline{f}u \arrow[d, Rightarrow, "\overline{\overline{k}}\,\overline{\mu}u"] & \overline{\Sigma}u & \overline{\overline{f'}}\,\overline{u'}\,\overline{h}u \arrow[d, Rightarrow, "\overline{\overline{f'}}\,\overline{\alpha}u"] \\
\overline{\overline{k}}\,\overline{\overline{f}}\overline{u}u \arrow[rr, Rightarrow, "\overline{\overline{\kappa}}\,\overline{u}u"] & & \overline{\overline{f'}}\,\overline{\overline{h}}\overline{u}u
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
\overline{v'}v'kf \arrow[rr, Rightarrow, "\overline{v'}v'\kappa"] \arrow[d, Rightarrow, "\overline{v'}\beta f"] & & \overline{v'}v'f'h \arrow[d, Rightarrow, "\overline{v'}\mu' h"] \\
\overline{v'}\,\overline{k}vf \arrow[d, Rightarrow, "\overline{\beta}vf"] & & \overline{v'}\,\overline{f'}u'h \arrow[d, Rightarrow, "\overline{\mu'}u'h"] \\
\overline{\overline{k}}\,\overline{v}\,vf \arrow[d, Rightarrow, "\overline{\overline{k}}\,\overline{v}\mu"] & \overline{\Sigma}\bullet\Sigma & \overline{\overline{f'}}\,\overline{u'}\,u'h \arrow[d, Rightarrow, "\overline{\overline{f'}}\,\overline{u'}\alpha"] \\
\overline{\overline{k}}\,\overline{v}\,\overline{f}u \arrow[d, Rightarrow, "\overline{\overline{k}}\,\overline{\mu}u"] & & \overline{\overline{f'}}\,\overline{u'}\,\overline{h}u \arrow[d, Rightarrow, "\overline{\overline{f'}}\,\overline{\alpha}u"] \\
\overline{\overline{k}}\,\overline{\overline{f}}\overline{u}u \arrow[rr, Rightarrow, "\overline{\overline{\kappa}}\,\overline{u}u"] & & \overline{\overline{f'}}\,\overline{\overline{h}}\overline{u}u
\end{tikzcd}
=
\begin{tikzcd}[column sep=small]
\overline{v'}v'kf \arrow[rr, Rightarrow, "\overline{v'}v'\kappa"] \arrow[d, Rightarrow, "(\overline{\beta}\bullet\beta)f"] & & \overline{v'}v'f'h \arrow[d, Rightarrow, "(\overline{\mu'}\bullet\mu')h"] \\
\overline{\overline{k}}\,\overline{v}\,vf \arrow[d, Rightarrow, "\overline{\overline{k}}(\overline{\mu}\bullet\mu)"] & \overline{\Sigma}\bullet\Sigma & \overline{\overline{f'}}\,\overline{u'}\,u'h \arrow[d, Rightarrow, "\overline{\overline{f'}}(\overline{\alpha}\bullet\alpha)"] \\
\overline{\overline{k}}\,\overline{\overline{f}}\overline{u}u \arrow[rr, Rightarrow, "\overline{\overline{\kappa}}\,\overline{u}u"] & & \overline{\overline{f'}}\,\overline{\overline{h}}\overline{u}u
\end{tikzcd}
$$

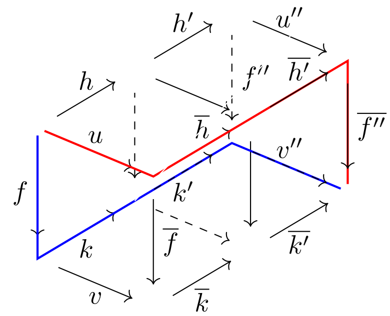
- Horizontal composition. Due to the fact that one has to choose a *type* of 2-cell for α and β (that is, horizontal or vertical, and we chose vertical), the horizontal composition of double cylinders is not entirely dual to the vertical one and merits a detailed computation. The composition of two horizontal arrows $f \xrightarrow{(h,\kappa,k)} f' \xrightarrow{(h',\kappa',k')} f''$ is the horizontal arrow $f \xrightarrow{(h'h,\kappa'\circ\kappa,k'k)} f''$,

where the horizontal 2-cell
$$
\begin{tikzcd}[row sep=small, column sep=small]
 & & \cdot \arrow[dd, "f''"] \\
 & \cdot \arrow[ur, "h'"] & \\
\cdot \arrow[dd, "f"'] \arrow[ur, "h"] & \Uparrow \kappa'\circ\kappa & \cdot \\
 & \cdot \arrow[ur, "k'"'] & \\
\cdot \arrow[ur, "k"'] & &
\end{tikzcd}
$$
is a horizontal pasting in the transpose of $\mathbb{Q}\mathcal{C}_h$, $\kappa'\circ\kappa = (\kappa' h)\circ(k'\kappa)$. The horizontal composition of double cylinders is an operation

$$
\begin{tikzcd}
f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[r, "{(h',\kappa',k')}"] \arrow[d, "{(u',\mu',v')}"] \arrow[dr, phantom, "{(\alpha',\Sigma',\beta')}"] & f'' \arrow[d, "{(u'',\mu'',v'')}"] \\
\overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \arrow[r, "{(\overline{h'},\overline{\kappa'},\overline{k'})}"'] & \overline{f''}
\end{tikzcd}
\quad\rightsquigarrow\quad
\begin{tikzcd}
f \arrow[r, "{(h'h,\kappa'\circ\kappa,k'k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha\circ\alpha',\Sigma'\circ\Sigma,\beta\circ\beta')}"] & f'' \arrow[d, "{(u'',\mu'',v'')}"] \\
\overline{f} \arrow[r, "{(\overline{h'h},\overline{\kappa'}\circ\overline{\kappa},\overline{k'k})}"'] & \overline{f''}
\end{tikzcd}
$$

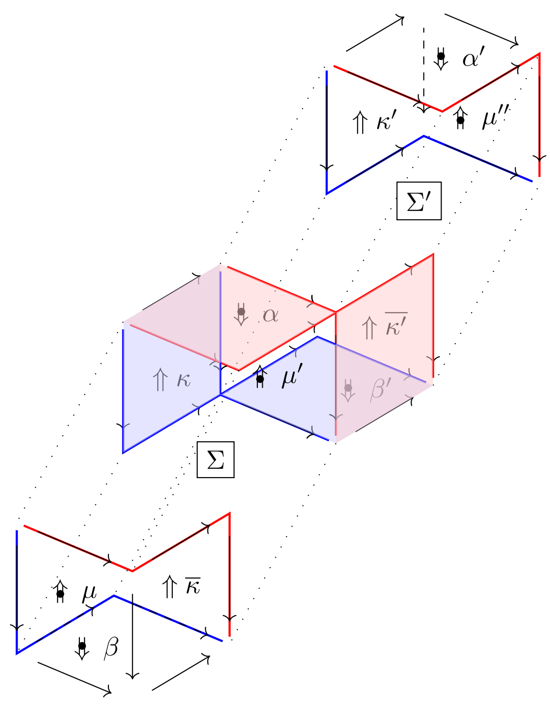
that is given by their *pasting* in the shared face μ'. Note that $\alpha\circ\alpha'$ and $\beta\circ\beta'$ are horizontal pastings of double cells in $\mathbb{Q}\mathcal{C}_v$. At the level of 1-cells:

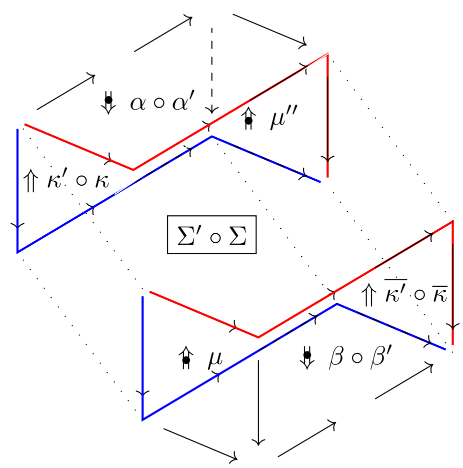




(4.5)

At the level of 2- and triple cells, we picture this as





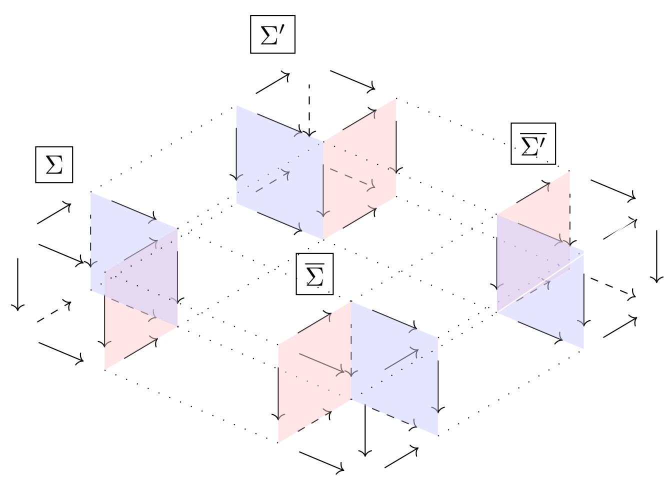
The triple cell $\Sigma'\circ\Sigma$ is defined as the following pasting diagram (recall Remark 4.1)

$$
\begin{tikzcd}
v''k'kf \arrow[r, Rightarrow, "v''k'\kappa"] \arrow[d, Rightarrow, "\beta' kf"'] \arrow[dr, phantom, "\beta'\kappa"] & v''k'f'h \arrow[r, Rightarrow, "v''\kappa' h"] \arrow[d, Rightarrow, "\beta' f'h"] \arrow[ddr, phantom, "\Sigma' h"] & v''f''h'h \arrow[d, Rightarrow, "\mu' h'h"] \\
\overline{k'}v'kf \arrow[r, Rightarrow, "\overline{k'}v'\kappa"] \arrow[d, Rightarrow, "\overline{k'}\beta f"'] \arrow[ddr, phantom, "\overline{k'}\Sigma"] & \overline{k'}v'f'h \arrow[d, Rightarrow, "\overline{k'}\mu' h"] & \overline{f''}u''h'h \arrow[d, Rightarrow, "\overline{f''}\alpha' h"] \\
\overline{k'}\ \overline{k}vf \arrow[d, Rightarrow, "\overline{k'}\ \overline{k}\mu"'] & \overline{k'}\ \overline{f'}u'h \arrow[r, Rightarrow, "\overline{\kappa'}u'h"] \arrow[d, Rightarrow, "\overline{k'}\ \overline{f'}\alpha"'] \arrow[dr, phantom, "\overline{\kappa'}\alpha"] & \overline{f''}\ \overline{h'}u'h \arrow[d, Rightarrow, "\overline{f''}\ \overline{h'}\alpha"] \\
\overline{k'}\ \overline{k}\ \overline{f}u \arrow[r, Rightarrow, "\overline{k'}\ \overline{\kappa}u"] & \overline{k'}\ \overline{f'}\ \overline{h}u \arrow[r, Rightarrow, "\overline{\kappa'}\ \overline{h}u"] & \overline{k'}\ \overline{f'}\ \overline{h}u
\end{tikzcd}
=
\begin{tikzcd}
v''k'kf \arrow[r, Rightarrow, "v''(\kappa'\circ\kappa)"] \arrow[d, Rightarrow, "(\beta\circ\beta')f"'] \arrow[dr, phantom, "\Sigma'\circ\Sigma"] & v''f''h'h \arrow[d, Rightarrow, "\mu' h'h"] \\
\overline{k'}\ \overline{k}vf \arrow[d, Rightarrow, "\overline{k'}\ \overline{k}\mu"'] & \overline{f''}u''h'h \arrow[d, Rightarrow, "f(\alpha\circ\alpha')"] \\
\overline{k'}\ \overline{k}\ \overline{f}u \arrow[r, Rightarrow, "(\overline{\kappa'}\circ\overline{\kappa})u"] & \overline{k'}\ \overline{f'}\ \overline{h}u
\end{tikzcd}
\tag{4.6}
$$

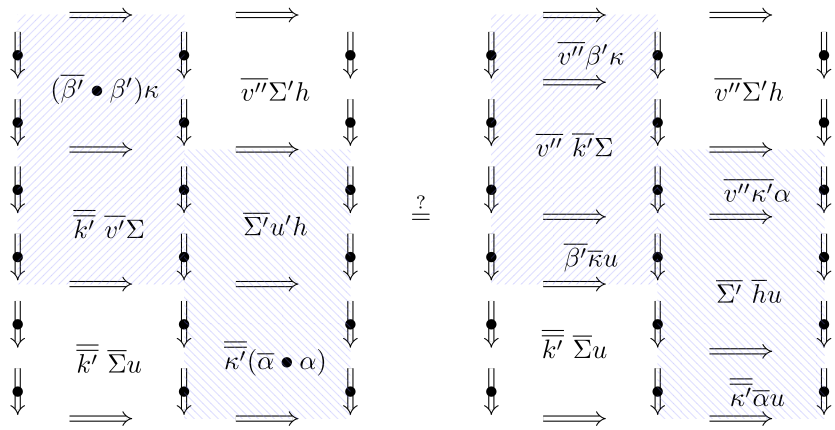
Proposition 4.7. *Given a* DblCat*-enriched category* $\mathcal{C}$*, the construction above gives a well-defined double category of vertical double cylinders* $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$*.*

Proof. The identities for both compositions are given by the respective identities of $\mathcal{C}$. It remains for us to prove associativity for the composition in each direction and the middle-four interchange law.

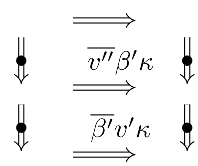
We begin by proving the interchange law in $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$. We consider thus four double cylinders as in the following configuration (the matching blue and red faces are the same 2-cells).



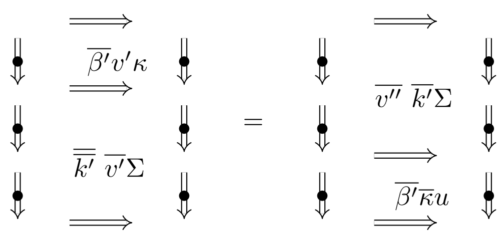
If we first compute the vertical pastings (of Σ with $\overline{\Sigma}$, and Σ' with $\overline{\Sigma'}$), and then we paste in the horizontal direction, we get the diagram on the left below. We will show that it coincides with the diagram on the right, corresponding to the other order for the pasting.



Clearly, it suffices to show that the shaded regions on the left and on the right are equal. We will show this for the top-left region, the computations are dual for the bottom-right one. We recall first that, by definition, $\overline{\beta'} \bullet \beta'$ is the composition of the vertical 2-cells $\overline{\beta'}v'$ and $\overline{v''}\beta'$. Since the composition of $\mathcal{C}$ distributes over this composition, we have that $(\overline{\beta'} \bullet \beta')\kappa$ equals

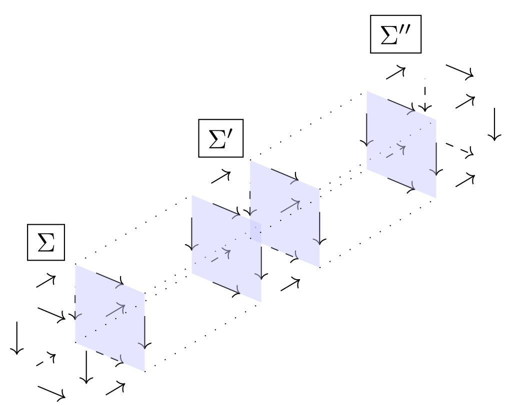


and thus it suffices to show the following equality:



We work with the diagram on the left. Using that the composition of $\mathcal{C}$ distributes over vertical composition, we have that this diagram is the composition in $\mathcal{C}$ of two vertical compositions, which reduce in turn to $\overline{\beta'}$ and Σ, so the whole diagram equals $\overline{\beta'}\Sigma$. A dual computation shows that the diagram on the right reduces to $\overline{\beta'}\Sigma$ as well.

To show associativity for the horizontal composition, we consider three double cylinders



A direct computation, using just the definition of the horizontal composition and similar properties to the ones used for the interchange property above, shows that both possible compositions $\Sigma'' \circ (\Sigma' \circ \Sigma)$ and $(\Sigma'' \circ \Sigma') \circ \Sigma$ yield the following triple cell

$$\begin{array}{ccccccc}
 & \Longrightarrow & & \Longrightarrow & & \Longrightarrow & \\
\overset{\bullet}{\Downarrow} & \beta''k'\kappa & \overset{\bullet}{\Downarrow} & \beta''\kappa'k & \overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} \\
 & \Longrightarrow & & \Longrightarrow & & \Sigma''h'h & \\
\overset{\bullet}{\Downarrow} & \overline{k''}\beta'\kappa & \overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} \\
 & \Longrightarrow & & \overline{k''}\Sigma'h & & \Longrightarrow & \\
\overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} & \kappa''\alpha'h & \overset{\bullet}{\Downarrow} \\
 & \overline{k''}\,\overline{k'}\Sigma & & \Longrightarrow & & \Longrightarrow & \\
\overset{\bullet}{\Downarrow} & & \overset{\bullet}{\Downarrow} & \overline{k''}\,\overline{\kappa'}\alpha & \overset{\bullet}{\Downarrow} & \overline{k''}\,\overline{h'}\alpha & \overset{\bullet}{\Downarrow} \\
 & \Longrightarrow & & \Longrightarrow & & \Longrightarrow &
\end{array}$$

The associativity for the vertical composition is a similar computation left to the reader. □

4.2 Doubly lax transformations and doubly lax cones

We begin by extending the notion of vertical double functor from Notation 3.10 to the case of an arbitrary DblCat-category $\mathcal{C}$ instead of $\mathcal{D}\mathbf{blCat}$.

Definition 4.8. Let $\mathbb{D}$ be a double category, and $\mathcal{C}$ be a category enriched in DblCat. A *vertical double functor* $F\colon \mathbb{D} \xrightarrow{v} \mathcal{C}$ is a double functor $F\colon \mathbb{D} \longrightarrow \mathbb{Q}\mathcal{C}_v$. Note that F maps objects of $\mathbb{D}$ to objects of $\mathcal{C}$, both horizontal and vertical arrows of $\mathbb{D}$ to arrows of $\mathcal{C}$ and double cells $\alpha : (u \; {}^{f}_{f'} \; v)$ of $\mathbb{D}$ to vertical 2-cells $F\alpha : FvFf \overset{\bullet}{\Longrightarrow} Ff'Fu$ of $\mathcal{C}$, strictly preserving all compositions.

Remark 4.9. Given a vertical double functor and a double functor fitting as $\mathbb{D} \xrightarrow{F} \mathbb{E} \xrightarrow[v]{G} \mathcal{C}$, their composition as double functors $\mathbb{D} \xrightarrow{F} \mathbb{E} \xrightarrow{G} \mathbb{Q}\mathcal{C}_v$ defines a vertical double functor $\mathbb{D} \xrightarrow[v]{GF} \mathcal{C}$.

Example 4.10. There are vertical double functors $d_0, d_1\colon \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \xrightarrow{v} \mathcal{C}$, that we call the *top* and *bottom* projections respectively, mapping a cylinder Σ as in (4.3) respectively to α and β.

We can now give the definition of a doubly lax transformation between vertical double functors, which amounts to a double functor into the cylinder double category. We give this definition in full detail as we will continue to build on its data and conditions in the definition of doubly lax colimit.

Definition 4.11. Let $F, G\colon \mathbb{D} \xrightarrow{v} \mathcal{C}$ be two vertical double functors. A *doubly lax transformation* $\theta\colon F \Rightarrow G$ is a double functor $\theta\colon \mathbb{D} \longrightarrow \mathbb{C}\mathsf{yl}_v(\mathcal{C})$ such that $d_0\theta = F$ and $d_1\theta = G$ (where d_0 and d_1 are the projections defined in Example 4.10). From Definition 4.2 we obtain that it is given by the following data:

- for each object A of $\mathbb{D}$, an arrow $FA \xrightarrow{\theta_A} GA$ in $\mathcal{C}$;
- for each vertical arrow $A \overset{u}{\longrightarrow\!\!\!\!\!\!\!\!\bullet\;\;\;} A'$ in $\mathbb{D}$, a vertical 2-cell $Gu\theta_A \overset{\theta_u}{\Longrightarrow\!\!\!\!\!\!\bullet\;\;} \theta_{A'}Fu$ in $\mathcal{C}$;
- for each horizontal arrow $A \xrightarrow{f} B$ in $\mathbb{D}$, a horizontal 2-cell $Gf\theta_A \overset{\theta_f}{\Longrightarrow} \theta_B Ff$ in $\mathcal{C}$;

- for each double cell $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$ of $\mathcal{D}$, a triple cell θ_α of $\mathcal{C}$ as in the following diagram[4]

$$\begin{tikzcd} GvGf\theta_A \arrow[r, Rightarrow, "Gv\theta_f"] \arrow[d, Rightarrow, "G\alpha\theta_A"'] \arrow[ddr, phantom, "\theta_\alpha"] & Gv\theta_B Ff \arrow[d, Rightarrow, "\theta_v Ff"] \\ Gf'Gu\theta_A \arrow[d, Rightarrow, "Gf'\theta_u"'] & \theta_{B'}FvFf \arrow[d, Rightarrow, "\theta_{B'}F\alpha"] \\ Gf'\theta_{A'}Fu \arrow[r, Rightarrow, "\theta_{f'}Fu"'] & \theta_{B'}Ff'Fu \end{tikzcd} \tag{4.12}$$

The fact that θ is a double functor can be expressed with the following conditions:

1. The vertical 2-cells preserve identities and compositions: $\theta_{1^\bullet_A} = 1^\bullet_{\theta_A}$, $\theta_{v\bullet u} = (\theta_v Fu) \bullet (Gv\theta_u)$ for each composable u, v.
2. Dually, for the horizontal 2-cells: $\theta_{1_A} = 1_{\theta_A}$ for each A, $\theta_{g\circ f} = (\theta_g Ff) \circ (Gg\theta_f)$.
3. The triple cells preserve identities: $\theta_{1_f} = 1_{\theta_f}, \theta_{1^\bullet_u} = 1^\bullet_{\theta_u}$.
4. The triple cells preserve vertical composition: for 2-cells $\alpha : (u \;{}^{f}_{f'}\; v)$, $\alpha' : (u' \;{}^{f'}_{f''}\; v')$,
$$\theta_{\frac{\alpha}{\alpha'}} = \frac{Gv'\theta_\alpha}{\theta_{\alpha'}Fu}$$
5. The triple cells preserve horizontal composition: for each configuration $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[r, "g"] \arrow[d, "v"] \arrow[dr, phantom, "\beta"] & C \arrow[d, "w"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C \end{tikzcd}$, the triple cell $\theta_{(\alpha|\beta)}$ equals the composition (as described in (4.6))

$$\begin{tikzcd} GwGgGf\theta_A \arrow[r, Rightarrow, "GwGg\theta_f"] \arrow[d, Rightarrow, "G\beta Gf\theta_A"'] \arrow[dr, phantom, "G\beta\theta_f"] & GwGg\theta_B Ff \arrow[r, Rightarrow, "Gw\theta_g Ff"] \arrow[d, Rightarrow, "G\beta\theta_B Ff"] \arrow[ddr, phantom, "\theta_\beta Ff"] & Gw\theta_C FgFf \arrow[d, Rightarrow, "\theta_w FgFf"] \\ Gg'GvGf\theta_A \arrow[r, Rightarrow, "Gg'Gv\theta_f"'] \arrow[d, Rightarrow, "Gg'G\alpha\theta_A"'] \arrow[ddr, phantom, "Gg'\theta_\alpha"] & Gg'Gv\theta_B Ff \arrow[d, Rightarrow, "Gg'\theta_v Ff"] & \theta_{C'}FwFgFf \arrow[d, Rightarrow, "\theta_{C'}F\beta Ff"] \\ Gg'Gf'Gu\theta_A \arrow[d, Rightarrow, "Gg'Gf'\theta_u"'] & Gg'\theta_{B'}FvFf \arrow[r, Rightarrow, "\theta_{g'}FvFf"] \arrow[d, Rightarrow, "Gg'\theta_{B'}F\alpha"'] \arrow[dr, phantom, "\theta_{g'}F\alpha"] & \theta_{C'}Fg'FvFf \arrow[d, Rightarrow, "\theta_{C'}Fg'F\alpha"] \\ Gg'Gf'\theta_{A'}Fu \arrow[r, Rightarrow, "Gg'\theta_{f'}Fu"'] & Gg'\theta_{B'}Ff'Fu \arrow[r, Rightarrow, "\theta_{g'}Ff'Fu"'] & \theta_{C'}Fg'Ff'Fu \end{tikzcd}$$

Notation 4.13. For a 2-category $\mathbf{B}$, we denote by $\widehat{\mathbf{Q}}(\mathbf{B})$ the DblCat-category obtained by applying $\mathbb{Q}$ to the hom-categories of $\mathbf{B}$, that is $\widehat{\mathbf{Q}}(\mathbf{B})(A, B) = \mathbb{Q}(\mathbf{B}(A, B))$. Note that $\widehat{\mathbf{Q}}(\mathbf{B})$ has as objects and arrows those of $\mathbf{B}$, as horizontal and vertical 2-cells the 2-cells of $\mathbf{B}$, and only identities as triple cells (that is, there is a unique triple cell filling a square of four 2-cells of $\mathbf{B}$ if and only if this diagram commutes in $\mathbf{B}$). Note also that $\widehat{\mathbf{Q}}(\mathbf{B})_v = \mathbf{B}$.

[4]For simplicity in reading, we have adopted the following notation: the *names* of non-identity 2-cells will be written in blue

Example 4.14. We now present the most obvious way in which doubly lax transformations generalize lax transformations between 2-functors. Let $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ be a 2-functor between 2-categories, and consider the induced double functor $\mathbb{Q}F\colon \mathbb{Q}\mathbf{A} \longrightarrow \mathbb{Q}\mathbf{B}$. In view of the above, $\mathbb{Q}F$ is equivalently a vertical double functor $\mathbb{Q}F\colon \mathbb{Q}\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}}(\mathbf{B})$. Given two such functors, $\mathbb{Q}F, \mathbb{Q}G\colon \mathbb{Q}\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}}(\mathbf{B})$, we examine now what a doubly lax transformation $\theta\colon \mathbb{Q}F \Longrightarrow \mathbb{Q}G$ amounts to.

Looking at Definition 4.11, we see that for each arrow f of $\mathbf{A}$ we have a priori two different 2-cells of $\mathbf{B}$ that should be called "θ_f", depending on whether we consider f as a vertical or horizontal arrow of $\mathbb{Q}\mathbf{A}$. However, we note that we have a double cell

$$\begin{tikzcd} \bullet \arrow[r, "f"] \arrow[d, "f"'] \arrow[dr, phantom, "\mathrm{id}_f"] & \bullet \arrow[d, "\mathrm{id}"] \\ \bullet \arrow[r, "\mathrm{id}"'] & \bullet \end{tikzcd}$$

in $\mathbb{Q}\mathbf{A}$ coming from the identity 2-cell id_f in $\mathbf{A}$. Now the existence of the triple cell θ_{id_f} implies that the two versions of 'θ_f' are the same 2-cell. Once this is observed, it is straightforward to check that the data and the axioms defining θ are precisely those of a classical lax transformation between the 2-functors F and G.

Example 4.15. If the previous example was to be expected, this one may be more surprising. It is a basic idea of 2-dimensional category theory that goes back to at least [13], to consider an arbitrary family Ω of 2-cells of a 2-category $\mathbf{B}$ (which we may assume to be closed under identities and compositions, though this doesn't really make a difference), and to ask for the structural 2-cells "θ_f" of a lax transformation to be in Ω. Let us call such a transformation ω-natural. In this way, for example, if Ω consists of the invertible 2-cells of $\mathbf{B}$, we get the notion of pseudonatural transformation, and if Ω consists only of identities, we get 2-naturality.

Given such a pair $(\mathbf{B},\Omega)$, *combining* the notions introduced in Notation 3.11 and Notation 4.13, we can define a DblCat-category $\widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})$ by the formula $\widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})(A,B) = \mathbb{Q}^{\Omega}(\mathbf{B}(A,B))$. In other words, $\widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})$ is defined by restricting the horizontal 2-cells in $\widehat{\mathbf{Q}}(\mathbf{B})$ to those in Ω.

Note that, for any choice of Ω, we still have $\widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})_v = \mathbf{B}$, and so 2-functors $F\colon \mathbf{A} \longrightarrow \mathbf{B}$ amount to vertical double functors $\mathbb{Q}F\colon \mathbb{Q}\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})$ just as above. But now, for vertical double functors $\mathbb{Q}F, \mathbb{Q}G\colon \mathbb{Q}\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}^{\Omega}}(\mathbf{B})$, the computation of Example 4.14 shows that doubly lax transformations $\theta\colon \mathbb{Q}F \Longrightarrow \mathbb{Q}G$ are the same as ω-natural transformations $F \Longrightarrow G$. Thus, the doubly lax transformations of Definition 4.11 also include pseudonatural and 2-natural transformations between 2-functors as special cases, by choosing appropriate *codomain* DblCat-categories.

In fact, it has also been known since [13] that we may not want to ask for *all* the 2-cells "θ_f" to be in Ω, but only those coming from a fixed family Σ of arrows of $\mathbf{A}$. By restricting the double functors $\mathbb{Q}F$, $\mathbb{Q}G$ to $\mathbb{Q}^{\Sigma}\mathbf{A}$, we also obtain this notion of naturality as a particular case of a doubly lax transformation. We mention in passing that the relevance of these notions of transformations between 2-functors lies in the fact that the corresponding notions of conical limit (originally called cartesian quasi-limit) that arise from them are as expressive as weighted limits. That is, for these notions of 2-dimensional limits, every weighted limit is conical. We refer to the classical [13], [23], and also to [7], [25] for a more modern treatment, where these limits are referred to as σ-limits and σ-ω-limits.

Given an object $E \in \mathcal{C}$, it is clear how to define a constant vertical double functor $\Delta E\colon \mathbb{D} \longrightarrow \mathcal{C}$. Taking either $F = \Delta E$, or $G = \Delta E$ in Definition 4.11, we get a notion of *doubly lax (co)cone*. For convenience we spell out the case $G = \Delta E$:

Definition 4.16. Let $F : \mathbb{D} \xrightarrow{v} \mathcal{C}$ be a vertical double functor, $E \in \mathcal{C}$. A *doubly lax cocone* for F, with nadir E, is a doubly lax transformation $\theta\colon F \Longrightarrow \Delta E$. As such, it is given by the following data:

- for each object A of $\mathbb{D}$, an arrow $FA \xrightarrow{\theta_A} E$ of $\mathcal{C}$;

- for each horizontal arrow $A \xrightarrow{f} B$ of $\mathbb{D}$, a horizontal 2-cell $\theta_A \overset{\theta_f}{\Longrightarrow} \theta_B F f$ of $\mathcal{C}$;
- for each vertical arrow $A \xrightarrow{u} A'$ of $\mathbb{D}$, a vertical 2-cell $\theta_A \overset{\theta_u}{\Longrightarrow} \theta_{A'} F u$ of $\mathcal{C}$;
- for each double cell $\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\alpha"] & B \arrow[d, "v"] \\ A' \arrow[r, "f'"'] & B' \end{tikzcd}$ of $\mathbb{D}$, a triple cell θ_α of $\mathcal{C}$ as in the following diagram:

$$\begin{tikzcd} \theta_A \arrow[r, Rightarrow, "\theta_f"] \arrow[dd, Rightarrow, "\theta_u"'] & \theta_B F f \arrow[d, Rightarrow, "\theta_v F f"] \\ \theta_\alpha & \theta_{B'} F v F f \arrow[d, Rightarrow, "\theta_{B'} F\alpha"] \\ \theta_{A'} F u \arrow[r, Rightarrow, "\theta_{f'} F u"'] & \theta_{B'} F f' F u \end{tikzcd} \tag{4.17}$$

satisfying Conditions 1-5 of Definition 4.11.

4.3 Higher structure of doubly lax transformations

doubly lax transformations between F and G (in particular doubly lax cones), being double functors, are at the same time objects of the double category $\mathcal{D}\mathbf{blCat}(\mathbb{D}, \mathbb{C}\mathsf{yl}_v(\mathcal{C}))$. We can thus define a double category of doubly lax transformations between F and G, $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$, as the full sub double category of $\mathcal{D}\mathbf{blCat}(\mathbb{D}, \mathbb{C}\mathsf{yl}_v(\mathcal{C}))$ given by those θ such that $d_0\theta = F$, $d_1\theta = G$. As we did in Section 4.2, we state precisely the axioms involved in these structures, since we will use them later.

Definition 4.18. Let $\mathbb{D}$ be a double category and $\mathcal{C}$ a DblCat-enriched category with two vertical double functors $F, G\colon \mathbb{D} \xrightarrow{v} \mathcal{C}$, as in Definition 4.8. The lax hom double category $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$ has

- *Objects* are the doubly lax transformations $F \overset{\theta}{\Longrightarrow} G$ of Definition 4.11, also considered as double functors $\mathbb{D} \xrightarrow{\theta} \mathbb{C}\mathsf{yl}_v(\mathcal{C})$.
- *Vertical arrows* are the vertical transformations $\theta \overset{U}{\Longrightarrow} \theta' \colon \mathbb{D} \rightrightarrows \mathbb{C}\mathsf{yl}_v(\mathcal{C})$. These are given by the following data (cf. Definition 2.1):
  - for each object A of $\mathbb{D}$, a vertical 2-cell $\theta_A \overset{U_A}{\Longrightarrow} \theta'_A$ in $\mathcal{C}$;
  - for each horizontal arrow $A \xrightarrow{f} B$ in $\mathbb{D}$, a triple cell U_f of $\mathcal{C}$ as in the following diagram:

$$\begin{tikzcd} G f \theta_A \arrow[r, Rightarrow, "\theta_f"] \arrow[d, Rightarrow, "G f U_A"'] \arrow[dr, phantom, "U_f"] & \theta_B F f \arrow[d, Rightarrow, "U_B F f"] \\ G f \theta'_A \arrow[r, Rightarrow, "\theta'_f"'] & \theta'_B F f \end{tikzcd}$$

  subject to the following conditions:

(v.1) $U_{1_A} = 1_{U_A}$ for each A,

(v.2) $U_{g \circ f} = (G g U_f | U_g F f)$ for composable f, g,

(v.3) $\frac{\theta_\alpha}{U_{f'} F u} = \frac{G v U_f}{\theta'_\alpha}$ for each $\alpha : (u \; {}^{f}_{f'} \; v)$.

- *Horizontal arrows* are the horizontal transformations $\theta \overset{H}{\Longrightarrow} \lambda$, which are dually given by a family of horizontal 2-cells $H_A\colon \theta_A \Longrightarrow \lambda_A$, and triple cells $H_u : ({\theta_u}\ {}^{GuH_A}_{H_{A'}Fu}\ {\lambda_u})$ satisfying the axioms:

  (h.1) $H_{1^\bullet_A} = 1^\bullet_{H_A}$ for each A,

  (h.2) $H_{v\bullet u} = \frac{GvH_u}{H_vFu}$ for composable u, v,

  (h.3) $(\theta_\alpha|\frac{HvFf}{H_{B'}F\alpha}) = (\frac{G\alpha H_A}{G_{f'}Hu}|\Lambda_\alpha)$ for each $\alpha : (u\ {}^{f}_{f'}\ v)$.

- *Double cells* are modifications $\begin{array}{ccc} \theta & \overset{H}{\Longrightarrow} & \Lambda \\ U\Downarrow & M & \Downarrow V \\ \theta' & \underset{H'}{\Longrightarrow} & \Lambda' \end{array}$, which are given by, for each object A of $\mathbb{D}$, a triple cell $M_A : (U_A\ {}^{H_A}_{H'_A}\ V_A)$ of $\mathcal{C}$ satisfying the axioms (cf. 2.2)

  (m.1) $(GfM_A|V_f) = (U_f|M_BFf)$ for each $A \xrightarrow{f} B$,

  (m.2) a symmetric condition for each $A \xrightarrow{u} A'$.

Remark 4.19. The definition of the double category $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$ is a *natural* one in the sense that it is the one induced by the double category structure of $\mathbb{C}\mathsf{yl}_v(\mathcal{C})^{\mathbb{D}}$. This will be the structure that we will use to define the universal property of the doubly lax colimit that is satisfied by the Grothendieck construction introduced in this paper (in diagram (5.4) below, the reader will see how all these notions fit together).

Remark 4.20. Note that the vertical double functors $F, G\colon \mathbb{D} \longrightarrow \mathcal{C}$ can be *restricted* to 2-functors $F^v, G^v\colon \mathbf{V}\mathbb{D} \longrightarrow \mathcal{C}_v$, where $\mathcal{C}_v$ is the 3-category considered in Notation 2.3, and similarly two doubly lax transformations $\theta, \theta'\colon F \Longrightarrow G$ as above induce two (op)lax transformations $\theta, \theta'\colon F^v \Longrightarrow G^v$ (we are *forgetting* about the horizontal structure of $\mathbb{D}$ and $\mathcal{C}$). This leads to considering the following *higher morphisms* between doubly lax transformations, different to the structures defined above, but relevant for the situation described in Example 4.15.

There is a notion of morphism $U : \theta \longrightarrow \theta'$, corresponding to a modification [10], [16, Def. 3.3.8] between the lax transformations $\theta, \theta'\colon F^v \Longrightarrow G^v$, that is given by vertical 2-cells $U_A\colon \theta_A \Longrightarrow \theta'_A$ (just as for a vertical transformation U as above), and (unlike for vertical transformations) for each vertical arrow $u\colon A \longrightarrow A'$ in $\mathbb{D}$, a *vertical* triple cell U_u of $\mathcal{C}$,

$$\begin{tikzcd}
Gu\theta_A \arrow[r, Rightarrow, "1_{Gu\theta_A}"] \arrow[d, Rightarrow, "GuU_A"'] & Gu\theta_A \arrow[d, Rightarrow, "\theta_u"] \\
Gu\theta'_A \arrow[r, phantom, "U_u"] \arrow[d, Rightarrow, "\theta'_u"'] & \theta_{A'}Fu \arrow[d, Rightarrow, "U_{A'}Fu"] \\
\theta'_{A'}Fu \arrow[r, Rightarrow, "1_{\theta'_{A'}Fu}"'] & \theta'_{A'}Fu
\end{tikzcd}$$

satisfying the modification axioms in op. cit., which in this case become similar to (v.1)-(v.3) above but with respect to only the vertical structure of $\mathbb{D}$.

Furthermore, just like we have perturbations [10], [16, Def. 3.3.9] between modifications, there are also morphisms $U \Longrightarrow V$, which are given by a family of double cells $(U_A\ {}^{1}_{1}\ V_A)$ satisfying the perturbation axiom in op. cit. (that in this case becomes similar to (m.2) above). These are the 2-cells of a 2-category $\mathbf{Hom}^v_{d\ell}(F, G)$, that we remark is different from $\mathbf{V}\mathbb{H}\mathsf{om}_{d\ell}(F, G)$.

Remark 4.21. Let $\mathbf{A}, \mathbf{B}$ be 2-categories. Going back to the situation of Example 4.14, recall that we have shown that the lax transformations between 2-functors $F, G\colon \mathbf{A} \to \mathbf{B}$ correspond to doubly lax transformations between the induced vertical double functors $\mathbb{Q}F, \mathbb{Q}G \colon \mathbb{Q}\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}}(\mathbf{B})$. This means that the objects of $\mathsf{Hom}_\ell(F, G)$ (the *usual* category of lax transformations and modifications) are in bijection with the ones of the double category $\mathbb{H}\mathsf{om}_{d\ell}(\mathbb{Q}F, \mathbb{Q}G)$. Observe that, by definition, in this case the horizontal and vertical arrows of this double category are the same as the arrows of $\mathsf{Hom}_\ell(F, G)$, that is modifications, and the double cells are just commutative squares of modifications (recall that $\widehat{\mathbf{Q}}(\mathbf{B})$ had only identities as triple cells). We conclude that we have an isomorphism of double categories $\mathbb{Q}\mathsf{Hom}_\ell(F, G) \cong \mathbb{H}\mathsf{om}_{d\ell}(\mathbb{Q}F, \mathbb{Q}G)$.

Remark 4.22. We consider now two 2-categories $\mathbf{A}, \mathbf{B}$, a fixed family Σ of arrows of $\mathbf{A}$ and a fixed family Ω of 2-cells of $\mathbf{B}$, as in Example 4.15. Recall that we have shown there that, by restricting the horizontal structure in the appropriate double categories of quintets, the σ-ω-natural transformations between 2-functors $F, G \colon \mathbf{A} \to \mathbf{B}$ correspond to doubly lax transformations between the induced vertical double functors $\mathbb{Q}F, \mathbb{Q}G \colon \mathbb{Q}^\Sigma\mathbf{A} \xrightarrow{v} \widehat{\mathbf{Q}^{\boldsymbol{\Omega}}}(\mathbf{B})$.

We note that the category $\mathsf{Hom}_{\sigma,\omega}(F, G)$ that is used in the definition of σ-ω-limit is a full subcategory of $\mathsf{Hom}_\ell(F, G)$; that is, it has σ-ω-natural transformations as objects, and *all* modifications as arrows (coming from considering σ-ω-natural transformations as lax transformations). In other words, the *restricted* horizontal structure of $\mathbb{Q}^\Sigma\mathbf{A}$ and $\widehat{\mathbf{Q}^{\boldsymbol{\Omega}}}(\mathbf{B})$ doesn't play a role in the definition of these modifications. Note that both $\mathbb{Q}^\Sigma\mathbf{A}$ and $\widehat{\mathbf{Q}^{\boldsymbol{\Omega}}}(\mathbf{B})$ still have the full structure of $\mathbf{A}$ and $\mathbf{B}$ vertically, so at this point it should be no surprise that such a modification between two σ-ω-natural transformations θ, θ' corresponds to a morphism $U : \theta \longrightarrow \theta'$ (between the corresponding doubly lax transformations), as defined in Remark 4.20. In this way we have $\mathsf{Hom}_{\sigma,\omega}(F, G) \cong \mathsf{Hom}^v_{d\ell}(\mathbb{Q}F, \mathbb{Q}G)$ (an isomorphism of categories, that is in fact an isomorphism of 2-categories with the definition in Remark 4.20, that is if *perturbations* are considered).

4.4 Doubly lax colimits

We will now define explicitly the notion of doubly lax colimit that corresponds to the notion of doubly lax transformation, spelling out the details for its use in the rest of the paper. Considering an object $E \in \mathcal{C}$, recalling Definition 4.16, and putting $G = \Delta E$ in Definition 4.18 we have a description of a double category of doubly lax cocones with nadir E, $\mathbb{L}\mathsf{C}(F, E) := \mathbb{H}\mathsf{om}_{d\ell}(F, \Delta E)$, that we spell out for convenience:

Definition 4.23. Let $F : \mathbb{D} \xrightarrow{v} \mathcal{C}$ be a vertical double functor, and E an object of $\mathcal{C}$. The double category $\mathbb{L}\mathsf{C}(F, E)$ has

- The doubly lax cones $F \overset{\theta}{\Longrightarrow} \Delta E$ of Definition 4.16 as objects.

- Vertical transformations $\theta \overset{U}{\bullet\!\!\Rightarrow} \theta'$ as vertical arrows, given by vertical 2-cells $\theta_A \overset{U_A}{\bullet\!\!\Rightarrow} \theta'_A$, and triple cells
$$\begin{tikzcd} \theta_A \arrow[r, Rightarrow, "\theta_f"] \arrow[d, Rightarrow, "U_A"'] \arrow[dr, phantom, "U_f"] & \theta_B Ff \arrow[d, Rightarrow, "U_B Ff"] \\ \theta'_A \arrow[r, Rightarrow, "\theta'_f"'] & \theta'_B Ff \end{tikzcd}$$
, satisfying the axioms coming from Definition 4.18:

(v.1) $U_{1_A} = 1_{U_A}$, (v.2) $U_{g \circ f} = (U_f | U_g Ff)$, (v.3) $\frac{\theta_\alpha}{U_{f'} Fu} = \frac{U_f}{\theta'_\alpha}$.

- Horizontal transformations $\theta \overset{H}{\Longrightarrow} \Lambda$ as horizontal arrows, dually given by $\theta_A \overset{H_A}{\Longrightarrow} \Lambda_A$, H_u : $(\theta_u \; {}^{H_A}_{H_{A'} Fu} \; \Lambda_u)$, satisfying:

(h.1) $H_{1^\bullet_A} = 1^\bullet_{H_A}$, (h.2) $H_{v \bullet u} = \frac{H_u}{H_v Fu}$, (h.3) $(\theta_\alpha | \frac{HvFf}{H_{B'} F\alpha}) = (Hu | \Lambda_\alpha)$.

- Modifications
$$\begin{tikzcd} \theta \arrow[r, Rightarrow, "H"] \arrow[d, Rightarrow, "U"'] \arrow[dr, phantom, "M"] & \Lambda \arrow[d, Rightarrow, "V"] \\ \theta' \arrow[r, Rightarrow, "H'"'] & \Lambda' \end{tikzcd}$$
as double cells, given by triple cells $M_A : (U_A \; {}^{H_A}_{H'_A} \; V_A)$ of $\mathcal{C}$ satisfying the conditions

(m.1) $(M_A | V_f) = (U_f | M_B Ff)$ for $A \xrightarrow{f} B$, (m.2) a symmetric condition for $A \overset{u}{\bullet\!\!\longrightarrow} A'$.

Remark 4.24. Note that, by Remark 2.2, the vertical (resp. horizontal) transformations also satisfy:

(v.*) $(U_{A'}Fu) \bullet \theta_u = \theta'_u \bullet U_A$ for each vertical arrow $A \overset{u}{\bullet\!\!\!\longrightarrow} A'$.

(h.*) $(H_B Fu) \circ \theta_f = \Lambda_f \circ H_A$ for each horizontal arrow $A \xrightarrow{f} B$.

We can use this double category $\mathbb{L}\mathsf{C}(F,E)$ to define the notion of doubly lax colimit of a vertical double functor $F\colon \mathbb{D} \overset{v}{\longrightarrow} \mathcal{C}$, just like lax (co)limits are defined for 2-categories. We observe first that composition with an arbitrary doubly lax cone gives a double functor into this double category:

Remark 4.25. Let $F\colon \mathbb{D} \overset{v}{\longrightarrow} \mathcal{C}$ be a vertical double functor, $E, L \in \mathcal{C}$, and a doubly lax cone $\lambda\colon F \Longrightarrow \Delta L$ as in Definition 4.16. Then composition with λ yields a double functor that we denote $\lambda^*\colon \mathcal{C}(L,E) \longrightarrow \mathbb{L}\mathsf{C}(F,E)$. More explicitly, we have that the formula $\lambda^*(-) = (-)\lambda$, holds for objects, both kinds of arrows and double cells in $\mathcal{C}(L,E)$, and with the notation as in Definition 4.23 we can write (the reader should recall the notation introduced in Remark 4.1, in particular for understanding the compositions $\mu\lambda_f$ and $h\lambda_u$ below):

- For $L \xrightarrow{\xi} E$, $(\lambda^*\xi)_A = \xi\lambda_A$, $(\lambda^*\xi)_f = \xi\lambda_f$, $(\lambda^*\xi)_u = \xi\lambda_u$, $(\lambda^*\xi)_\alpha = \xi\lambda_\alpha$,
- For $\xi \overset{\mu}{\bullet\!\!\!\Longrightarrow} \xi'$, $(\lambda^*\mu)_A = \mu\lambda_A$, $(\lambda^*\mu)_f = \mu\lambda_f$,
- For $\xi \overset{h}{\Longrightarrow} \eta$, $(\lambda^*h)_A = h\lambda_A$, $(\lambda^*h)_u = h\lambda_u$,
- For $m : (\mu \; {}^{h}_{h'} \; \mu)$, $(\lambda^*m)_A = m\lambda_A$.

All the verifications are straightforward (see also Remark 6.4).

We are ready now to give the definition of doubly lax (co)limit.

Definition 4.26. Let $F\colon \mathbb{D} \overset{v}{\longrightarrow} \mathcal{C}$ be a vertical double functor. We say that an object $L \in \mathcal{C}$, or more precisely a doubly lax cocone $\lambda\colon F \Longrightarrow \Delta L$ as in Definition 4.16, is the doubly lax colimit of F if, for every $E \in \mathcal{C}$, $\lambda^*\colon \mathcal{C}(L,E) \longrightarrow \mathbb{L}\mathsf{C}(F,E)$ is an isomorphism of double categories.

In view of the formulas in Remark 4.25, this amounts to the following statements:

LC1 For each $F \overset{\theta}{\Longrightarrow} \Delta E$ as in Definition 4.16, there is a unique arrow $L \xrightarrow{\xi} E$ of $\mathcal{C}$ such that **(a)** $\theta_A = \xi\lambda_A$, **(b)** $\theta_f = \xi\lambda_f$, **(c)** $\theta_u = \xi\lambda_u$, **(d)** $\theta_\alpha = \xi\lambda_\alpha$,

LC2v For each $\lambda^*\xi \overset{U}{\bullet\!\!\!\Longrightarrow} \lambda^*\xi'$ as in Definition 4.23, there is a unique vertical 2-cell $\lambda^*\xi \overset{\mu}{\bullet\!\!\!\Longrightarrow} \lambda^*\xi'$ of $\mathcal{C}$ such that **(a)** $U_A = \mu\lambda_A$, **(b)** $U_f = \mu\lambda_f$,

LC2h For each $\lambda^*\xi \overset{H}{\Longrightarrow} \lambda^*\eta$ as in Definition 4.23, there is a unique horizontal 2-cell $\xi \overset{h}{\Longrightarrow} \eta$ of $\mathcal{C}$ such that **(a)** $H_A = h\lambda_A$, **(b)** $H_u = h\lambda_u$,

LC3 For each $M : (\lambda^*\mu \; {}^{\lambda^*h}_{\lambda^*h'} \; \lambda^*\nu)$ as in Definition 4.23, there is a unique triple cell $m : (\mu \; {}^{h}_{h'} \; \nu)$ of $\mathcal{C}$ such that $M_A = m\lambda_A$.

Remark 4.27. We recall that there is a notion of *trilimit* for tricategories [10], [16], so in particular, for strict 3-categories (one concise way of defining lax trilimits is as triadjoints, see for example [6, Th 3.1]). Roughly, the "lax tricones" now involve a triple cell instead of an equality of 2-cells as in the 2-dimensional lax cones. It should be remarked that in the present paper, we are considering limits whose universal property is given by an isomorphism, and not an equivalence, so we will think of lax trilimits in this way, as a 3-dimensional version of the 2-dimensional lax limits.

Thinking that DblCat-categories extend 3-categories in one (horizontal) direction, the notion of doubly lax limit above extends the notion of "lax trilimit" in that same direction. When $\mathbb{D}$ is a vertical double category, that is of the form $\mathbb{V}\mathbf{A}$, vertical double functors $F\colon \mathbb{V}\mathbf{A} \overset{v}{\longrightarrow} \mathcal{C}$ correspond to 2-functors $\overline{F}\colon \mathbf{A} \longrightarrow \mathcal{C}_v$, and the doubly lax colimit of F is the same as the lax tricolimit of $\overline{F}$.

Example 4.28. Let $\mathbf{A}, \mathbf{B}$ be 2-categories, and consider a 2-functor $F\colon \mathbf{A} \to \mathbf{B}$ and an arbitrary object $E \in \mathbf{B}$ with the induced constant 2-functor $\Delta E\colon \mathbf{A} \to \mathbf{B}$. In the notation of Remark 4.21 with $G = \Delta E$, we note that $\mathbb{Q}\Delta E$, as constructed there, is the vertical functor constant at E and we denote this also by $\Delta E\colon \mathbb{Q}\mathbf{A} \xrightarrow{\bullet} \hat{\mathbf{Q}}(\mathbf{B})$. So we obtain

$$\mathbb{L}\mathsf{C}(\mathbb{Q}F, E) = \mathbb{H}\mathsf{om}_{d\ell}(\mathbb{Q}F, \Delta E) \cong \mathbb{Q}\mathsf{Hom}_\ell(F, \Delta E). \tag{4.29}$$

Given $L \in \mathbf{B}$ and a lax cocone $F \overset{\mu}{\Longrightarrow} \Delta L$, we have an induced functor $\mathbf{B}(L, E) \xrightarrow{\mu^*} \mathsf{Hom}_\ell(F, \Delta E)$, and by definition L is the lax colimit of F (in the sense of 2-category theory) if this functor is an isomorphism of categories for each E. It is immediate to check that, if we write λ for the corresponding doubly lax cocone $\mathbb{Q}F \Longrightarrow \Delta L$ (as in (4.29)), we have that the double functor λ^* in Definition 4.26 is $\mathbb{Q}(\mu^*)$; thus, L is the lax colimit of F if and only if it is the doubly lax colimit of $\mathbb{Q}F$.

Example 4.30. We consider finally the situation in Remark 4.22, and recall that the (2-)category "$\mathbf{Hom}^v_{d\ell}$" that is used for the definition of σ-ω-limit is not the one that we have considered for the definition of doubly lax limit. Without getting into the details, as this is not relevant for the subject of this paper, let us just mention that we could define a different notion of double limit, by requiring the (2-)functor $\lambda^*\colon \mathcal{C}_v(L, E) \longrightarrow \mathbf{Hom}^v_{d\ell}(F, \Delta E)$ to be an isomorphism of (2-)categories. Let us momentarily call this notion "vertical doubly lax limit". Then a similar reasoning to the one of Example 4.28 would show that, with the constructions in Remark 4.22, L is the σ-ω-limit of a 2-functor F if and only if it is the vertical doubly lax limit of the induced vertical double functor $\mathbb{Q}F\colon \mathbb{Q}^\Sigma\mathbf{A} \xrightarrow{\bullet} \widehat{\mathbf{Q}^\Omega}(\mathbf{B})$.

5 The Grothendieck construction as a doubly lax colimit

The double Grothendieck construction of Section 3 provides a first example of a doubly lax colimit.

Theorem 5.1. *Given a vertical double functor $F\colon \mathbb{D} \xrightarrow{\bullet} \mathcal{D}\mathbf{blCat}$, the double Grothendieck construction $\mathbb{G}\mathrm{r}(F)$ is the doubly lax colimit of F in $\mathcal{D}\mathbf{blCat}$, as defined in Definition 4.26.*

Proof. There is a doubly lax cone, i.e. a doubly lax transformation $F \overset{\lambda}{\Longrightarrow} \Delta\mathbb{G}\mathrm{r}(F)$, defined by:

- The double functors $FA \xrightarrow{\lambda_A} \mathbb{G}\mathrm{r}(F)$ are given by $\lambda_A(-) = (A, -)$. That is, for objects x, horizontal arrows φ, vertical arrows ρ, and double cells Φ of FA, we have respectively
  $$\lambda_A(x) = (A, x), \quad \lambda_A(\varphi) = (1_A, \varphi), \quad \lambda_A(\rho) = (1^\bullet_A, \rho), \quad \lambda_A(\Phi) = (1^\square_A, \Phi).$$
- The horizontal transformations $\lambda_A \overset{\lambda_f}{\Longrightarrow} \lambda_B \otimes Ff$ are given by: for each $x \in FA$, resp. each vertical arrow $x \xrightarrow{\rho} x'$ in FA:
  $$(A,x) \xrightarrow{(\lambda_f)_x = (f, 1_{Ffx})} (B, Ffx), \qquad \begin{tikzcd} (A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[d, "{(1^\bullet_A,\rho)}"'] \arrow[dr, phantom, "{(\lambda_f)_\rho = (1^\bullet_f, 1_{Ff\rho})}"] & (B, Ffx) \arrow[d, "{(1^\bullet_B, Ff\rho)}"] \\ (A,x') \arrow[r, "{(f,1_{Ffx'})}"'] & (B, Ffx') \end{tikzcd}$$
- The vertical transformations $\lambda_A \overset{\lambda_u}{\Longrightarrow} \lambda_{A'} \otimes Fu$ are given by: for each $x \in FA$, resp. each horizontal arrow $x \xrightarrow{\varphi} y$ in FA:
  $$\begin{tikzcd} (A,x) \arrow[d, "{(\lambda_u)_x = (u, 1^\bullet_{Fux})}"] \\ (A', Fux) \end{tikzcd}, \qquad \begin{tikzcd} (A,x) \arrow[r, "{(1_A,\varphi)}"] \arrow[d, "{(u,1^\bullet_{Fux})}"'] \arrow[dr, phantom, "{(\lambda_u)_\varphi = (1_u, 1^\bullet_{Fu\varphi})}"] & (A,y) \arrow[d, "{(u,1^\bullet_{Fuy})}"] \\ (A', Fux) \arrow[r, "{(1_{A'}, Fu\varphi)}"'] & (A', Fuy) \end{tikzcd}$$

- The modifications λ_α, as in the left diagram below, are given for each $x \in FA$ by the diagram on the right:

$$
\begin{tikzcd}
\lambda_A \arrow[r, Rightarrow, "\lambda_f"] \arrow[dd, Rightarrow, "\lambda_u"'] & \lambda_B \otimes Ff \arrow[d, Rightarrow, "\lambda_v \otimes Ff"] \\
\arrow[r, phantom, "\lambda_\alpha"] & \lambda_{B'} \otimes Fv \otimes Ff \arrow[d, Rightarrow, "\lambda_{B'} \otimes F\alpha"] \\
\lambda_{A'} \otimes Fu \arrow[r, Rightarrow, "\lambda_{f'} \otimes Fu"'] & \lambda_{B'} \otimes Ff' \otimes Fu
\end{tikzcd}
\qquad
\begin{tikzcd}
(A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[dd, "{(u,1^\bullet_{Fux})}"'] & (B,Ffx) \arrow[d, "{(v,1^\bullet_{F(vf)x})}"] \\
\arrow[r, phantom, "{(\lambda_\alpha)_x=(\alpha,1_{(F\alpha)_x})}"] & (B',FvFfx) \arrow[d, "{(1^\bullet_{B'},(F\alpha)_x)}"] \\
(A',Fux) \arrow[r, "{(f',1_{F(f'u)x})}"'] & (B',Ff'Fux)
\end{tikzcd}
$$

We will now show that this cone is universal as described in Definition 4.26. So let $\mathbb{E}$ be a double category. We need to check that the data for the cocone $\lambda\colon F \Rightarrow \Delta\mathbb{G}\mathrm{r}(F)$ satisfies the statements LC1, LC2v, LC2h and LC3 with respect to $\mathbb{E}$. Each of these statements amounts to the existence of a unique arrow (or cell) satisfying certain conditions, and our strategy for each proof will be the same: we will show that these conditions uniquely determine an arrow (or cell), and that this arrow or cell satisfies the required conditions.

- [Proof of LC1] We will show: for each doubly lax transformation $\theta\colon F \Longrightarrow \Delta\mathbb{E}$, there is a unique arrow $\xi\colon \mathbb{G}\mathrm{r}F \longrightarrow \mathbb{E}$ of $\mathcal{C}$ such that

  $$\textbf{(a)}\ \theta_A = \xi\lambda_A, \qquad \textbf{(b)}\ \theta_f = \xi\lambda_f, \qquad \textbf{(c)}\ \theta_u = \xi\lambda_u, \qquad \textbf{(d)}\ \theta_\alpha = \xi\lambda_\alpha.$$

  Let $\theta\colon F \Longrightarrow \Delta\mathbb{E}$. We note first that the equalities **(a)**-**(d)**, in view of the definition of λ above, are the following conditions

  $$
  \begin{array}{lcccc}
  & \textbf{(1)} & \textbf{(2h)} & \textbf{(2v)} & \textbf{(3)} \\
  \textbf{(a)} & \xi(A,x) = \theta_A(x), & \xi(1_A,\varphi) = \theta_A(\varphi), & \xi(1^\bullet_A,\rho) = \theta_A(\rho), & \xi(1^\Box_A,\Phi) = \theta_A(\Phi), \\
  \textbf{(b)} & & \xi(f,1_{Ffx}) = (\theta_f)_x, & & \xi(1^\bullet_f,1_{Ff\rho}) = (\theta_f)_\rho, \\
  \textbf{(c)} & & & \xi(u,1^\bullet_{Fux}) = (\theta_u)_x, & \xi(1_u,1^\bullet_{Fu\varphi}) = (\theta_u)_\varphi, \\
  \textbf{(d)} & & & & \xi(\alpha,1_{(F\alpha)_x}) = (\theta_\alpha)_x.
  \end{array}
  \tag{5.2}
  $$

  Recall the factorizations given in Remark 3.6 (we include below a copy of diagram (3.9) for the reader's convenience). An arbitrary horizontal arrow (f,φ) is factored as the top composition in the diagram below, which is formed by arrows of the form appearing in the column **(2h)**. Similarly, an arbitrary vertical arrow (u,ρ) is factored as the left composition, which is formed by arrows in **(2v)**. And an arbitrary double cell (α,Φ) is factored as follows, using double cells in **(3)**:

  $$
  \begin{tikzcd}
  (A,x) \arrow[r, "{(f,1_{Ffx})}"] \arrow[dd, "{(u,1^\bullet_{Fux})}"'] & (B,Ffx) \arrow[r, "{(1_B,\varphi)}"] \arrow[d, "{(v,1^\bullet_{F(vf)x})}"'] \arrow[dr, phantom, "{(1_v,1^\bullet_{Fv\varphi})}"] & (B,y) \arrow[d, "{(v,1^\bullet_{Fvy})}"] \\
  \arrow[r, phantom, "{(\alpha,1_{(F\alpha)_x})}"] & (B',FvFfx) \arrow[r, "{(1_{B'},Fv\varphi)}"'] \arrow[d, "{(1^\bullet_{B'},(F\alpha)_x)}"] & (B',Fvy) \arrow[dd, "{(1^\bullet_{B'},\lambda)}"] \\
  (A',Fux) \arrow[r, "{(f',1_{F(f'u)x})}"] \arrow[d, "{(1^\bullet_{A'},\rho)}"'] \arrow[dr, phantom, "{(1^\bullet_{f'},1_{Ff'\rho})}"] & (B',Ff'Fux) \arrow[d, "{(1^\bullet_{B'},Ff'\rho)}"] \arrow[r, phantom, "{(1^\Box_{B'},\Phi)}"] & \\
  (A',x') \arrow[r, "{(f',1_{Ff'x'})}"'] & (B',Ff'x') \arrow[r, "{(1_{B'},\varphi')}"'] & (B',y')
  \end{tikzcd}
  \tag{5.3}
  $$

This implies that the only possible definition of a double functor $\xi\colon \mathbb{G}\mathrm{r}(F) \to \mathbb{E}$ satisfying the conditions in (5.2) is as follows:

(1) on objects (A,x): $\xi(A,x) = \theta_A(x)$;

(2h) on horizontal arrows $(f,\varphi)\colon (A,x) \to (B,y)$: $\xi(f,\varphi) = \theta_B(\varphi) \circ (\theta_f)_x$;

(2v) on vertical arrows $(u,\rho)\colon (A,x) \longrightarrow\!\!\!\!\!\bullet\,\, (A',x')$: $\xi(u,\rho) = \theta_{A'}(\rho) \bullet (\theta_u)_x$;

(3) on double cells: $\xi(\alpha,\Phi)$ equals the following pasting

$$\begin{tikzcd}
\theta_A(x) \arrow[rr, "(\theta_f)_x"] \arrow[dd, "(\theta_u)_x"' , "\bullet" marking] & & \theta_B(Ffx) \arrow[rr, "\theta_B(\varphi)"] \arrow[d, "(\theta_v)_{Ffx}", "\bullet" marking] \arrow[drr, phantom, "(\theta_v)_\varphi"] & & \theta_B(y) \arrow[d, "(\theta_v)_y", "\bullet" marking] \\
 & (\theta_\alpha)_x & \theta_{B'}(FvFfx) \arrow[rr, "\theta_{B'}(Fv\varphi)"'] \arrow[d, "\theta_{B'}((F\alpha)_x))", "\bullet" marking] & & \theta_{B'}(Fvy) \arrow[dd, "\theta_{B'}(\lambda)", "\bullet" marking] \\
\theta_{A'}(Fux) \arrow[rr, "(\theta_{f'})_{Fux}"] \arrow[d, "\theta_{A'}(\rho)"', "\bullet" marking] \arrow[drr, phantom, "(\theta_{f'})_\rho"] & & \theta_{B'}(Ff'Fux) \arrow[d, "\theta_{B'}(Ff'\rho)", "\bullet" marking] & \theta_{B'}(\Phi) & \\
\theta_{A'}(x') \arrow[rr, "(\theta_{f'})_{x'}"'] & & \theta_{B'}(Ff'x') \arrow[rr, "\theta_{B'}(\varphi')"'] & & \theta_{B'}(y')\,.
\end{tikzcd} \tag{5.4}$$

Replacing $f, \varphi, u, \rho, \alpha$, and/or Φ by the appropriate identities in each of the formulas (**2h**), (**2v**), (**3**), and using that θ preserves identities (items 1,2, and 3 in Definition 4.11, and the fact that each θ_A is a double functor), we recover the formulas in each of the respective three columns (this shows that ξ, as defined above, does indeed satisfy the conditions **(a)-(d)** in LC1). By further substituting identities for the variables in the formulas (**2h**), (**2v**), (**3**), it follows that ξ preserves identities. To finish the proof of LC1, it only remains then to show that ξ preserves compositions. Since this is a long task, we defer this computation to Appendix A.

- [Proof of LC2v] Let $U\colon \lambda^*\xi \Longrightarrow\!\!\!\!\!\bullet\,\, \lambda^*\xi'$. Similarly to the case of LC1 above, the equalities **(a)** and **(b)** in LC2v correspond to the conditions

$$\textbf{(a)}\ \mu_{(A,x)} = (U_A)_x, \quad \mu_{(1_A,\varphi)} = (U_A)_\varphi, \qquad \textbf{(b)}\ \mu_{(f,1_{Ffx})} = (U_f)_x. \tag{5.5}$$

By the factorization of (f,φ) in (3.7) or in the top row of (5.3), and in view of axiom (v.2) in Definition 2.1, this gives a unique possible definition of μ such that (5.5) holds, by the formula $\mu_{(f,\varphi)} = ((U_f)_x|(U_B)_\varphi)$: (where we denote $\theta = \lambda^*\xi$, $\theta' = \lambda^*\xi'$)

$$\begin{tikzcd}
\xi(A,x) \arrow[r, "\xi(f,\varphi)"] \arrow[d, "\mu_{(A,x)}"', "\bullet" marking] \arrow[dr, phantom, "\mu_{(f,\varphi)}"] & \xi(B,y) \arrow[d, "\mu_{(B,y)}", "\bullet" marking] \\
\xi'(A,x) \arrow[r, "\xi'(f,\varphi)"'] & \xi'(B,y)
\end{tikzcd}
=
\begin{tikzcd}
\theta_A(x) \arrow[r, "(\theta_f)_x"] \arrow[d, "(U_A)_x"', "\bullet" marking] \arrow[dr, phantom, "(U_f)_x"] & \theta_B(Ffx) \arrow[r, "\theta_B(\varphi)"] \arrow[d, "(U_B)_{Ffx}"', "\bullet" marking] \arrow[dr, phantom, "(U_B)_\varphi"] & \theta_B(y) \arrow[d, "(U_B)_y", "\bullet" marking] \\
\theta'_A(x) \arrow[r, "(\theta'_f)_x"'] & \theta'_B(Ffx) \arrow[r, "\theta'_B(\varphi)"'] & \theta'_B(y)
\end{tikzcd} \tag{5.6}$$

Replacing either f or φ by a horizontal identity, and using respectively either condition (v.1) in Definition 4.23 for U, or (v.1) in Definition 2.1 for the vertical transformation U_B, we recover the formulas in (5.5). Replacing both f and φ by identities gives axiom (v.1) in Definition 2.1 for $\mu : \xi \Longrightarrow\!\!\!\!\!\bullet\,\, \xi'$. The verification of (v.2) and (v.3), showing that μ is indeed a vertical transformation and finishing the proof is deferred to Section A.2 in Appendix A.

- [Proof of LC2h] Except for the verification of (h.3) at the very end of the proof, this proof is formally dual to the proof of LC2v and it is deferred to Section A.3 in Appendix A.

- [Proof of LC3] Let $M : (\lambda^*\mu \; {}^{\lambda^* h}_{\lambda^* h'} \; \lambda^*\nu) = (U \; {}^{H}_{H'} \; V)$, then the formula $M_A = m\lambda_A$ forces the definition $m_{(A,x)} = (M_A)_x$, and it only remains to check that m is indeed a modification. To show axiom (m.1), we consider an arrow $(f,\varphi)\colon (A,x) \to (B,y)$ of $\mathbb{G}\mathrm{r}(F)$, and we will show the equality $(m_{(A,x)}|\nu_{(f,\varphi)}) = (\mu_{(f,\varphi)}|m_{(B,y)})$. Replacing each double cell by its definition, and using in turn (m.1) in Definition 4.23 and (m.1) in 2.2 for the modification M_B, we compute:

$$(m_{(A,x)}|\nu_{(f,\varphi)}) = ((M_A)_x|(V_f)_x|(V_B)_\varphi) = ((U_f)_x|(M_B)_{Ffx}|(V_B)_\varphi) = \\ ((U_f)_x|(U_B)_\varphi|(M_B)_y) = (\mu_{(f,\varphi)}|m_{(B,y)}).$$

  Axiom (m.2) is dual and left to the reader. □

Example 5.7. We consider, as in Example 3.12, a 2-functor $\mathbf{A} \xrightarrow{F} \mathbf{2\text{-}Cat}$, and recall from (3.13) that $\int_{\mathbb{V}\mathbf{A}} \mathbb{V}(\mathbb{V}F) = \mathbb{V}\int_{\mathbf{A}} F$. From Theorem 5.1 and Remark 4.27, it follows that $\mathbb{V}\int_{\mathbf{A}} F$ is the lax tricolimit of $\mathbb{V}F$ in $\mathcal{D}\mathbf{blCat}_v$. Since $\mathbb{V}$ is an isomorphism of 3-categories between **2-Cat** and its image in $\mathcal{D}\mathbf{blCat}_v$, we conclude that $\int_{\mathbf{A}} F$ is the lax tricolimit of F in **2-Cat** (for a proof of a similar fact for diagrams of bicategories indexed by a category, see [6]).

Considering the other constructions in Example 3.12, Theorem 5.1 leads to other universal properties that these constructions have, which may be interesting to explore.

5.1 An application to 2-colimits in Cat

Consider a Set-valued functor F. It is known that one can compute the colimit of F as the connected components of its category of elements (that is, the usual Grothendieck construction applied to F when seen as taking values in Cat). The fact that this set is indeed the colimit of F can be seen to follow from the *usual* lax colimit property of the Grothendieck construction (see **A** in Section 1). Note also that taking connected components is the left adjoint to the inclusion of Set in Cat. We will show now that there is a double categorical analogue of this situation, which allows to compute 2-colimits of categories, where the result in Theorem 5.1 plays the role of the result **A**. For the basic definitions and results on 2-limits and 2-colimits we refer to [20].

Recall Example 3.12, item 3, where for any 2-functor $F\colon \mathbf{A} \longrightarrow \mathbf{Cat}$ we constructed the vertical double functor $\mathbb{Q}(\mathbb{V}\circ F)\colon \mathbb{H}\mathbf{A} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$ (note the restriction of the domain to $\mathbb{H}\mathbf{A}$), and we showed that its double Grothendieck construction, as defined in Definition 3.1, is the double category of elements of F, $\mathbb{E}\mathsf{l}(F)$, as originally constructed by Pare.

Lemma 5.8. *Let* $\mathbf{C}$ *be a category. In the situation of Example 3.12, item 3, a doubly lax cone for* $\mathbb{Q}(\mathbb{V}\circ F)$ *with vertex* $\mathbb{V}\mathbf{C}$ *corresponds naturally to a 2-natural transformation* $F \Longrightarrow \triangle\mathbf{C}$ *(i.e. a* strict 2-cone *for* F *with vertex* $\mathbf{C}$*).*

Proof. Consider the description of an arbitrary doubly lax cone in Definition 4.16. Note that, since the domain of the vertical double functor $\mathbb{Q}(\mathbb{V}\circ F)$ is a horizontal double category, a doubly lax cone in this case is given by the following data (there are no non-trivial θ_u)

- for each object A of $\mathbf{A}$, a double functor $\mathbb{V}FA \xrightarrow{\theta_A} \mathbb{V}\mathbf{C}$;
- for each arrow $A \xrightarrow{f} B$, a horizontal transformation
$$\begin{tikzcd} \mathbb{V}FA \arrow[d, "\mathbb{V}F(f)"'] \arrow[dr, "\theta_A"] \arrow[d, phantom, shift left=4ex, "\Downarrow\theta_f"] & \\ \mathbb{V}FB \arrow[r, "\theta_B"'] & \mathbb{V}\mathbf{C} \end{tikzcd}$$
such that
- for each 2-cell $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$ the following diagram (corresponding to (4.17)) commutes

$$
\begin{tikzcd}
\theta_A \arrow[r, Rightarrow, "\theta_f"] \arrow[d, Rightarrow, "1^\bullet_{\theta_A}"'] & \theta_B \mathbb{V}F(f) \arrow[d, Rightarrow, "\theta_B \mathbb{V}F(\alpha)"] \\
\theta_A \arrow[r, Rightarrow, "\theta_{f'}"'] & \theta_B \mathbb{V}F(f')
\end{tikzcd}
\tag{5.9}
$$

Recall also that a 2-natural transformation $F \Longrightarrow \triangle\mathbf{C}$ has components which are functors $\mu_A\colon FA \longrightarrow \mathbf{C}$, satisfying that the diagram on the left is commutative for each f, and that the equality on the right holds for each α

$$
\begin{tikzcd}
FA \arrow[d, "F(f)"'] \arrow[dr, "\mu_A"] & \\
FB \arrow[r, "\mu_B"'] & \mathbf{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
FA \arrow[d, bend right, "F(f')"'] \arrow[d, bend left, "F(f)"] \arrow[d, phantom, "\overset{F\alpha}{\Leftarrow}"] \arrow[dr, "\mu_A"] & \\
FB \arrow[r, "\mu_B"'] & \mathbf{C}
\end{tikzcd}
=
\begin{tikzcd}
FA \arrow[d, "F(f')"'] \arrow[dr, "\mu_A"] & \\
FB \arrow[r, "\mu_B"'] & \mathbf{C}
\end{tikzcd}
\tag{5.10}
$$

The proof finishes by recalling that $\mathbb{V}$ is an isomorphism with its image, so the double functors θ_A correspond to the functors μ_A, and noting that a horizontal transformation between double functors whose codomain is $\mathbb{V}\mathbf{C}$ can only be an identity ((5.9) corresponds then to the equality in (5.10)). ☐

We consider now the left adjoint $L\colon \mathsf{DblCat} \longrightarrow \mathsf{Cat}$ of the functor $\mathbb{V}$.

Theorem 5.11. *Let $F\colon \mathbf{A} \longrightarrow \mathbf{Cat}$ be a 2-functor. Then $L\mathbb{E}\mathsf{l}(F)$ is the 2-colimit of F.*

Proof. Since **Cat** is known to be 2-cocomplete, it suffices to show the 1-dimensional universal property (see [20, p. 306]). Note that we have the natural chain of bijections given in turn by Lemma 5.8, Theorem 5.1 (recalling from Example 3.12, item 3 that $\mathbb{G}\mathrm{r}(\mathbb{Q}(\mathbb{V} \circ F)) = \mathbb{E}\mathsf{l}(F)$), and by the adjunction $L \dashv \mathbb{V}$

$F \Longrightarrow \triangle\mathbf{C}$	2-natural transformation
$\mathbb{Q}(\mathbb{V} \circ F) \Longrightarrow \triangle\mathbb{V}\mathbf{C}$	doubly lax transformation
$\mathbb{E}\mathsf{l}(F) \longrightarrow \mathbb{V}\mathbf{C}$	double functor
$L\mathbb{E}\mathsf{l}(F) \longrightarrow \mathbf{C}$	functor

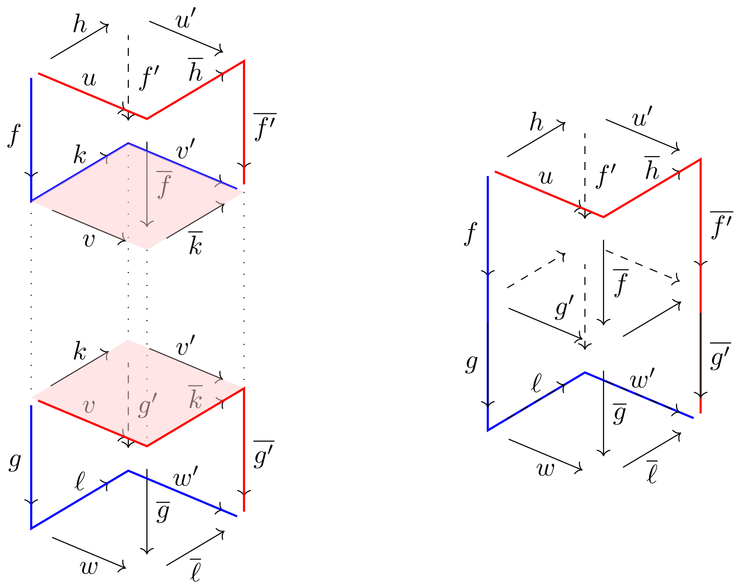
This finishes the proof that $L\mathbb{E}\mathsf{l}(F)$ is the 2-colimit of F. ☐

6 The *higher structure* isomorphism of double categories

6.1 Further structure of double cylinders

Note that in Section 4.1 we have given $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$ a double category structure, in which horizontal and vertical composition is given by pasting the cylinders in the corresponding directions. Here we will show that cylinders can also be pasted in the remaining (third) direction, and that this operation is in fact the composition of a category object $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$ in DblCat, which has the double cylinders as arrows.

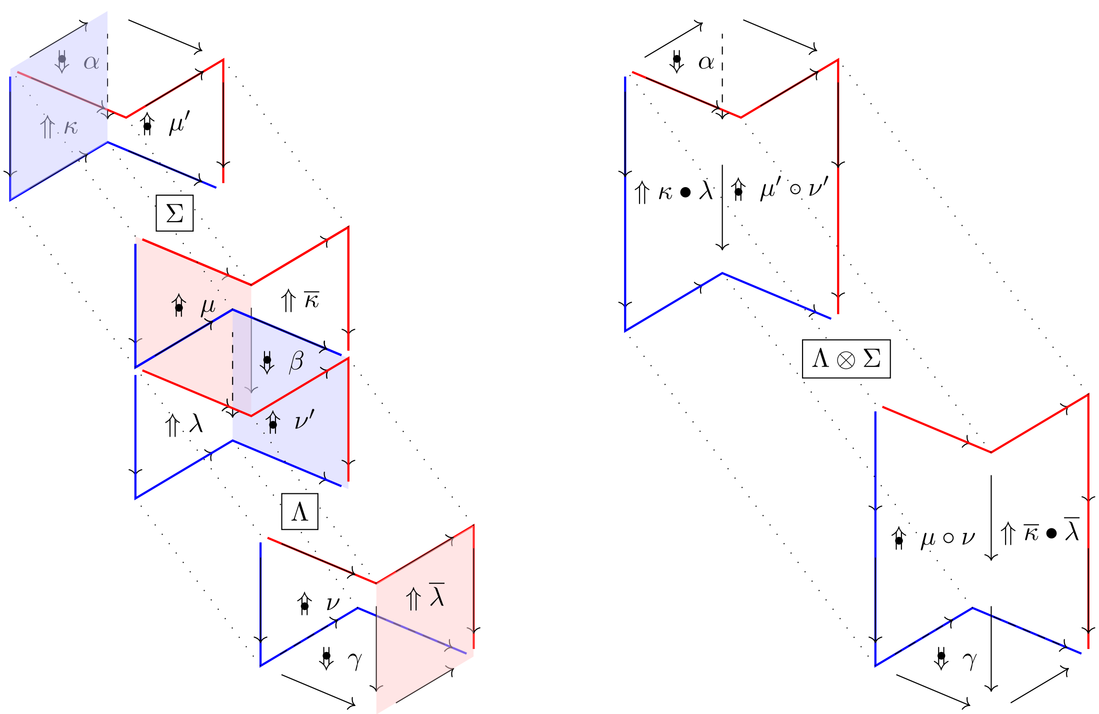
Definition 6.1 (z-pasting, tensor of cylinders)**.** Consider two cylinders as in (4.3), sharing a face β as in the diagram below.



In such a situation we can define a tensoring of cylinders, that is an operation

$$\left(\begin{tikzcd} f \arrow[r, "{(h,\kappa,k)}"] \arrow[d, "{(u,\mu,v)}"'] \arrow[dr, phantom, "{(\alpha,\Sigma,\beta)}"] & f' \arrow[d, "{(u',\mu',v')}"'] \\ \overline{f} \arrow[r, "{(\overline{h},\overline{\kappa},\overline{k})}"'] & \overline{f'} \end{tikzcd} , \begin{tikzcd} g \arrow[r, "{(k,\lambda,\ell)}"] \arrow[d, "{(v,\nu,w)}"] \arrow[dr, phantom, "{(\beta,\Lambda,\gamma)}"] & g' \arrow[d, "{(v',\nu',w')}"] \\ \overline{g} \arrow[r, "{(\overline{k},\overline{\lambda},\overline{\ell})}"'] & \overline{g'} \end{tikzcd}\right) \quad \rightsquigarrow \quad \begin{tikzcd} gf \arrow[r, "{(h,\lambda\bullet\kappa,\ell)}"] \arrow[d, "{(u,\nu\bullet\mu,w)}"'] \arrow[dr, phantom, "{(\alpha,\Lambda\otimes\Sigma,\gamma)}"] & g'f' \arrow[d, "{(u',\nu'\bullet\mu',w')}"'] \\ \overline{g}\,\overline{f} \arrow[r, "{(\overline{h},\overline{\lambda}\bullet\overline{\kappa},\overline{\ell})}"'] & \overline{g'}\,\overline{f'} \end{tikzcd}$$

given by pasting in the shared face β. We picture this as



The triple cell $\Lambda \otimes \Sigma$ is defined as the following pasting diagram (recall Remark 4.1):

$$
\begin{tikzcd}[column sep=small]
w'\ell g f \arrow[rr, "w'\lambda f"] \arrow[d, "\gamma g f"] & & w'g'kf \arrow[rr, "w'g'\kappa"] \arrow[d, "\nu' k f"'] \arrow[rr, phantom, shift right=6ex, "\nu'\kappa"] & & w'g'f'h \arrow[d, "\nu' f' h"] \\
\overline{\ell} w g f \arrow[d, "\overline{\ell}\nu f"] \arrow[rr, phantom, "\Lambda f"] & & \overline{g'} v' k f \arrow[rr, "\overline{g'} v' \kappa"] \arrow[d, "\overline{g'}\beta f"] \arrow[rr, phantom, shift right=6ex, "\overline{g'}\Sigma"] & & \overline{g'} v' f' h \arrow[d, "\overline{g'}\mu' h"] \\
\overline{\ell}\,\overline{g} v f \arrow[rr, "\overline{\lambda} v f"] \arrow[d, "\overline{\ell}\,\overline{g}\mu"'] \arrow[rr, phantom, shift right=6ex, "\overline{\lambda}\mu"] & & \overline{g'}\,\overline{k} v f \arrow[d, "\overline{g'}\,\overline{k}\mu"] & & \overline{g'}\,\overline{f'} u' h \arrow[d, "\overline{g'}\,\overline{f'}\alpha"] \\
\overline{\ell}\,\overline{g}\,\overline{f} u \arrow[rr, "\overline{\lambda}\,\overline{f} u"] & & \overline{g'}\,\overline{k}\,\overline{f} u \arrow[rr, "\overline{g'}\overline{\kappa} u"] & & \overline{g'}\,\overline{f'}\,\overline{h} u
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
w'\ell g f \arrow[r, "w'(\kappa\bullet\lambda)"] \arrow[d, "\gamma g f"'] & w'g'f'h \arrow[d, "(\mu'\circ\nu')h"] \\
\overline{\ell} w g f \arrow[d, "\overline{\ell}(\mu\circ\nu)"'] \arrow[r, phantom, "\Lambda\otimes\Sigma"] & \overline{g'}\,\overline{f'} u' h \arrow[d, "\overline{g'}\,\overline{f'}\alpha"] \\
\overline{\ell}\,\overline{g}\,\overline{f} u \arrow[r, "(\overline{\kappa}\bullet\overline{\lambda})u"] & \overline{g'}\,\overline{f'}\,\overline{h} u
\end{tikzcd}
$$

Remark 6.2. Recall the *top* and *bottom* projections from Example 4.10, that are double functors $d_0, d_1 \colon \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \longrightarrow \mathbb{Q}\mathcal{C}_v$, mapping a cylinder Σ as in (4.3) respectively to α, β. There is also a double functor $i \colon \mathbb{Q}\mathcal{C}_v \longrightarrow \mathbb{C}\mathsf{yl}_v(\mathcal{C})$ that maps objects, horizontal arrows, vertical arrows and double cells to their respective identities. All the verifications are straightforward. For example, for a double cell

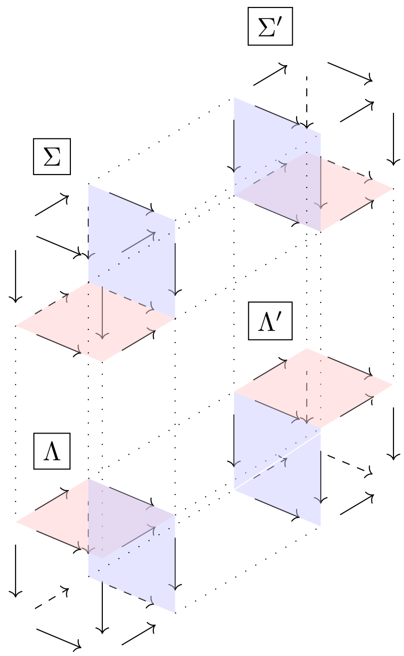
$$
\begin{array}{ccc}
 & \xrightarrow{h} & \\
u\downarrow & \Downarrow\ \alpha & \downarrow u' \\
 & \xrightarrow[\overline{h}]{} &
\end{array}
$$

$i\alpha$ is a cylinder as in (4.3) in which $\Sigma = id_\alpha$ ($\beta = \alpha$, and all the other 2-cells are identities).

Proposition 6.3. *The z-pasting of Definition 6.1 yields a double functor* $\mathbb{C}\mathsf{yl}_v(\mathcal{C}) \times_{\mathbb{Q}\mathcal{C}_v} \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \xrightarrow{\otimes} \mathbb{C}\mathsf{yl}_v(\mathcal{C})$ *that, together with* d_0, d_1*, and* i*, defines a category object* $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$ *in* DblCat

$$
\begin{tikzcd}
\mathbb{C}\mathsf{yl}_v(\mathcal{C}) \times_{\mathbb{Q}\mathcal{C}_v} \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \arrow[r, "\otimes"] & \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & \mathbb{Q}\mathcal{C}_v \arrow[l, "i"']
\end{tikzcd}
$$

Proof. Most verifications are immediate. Others are lengthy but straightforward, so some details are left to the reader. To show that the double functor $\otimes$ distributes with respect to horizontal pasting, we consider four cylinders fitting as



If we paste first horizontally and then in the z-direction, or if we do it the other way around, just by unwinding the definitions we obtain two composites of triple cells whose boundaries are 4 by 4 squares. If we cut these squares into four equal "1 by 4" parts with vertical lines, then the leftmost and rightmost parts of each square are equal, and the two middle parts are respectively the left and right diagrams below:

$$
\begin{array}{ccc}
\Lambda' k f & & \nu'' k' \kappa \\
 & & \overline{g''}\beta'\kappa \\
\overline{\lambda'}\beta f & & \overline{g''}\ \overline{k'}\Sigma \\
\overline{\lambda'}\ \overline{k}\mu & &
\end{array}
\qquad\qquad
\begin{array}{ccc}
\gamma' g' \kappa & & \Lambda' f' h \\
\overline{\ell'}\nu'\kappa & & \\
\overline{\ell'}\ \overline{g'}\Sigma & & \overline{\lambda'}\mu' h \\
 & & \overline{\lambda'}\ \overline{f'}\alpha
\end{array}
$$

Both diagrams can be seen to be the vertical pasting of the triple cells $\Lambda'\kappa$ and $\overline{\lambda'}\Sigma$. The computations showing that the double functor $\otimes$ distributes with respect to vertical pasting is dual to the one we did for proving the interchange property in $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$. The computation showing that composition in this internal category is associative is dual to the one we did for proving the associativity for the horizontal composition in $\mathbb{C}\mathsf{yl}_v(\mathcal{C})$. $\square$

Remark 6.4. Note that a double category $\mathbb{D}$ can be thought of as a discrete category object $\mathbb{D}_{\text{disc}}$ in DblCat. Vertical double functors $\mathbb{D} \xrightarrow{v} \mathcal{C}$ amount to internal functors from $\mathbb{D}_{\text{disc}}$ to $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$, and doubly lax transformations amount to internal natural transformations. The composition of $\mathcal{C}\mathsf{yl}_v(\mathcal{C})$ as an internal category, which is the (vertical) tensor of cylinders developed in Definition 6.1, allows thus to compose doubly lax transformations, given $F \overset{\theta}{\Longrightarrow} G \overset{\eta}{\Longrightarrow} H$,

$$\begin{tikzcd}
 & & \mathbb{D} \arrow[dll, bend right, "{(\eta,\theta)}"'] \arrow[dl, shift right, "\theta"'] \arrow[dl, shift left, "\eta"] \arrow[d, shift right=2, "F"'] \arrow[d, "G"] \arrow[d, shift left=2, "H"] \\
\mathbb{C}\mathsf{yl}_v(\mathcal{C}) \times_{\mathbb{Q}\mathcal{C}_v} \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \arrow[r, "\otimes"] & \mathbb{C}\mathsf{yl}_v(\mathcal{C}) \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & \mathbb{Q}\mathcal{C}_v
\end{tikzcd}$$

$\eta \otimes \theta = \otimes(\eta, \theta)$. Fixing θ and letting η vary, we get a double functor $\theta^*\colon \mathbb{H}\mathsf{om}_{d\ell}(G, H) \longrightarrow \mathbb{H}\mathsf{om}_{d\ell}(F, H)$. Putting $\theta = \lambda$, we note that the double functor $\lambda^*\colon \mathcal{C}(L, E) \longrightarrow \mathbb{L}\mathsf{C}(F, E)$ considered in Remark 4.25 is in fact the composition

$$\mathcal{C}(L, E) \xrightarrow{\Delta} \mathbb{H}\mathsf{om}_{d\ell}(\Delta L, \Delta E) \xrightarrow{\lambda^*} \mathbb{H}\mathsf{om}_{d\ell}(F, \Delta E)$$

of two double functors (note the abuse in denoting both double functors by λ^*).

In the present paper, we will not use the composition of two arbitrary doubly lax transformations θ, η as above explicitly, but we note nevertheless that this is in fact the composition of a DblCat-enriched category, that we may call $\mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{C})_{d\ell}$, in which objects are vertical double functors $\mathbb{D} \xrightarrow{v} \mathcal{C}$ and $\mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{C})_{d\ell}(F, G) = \mathbb{H}\mathsf{om}_{d\ell}(F, G)$.

Given any vertical double functor $F\colon \mathbb{D} \xrightarrow{v} \mathcal{C}$, one can then consider the DblCat-functor

$$\mathcal{C} \xrightarrow{\Delta -} \mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{C})_{d\ell} \xrightarrow{\mathcal{H}\mathsf{om}_v(\mathbb{D},\mathcal{C})_{d\ell}(F,-)} \mathcal{D}\mathbf{blCat},$$

mapping E to $\mathbb{H}\mathsf{om}_{d\ell}(F, \Delta E)$, and define the doubly lax colimit of F as its representation (in the sense of DblCat-enriched category theory). The equivalence between this notion of colimit coming from the general enriched category theory, which amounts to having natural isomorphisms of double categories $\mathbb{H}\mathsf{om}_{d\ell}(F, \Delta E) \cong \mathcal{C}(L, E)$, and the one given in Definition 4.26 in terms of cones is given as usual by considering id_L on the right side.

6.2 The higher isomorphism of double categories given by $\mathbb{G}\mathrm{r}$

Recall the *slice* DblCat-category $\mathcal{D}\mathbf{blCat}/\mathbb{D}$ from Definition 2.4. In this section we will prove:

Theorem 6.5. *The double Grothendieck construction of Section 3 is the value on objects of a* DblCat*-functor*

$$\mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{D}\mathbf{blCat})_{d\ell} \xrightarrow{\mathbb{G}r} \mathcal{D}\mathbf{blCat}/\mathbb{D},$$

which is locally an isomorphism of double categories $\mathbb{H}\mathsf{om}_{d\ell}(F, G) \xrightarrow{\mathbb{G}r} (\mathcal{D}\mathbf{blCat}/\mathbb{D})(\mathbb{G}rF, \mathbb{G}rG)$.

Before proving this theorem, it is convenient to consider the following situation. Let $F, G : \mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$. Consider the colimit doubly lax cone $\lambda : G \longrightarrow \Delta\mathbb{G}\mathrm{r}G$, and the double functor given by postcomposition with λ, that we have by Remark 6.4:

$$\mathbb{H}\mathsf{om}_{d\ell}(F, G) \xrightarrow{\lambda_*} \mathbb{H}\mathsf{om}_{d\ell}(F, \Delta\mathbb{G}\mathrm{r}G)$$

We consider:

(1) objects of $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$; that is, doubly lax transformations $\theta\colon F \Longrightarrow G$ as described in Definition 4.11,

(2v) vertical arrows $U : \theta \Rightarrow \theta'$ of $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$, that is vertical transformations as described in Definition 4.18, as well as

(2h) horizontal arrows H and **(3)** double cells M of $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$, again as in 4.18.

We apply λ_* to each of these data, and we observe:

(1) For each $\theta : F \Longrightarrow G$, $\mu := \lambda_* \theta : F \Longrightarrow \Delta\mathbb{G}\mathrm{r}G$ has the following components:

(as above, for $A \in \mathbb{D}$, we denote x, φ, ρ, Φ an arbitrary object, arrow, or double cell of FA)

$$\mu_A(x) = (A, \theta_A(x)), \quad \mu_A(\varphi) = (1_A, \theta_A(\varphi)), \quad \mu_A(\rho) = (1^\bullet_A, \theta_A(\rho)), \quad \mu_A(\Phi) = (1^\square_A, \theta_A(\Phi))$$

For each $A \overset{u}{\bullet\!\!\longrightarrow} A'$, $\qquad (\mu_u)_x = (u, (\theta_u)_x), \qquad (\mu_u)_\varphi = (1_u, (\theta_u)_\varphi)$,

For each $A \xrightarrow{f} B$, $\qquad (\mu_f)_x = (f, (\theta_f)_x), \qquad (\mu_f)_\rho = (1^\bullet_f, (\theta_f)_\rho)$,

For each $\alpha : (u \ {}^{f}_{f'} \ v)$, $\quad (\mu_\alpha)_x = (\alpha, (\theta_\alpha)_x)$.

We display this last cell for the reader's convenience:

$$\begin{tikzcd}
(A, \theta_A(x)) \arrow[r, Rightarrow, "{(f,(\theta_f)_x)}"] \arrow[dd, Rightarrow, "{(u,(\theta_u)_x)}"'] & (B, \theta_B Ff(x)) \arrow[d, Rightarrow, "{(v,(\theta_v)_{Ff(x)})}"] \\
{} \arrow[r, phantom, "(\mu_\alpha)_x"] & (B', \theta_{B'} FvFf(x)) \arrow[d, Rightarrow, "{(1^\square_{B'},(F\alpha)_x)}"] \\
(A', \theta_{A'} Fu(x)) \arrow[r, Rightarrow, "{(f',(\theta_{f'})_{Fu(x)})}"'] & (B', \theta_{B'} Ff'Fu(x))
\end{tikzcd}
\qquad
\begin{tikzcd}
GvGf\theta_A(x) \arrow[r, Rightarrow, "Gv(\theta_f)_x"] \arrow[d, Rightarrow, "(G\alpha)_{\theta_A(x)}"'] & Gv\theta_B Ff(x) \arrow[d, Rightarrow, "\theta_{vFf(x)}"] \\
Gf'Gu\theta_A(x) \arrow[r, phantom, "(\theta_\alpha)_x"] \arrow[d, Rightarrow, "Gf'(\theta_u)_x"'] & \theta_{B'} FvFf(x) \arrow[d, Rightarrow, "\theta_{B'}(F\alpha)_x"] \\
Gf'\theta_{A'} Fu(x) \arrow[r, Rightarrow, "(\theta_{f'})_{Fu(x)}"'] & \theta_{B'} Ff'Fu(x)
\end{tikzcd}$$

The diagram on the left can be obtained from (4.17) by putting $\theta = \mu$, instantiating at x, and using the definitions above. The one on the right can be obtained from (4.12) by instantiating at x. Comparing with (3.2), one can verify that $(\theta_\alpha)_x$ has the correct *shape* to make the pair $(\alpha, (\theta_\alpha)_x)$ a double cell $(\mu_\alpha)_x$ of $\mathbb{G}\mathrm{r}G$.

(2v) For each $U : \theta \overset{\bullet}{\Longrightarrow} \theta'$, $u := \lambda_* U : \mu \overset{\bullet}{\Longrightarrow} \mu'$ has components:

For each $A \in \mathbb{D}$, $\qquad (u_A)_x = (1^\bullet_A, (U_A)_x), \qquad (u_A)_\varphi = (1^\square_A, (U_A)_x)$

For each $A \xrightarrow{f} B$, $\qquad (u_f)_x = (1_f, (U_f)_x)$

(2h) For each $H : \theta \Longrightarrow \Lambda$, $h := \lambda_* H : \mu \Longrightarrow \nu$, the components are dual to the ones in **(2v)**.

(3) For each $\begin{tikzcd} \theta \arrow[r, Rightarrow, "H"] \arrow[d, Rightarrow, "U"'] \arrow[dr, phantom, "M"] & \Lambda \arrow[d, Rightarrow, "V"] \\ \theta' \arrow[r, Rightarrow, "H'"'] & \Lambda' \end{tikzcd}$, we have $m = \lambda_* M$, $\begin{tikzcd} \mu \arrow[r, Rightarrow, "h"] \arrow[d, Rightarrow, "u"'] \arrow[dr, phantom, "m"] & \nu \arrow[d, Rightarrow, "v"] \\ \mu' \arrow[r, Rightarrow, "h'"'] & \nu' \end{tikzcd}$, $\quad (m_A)_x = (1^\square_A, (M_A)_x)$.

We will say that an arbitrary object μ of $\mathbb{H}\mathsf{om}_{d\ell}(F, \Delta\mathbb{G}\mathrm{r}G)$ as in item **(1)** above (resp. vertical or horizontal arrow u or h in item **(2v)** or **(2h)**, resp. double cell m in item **(3)**) is *special* if its components are as in the respective item, more precisely:

A doubly lax transformation $\mu : F \Longrightarrow \Delta\mathbb{G}\mathrm{r}G$ is *special* if it satisfies the **nine** conditions coming from item **(1)** above:

$$\mu_A(x) = (A, -), \qquad \mu_A(\varphi) = (1_A, -), \quad (...), \quad (\mu_u)_x = (u, -), \quad \text{etc.}$$

Note that, in this case, we can uniquely define a doubly lax transformation $\theta : F \Longrightarrow G$ (i.e. an object of $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$) such that $\mu = \lambda_* \theta$, by these nine formulas, that is

$$\mu_A(x) = (A, \theta_A(x)), \qquad \mu_A(\varphi) = (1_A, \theta_A(\varphi)), \quad (...), \quad (\mu_u)_x = (u, (\theta_u)_x), \quad \text{etc.}$$

We let the reader verify that θ satisfies the doubly lax transformation axioms, and observe that exactly the same happens for the other items **(2v)**, **(2h)**, and **(3)**. In this way, any special object, arrow, or double cell in $\mathbb{H}\mathsf{om}_{d\ell}(F, \Delta\mathbb{G}\mathrm{r}G)$ is the image of a unique object, arrow, or double cell of $\mathbb{H}\mathsf{om}_{d\ell}(F, G)$ by λ_*.

We consider now the canonical projections $\mathbb{G}\mathrm{r}F \longrightarrow \mathbb{D}$, $\mathbb{G}\mathrm{r}G \longrightarrow \mathbb{D}$, and the isomorphism of double categories $\mathbb{H}\mathsf{om}_{d\ell}(F, \Delta\mathbb{G}\mathrm{r}G) \cong \mathcal{D}\mathbf{blCat}(\mathbb{G}\mathrm{r}F, \mathbb{G}\mathrm{r}G)$ given by the universal property of the doubly lax colimit $\mathbb{G}\mathrm{r}F$, and constructed explicitly in the proof of Theorem 5.1. It can then be easily checked that an object μ (resp. vertical or horizontal arrow u or h, resp. double cell m) as above is special if and only if its image by this isomorphism is in $\mathcal{D}\mathbf{blCat}/\mathbb{D}$ (see Definition 2.4).

Proof of Theorem 6.5. We can define the DblCat-functor $\mathbb{G}r\colon \mathcal{H}\mathsf{om}_v(\mathbb{D}, \mathcal{D}\mathbf{blCat})_{d\ell} \longrightarrow \mathcal{D}\mathbf{blCat}$, with a standard technique, by using the universal property of the doubly lax colimit. To do so, for each $F, G : \mathbb{D} \xrightarrow{v} \mathcal{D}\mathbf{blCat}$, we consider the situation in the following diagram, showing how $\xi = \mathbb{G}r(\theta)$ is defined for each doubly lax transformation $\theta : F \Longrightarrow G$ (the higher structure is defined analogously, using the higher-dimensional universal property of the doubly lax colimit proved in Theorem 5.1):

$$\begin{tikzcd}
F \arrow[r, Rightarrow, "\lambda"] \arrow[d, Rightarrow, "\theta"'] & \Delta\mathbb{G}\mathrm{r}F \arrow[d, Rightarrow, "\Delta\xi"] & \mathbb{G}\mathrm{r}F \arrow[d, dotted, "\exists!\xi"] \\
G \arrow[r, Rightarrow, "\lambda"'] & \Delta\mathbb{G}\mathrm{r}G & \mathbb{G}\mathrm{r}G
\end{tikzcd}$$

The fact that ξ (as well as the higher structure) is in $\mathcal{D}\mathbf{blCat}/\mathbb{D}$, and that furthermore, $\mathbb{G}\mathrm{r}$ is locally an isomorphism when co-restricted to $\mathcal{D}\mathbf{blCat}/\mathbb{D}$ follows at once from the considerations we did before this proof. $\square$

A Part of the Proof of Theorem 5.1

A.1 The double functor ξ

We complete here the proof of the LC1 property in Theorem 5.1, by verifying that ξ, as defined there, preserves horizontal compositions of both arrows and double cells. We note that, as can be observed in the explicit computation below, showing that ξ preserves the horizontal composition of double cells involves working both with horizontal and vertical compositions of double cells. This is so because, as can be seen in (5.4), the value of ξ at a double cell can generally not be decomposed as a non-trivial vertical pasting of double cells. This double cell can however be decomposed as a horizontal pasting of two double cells; this makes the verification that ξ preserves vertical composition simpler (and we shall omit this). Note that a similar phenomenon is observed in the proofs of the properties LC2v and LC2h in sections A.2 and A.3 below (we can prove the vertical naturality of the vertical transformation in (A.1) by showing two equalities of smaller diagrams separately, while it is not possible to prove the horizontal naturality of the horizontal transformation in (A.4) in a similar way).

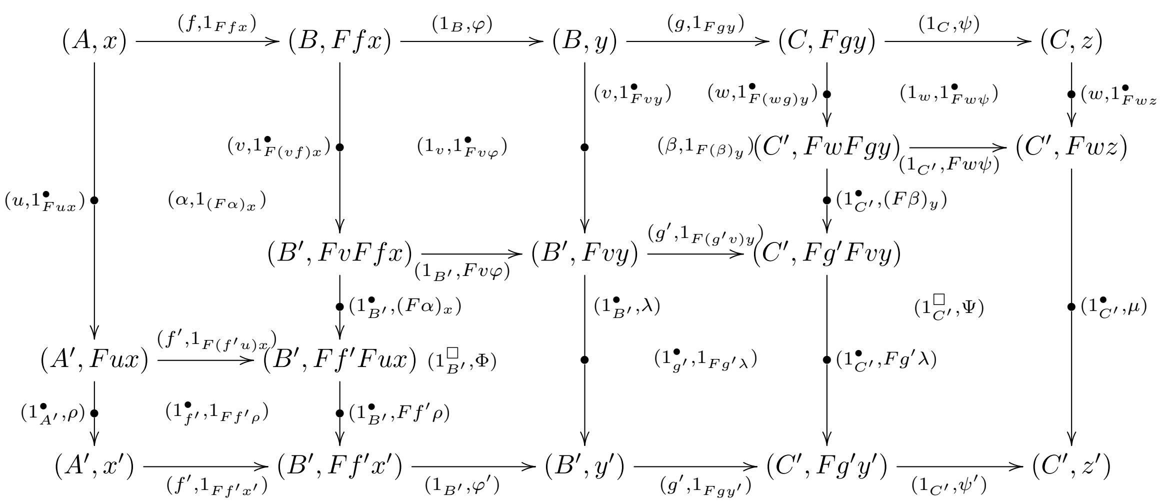
We begin with the horizontal composition of arrows. Consider the following commutative diagram, showing how the horizontal composition of two arrows is factored as in Remark 3.6:

$$\begin{tikzcd}
(A,x) \arrow[rr, "{(f,\varphi)}"] \arrow[dr, "{(f,1_{Ffx})}"] \arrow[ddrr, bend right, "{(g\circ f,1_{F(g\circ f)x})}"'] & & (B,y) \arrow[rr, "{(g,\psi)}"] \arrow[dr, "{(g,1_{Fgy})}"] \arrow[d, phantom, "(\mathbf{1})"] & & (C,z) \\
& (B,Ffx) \arrow[ur, "{(1_B,\varphi)}"] \arrow[rr, "{(g,Fg\varphi)}"] \arrow[dr, "{(g,1_{F(gf)x})}"] \arrow[dl, phantom, "(\mathbf{2})"] & & (C,Fgy) \arrow[ur, "{(1_C,\psi)}"] \arrow[dr, phantom, "(\mathbf{3})"] & \\
& & (C,FgFfx) \arrow[ur, "{(1_C,Fg\varphi)}"] \arrow[uurr, bend right, "{(1_C,\psi\circ Fg\varphi)}"'] & &
\end{tikzcd}$$

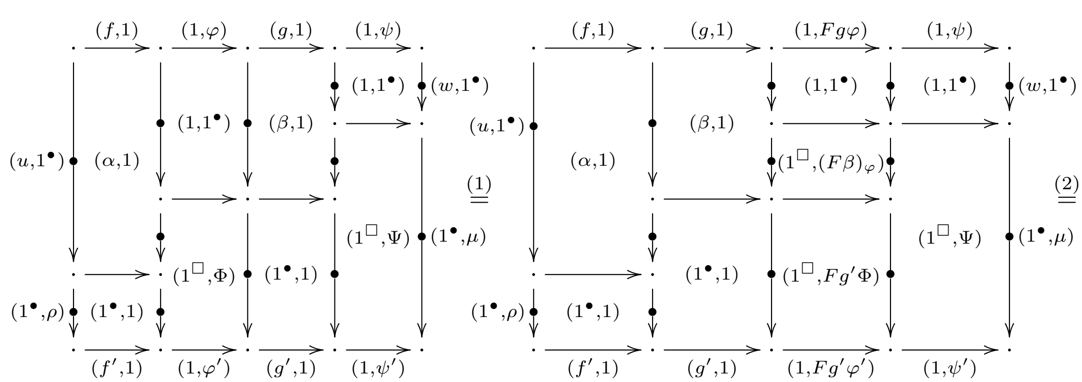
Note that, by definition, ξ maps the three unlabeled small commutative triangles in the diagram, as well as the big outer commutative triangle, to three commutative triangles. Then, in view of this diagram, the fact that ξ respects the horizontal composition of these two arrows follows using the three statements on the left of the following table, each expressing that ξ maps the corresponding commutative diagram **(1)**, **(2)**, and **(3)**, to a commutative diagram. Recalling that we have $\xi(f, 1_{Ffx}) = (\theta_f)_x$ and $\xi(1_A, \varphi) = \theta_A(\varphi)$ (for arbitrary f, φ, see column **(2h)** in (5.2)), we observe that each of these three statements is equivalent to the corresponding one on the right, which holds by one of the properties of the cone θ.

	ξ preserves a composition of the form:	θ satisfies:
(1)	$(A,x) \xrightarrow{(1_A,\varphi)} \cdot \xrightarrow{(f,1_{Ffy})} \cdot$	θ_f natural in the horizontal direction ((h.*) in Remark 2.2, at φ)
(2)	$(A,x) \xrightarrow{(f,1_{Ffx})} \cdot \xrightarrow{(g,1_{F(gf)x})} \cdot$	Axiom 2 in Definition 4.11, at x (that is, $(\theta_{g\circ f})_x = (\theta_g)_{Ffx} \circ (Gg(\theta_f)_x)$)
(3)	$(A,x) \xrightarrow{(1_A,\varphi)} \cdot \xrightarrow{(1_A,\psi)} \cdot$	θ_A preserves horizontal composition of arrows

The idea for the horizontal composition of double cells is similar, but more involved. We consider now two double cells that can be horizontally composed, where each is factored as in Remark 3.6:



We can rewrite this pasting diagram in the following way, showing how the horizontal composition of these two double cells is factored (we are omitting some labels to improve readability):



$$
\begin{tikzcd}[column sep=small]
\cdot \arrow[r,"{(f,1)}"] \arrow[ddd,"{(u,1^\bullet)}"'] \arrow[dddr, phantom, "{(\alpha,1)}"] & \cdot \arrow[r,"{(g,1)}"] \arrow[dd] \arrow[ddr, phantom, "{(\beta,1)}"] & \cdot \arrow[r,"{(1,Fg\varphi)}"] \arrow[d] \arrow[dr, phantom, "{(1,1^\bullet)}"] & \cdot \arrow[r,"{(1,\psi)}"] \arrow[d] \arrow[dr, phantom, "{(1,1^\bullet)}"] & \cdot \arrow[d,"{(w,1^\bullet)}"] & & \cdot \arrow[r,"{(f,1)}"] \arrow[ddd,"{(u,1^\bullet)}"'] \arrow[dddrr, phantom, "{(\alpha|\beta,1)}"] & \cdot \arrow[r,"{(g,1)}"] & \cdot \arrow[r,"{(1,Fg\varphi)}"] \arrow[d] \arrow[drr, phantom, "{(1,1^\bullet)}"] & \cdot \arrow[r,"{(1,\psi)}"] & \cdot \arrow[d] \\
 & & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{(1^\square,(F\beta)_\varphi)}"] & \cdot \arrow[r] \arrow[d] \arrow[dddr, phantom, "{(1^\square,\Psi)}"] & \cdot \arrow[ddd,"{(1^\bullet,\mu)}"] & & & & \cdot \arrow[r] \arrow[d] \arrow[dddrr, phantom, "{(1^\square,\Lambda)}"] & \cdot \arrow[r] & \cdot \arrow[ddd] \\
 & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[r] \arrow[d] \arrow[ddr, phantom, "{(1^\square,Fg'\Phi)}"] & \cdot \arrow[dd] & & \overset{(3)}{=} & & & \cdot \arrow[d] & & \\
\cdot \arrow[r] \arrow[d,"{(1^\bullet,\rho)}"'] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] & & & & \cdot \arrow[r] \arrow[d,"{(1^\bullet,\rho)}"'] \arrow[drr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[r] & \cdot \arrow[d] & & \\
\cdot \arrow[r,"{(f',1)}"'] & \cdot \arrow[r,"{(g',1)}"'] & \cdot \arrow[r,"{(1,Fg'\varphi')}"'] & \cdot \arrow[r,"{(1,\psi')}"'] & \cdot & & \cdot \arrow[r,"{(f',1)}"'] & \cdot \arrow[r,"{(g',1)}"'] & \cdot \arrow[r,"{(1,Fg'\varphi')}"'] & \cdot \arrow[r,"{(1,\psi')}"'] & \cdot
\end{tikzcd}
$$

where Λ is the double cell $\left(\frac{(F\beta)_\varphi}{Fg'\Phi}\,\middle|\,\Psi\right)$, written explicitly in (3.3). Now, with the same reasoning we have used for the case of the horizontal composition of arrows above (but with equalities of double cells instead of commutative diagrams), it follows that ξ will preserve the horizontal composition of these two double cells if (and only if) it "preserves the compositions" listed in the middle column of the following table. By this we mean precisely the following: note first that each of these compositions of double cells in the middle column can be found as part of the diagram on the left hand side of the equality marked with the label indicated by the left column of the table. When we say that ξ "preserves a composition of these double cells" we mean that if we first apply ξ to each of the double cells, and then compose, that double cell coincides with the value of ξ applied to the corresponding double cell that is part of the diagram on the right hand side of the respective equality (note that, for all these cases, the double cell on the right hand side is already "factored", so we know already that ξ preserves this composition). As before, each of these "ξ preserves a composition" statements is equivalent to the one on the right column of the table, under the following equalities coming from column **(3)** of (5.2), that we recall here for the reader's convenience:

$$\xi(1^\square_A,\Phi)=\theta_A(\Phi),\quad \xi(1^\bullet_f,1_{Ff\rho})=(\theta_f)_\rho,\quad \xi(1_u,1^\bullet_{Fu\varphi})=(\theta_u)_\varphi,\quad \xi(\alpha,1_{(F\alpha)_x})=(\theta_\alpha)_x$$

	ξ preserves a composition:	θ satisfies:
(1)	$\begin{tikzcd} \cdot \arrow[r, "{(1,\varphi)}"] \arrow[dd, "{(u,1^\bullet)}"'] & \cdot \arrow[r, "{(f,1)}"] \arrow[dd] & \cdot \arrow[d] \\ \arrow[r, phantom, "{(1,1^\bullet)}"] & \arrow[r, phantom, "{(\alpha,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{(1,\varphi)}"'] & \cdot \arrow[r] & \cdot \end{tikzcd}$	Axiom (m.1) in 2.2 for θ_α (with respect to φ)
(1)	$\begin{tikzcd} \cdot \arrow[r, "{(1,\varphi)}"] \arrow[d] \arrow[dr, phantom, "{(1^\square,\Phi)}"] & \cdot \arrow[r, "{(f,1)}"] \arrow[d] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{(1,\varphi')}"'] & \cdot \arrow[r, "{(f,1)}"'] & \cdot \end{tikzcd}$	Axiom (h.3) in Def. 2.1 for θ_f (with respect to Φ)
(2)	$\begin{tikzcd} \cdot \arrow[r, "{(f,1)}"] \arrow[d, "{(1^\bullet,\rho)}"'] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r] \arrow[d, "{(1^\bullet,\rho')}"'] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{(f,1)}"'] & \cdot \end{tikzcd}$	Axiom (h.2) in Def. 2.1 for θ_f (with respect to ρ, ρ')
(3)	$\begin{tikzcd} \cdot \arrow[r, "{(f,1)}"] \arrow[ddd, "{(u,1^\bullet)}"'] & \cdot \arrow[r, "{(g,1)}"] \arrow[dd] & \cdot \arrow[d] \\ \arrow[r, phantom, "{(\alpha,1)}"] & \arrow[r, phantom, "{(\beta,1)}"] & \cdot \arrow[d] \\ & \cdot \arrow[r, "{(g',1)}"] \arrow[d] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{(f',1)}"'] & \cdot \arrow[r, "{(g',1)}"'] & \cdot \end{tikzcd}$	Axiom 5 in Definition 4.11
(3)	$\begin{tikzcd} \cdot \arrow[r, "{(f,1)}"] \arrow[d, "{(1^\bullet,\rho)}"'] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[r, "{(g,1)}"] \arrow[d] \arrow[dr, phantom, "{(1^\bullet,1)}"] & \cdot \arrow[d] \\ \cdot \arrow[r, "{(f,1)}"'] & \cdot \arrow[r, "{(g,1)}"'] & \cdot \end{tikzcd}$	Axiom 2 in Definition 4.11, at ρ
(3)	$\begin{tikzcd} \cdot \arrow[r, "{(1,\varphi)}"] \arrow[d, "{(u,1^\bullet)}"'] \arrow[dr, phantom, "{(1,1^\bullet)}"] & \cdot \arrow[r, "{(1,\psi)}"] \arrow[d] \arrow[dr, phantom, "{(1,1^\bullet)}"] & \cdot \arrow[d, "{(u,1^\bullet)}"] \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$	Axiom (v.2) in Def. 2.1 for θ_u (with respect to φ, ψ)
(3)	$\begin{tikzcd} (A,x) \arrow[r, "{(1,\varphi)}"] \arrow[d, "{(1^\bullet,\rho)}"'] \arrow[dr, phantom, "{(1^\square,\Phi)}"] & \cdot \arrow[r, "{(1,\psi)}"] \arrow[d] & \cdot \arrow[dd] \\ \cdot \arrow[r] \arrow[d, "{(1^\bullet,\rho')}"'] \arrow[dr, phantom, "{(1^\square,\Phi')}"] & \cdot \arrow[d] \arrow[r, phantom, "{(1^\square,\Psi)}"] & \\ \cdot \arrow[r] & \cdot \arrow[r] & \cdot \end{tikzcd}$	θ_A preserves both compositions of double cells

A.2 The vertical transformation μ

We complete here the proof of the LC2v property in Theorem 5.1, by verifying that μ, as defined there, satisfies axioms (v.2) and (v.3) in Definition 2.1. Axiom (v.2) reduces to showing that $(U_B)_\varphi|(U_g)_y) = ((U_g)_{Ffx}|(U_C)_{Fg\varphi})$, which holds by axiom (m.1) in 2.2 for the modification U_g (with respect to the horizontal arrow φ). To show axiom (v.3), we consider (α, Φ) a double cell of $\mathbb{G}\mathrm{r}(F)$, and we will show the equality

$$
\begin{tikzcd}
\xi(A,x) \arrow[r, "\xi(f,\varphi)"] \arrow[d, "\xi(u,\rho)"'] \arrow[dr, phantom, "\xi(\alpha,\Phi)"] & \xi(B,y) \arrow[d, "\xi(v,\lambda)"] \\
\xi(A',x') \arrow[r, "\xi(f',\varphi')"'] \arrow[d, "\mu_{(A',x')}"'] \arrow[dr, phantom, "\mu_{(f',\varphi')}"] & \xi(B',y') \arrow[d, "\mu_{(B',y')}"] \\
\xi'(A',x') \arrow[r, "\xi'(f',\varphi')"'] & \xi'(B',y')
\end{tikzcd}
\quad = \quad
\begin{tikzcd}
\xi(A,x) \arrow[r, "\xi(f,\varphi)"] \arrow[d, "\mu_{(A,x)}"'] \arrow[dr, phantom, "\mu_{(f,\varphi)}"] & \xi(B,y) \arrow[d, "\mu_{(B,y)}"] \\
\xi'(A,x) \arrow[r, "\xi'(f,\varphi)"'] \arrow[d, "\xi'(u,\rho)"'] \arrow[dr, phantom, "\xi'(\alpha,\Phi)"] & \xi'(B,y) \arrow[d, "\xi'(v,\lambda)"] \\
\xi'(A',x') \arrow[r, "\xi'(f',\varphi')"'] & \xi'(B',y')
\end{tikzcd}
\tag{A.1}
$$

Replacing each double cell by its definition (in equations (5.4), (5.6)), we see that the equality we have to show is:

$$
\begin{tikzcd}
\theta_A(x) \arrow[r, "(\theta_f)_x"] \arrow[dd, "(\theta_u)_x"'] \arrow[ddr, phantom, "(\theta_\alpha)_x"] & \theta_B(Ffx) \arrow[r, "\theta_B(\varphi)"] \arrow[d, "(\theta_v)_{Ffx}"] \arrow[dr, phantom, "(\theta_v)_\varphi"] & \theta_B(y) \arrow[d, "(\theta_v)_y"] \\
 & \theta_{B'}(FvFfx) \arrow[r, "\theta_{B'}(Fv\varphi)"'] \arrow[d, "\theta_{B'}((F\alpha)_x)"] \arrow[ddr, phantom, "\theta_{B'}(\Phi)"] & \theta_{B'}(Fvy) \arrow[dd, "\theta_{B'}(\lambda)"] \\
\theta_{A'}(Fux) \arrow[r, "(\theta_{f'})_{Fux}"] \arrow[d, "\theta_{A'}(\rho)"'] \arrow[dr, phantom, "(\theta_{f'})_\rho"] & \theta_{B'}(Ff'Fux) \arrow[d, "\theta_{B'}(Ff'\rho)"] & \\
\theta_{A'}(x') \arrow[r, "(\theta_{f'})_{x'}"] \arrow[d, "(U_{A'})_{x'}"'] \arrow[dr, phantom, "(U_{f'})_{x'}"] & \theta_{B'}(Ff'x') \arrow[r, "\theta_{B'}(\varphi')"] \arrow[d, "(U_{B'})_{Ff'x'}"] \arrow[dr, phantom, "(U_{B'})_{\varphi'}"] & \theta_{B'}(y') \arrow[d, "(U_{B'})_{y'}"] \\
\theta'_{A'}(x') \arrow[r, "(\theta'_{f'})_{x'}"'] & \theta'_{B'}(Ff'x') \arrow[r, "\theta'_{B'}(\varphi')"'] & \theta'_{B'}(y')
\end{tikzcd}
=
\begin{tikzcd}
\theta_A(x) \arrow[r, "(\theta_f)_x"] \arrow[d, "(U_A)_x"'] \arrow[dr, phantom, "(U_f)_x"] & \theta_B(Ffx) \arrow[r, "\theta_B(\varphi)"] \arrow[d, "(U_B)_{Ffx}"] \arrow[dr, phantom, "(U_B)_\varphi"] & \theta_B(y) \arrow[d, "(U_B)_y"] \\
\theta'_A(x) \arrow[r, "(\theta'_f)_x"] \arrow[dd, "(\theta'_u)_x"'] \arrow[ddr, phantom, "(\theta'_\alpha)_x"] & \theta'_B(Ffx) \arrow[r, "\theta'_B(\varphi)"] \arrow[d, "(\theta'_v)_{Ffx}"] \arrow[dr, phantom, "(\theta'_v)_\varphi"] & \theta'_B(y) \arrow[d, "(\theta'_v)_y"] \\
 & \theta'_{B'}(FvFfx) \arrow[r, "\theta'_{B'}(Fv\varphi)"'] \arrow[d, "\theta'_{B'}((F\alpha)_x)"] \arrow[ddr, phantom, "\theta'_{B'}(\Phi)"] & \theta'_{B'}(Fvy) \arrow[dd, "\theta'_{B'}(\lambda)"] \\
\theta'_{A'}(Fux) \arrow[r, "(\theta'_{f'})_{Fux}"] \arrow[d, "\theta'_{A'}(\rho)"'] \arrow[dr, phantom, "(\theta'_{f'})_\rho"] & \theta'_{B'}(Ff'Fux) \arrow[d, "\theta'_{B'}(Ff'\rho)"] & \\
\theta'_{A'}(x') \arrow[r, "(\theta'_{f'})_{x'}"'] & \theta'_{B'}(Ff'x') \arrow[r, "\theta'_{B'}(\varphi')"'] & \theta'_{B'}(y')
\end{tikzcd}
$$

We will show that the *left parts* of the two diagrams are equal, and that so are the *right parts*. For the left part, using in turn axiom (m.2) in 2.2 for the modification $U_{f'}$ (with respect to the vertical arrow ρ), and axiom (v.3) in Definition 4.23 we compute:

$$
\begin{tikzcd}
\cdot \arrow[r, "(\theta_f)_x"] \arrow[dd, "(\theta_u)_x"'] \arrow[ddr, phantom, "(\theta_\alpha)_x"] & \cdot \arrow[d, "(\theta_v)_{Ffx}"] \\
 & \cdot \arrow[d, "\theta_{B'}((F\alpha)_x)"] \\
\cdot \arrow[r, "(\theta_{f'})_{Fux}"] \arrow[d, "\theta_{A'}(\rho)"'] \arrow[dr, phantom, "(\theta_{f'})_\rho"] & \cdot \arrow[d, "\theta_{B'}(Ff'\rho)"] \\
\cdot \arrow[r, "(\theta_{f'})_{x'}"] \arrow[d, "(U_{A'})_{x'}"'] \arrow[dr, phantom, "(U_{f'})_{x'}"] & \cdot \arrow[d, "(U_{B'})_{Ff'x'}"] \\
\cdot \arrow[r, "(\theta'_{f'})_{x'}"'] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "(\theta_f)_x"] \arrow[dd, "(\theta_u)_x"'] \arrow[ddr, phantom, "(\theta_\alpha)_x"] & \cdot \arrow[d, "(\theta_v)_{Ffx}"] \\
 & \cdot \arrow[d, "\theta_{B'}((F\alpha)_x)"] \\
\cdot \arrow[r, "(\theta_{f'})_{Fux}"] \arrow[d, "(U_{A'})_{Fux}"'] \arrow[dr, phantom, "(U_{f'})_{Fux}"] & \cdot \arrow[d, "(U_{B'})_{Ff'Fux}"] \\
\cdot \arrow[r, "(\theta_{f'})_{Fux}"] \arrow[d, "\theta'_{A'}(\rho)"'] \arrow[dr, phantom, "(\theta'_{f'})_\rho"] & \cdot \arrow[d, "\theta'_{B'}(Ff'\rho)"] \\
\cdot \arrow[r, "(\theta'_{f'})_{x'}"'] & \cdot
\end{tikzcd}
=
\begin{tikzcd}
\cdot \arrow[r, "(\theta_f)_x"] \arrow[d, "(U_A)_x"'] \arrow[dr, phantom, "(U_f)_x"] & \cdot \arrow[d, "(U_B)_{Ffx}"] \\
\cdot \arrow[r, "(\theta'_f)_x"] \arrow[dd, "(\theta'_u)_x"'] \arrow[ddr, phantom, "(\theta'_\alpha)_x"] & \cdot \arrow[d, "(\theta'_v)_{Ffx}"] \\
 & \cdot \arrow[d, "\theta'_{B'}((F\alpha)_x)"] \\
\cdot \arrow[r, "(\theta'_{f'})_{Fux}"] \arrow[d, "\theta'_{A'}(\rho)"'] \arrow[dr, phantom, "(\theta'_{f'})_\rho"] & \cdot \arrow[d, "\theta'_{B'}(Ff'\rho)"] \\
\cdot \arrow[r, "(\theta'_{f'})_{x'}"'] & \cdot
\end{tikzcd}
$$

For the right part, we proceed in a similar fashion using first axiom (v.3) in Definition 2.1 for the vertical transformation $U_{B'}$ (with respect to the double cell Φ), and then the vertical naturality (v.*) in Definition 4.23 (with respect to the vertical arrow v).

A.3 The horizontal transformation h

As mentioned in the proof of Theorem 5.1, the first part of the proof of LC2h is dual to the one of LC2v. To set up the notation, we give here the explicit formulas (A.2), (A.3) corresponding to

(5.5), (5.6). Let $\lambda^*\xi \overset{H}{\Longrightarrow} \lambda^*\eta$, then the equalities **(a)** and **(b)** in LC2h correspond to the conditions

$$\textbf{(a)}\ h_{(A,x)} = (H_A)_x, \quad h_{(1^\bullet_A,\rho)} = (H_A)_\rho, \qquad \textbf{(b)} h_{(u,1^\bullet_{Fux})} = (H_u)_x. \tag{A.2}$$

Just as in the proof of LC2v, this gives a unique possible definition of $h : \xi \Longrightarrow \eta$, $h_{(u,\rho)} = (H_{A'})_\rho \bullet (H_u)_x$: (we denote $\theta = \lambda^*\xi$, $\Lambda = \lambda^*\eta$)

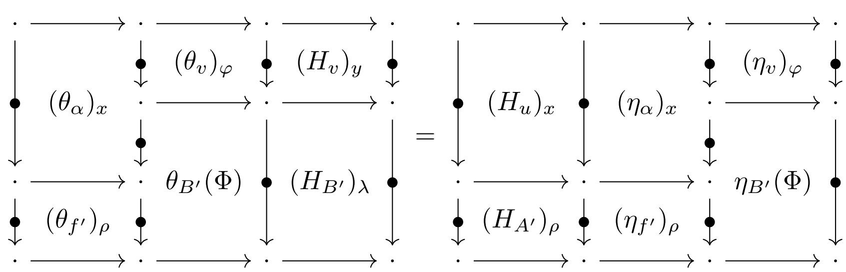
$$\begin{tikzcd}
\xi(A,x) \arrow[r, "h_{(A,x)}"] \arrow[d, "\xi(u,\rho)"'] \arrow[dr, phantom, "h_{(u,\rho)}"] & \eta(A,x) \arrow[d, "\eta(u,\rho)"] \\
\xi(A',x') \arrow[r, "h_{(A',x')}"'] & \eta(A',x')
\end{tikzcd}
=
\begin{tikzcd}
\theta_A(x) \arrow[r, "(H_A)_x"] \arrow[d, "(\theta_u)_x"'] \arrow[dr, phantom, "(H_u)_x"] & \Lambda_A(x) \arrow[d, "(\eta_u)_x"] \\
\theta_{A'}(Fux) \arrow[r, "(H_{A'})_{Fux}"] \arrow[d, "\theta_{A'}(\rho)"'] \arrow[dr, phantom, "(H_{A'})_\rho"] & \Lambda_{A'}(Fux) \arrow[d, "\eta_{A'}(\rho)"] \\
\theta_{A'}(x') \arrow[r, "(H_{A'})_{x'}"'] & \Lambda_{A'}(x'),
\end{tikzcd} \tag{A.3}$$

that satisfies the required conditions, as well as axioms (h.1) and (h.2) in Definition 2.1.

To show axiom (h.3), we consider (α, Φ) a double cell of $\mathbb{G}\mathrm{r}F$, and we will show the equality

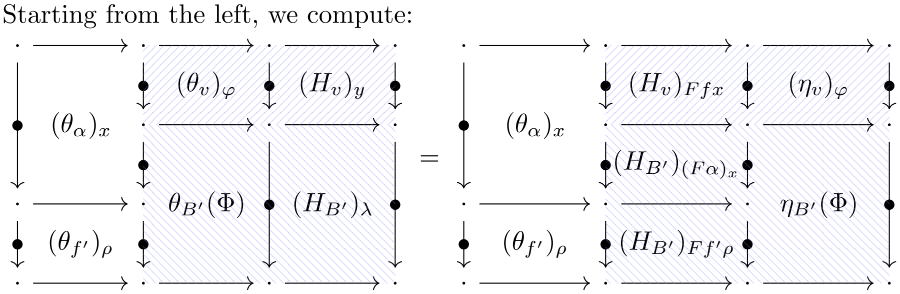
$$\begin{tikzcd}
\xi(A,x) \arrow[r, "\xi(f,\varphi)"] \arrow[d, "\xi(u,\rho)"'] \arrow[dr, phantom, "\xi(\alpha,\Phi)"] & \xi(B,y) \arrow[r, "h_{(B,y)}"] \arrow[d, "\xi(v,\lambda)"] \arrow[dr, phantom, "h_{(v,\lambda)}"] & \eta(B,y) \arrow[d, "\eta(v,\lambda)"] \\
\xi(A',x') \arrow[r, "\xi(f',\varphi')"'] & \xi(B',y') \arrow[r, "h_{(B',y')}"'] & \xi(B',y')
\end{tikzcd}
=
\begin{tikzcd}
\xi(A,x) \arrow[r, "h_{(A,x)}"] \arrow[d, "\xi(u,\rho)"'] \arrow[dr, phantom, "h_{(u,\rho)}"] & \eta(A,x) \arrow[r, "\eta(f,\varphi)"] \arrow[d, "\eta(u,\rho)"] \arrow[dr, phantom, "\eta(\alpha,\Phi)"] & \eta(B,y) \arrow[d, "\eta(v,\lambda)"] \\
\xi(A',x') \arrow[r, "h_{(A',x')}"'] & \eta(A',x') \arrow[r, "\eta(f',\varphi')"'] & \eta(B',y')
\end{tikzcd}$$

Replacing each double cell by its definition (in equations (5.4), (A.3)), we see that the equality we want to establish is:



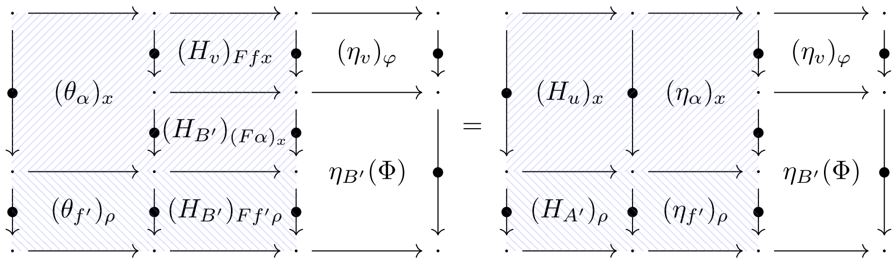
(A.4)

Starting from the left, we compute:



The equality between the top shaded regions is given by axiom (m.1) in 2.2 for the modification H_v (with respect to the horizontal arrow ϕ). The equality between the bottom shaded regions is given by first applying axiom (h.3) in Definition 2.1 for the horizontal transformation $H_{B'}$ (with respect to the double cell Φ as in (3.2)) and then applying axiom (h.2) in the same Definition.

We finish proving (A.4) by using the equality below:



The equality between the top shaded regions is now given by axiom (h.3) in Definition 4.23 (instantiate at x), and the equality between the bottom shaded ones by (h.*) in the same Definition (for f', instantiate at ρ).

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