Model bicategories and their homotopy bicategories

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Abstract

We give the definitions of model bicategory and q-homotopy, which are natural generalizations of the notions of model category and homotopy to the context of bicategories. For any model bicategory $\mathcal{C}$, denote by $\mathcal{C}_{fc}$ the full sub-bicategory of the fibrant-cofibrant objects. We prove that the 2-dimensional localization of $\mathcal{C}$ at the weak equivalences can be computed as a bicategory $\mathcal{H}o(\mathcal{C})$ whose objects and arrows are those of $\mathcal{C}_{fc}$ and whose 2-cells are classes of q-homotopies up to an equivalence relation. When considered for a model category, q-homotopies coincide with the homotopies as considered by Quillen. The pseudofunctor $\mathcal{C} \xrightarrow{q} \mathcal{H}o(\mathcal{C})$ which yields the localization is constructed by using a notion of fibrant-cofibrant replacement in this context. We include an appendix with a general result of independent interest on a transfer of structure for lax functors, that we apply to obtain a pseudofunctor structure for the fibrant-cofibrant replacement.

1 Introduction

The notion of model category, originally introduced in [27], is the basis for a great deal of modern homotopy theory. A basic feature of a model category is that it allows the construction of its localization at a class of arrows, the *weak equivalences*, as a quotient by the congruence determined by *homotopies* between arrows. In addition, the lifting properties relating the three distinguished families of arrows of a model category are at the very core of this theory and its applications, perhaps most notably in the definition of quasi-categories [17], or ∞-categories [21].

On the other hand, basic 2-dimensional categorical structures such as 2-categories and bicategories [2] have extensively been used since the '60s to explore the higher structure *hiding* behind the equality of arrows of a category. Many basic categorical concepts such as limit, kan extension, adjunction, among many others have been fruitfully adapted to the theory of 2-categories and bicategories. In particular the bicategories, for which the identity and associativity axioms for arrows do not hold strictly, are a natural context in which to consider that diagrams of arrows should in general be required to commute only "up to an invertible 2-cell".

The idea of considering a notion of model 2-category or model bicategory, where the diagrams in the axioms defining a model category are required to commute up to invertible 2-cells, seems then like a natural one to explore. As far as we know, this idea first appeared for 2-categories in print as an open question in [16, Problem 8.1]. A set of axioms for a 2-category $\mathcal{C}$ to be a model 2-category was given in [5], and more recently equivalent axioms are considered in [1, Chapter 7] (for further information regarding their motivation for considering such a notion, we refer the reader to [16, §8] and to [1, Introduction]). We note explicitly that these axioms deal only with the underlying $(2, 1)$-category structure of $\mathcal{C}$, that is, they involve only its invertible 2-cells.

The main contributions of this article are a generalization to the context of bicategories of the concepts of lifting property and model category, which do not only involve the invertible 2-cells of the bicategory, as well as a construction of the homotopy bicategory of a model bicategory, that is its localization at the weak equivalences.

As in dimension 1, a model bicategory consists of a bicategory together with three families of arrows, namely fibrations, cofibrations and weak equivalences, satisfying a set of axioms. The axioms we give are a natural generalization to bicategories of those given by Quillen in the sense that they are obtained by requiring the diagrams to commute up to invertible 2-cells, and by considering a 2-dimensional aspect of the lifting properties which relate these families of arrows. In particular, when we consider a category and three families of arrows as a trivial bicategory with the same families of arrows, the two notions coincide: it will be a model bicategory if and only if it is a model category.

Note that in this case the homotopy bicategory is a 2-category that has the usual homotopies as arrows, but now it also has new 2-cells between them. A homotopy bicategory carries in this way a richer structure than a homotopy category, which could be used in particular to compute finer homotopical invariants, and we describe such a potential application in the last paragraph of this introduction. Further applications of model bicategories that we are currently exploring involve studying the higher structure which could be hidden in the many examples of model categories, such as [20], [17].

We describe now precisely how the axioms for a model bicategory that we introduce here differ from the ones considered in [5] and [1]. No significant work is required in order to consider these same axioms for a bicategory instead of a 2-category. In particular, regarding the *lifting properties* relating the families of arrows, it is natural to consider the following situation (we refer to Section 3 for details):

Let $\mathcal{C}$ be a bicategory and $A \xrightarrow{i} X$, $Y \xrightarrow{p} B$ be two morphisms in $\mathcal{C}$. They determine the following diagram

$$\begin{tikzcd}
\mathcal{C}(X, Y) \arrow[rd, "h"] \arrow[rrd, bend left, "i^*"] \arrow[rdd, bend right, "p_*"] & & \\
& \mathbf{P} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[rd, phantom, "\Downarrow\cong"] & \mathcal{C}(A, Y) \arrow[d, "p_*"] \\
& \mathcal{C}(X, B) \arrow[r, "i^*"] & \mathcal{C}(A, B)
\end{tikzcd}$$

where $\mathbf{P}$ is the pseudo pullback of categories. In [5] and [1], the lifting property is taken to mean that h is essentially surjective (see for example [1, Prop. 7.1.12]).

The only substantial modification we make to the aforementioned set of axioms, and that deals with non-invertible 2-cells of $\mathcal{C}$, is that we require the functor h to be also essentially full, see Definition 2.6. Note that when all the 2-cells of $\mathcal{C}$ are invertible, if h is essentially surjective then it is automatically essentially full.

Notably, this single change suffices for our purpose of constructing the *homotopy bicategory of* $\mathcal{C}$, that is its 2-dimensional localization at the weak equivalences. We also show that this homotopy bicategory is locally small (resp. a 2-category) when $\mathcal{C}$ is so.

We give now a description of the main results in this paper. Given a model bicategory $\mathcal{C}$, denote by $\mathcal{C}_{fc}$ the full sub-bicategory given by the fibrant-cofibrant objects of $\mathcal{C}$. We construct a bicategory $\mathcal{H}\mathrm{o}(\mathcal{C})$ whose objects and arrows are those of $\mathcal{C}_{fc}$ and whose 2-cells are the classes of q-homotopies (a 2-dimensional version of Quillen's notion of homotopy) up to an equivalence relation. We prove that the 2-dimensional localization of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of weak equivalences can be computed as $\mathcal{H}\mathrm{o}(\mathcal{C})$. More specifically, our main theorem (Theorem 6.4) asserts that there is a pseudofunctor $\mathcal{C} \xrightarrow{q} \mathcal{H}\mathrm{o}(\mathcal{C})$ which sends weak equivalences to equivalences and has the following universal property:

$$Hom(\mathcal{H}\mathrm{o}(\mathcal{C}), \mathcal{D}) \xrightarrow{q^*} Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D}) \tag{1.1}$$

is a biequivalence of bicategories for every bicategory $\mathcal{D}$; where $Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ stands for the full sub-bicategory of $Hom(\mathcal{C}, \mathcal{D})$ given by those pseudofunctors that send weak equivalences into equivalences. We also show that the homotopy bicategory $\mathcal{H}\mathrm{o}(\mathcal{C})$ is locally small (resp. a 2-category) when $\mathcal{C}$ is so.

We define q-homotopies as w-homotopies which satisfy some extra conditions related to fibrations and cofibrations. The notion of w-homotopy was introduced in [6] and is taken with respect only to the class $\mathcal{W}$. Note that in op. cit. w-homotopies are called homotopies, but here we chose to add the prefix w- to explicitly distinguish them from q-homotopies. This definition allows to use previous results of [6], where for an arbitrary pair $(\mathcal{A}, \mathcal{W})$ given by a family of arrows of a bicategory we construct a bicategory $\mathcal{H}\mathrm{o}(\mathcal{A}, \mathcal{W})$ whose objects and arrows are those of $\mathcal{A}$ and whose 2-cells are constructed with w-homotopies. In this general case there is no vertical composition of w-homotopies, and thus the 2-cells of $\mathcal{H}\mathrm{o}(\mathcal{A}, \mathcal{W})$ are given by classes of finite sequences of w-homotopies by an appropriate equivalence relation. Associated with $\mathcal{H}\mathrm{o}(\mathcal{A}, \mathcal{W})$ there is a *canonical* 2-functor $\mathcal{A} \xrightarrow{i} \mathcal{H}\mathrm{o}(\mathcal{A}, \mathcal{W})$.

A natural way in which a proof of Theorem 6.4 can be attempted is to consider Quillen's proof of the localization theorem for model categories in [27], and to try to generalize it to this context using q-homotopies. In doing so one is faced to a first obstacle, which is that arbitrary q-homotopies can't be composed (whiskered) with arrows in a natural way. This is already so for the 1-dimensional case in [27], as is well known: for arbitrary arrows $A \underset{g}{\overset{f}{\rightrightarrows}} B$ in a model category, $f \overset{\ell}{\sim} g$ doesn't imply $uf \overset{\ell}{\sim} ug$ (where $\overset{\ell}{\sim}$ is the left homotopy relation). The other composition poses no problem, $fu \overset{\ell}{\sim} gu$ is easily seen to hold. The way in which this obstacle is avoided in [27] is by considering A to be a cofibrant object, and B fibrant, so that there is a left homotopy between two arrows if and only if there is a right homotopy. An important feature of (left) w-homotopies is that both compositions (whiskerings) are naturally defined, and so this obstacle disappears.

There is a rich interplay between w- and q-homotopies, and that was the key we have found to overcoming the obstacle described above for the q-homotopies and furthermore proving Theorem 6.4. On the one hand, w-homotopies allow to define a homotopy bicategory. On the other hand, q-homotopies satisfy several results analogous to the ones that can be found in [27], that we establish and use in this article.

More explicitly, to prove Theorem 6.4 we will use the following results:

A. $\mathcal{H}\text{o}(\mathcal{C})$ inherits the bicategory structure of $\mathcal{H}\text{o}(\mathcal{C},\mathcal{W})$ constructed in [6], in a way that $\mathcal{H}\text{o}(\mathcal{C}) = \mathcal{H}\text{o}_{fc}(\mathcal{C},\mathcal{W}) \subset \mathcal{H}\text{o}(\mathcal{C},\mathcal{W})$, that is the full sub-bicategory of $\mathcal{H}\text{o}(\mathcal{C},\mathcal{W})$ given by the fibrant-cofibrant objects. We will deduce this from the following two facts:

A1. q-homotopies can be composed vertically: for any pair of composable q-homotopies there is a single q-homotopy representing the composition.

A2. For any w-homotopy in $\mathcal{C}_{fc}$ there is a fibrant w-homotopy in the same class. For any fibrant w-homotopy there is a fibrant q-homotopy in the same class (we say that a homotopy is fibrant if its cylinder is so, this is made precise in Definition 4.1).

In fact, we show more generally that, when X is cofibrant and Y is fibrant, all possible notions of homotopy (right or left, w or q, fibrant or cofibrant) coincide in the sense that for any two choices of these concepts, for any homotopy of that kind there is one of the other kind in the same class. In particular, note that $\mathcal{H}o(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C},\mathcal{W})$ could be defined using any of these concepts and this would lead to the same homotopy bicategory.

B. There is a fibrant-cofibrant replacement for model bicategories, that is an assignment from $\mathcal{C}$ to $\mathcal{C}$, defined on objects and arrows (but not on 2-cells) $X \overset{f}{\underset{g}{\rightrightarrows}} Y \rightsquigarrow RQX \overset{RQf}{\underset{RQg}{\rightrightarrows}} RQY$, satisfying the expected conditions for a replacement: all the objects QX are cofibrant, all the objects RQX are fibrant-cofibrant, and there are trivial fibrations $QX \xrightarrow{\rho_X} X$, trivial cofibrations $QX \xrightarrow{\lambda_{QX}} RQX$, and invertible 2-cells

$$\begin{tikzcd}
RQX \arrow[d, "RQf"'] \arrow[dr, phantom, "\Downarrow\lambda_{Qf}"] & QX \arrow[l, "\lambda_{QX}"'] \arrow[r, "\rho_X"] \arrow[d, "Qf"'] \arrow[dr, phantom, "\Downarrow\rho_f"] & X \arrow[d, "f"] \\
RQY & QY \arrow[l, "\lambda_{QY}"] \arrow[r, "\rho_Y"'] & Y
\end{tikzcd}$$

.

Note that Q and RQ are just assignments on objects and arrows and not necessarily pseudofunctors, because we are not assuming that there is a (pseudo)functorial factorization, just as in Quillen's original axioms in [27]. Nevertheless, we will show that there are structural 2-cells (homotopies) which make Q and RQ, suitably defined, into pseudofunctors $\mathcal{H}o^f(\mathcal{C},\mathcal{W}) \longrightarrow \mathcal{H}o^f(\mathcal{C},\mathcal{W})$, and ρ and λ into pseudonatural transformations.

In the above, $\mathcal{H}o^f(\mathcal{C},\mathcal{W})$ denotes the bicategory determined by the fibrant left homotopies (indistinctly w- or q-, see A2 above). It is this bicategory that plays a fundamental role in our proof of Theorem 6.4. Note that, even though its 2-cells can be given by q-homotopies, the composition (whiskering) in the bicategory $\mathcal{H}o^f(\mathcal{C},\mathcal{W})$ of a homotopy with an arrow on the left has to be defined by using the general composition of w-homotopies, which as mentioned above does not need the consideration of right homotopies.

Although ultimately for the fibrant-cofibrant replacement both left and right homotopies are necessary, the latter become implicit in our construction. This allows for a concise, clear proof of the localization theorem, where many of the technical difficulties of working with bicategories are avoided.

C. The 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H}\mathrm{o}(\mathcal{C},\mathcal{W})$ factors through $\mathcal{H}o^f(\mathcal{C},\mathcal{W})$, and we define $q = RQ\,i$ (more precisely, as its corestriction to $\mathcal{H}\mathrm{o}(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C},\mathcal{W})$). Its universal property is deduced from the properties that RQ and i have separately.

D. The pseudofunctor structures for Q and R above are obtained by a *transfer of pseudofunctor structure*, a novel technique introduced in this paper. A natural context in which to state and prove the transfer results is the more general one of lax functors between bicategories, and this is done in Appendix A.

E. The definition of $\mathcal{H}o^f(\mathcal{C},\mathcal{W}) \xrightarrow{Q} \mathcal{H}o^f(\mathcal{C},\mathcal{W})$ on 2-cells and its pseudofunctor structure are obtained as the unique ones for which ρ_X, ρ_f form a pseudonatural transformation $Q \overset{\rho}{\Rightarrow} id$ (we transfer to Q the pseudofunctor structure of the identity pseudofuntor). A dual transfer yields $id \overset{\lambda}{\Rightarrow} R$, which is now defined in the right homotopies. A careful analysis of the relation between the right and left homotopies, generalizing to dimension 2 the situation in [27], allows to consider R to be defined in the left homotopies, which allows to compose these two pseudofunctors, leading to the pseudonatural transformations $RQ \overset{\lambda Q}{\Leftarrow} Q \overset{\rho}{\Rightarrow} id$ which are used for proving the localization theorem.

Despite the independent interest that the results in this article may have, our motivation to develop a theory of model bicategories comes from potential applications in the homotopy theory of topoi. Given a site $\mathcal{C}$, the category of coverings with refinements as arrows fails to be filtered, but it underlies a 2-category which is 2-filtered in the sense of [7]. The Čech nerve $\mathsf{COV}(\mathcal{C}) \xrightarrow{\check{C}} \mathsf{SS}$ into the category of simplicial sets followed by the functor $\mathsf{SS} \longrightarrow Ho(\mathsf{SS})$ into the homotopy category factors through the poset $cov(\mathcal{C})$ of coverings under refinements, which is filtered, and so it determines a pro-object $cov(\mathcal{C}) \longrightarrow \mathsf{SS}$, that is, it determines an object of the category $Pro(Ho(\mathsf{SS}))$, where the information coded into the explicit homotopies is lost. In [4], the first two authors developed a 2-dimensional theory of pro-objects which generalizes Grothendieck's pro-objects theory, and in [5] it is proven that a 2-model structure in a 2-category $\mathcal{C}$ can be lifted to the 2-category 2-$Pro(\mathcal{C})$, a 2-dimensional generalization of the construction in [11]. This paper, in particular, complements [4] and [5]. The Čech nerve can be seen as a simplicial 2-pro-object, and thus it determines an object of the 2-category $Ho(2\text{-}Pro(\mathsf{SS}))$. In particular, shape theory of topological spaces discards the explicit homotopies and works with the Čech nerve in the category $Pro(Ho(\mathsf{SS}))$. Strong shape theory works in the category $Ho(Pro(\mathsf{SS}))$, that keeps the information coded in the explicit homotopies, but has the difficulty that the Čech nerve is not an object of $Pro(Ho(\mathsf{SS}))$. Our results provide a conceptual framework to use the Čech nerve in strong shape theory as a object in $\mathcal{H}o(2\text{-}Pro(\mathsf{SS}))$, as it is used in shape theory. Note that the results in [5] are applied to lift the model structure in the category SS into a 2-model structure in the 2-category 2-Pro(SS).

Organization

The paper is structured as follows. Section 2 contains some preliminaries which is convenient to have explicitly at hand. In Section 3 we give the basic definitions of the theory of model bicategories. In Section 4 we define the homotopy bicategory $\mathcal{H}o(\mathcal{C})$ and prove the results explained in **A** above, as well as other relevant results for w- and q-homotopies such as the relation between right and left homotopies mentioned in **E**. In Section 5 we construct the fibrant-cofibrant replacement, that is the pseudofunctors Q and RQ, and the pseudonatural transformations ρ, λ as explained in **B** and **E**. Section 6 consists of Theorem 6.4 and its proof as outlined in **C**. Appendix A establishes the transfer of structure for bicategories and pseudofunctors mentioned in **D** that we apply in the paper.

2 Preliminaries

The concepts of bicategory, pseudofunctor, pseudonatural transformation and modification are ubiquitous in the literature, so we have decided to omit their definitions here. The reader can check Appendix A where we found it necessary to recall these definitions because the axioms are explicitly used, contrary to the case of the body of the paper.

2.1. Notation. For a bicategory $\mathcal{C}$ and objects $X, Y \in \mathcal{C}$, $\mathcal{C}(X, Y)$ denotes the hom-category of arrows and 2-cells. For two bicategories $\mathcal{C}$, $\mathcal{D}$, $Hom(\mathcal{C}, \mathcal{D})$ denotes the bicategory of pseudofunctors, pseudonatural transformations and modifications. Vertical composition is denoted by "$\circ$", and horizontal composition by "$*$". We consider "$*$" more binding than "$\circ$" .

Coherence. There is a well-known coherence theorem (see for example [22]) which generalizes the coherence theorem for tensor categories. Given any sequence of composable arrows in a bicategory, the parentheses determine the order in which the compositions are performed. The coherence theorem states that the arrows resulting of any choice of parentheses (and adding or subtracting identities) are canonically isomorphic by an unique 2-cell built with the associators and the unitors. This justifies the following abuse of notation which greatly simplifies the computations:

2.2. *We write any horizontal composition of arrows omitting the parentheses and the identities. In this way, the associator and the unitors disappear in the diagrams of 2-cells.*

2.3. Dual bicategory. Given a bicategory $\mathcal{C}$, the *dual* bicategory, denoted $\mathcal{C}^{op}$, consists of the bicategory with the same objects, arrows and 2-cells, but formally reversing the direction of the arrows (while retaining the direction of the 2-cells).

2.4. Equivalences.

1. An arrow $X \xrightarrow{f} Y$ of a bicategory is an *equivalence* if there exists an arrow $Y \xrightarrow{g} X$ (which we call a *quasiinverse* of f) and isomorphisms $g * f \cong id_X$, $f * g \cong id_Y$. It is well-known that these isomorphisms can be taken satisfying the usual triangular identities, and we will assume that this is the case when needed.

2. It is well known that $X \xrightarrow{f} Y$ is an equivalence if and only if for every object Z the functor $\mathcal{C}(Z,X) \xrightarrow{f_*} \mathcal{C}(Z,Y)$ is an equivalence of categories, and if and only if the functor $\mathcal{C}(Y,Z) \xrightarrow{f^*} \mathcal{C}(X,Z)$ is so.

3. Recall that a pseudonatural transformation $\theta : F \Rightarrow G : \mathcal{C} \longrightarrow \mathcal{D}$ is an equivalence in the bicategory $Hom(\mathcal{C},\mathcal{D})$, i.e. there exists $G \overset{\mu}{\Rightarrow} F$ and invertible modifications $\theta \circ \mu \cong id_G$, $\mu \circ \theta \cong id_F$, if and only if each θ_X is an equivalence in $\mathcal{D}$. For a proof, see for example [26, 1.10] or [12, 1.17]. It is worth mentioning that this is not true for 2-natural transformations.

4. A pseudofunctor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ is a *biequivalence of bicategories* if there exist a pseudofunctor $\mathcal{D} \xrightarrow{G} \mathcal{C}$ (which we call a *bi-inverse* or a *pseudoinverse* of F) and pseudonatural transformations $GF \overset{\alpha}{\Rightarrow} id_{\mathcal{C}}$, $FG \overset{\beta}{\Rightarrow} id_{\mathcal{D}}$ which are equivalences.

2.5. On a weak notion of full functor. For a category $\mathbf{X}$, we denote by $\mathbf{X^2}$ the category that has the arrows of $\mathbf{X}$ as objects and the commutative squares as arrows. Note that a functor between categories $\mathbf{X} \xrightarrow{F} \mathbf{A}$ is full if and only if the induced functor $\mathbf{X^2} \xrightarrow{F^2} (\mathbf{Im_F})^\mathbf{2} \subset \mathbf{A^2}$ is surjective on objects, where $\mathbf{Im_F}$ denotes the full image of F.

Definition 2.6. *We say that a functor between categories* $\mathbf{X} \xrightarrow{F} \mathbf{A}$ *is essentially full if and only if the induced functor* $\mathbf{X^2} \xrightarrow{F^2} (\mathbf{Im_F})^\mathbf{2} \subset \mathbf{A^2}$ *is essentially surjective.*

That is, given any $FX \xrightarrow{g} FY$ *there exists* X', Y', $X' \xrightarrow{f} Y'$, *and isomorphisms* $FX' \xrightarrow{a} FX$, $FY' \xrightarrow{b} FY$ *such that* $g \circ a = b \circ Ff$.

Remark 2.7. Note that a notion of *essentially full* is usually considered for a pseudofunctor F between bicategories, and it means that the induced functors between the Hom-categories are essentially surjective. This means that F is surjective on 1-cells, up to invertible 2-cells, that is, given any $FX \xrightarrow{g} FY$ there exists $X \xrightarrow{f} Y$, and an invertible 2-cell $Ff \overset{\alpha}{\Rightarrow} g$. This is are different notion than the one we consider here for a functor between categories, where *"essentially"* should be considered to mean up to the isomorphisms a and b. A notion of essentially full functor between categories which is similar to the one in the present article was considered in [29, p.469].

The following can be easily checked:

Remark 2.8. A functor between categories $\mathbf{X} \xrightarrow{F} \mathbf{A}$ is essentially surjective and essentially full if and only if $\mathbf{X^2} \xrightarrow{F^2} \mathbf{A^2}$ is essentially surjective, that is for every A, B and $A \xrightarrow{g} B$ in $\mathbf{A}$, there exist X, Y, isomorphisms $FX \xrightarrow{a} A$, $FY \xrightarrow{b} B$ and $X \xrightarrow{f} Y$ in $\mathbf{X}$, such that $g \circ a = b \circ Ff$.

2.9. Terminology on Limits. If $\mathcal{C}$ is a bicategory, by the term *biLimit* we refer to the object of $\mathcal{C}$ defined by the universal property established by postulating an equivalence between the appropriate hom-categories. By the term *psLimit* ("ps" should be read as "pseudo"), we require the equivalence to be an isomorphism. The use of an initial capital letter was adopted following [3]. We will consider in $\mathcal{C}$ the concepts of initial object, product, pullback, comma-object, tensor and their dual versions terminal object, coproduct, pushout, cocomma-object, cotensor. These bicategorical limits and their terminology are well established, and we refer for example to [18] for details if needed. We specify a case which differs from the terminology in op. cit. By *biPullback* we refer to the *bi-iso-comma object*, and by *lax-biPullback* we refer to the *bi-comma object*.

We denote the bicoProduct of X and Y by $X \amalg Y$ and its inclusions by i_0, i_1. Given arrows $X \xrightarrow{f} Z$, $Y \xrightarrow{g} Z$, we denote the induced arrow (note that this is an abuse of notation since the arrow is not unique) by $\binom{f}{g} : X \amalg Y \longrightarrow Z$. We leave unnamed the invertible 2-cells $f \cong \binom{f}{g} * i_0$, $g \cong \binom{f}{g} * i_1$. Reciprocally, given an arrow $X \amalg Y \longrightarrow Z$ we denote it by $\binom{f}{g}$ to indicate that we define $f = \binom{f}{g} * i_0$, $g = \binom{f}{g} * i_1$. Note that if we have in addition $Z \xrightarrow{h} W$, we have $h * \binom{f}{g} = \binom{h*f}{h*g}$. Given 2-cells $X \underset{f'}{\overset{f}{\underset{\alpha\Downarrow}{\rightrightarrows}}} Z$, $Y \underset{g'}{\overset{g}{\underset{\beta\Downarrow}{\rightrightarrows}}} Z$, we denote the induced 2-cell by $\binom{\alpha}{\beta} : \binom{f}{g} \Rightarrow \binom{f'}{g'}$. When $Y = X$ we denote the codiagonal $\binom{id_X}{id_X}$ by ∇_X. Similarly we denote the biProduct of X and Y by $X \times Y$, and given arrows $Z \xrightarrow{f} X$, $Z \xrightarrow{g} Y$, we denote the induced arrow by $(f,\, g) : Z \longrightarrow X \times Y$. The diagonal is denoted Δ_X.

3 Model bicategories

In this section we will give the definition of a model bicategory. We consider Quillen's original axioms as were introduced in [27, §1]. Since in the known examples Quillen's axioms of *closed* model category hold, it is usual nowadays to consider these stronger axioms as the definition of model category. Especially since this notion (that of a closed model category) admits a neat and compact presentation in terms of *weak factorization systems*. In this paper, however, in an attempt to be closer to the original construction of the homotopy category in [27, §1], we won't assume these stronger axioms. We think it is still convenient (and important) to consider Quillen's original axioms as a guide for the model bicategory axioms.

We now give the notion of lifting property for a pair of arrows in a bicategory. Note that a stronger notion in which the lifting is required to be *universal* is considered for *factorization systems* (not weak) in 2-categories in [8, 1.3,1.4]. This notion would correspond to asking the functor h in the diagram below to be an equivalence of categories, instead of just essentially surjective and essentially full as in our definition.

Let $\mathcal{C}$ be a bicategory and $A \xrightarrow{i} X$, $Y \xrightarrow{p} B$ be two morphisms in $\mathcal{C}$, they determine the following diagram (note that the associator defines a natural transformation $p_* \, i^* \overset{\theta}{\Rightarrow} i^* \, p_*$).

$$\begin{tikzcd}
\mathcal{C}(X,\,Y) \arrow[rrd, bend left, "i^*"] \arrow[rd, "h"] \arrow[rdd, bend right, "p_*"'] & & \\
& \mathbf{P} \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] \arrow[rd, phantom, "\Downarrow \cong"] & \mathcal{C}(A,\,Y) \arrow[d, "p_*"] \\
& \mathcal{C}(X,\,B) \arrow[r, "i^*"] & \mathcal{C}(A,\,B)
\end{tikzcd}$$

where $\mathbf{P}$ is the biPullback of categories (in fact a psPullback), $\pi_1 \, h = i^*$, $\pi_2 \, h = p_*$.

Definition 3.1. *We say that a pair* (i, p) *as above has the* lifting property $\mathbf{L}$ *if the functor* $h = (i^*,\, p_*)$ *in the diagram above is essentially surjective and essentially full*[1].

Note that an object of $\mathbf{P}$ is a triplet $(a,\, \gamma,\, b)$ as in the left below, and an arrow between two such objects is a pair of 2-cells $(a,\, \gamma,\, b) \xrightarrow{(\alpha,\beta)} (a',\, \gamma',\, b')$ as in the middle diagram, satisfying the equation on the right:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rd, phantom, "\cong\Downarrow\ \gamma"] & Y \arrow[d, "p"] \\
X \arrow[r, "b"] & B
\end{tikzcd} , \qquad
\begin{tikzcd}
A \arrow[r, shift left=1.5, "a"] \arrow[r, shift right=1.5, "a'\ \alpha\Downarrow"'] \arrow[d, "i"'] \arrow[rd, phantom, "\cong\Downarrow\ \gamma\,\gamma'"] & Y \arrow[d, "p"] \\
X \arrow[r, shift left=1.5, "b"] \arrow[r, shift right=1.5, "b'\ \beta\Downarrow"'] & B
\end{tikzcd} , \qquad \beta * i \circ \gamma = \gamma' \circ p * \alpha \tag{3.2}$$

We now unfold this definition so as to have at hand the statements to be used in practice.

3.3. We found it convenient to set aside what it means for h to be essentially surjective, because it has independent interest and it is sufficient in many applications of the lifting property $\mathbf{L}$.

Ls. (h is essentially surjective). For each triplet $(a,\, \gamma,\, b)$ as in the left diagram below, there exist a morphism $X \xrightarrow{f} Y$ and invertible 2-cells λ, ρ as in the middle diagram, satisfying the equation on the right:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[rd, phantom, "\cong\Downarrow\ \gamma"] & Y \arrow[d, "p"] \\
X \arrow[r, "b"] & B
\end{tikzcd} , \qquad
\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[d, phantom, shift left=4, "\cong\Uparrow\lambda"] & Y \arrow[d, "p"] \arrow[d, phantom, shift right=4, "\cong\Downarrow\rho"'] \\
X \arrow[r, "b"] \arrow[ru, "f"] & B
\end{tikzcd} , \qquad \rho * i = \gamma \circ p * \lambda \tag{3.4}$$

We say that $(f,\, \lambda,\, \rho)$ is a *filler* for the square $(a,\, \gamma,\, b)$.

(Lifting property L.) By the description of the category $\mathbf{P}$ above, and Remark 2.8, the fact that h is essentially surjective and essentially full means:

---

[1] In a preliminary version of this article a stronger axiom was required for the lifting property which implied the uniqueness of the filler. We are grateful to Valery Isaev for pointing this out to us.

L. For each $(a,\,\gamma,\,b)$, $(a',\,\gamma',\,b')$ and $(a,\,\gamma,\,b) \xrightarrow{(\alpha,\beta)} (a',\,\gamma',\,b')$ as in (3.2), there exist fillers $(f,\,\lambda,\,\rho)$, $(f',\,\lambda',\,\rho')$ as in (3.4), and $f \overset{\delta}{\Rightarrow} f'$ such that

$$\lambda' \circ \delta * i = \alpha \circ \lambda \ , \qquad \rho' \circ p * \delta = \beta \circ \rho \tag{3.5}$$

Remark 3.6. For α and β invertible, it is easy to check that axiom **L** follows from **Ls**. Indeed, taking a filler (f, λ, ρ) for the square $(a,\,\gamma,\,b)$ and constructing the filler $(f, \alpha^{-1} \circ \lambda, \beta \circ \rho)$ for the square $(a',\,\gamma',\,b')$ does the trick.

Note that the lifting property L is self-dual, equivalent but different to its dual. The equivalence holds because the 2-cell γ is invertible. In the dual formulation the biPullback of categories is replaced by the op-biPullback.

It is convenient to spell out the dual statement:

Ls^{op}. For each invertible 2-cell $(a,\,\gamma,\,b)$ as in the left diagram below, there exist a morphism f and invertible 2-cells λ, ρ as in the middle diagram, satisfying the equation on the right:

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong\Uparrow\,\gamma"] & Y \arrow[d, "p"] \\ X \arrow[r, "b"'] & B \end{tikzcd} \ , \qquad \begin{tikzcd} A \arrow[r, "a"] \arrow[d, "i"'] & Y \arrow[d, "p"] \\ X \arrow[r, "b"'] \arrow[ur, "f"] & B \end{tikzcd} \ , \qquad \gamma \circ \rho * i = p * \lambda \tag{3.7}$$

with $\cong\Uparrow\lambda$ in the upper triangle and $\cong\Downarrow\rho$ in the lower triangle of the middle diagram.

Definition 3.8. *We say that a bicategory $\mathcal{C}$ is a* model bicategory *if it is equipped with three classes of morphisms $\mathcal{F}, co\mathcal{F}$ and $\mathcal{W}$ called respectively fibrations, cofibrations and weak equivalences satisfying the following set of axioms:*

M0. *$\mathcal{C}$ has biterminal objects, biPullbacks, and their dual coLimit versions.*

M1. *Given a cofibration i and a fibration p, if one of them is a weak equivalence, then (i, p) has the lifting property* **L**.

M2. *Every morphism f can be factored up to an invertible 2-cell as $f \cong p * i$ with i a cofibration and p a fibration. In addition, we may require either i or p to be a weak equivalence.*

M3. *Fibrations (respectively cofibrations) are closed under composition and biPullbacks (respectively biPushouts). Every equivalence is a fibration and a cofibration. If there is an invertible 2-cell $f \cong g$ and f is a fibration (resp. a cofibration), then so is g.*

M4. *If a morphism f is the biPullback (resp. biPushout) of a fibration (resp. cofibration) which is also a weak equivalence, then f is a weak equivalence.*

M5. *The class of weak equivalences satisfies the "3 for 2" axiom: for every three arrows f, g, h such that there is an invertible 2-cell $gf \cong h$, whenever two of the three arrows are weak equivalences, so is the third one. Also, every equivalence is a weak equivalence.*

M1s. A notion of 2-model 2-category was introduced in [5] where the lifting axiom **M1** was considered with the essentially surjective item **Ls** in 3.3. We denote by **M1s** this weaker axiom, which as stated before is sufficient in many applications of axiom **M1**. The full strength of axiom **M1** is nevertheless necessary to deal with arbitrary, non-invertible 2-cells in the fibrant-cofibrant replacement.

Remark 3.9. By interchanging fibrations with cofibrations and keeping the same weak equivalences it is clear that the bicategory dual to a model bicategory is also a model bicategory, so that each valid statement has a valid dual statement.

Remark 3.10. Model categories are usually required to be complete and cocomplete, so axiom **M0** above may seem to be too weak a condition regarding limit existence. One could also consider stronger completeness axioms depending on the context:

MC0. $\mathcal{C}$ has all conical finite biLimits and biColimits

MW0. $\mathcal{C}$ has all weighted finite biLimits and biColimits.

In [5] axiom **MW0** was assumed. Our reason for not considering this axiom here is that the coTensor $\{\mathbf{I}, X\}$ with the generic isomorphism (that is the category $\mathbf{I}$ of two objects and an isomorphism between them) fits in a diagram

$$\begin{tikzcd}
 & \{\mathbf{I}, X\} \arrow[dr] & \\
X \arrow[ur] \arrow[rr] & & X \times X = \{\{0,1\}, X\}
\end{tikzcd}$$

given by the inclusion $\{0,1\} \subset \mathbf{I}$.

Under certain assumptions this is a functorial path-object for X (see [28]), which conflicts with our purpose to assume only Quillen's original axioms. We mention that functorial factorizations, tensors, cotensors and a stronger lifting property (Quillen's axiom SM7) are assumed in Quillen's simplicial model categories as well as in its generalization as enriched model categories considered in [14], though we have not investigated this line of research.

Finally, we note that in this paper we only use biPullbacks of fibrations (along arbitrary arrows), as well as the dual biPushouts of cofibrations, and that only the 1-dimensional property of these limits is used (in the sense of [18]), so axiom **M0** could even be further weakened and the constructions and results of this paper would still hold.

Remark 3.11. A model category $\mathbf{C}$ ([27]) can be regarded as a model bicategory (2-category) in which every 2-cell is the identity.

Taking the set of connected components in the hom-categories of the homotopy bicategories constructed in this paper yields the constructions and results of [27] for $\mathbf{C}$, see [12] for a full account of this.

Remark 3.12. It can be checked that any bicategory satisfying **M0** is a model bicategory with $\mathcal{W}$ the equivalences, and every arrow a fibration and a cofibration.

For the rest of the article, except when we consider the more general case ($\mathcal{C}$, $\mathcal{W}$) of a category and a single class $\mathcal{W}$, $\mathcal{C}$ will denote an arbitrary model bicategory ($\mathcal{C}$, $\mathcal{F}$, $co\mathcal{F}$, $\mathcal{W}$).

Definition 3.13. *We say that an arrow in $\mathcal{C}$ is a trivial (co)fibration if it is simultaneously a (co)fibration and a weak equivalence. We will use the following notation:*

1. $\cdot \xrightarrow{\cong} \cdot$ *is an equivalence.*

2. $\cdot \xrightarrow{\sim} \cdot$ *is a weak equivalence.*

3. $\cdot \twoheadrightarrow \cdot$ *is a fibration,* $\cdot \overset{\sim}{\twoheadrightarrow} \cdot$ *is a trivial fibration,*

4. $\cdot \hookrightarrow \cdot$ *is a cofibration,* $\cdot \overset{\sim}{\hookrightarrow} \cdot$ *is a trivial cofibration*

Definition 3.14. *Let X be an object of $\mathcal{C}$.*

1. *We say that X is a fibrant object if the morphism $X \longrightarrow *$ is a fibration.*

2. *We say that X is a cofibrant object if the morphism $0 \longrightarrow X$ is a cofibration.*

We denote by $\mathcal{C}_f$, $\mathcal{C}_c$, $\mathcal{C}_{fc}$ the full sub-bicategories of fibrant, cofibrant and fibrant-cofibrant objects (i.e. objects that are both fibrant and cofibrant) respectively. We denote with the same letters $\mathcal{F}$, co$\mathcal{F}$, $\mathcal{W}$ the restrictions of these families of arrows to the three bicategories.

Remark 3.15. Note that 0 and $*$ are denoting the Initial and the Terminal object respectively given by axiom **M0**. More explicitly, 0 satisfies that for each $X \in \mathcal{C}$, there exists a morphism $0 \longrightarrow X \in \mathcal{C}$ up to unique invertible 2-cell, and dually for $*$. In the previous definition the abuse of saying "the morphism" is justified by axiom **M3**. $\square$

A key fact in the theory of model categories is that any weak equivalence between fibrant-cofibrant objects can be factored as a section followed by a retraction, both of them weak equivalences, and that this fact can be used to prove Whitehead's theorem (see [13, Th. 1.10], and also [30, Prop. 3.1.21]). We now show the bicategorical equivalent of this statement.

Definition 3.16. *Let $X \xrightarrow{s} Y$, $Y \xrightarrow{r} X$ be arrows of a bicategory. If there is an invertible 2-cell $rs \cong id_X$, s is called a section for r, and r is called a retraction for s. An arrow $X \xrightarrow{s} Y$ is called a section if there exists r such that s is a section for r and dually an arrow is called a retraction if it admits a section. An arrow that is either a section or a retraction is called a* split *arrow.*

Proposition 3.17. *Let $X \xrightarrow{f} Y \in \mathcal{C}$ be a weak equivalence, with X fibrant and Y cofibrant. Then axiom* **M2** *yields a factorization of f as a composition $X \xrightarrow{i} Z \xrightarrow{p} Y$ of a section and a retraction, both of them weak equivalences. If furthermore both X and Y are fibrant-cofibrant, then so is Z.*

Proof. Let $X \xrightarrow{f} Y$ be such a weak equivalence. By axioms **M2** and **M5** we have

$$\begin{tikzcd} X \arrow[rr, "f", "\sim"'] \arrow[dr, hook, "\sim", "i"'] & \cong & Y \\ & Z \arrow[ur, two heads, "\sim"', "p"'] & \end{tikzcd}$$

. The last statement of the proposition clearly holds by axiom **M3**.

We will show that i is a section, dually it follows that p is a retraction. Using axiom **M1**, we have

$$\begin{tikzcd} X \arrow[r, "id_X"] \arrow[d, "i"'] \arrow[dr, phantom, "\cong"] & X \arrow[d] \\ Z \arrow[r] \arrow[ur, dotted, "r"'] & 1 \end{tikzcd}$$

as desired. □

4 The homotopy bicategory of a model bicategory

This section depends heavily on the results of [6], and we recommend the reader to have at hand this reference. Nevertheless we will recall here some of the content of [6] to improve readability. We fix a model bicategory $\mathcal{C}$. We will define a notion of q-homotopy between arrows of $\mathcal{C}$, which is analogous to Quillen's notion of homotopy for model categories. Our objective is to construct a bicategory $\mathcal{H}o(\mathcal{C})$ whose objects and arrows are those of $\mathcal{C}_{fc}$, and whose 2-cells are classes of q-homotopies under an equivalence relation, together with a pseudofunctor $\mathcal{C} \xrightarrow{q} \mathcal{H}o(\mathcal{C})$ which is the 2-dimensional localization of $\mathcal{C}$ with respect to the class $\mathcal{W}$ of weak equivalences in the sense of the following universal property: $\mathcal{C} \xrightarrow{q} \mathcal{H}o(\mathcal{C})$ sends weak equivalences to equivalences and q-precomposition $Hom(\mathcal{H}o(\mathcal{C}), \mathcal{D}) \xrightarrow{q^*} Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ determines a biequivalence of bicategories for every bicategory $\mathcal{D}$, where $Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ stands for the full sub-bicategory of $Hom(\mathcal{C}, \mathcal{D})$ given by those pseudofunctors that send weak equivalences into equivalences (Theorem 6.4).

The localization of a bicategory at a family of arrows was first considered in [24] by means of a 2-dimensional version of Gabriel-Zisman's calculus of fractions. There it is considered a bicategory $Hom_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$ with the same objects as $Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$, but whose arrows are the pseudonatural transformations that map the arrows of $\mathcal{W}$ to equivalences (when interpreted as pseudofunctors from $\mathcal{C}$ into a cylinder bicategory). However, we have recently noticed that a simple verification shows that $Hom_{\mathcal{W}}(\mathcal{C}, \mathcal{D}) = Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ (more details can be found in [25, Remark 3.7]), so the universal property that can be found in [24, Th. 21], involving $Hom_{\mathcal{W}}(\mathcal{C}, \mathcal{D})$, is indeed the same one of Theorem 6.4 but stated differently.

We will introduce q-homotopies as w-homotopies (a notion which depends only of the class $\mathcal{W}$) that satisfy some extra conditions with respect to the classes $\mathcal{F}$ and $co\mathcal{F}$. This allows to use previous results of [6], where for an arbitrary pair $(\mathcal{A}, \Sigma)$ given by a family Σ of arrows of a bicategory $\mathcal{A}$ there is a construction of a bicategory $\mathcal{H}o(\mathcal{A}, \Sigma)$ whose 2-cells are formed with w-homotopies. However, unless a vertical composition can be defined, the 2-cells of $\mathcal{H}o(\mathcal{A}, \Sigma)$ consist of the classes of finite composable sequences of w-homotopies.

In §4.1 we will show that q-homotopies can be vertically composed, and in §4.2 we will show that for any w-homotopy between arrows of $\mathcal{C}_{fc}$ there is a q-homotopy in the same class. This will allow to give $\mathcal{H}o(\mathcal{C})$ the bicategory structure of $\mathcal{H}o(\mathcal{C}, \mathcal{W})$.

Definition 4.1. *We consider any pair $(\mathcal{C}, \mathcal{W})$ given by a family that we call* weak equivalences *of a bicategory $\mathcal{C}$. Let $X \in \mathcal{C}$. A w-cylinder C (for X) is given by the data*

$C = (W, Z, d_0, d_1, x, s, \alpha_0, \alpha_1)$, fitting in

$$\begin{tikzcd}
X \arrow[r, "d_0"] \arrow[d, "d_1"'] \arrow[rd, "x"] \arrow[rd, phantom, shift left=3ex, "\cong \Downarrow \alpha_0"] \arrow[rd, phantom, shift right=3ex, "\cong \Uparrow \alpha_1"'] & W \arrow[d, "s", "\wr"'] \\
W \arrow[r, "s", "\sim"'] & Z
\end{tikzcd}$$

(s is a weak equivalence, α_0 and α_1 are invertible 2-cells). We denote $\alpha = \alpha_1^{-1} \circ \alpha_0$.

If $(\mathcal{C}, \mathcal{W})$ is part of a model bicategory structure, then a q-cylinder is a w-cylinder such that $Z = X$, $x = id_X$, and the arrow $\binom{d_0}{d_1} : X \amalg X \longrightarrow W$ is a cofibration. In this case we write $C = (W, d_0, d_1, s, \alpha_0, \alpha_1)$.

Let $f, g : X \to Y \in \mathcal{C}$. A left w-homotopy H from f to g, which we will denote by $f \overset{H}{\rightsquigarrow} g$, is given by the data $H = (C, h, \eta, \varepsilon)$, where C is a w-cylinder for X as above, h is an arrow $W \xrightarrow{h} Y$ and η, ε are 2-cells $f \overset{\eta}{\Rightarrow} h * d_0$, $h * d_1 \overset{\varepsilon}{\Rightarrow} g$. We organize these data as follows:

$$f \overset{H}{\rightsquigarrow} g: \quad \begin{tikzcd}
X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rd, "x"'] & W \arrow[r, "h"] \arrow[d, "s"', "\wr"] & Y \\
& Z &
\end{tikzcd} \qquad \begin{array}{c} f \overset{\eta}{\Rightarrow} h * d_0 \\ s * d_0 \overset{\alpha_0}{\Rightarrow} x \overset{\alpha_1}{\Leftarrow} s * d_1 \\ h * d_1 \overset{\varepsilon}{\Rightarrow} g \end{array} \tag{4.2}$$

We say that H has invertible cells if η and ε are invertible.

A left q-homotopy is a left w-homotopy in which C is a q-cylinder.

We say that a w-cylinder (in particular a q-cylinder) is fibrant *if the arrow s is a fibration. We use the same terminology for left w-homotopies.*

Note that the abuse of saying "the arrow $\binom{d_0}{d_1}$ is a cofibration" is justified by axiom **M3**.

We record for convenience the dual structures of right w- and q-homotopies. These notions arise from considering the *opposite model structure* on the bicategory $\mathcal{C}^{op}$.

Definition 4.3. *A w-path-object P (for Y) is given by the data $P = (V, Z, c_0, c_1, y, t, \beta_0, \beta_1)$, fitting in*

$$\begin{tikzcd}
Z \arrow[r, "t\ \sim"] \arrow[d, "t", "\wr"'] \arrow[rd, "y"] \arrow[rd, phantom, shift left=3ex, "\cong \Downarrow \beta_0"] \arrow[rd, phantom, shift right=3ex, "\cong \Uparrow \beta_1"'] & V \arrow[d, "c_0"] \\
V \arrow[r, "c_1"] & Y
\end{tikzcd}$$

. We denote $\beta = \beta_1^{-1} \circ \beta_0$. A q-path-object is a w-path-object such that $Z = Y$, $y = id_Y$, and the arrow $(c_0,\, c_1) : V \longrightarrow Y$ is a fibration. In this case we write $P = (V, c_0, c_1, t, \beta_0, \beta_1)$.

A right w-homotopy K from f to g, which we will denote by $f \overset{K}{\rightsquigarrow} g$, is given by the data $K = (P, k, \delta, \epsilon)$, where P is a w-path-object for Y, k is an arrow $X \xrightarrow{k} V$ and δ, ϵ are 2-cells $f \overset{\delta}{\Rightarrow} c_0 * k$, $c_1 * k \overset{\epsilon}{\Rightarrow} g$.

A right q-homotopy is a right w-homotopy in which P is a q-path-object.

We say that a w-path object (in particular a q-path object) is cofibrant *if the arrow t is a cofibration. We use the same terminology for right w-homotopies.*

In this section we will work only with left w-homotopies, and thus omit to specify it.

Remark 4.4. For any $X \in \mathcal{C}$, we can construct a fibrant q-cylinder for X as follows. We use axiom **M2** and factorize

$$\begin{tikzcd}
X \amalg X \arrow[rr, "\nabla_X"] \arrow[dr, hook, "\binom{d_0}{d_1}"'] & {} \arrow[d, phantom, "\Uparrow\cong"] & X \\
& W \arrow[ur, two heads, "\sim", "s"'] &
\end{tikzcd}$$

. By the universal property of the coProduct $\binom{s*d_0}{s*d_1} \Rightarrow \nabla_X$ corresponds to two invertible 2-cells $s * d_0 \xRightarrow{\alpha_0} id_X$, $s * d_1 \xRightarrow{\alpha_1} id_X$ and thus we have a fibrant q-cylinder $C = (W, d_0, d_1, s, \alpha_0, \alpha_1)$.

The concepts of q-cylinder and q-homotopy are the bicategorical analogues of Quillen's notions in [27]. The following is the bicategorical equivalent of Quillen's [27, 1, Lemma 2].

Lemma 4.5. *Let $X \in \mathcal{C}$ be a cofibrant object, and let C be a q-cylinder for X as in Definition 4.1. Then both d_0 and d_1 are trivial cofibrations, and W is a cofibrant object.*

Proof. By definition of $X \amalg X$, the fact that X is cofibrant and axiom **M3**, we have that $X \xrightarrow{i_0} X \amalg X$ is a cofibration. Then, since $d_0 \cong \binom{d_0}{d_1} * i_0$, by axiom **M3** d_0 is a cofibration. Also, since $s * d_0 \cong id_X$, by axiom **M5** d_0 is a weak equivalence. A similar reasoning shows that d_1 is also a trivial cofibration. Since X is cofibrant, it is clear that then so is W. □

An important consequence of the previous lemma is that any fibrant q-homotopy between objects of $\mathcal{C}_{fc}$ is *inside* $\mathcal{C}_{fc}$:

Corollary 4.6. *Let $f, g : X \to Y \in \mathcal{C}$, and $f \overset{H}{\rightsquigarrow} g$ be a fibrant q-homotopy as in Definition 4.1. If X is cofibrant and Y is fibrant, then W is a fibrant-cofibrant object.* □

We consider now an equivalence relation between w-homotopies (in particular q-homotopies) in a way such that its classes will form the 2-cells of a bicategory. This relation is defined in [6], we recall it now:

4.7. We say that an arrow $X \xrightarrow{f} Y$ of a bicategory is a *quasiequivalence*, a concept weaker than that of equivalence, if for every object Z the functors $\mathcal{C}(Z, X) \xrightarrow{f_*} \mathcal{C}(Z, Y)$ and $\mathcal{C}(Y, Z) \xrightarrow{f^*} \mathcal{C}(X, Z)$ are full and faithful.

The w-homotopies can be thought of something that would be an actual 2-cell if the weak equivalences were quasiequivalences. More precisely, given a w-homotopy $f \overset{H}{\rightsquigarrow} g$ as in Definition 4.1 for which s is a quasiequivalence, we can associate to it the 2-cell $f \xRightarrow{\widehat{H}} g$ given by the composition $f \overset{\eta}{\Rightarrow} h * d_0 \xRightarrow{h*\widehat{C}} h * d_1 \overset{\varepsilon}{\Rightarrow} g$, where $d_0 \overset{\widehat{C}}{\Rightarrow} d_1$ is the unique 2-cell such that $s * \widehat{C} = \alpha$. For any 2-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$, we denote by FH the structure obtained applying F to each of the components of H. In particular, if F maps all the weak equivalences to quasiequivalences, it follows that there is a 2-cell $\widehat{FH}$ of $\mathcal{D}$ for each w-homotopy H.

For any bicategory $\mathcal{D}$, we denote by $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$ a 2-functor which maps the weak equivalences to quasiequivalences[2].

Definition 4.8. *We say that two w-homotopies (in particular two q-homotopies) $f \overset{H,K}{\rightsquigarrow} g$ are in the same class if $\widehat{FH} = \widehat{FK}$ for all $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$ (for all bicategories $\mathcal{D}$). Note that the construction in 4.7 can be easily dualized for a right w-homotopy, and thus this definition makes sense when either one or both H, K are right w-homotopies.*

Remark 4.9. It is the composition $\alpha = \alpha_1^{-1} \circ \alpha_0$ which is used in order to determine the class of a w-homotopy. Different pairs of 2-cells α_0, α_1 can yield the same 2-cell α. As an example (which will be relevant later) consider any w-homotopy $f \overset{H}{\rightsquigarrow} g$ as in Definition 4.1, and define w-cylinders for X,

$$C_0 = (W, Z, d_0, d_1, s * d_0, s, s * d_0, \alpha^{-1}), \quad C_1 = (W, Z, d_0, d_1, s * d_1, s, \alpha, s * d_1),$$

and w-homotopies $f \overset{H_i}{\rightsquigarrow} g$, $H_i = (C_i, h, \eta, \varepsilon)$, $i = 0, 1$. In particular $[H] = [H_0] = [H_1]$.

4.10. On the composition of homotopies with arrows. Consider arrows $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ and a w-homotopy $H = (C, h, \eta, \varepsilon) : f \rightsquigarrow g$, as in (4.2). We want to define the composition of H with an arrow $Y \xrightarrow{r} Y'$ (on the right) and with an arrow $X' \xrightarrow{\ell} X$ (on the left):

$$\begin{tikzcd} X' \arrow[r, "\ell"] & X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[dr, "x"'] & W \arrow[r, "h"] \arrow[d, "s"', "\wr"] & Y \arrow[r, "r"] & Y' \\ & & Z & & \end{tikzcd}
\qquad
\begin{gathered} f \overset{\eta}{\Rightarrow} h * d_0 \\ s * d_0 \overset{\alpha_0}{\Rightarrow} x \overset{\alpha_1}{\Leftarrow} s * d_1 \\ h * d_1 \overset{\varepsilon}{\Rightarrow} g \end{gathered}$$

Given $Y \xrightarrow{r} Y'$ we can define $r * H = (C, r * h, r * \eta, r * \varepsilon) : r * f \rightsquigarrow r * g$, and it is immediate to show that $F\widehat{(r * H)} = Fr * \widehat{FH}$ for any $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$ as in Definition 4.8 (see [6, Prop. 3.20, 3] for a proof). Note that $r * H$ has the same w-cylinder C as H. In particular, if H is a q-homotopy, then so is $r * H$, and thus this can be used to define a composition $r * [H] = [r * H]$ in $\mathcal{H}o(\mathcal{C})$.

Given $X' \xrightarrow{\ell} X$, we can define a w-cylinder $C * \ell = (W, d_0 * \ell, d_1 * \ell, s, \alpha_0 * \ell, \alpha_1 * \ell)$ for X', note that $C * \ell$ is not in general a q-cylinder, even if so was C. We can also define a w-homotopy $H * \ell = (C * \ell, h, \varepsilon * \ell, \eta * \ell)$ and show that $F\widehat{(H * \ell)} = \widehat{FH} * F\ell$ for any $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$ (see [6, Prop. 3.20, 4]). But, if H is a q-homotopy it is not clear how to define a q-homotopy $H * \ell$ satisfying this equation. In the 1-dimensional case [27], this is solved using the dual notion of right homotopy, but here we take a different approach. In §4.2 below we will show that, for fibrant-cofibrant objects, any w-homotopy admits a

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[2] Working with the notion of quasiequivalence instead of equivalence is what allows to consider here a 2-functor F instead of a general pseudofunctor, see [6, 2.10] for details.

q-homotopy in the same class. Thus this is the way in which the composition $[H] * \ell$ in $\mathcal{H}o(\mathcal{C})$ is defined, as a q-homotopy in the same class of $H * \ell$.

4.11. On the bicategory of w-homotopies. For an arbitrary pair $(\mathcal{C}, \mathcal{W})$ as in Definition 4.1, the definition in 4.7 can be extended to finite sequences of composable w-homotopies $f \overset{H^1}{\rightsquigarrow} f_1 \overset{H^2}{\rightsquigarrow} f_2 \cdots f_{n-1} \overset{H^n}{\rightsquigarrow} g$ and in this way a bicategory $\mathcal{H}o(\mathcal{C}, \mathcal{W})$ is defined. Its objects and arrows are those of $\mathcal{C}$, the 2-cells are the classes of finite sequences of composable w-homotopies, $[H^n, \dots, H^2, H^1]$. Horizontal composition (whiskering) is defined as in 4.10 and vertical composition is given by juxtaposition. Together with $\mathcal{H}o(\mathcal{C}, \mathcal{W})$ there is a *projection* 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H}o(\mathcal{C}, \mathcal{W})$, which is the identity on objects and arrows and maps a 2-cell μ of $\mathcal{C}$ to the class of a w-homotopy I^μ which satisfies that $\widehat{FI^\mu} = F\mu$ for any $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$. We refer the interested reader to [6] for more details on this construction.

We mention that I^μ can be chosen to be either one of the following two w-homotopies:

$$H_0^\mu : \begin{tikzcd} X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "id_X"'] \arrow[dr, "id_X"'] & X \arrow[r, "g"] \arrow[d, "id_X"] & Y \\ & X & \end{tikzcd} \qquad H_1^\mu : \begin{tikzcd} X \arrow[r, shift left, "id_X"] \arrow[r, shift right, "id_X"'] \arrow[dr, "id_X"'] & X \arrow[r, "f"] \arrow[d, "id_X"] & Y \\ & X & \end{tikzcd}$$

$$\eta = \mu,\ \alpha_0 = \alpha_1 = id_X,\ \varepsilon = g \qquad \eta = f,\ \alpha_0 = \alpha_1 = id_X,\ \varepsilon = \mu$$

Remark 4.12. It clearly follows that I^μ can be choosen to be a fibrant w-homotopy, and such that if μ is invertible, I^μ has invertible cells.

A dual construction allows to define a right w-homotopy I^μ such that $\widehat{FI^\mu} = F\mu$ for any $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$. It follows:

Remark 4.13. The w-homotopy I^μ can be considered to be a left or a right w-homotopy as needed.

Remark 4.14. Any w-cylinder C as in Definition 4.1 determines a w-homotopy with invertible cells:

$$d_0 \overset{H^C}{\rightsquigarrow} d_1 : \quad \begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[dr, "x"'] & W \arrow[r, "id_W"] \arrow[d, "s"', "\wr"] & W \\ & Z & \end{tikzcd} \qquad \begin{array}{c} d_0 \overset{d_0}{\Rightarrow} d_0 \\ s * d_0 \overset{\alpha_0}{\Rightarrow} x \overset{\alpha_1}{\Leftarrow} s * d_1 \\ d_1 \overset{d_1}{\Rightarrow} d_1 \end{array}$$

Remark 4.15. [Composing homotopies with 2-cells] Consider a w-homotopy $f \overset{H}{\rightsquigarrow} g$ as in (4.2), and 2-cells $f' \overset{\nu}{\Rightarrow} f$, $g \overset{\mu}{\Rightarrow} g'$. Simply composing ν with η, and μ with ε, yields a new w-homotopy $f' \overset{H'}{\rightsquigarrow} g'$, $H' = \mu \circ H \circ \nu$, with the same cylinder as H, satisfying $[H'] = [I^\mu] \circ [H] \circ [I^\nu] = [I^\mu, H, I^\nu]$. Details can be found in [6, 3.6-3.7] if needed.

We recall now a basic decomposition result for w-homotopies proved in [6] that we will use later.

Proposition 4.16. *([6, Proposition 3.31]) Let H be any w-homotopy as in Definition 4.1. Then $[H]$ can be decomposed as:*

$$[H] = [\varepsilon \circ (h * H^C) \circ \eta] = [I^\varepsilon] \circ [h * H^C] \circ [I^\eta] = [I^\varepsilon,\, h * H^C,\, I^\eta]. \qquad \square$$

Proposition 4.17. *([6, Corollary 3.33]) The class $[H]$ of any w-homotopy with invertible cells is an invertible 2-cell in $\mathcal{H}o(\mathcal{C}, \Sigma)$.* $\square$

Lemma 4.18. *Let $X \substack{\xrightarrow{f} \\ \mu\Downarrow \\ \xrightarrow[g]{}} Y \in \mathcal{C}$ be a 2-cell and $C = (W, d_0, d_1, s, \alpha_0, \alpha_1)$ be a q-cylinder for X. Then we can construct $f \overset{I^\mu}{\rightsquigarrow} g$ with cylinder C, and such that if μ is invertible then I^μ has invertible cells.*

Proof. Let $H = (C, f * s, f * \alpha_0^{-1}, \mu \circ f * \alpha_1)$. To verify $H = I^\mu$, let $\mathcal{C} \xrightarrow{F} \mathcal{D}$ be a 2-functor that maps the weak equivalences to equivalences, then $\widehat{FH}$ is the composition

$$Ff \xRightarrow{Ff*F\alpha_0^{-1}} Ff * Fs * Fd_0 \xRightarrow{Ff*Fs*\widehat{FC}} Ff * Fs * Fd_1 \xRightarrow{Ff*F\alpha_1} Ff \xRightarrow{F\mu} Fg.$$

Since by definition we have $Fs * \widehat{FC} = F\alpha_1^{-1} \circ F\alpha_0$, the composition clearly reduces to $F\mu$ as desired. $\square$

For an arbitrary pair $(\mathcal{C}, \mathcal{W})$, i is not the localization of $\mathcal{C}$ at $\mathcal{W}$, since $i(f)$ will not be in general an equivalence for each weak equivalence. However $i(f)$ will always satisfy the "faithful" part in the definition of quasiequivalence:

Proposition 4.19. *For any weak equivalence $X \xrightarrow{f} Y$ and any object Z, the functors $\mathcal{H}o(\mathcal{C}, \mathcal{W})(Z, X) \xrightarrow{f_*} \mathcal{H}o(\mathcal{C}, \mathcal{W})(Z, Y)$ and $\mathcal{H}o(\mathcal{C}, \mathcal{W})(Y, Z) \xrightarrow{f^*} \mathcal{H}o(\mathcal{C}, \mathcal{W})(X, Z)$ are faithful.*

Proof. We deal with f_* first, we consider arrows g, h and 2-cells α, β as follows $Z \substack{\xrightarrow{g} \\ \Downarrow\alpha \;\; \Downarrow\beta \\ \xrightarrow[h]{}} X$, which are determined by sequences of composable w-homotopies, $\alpha = [H^n, \, \dots \, , H^2, \, H^1]$, $\beta = [K^n, \, \dots \, , K^2, \, K^1]$. We have to show that $f * \alpha = f * \beta$ implies $\alpha = \beta$. We note that by definition $f * \alpha = f * \beta$ means that for any $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$ as in Definition 4.8 we have

$$Ff * (\widehat{FH_n} \circ ... \circ \widehat{FH_2} \circ \widehat{FH_1}) = Ff * (\widehat{FK_m} \circ ... \circ \widehat{FK_2} \circ \widehat{FK_1}).$$

Since Ff is a quasiequivalence, the equality $\widehat{FH_n} \circ ... \circ \widehat{FH_2} \circ \widehat{FH_1} = \widehat{FK_m} \circ ... \circ \widehat{FK_2} \circ \widehat{FK_1}$ holds for arbitrary $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$, that is $\alpha = \beta$. The case of f^* is dual. $\square$

4.1 Vertical composition of q-homotopies

A situation in which w-homotopies can be vertically composed has been considered in [6, Appendix B]:

Lemma 4.20. *Assume that we have* $X \xrightarrow{f_1, f_2, f_3} Y$ *, and w-homotopies* $f_1 \overset{H^1}{\rightsquigarrow} f_2 \overset{H^2}{\rightsquigarrow} f_3$ *as in Definition 4.1, with* $Z^1 = Z^2 = Z$*,* $x^1 = x^2 = x$ *fitting in the following diagram, where* ν^1*,* ν^2*,* γ^1*,* γ^2 *are invertible 2-cells, and* s, s^1, s^2 *are weak equivalences:*

$$\begin{tikzcd}
 & W^1 \arrow[dr, "b^1"] \arrow[rr, bend left, "s^1"] \arrow[rrr, bend left=40, "h^1"] & & & \\
X \arrow[ur, "d_1^1"] \arrow[dr, "d_0^2"'] & \Downarrow\delta & W \arrow[r, "s"] \arrow[rr, bend right, "h"] & Z & Y \\
 & W^2 \arrow[ur, "b^2"] \arrow[rr, bend right, "s^2"'] \arrow[rrr, bend right=40, "h^2"'] & & &
\arrow[from=1-2, to=2-4, phantom, "\cong \nu^1"]
\arrow[from=1-2, to=2-5, phantom, "\cong \gamma^1"]
\arrow[from=3-2, to=2-4, phantom, "\cong \nu^2"]
\arrow[from=3-2, to=2-5, phantom, "\cong \gamma^2"]
\end{tikzcd} \tag{4.21}$$

Assume also that

The 2-cell $h^1 {*} d_1^1 \overset{\varepsilon^1}{\Rightarrow} f_2 \overset{\eta^2}{\Rightarrow} h^2 {*} d_0^2$ *equals* $h^1 {*} d_1^1 \xRightarrow{\gamma^1 * d_1^1} h {*} b^1 {*} d_1^1 \xRightarrow{h * \delta} h {*} b^2 {*} d_0^2 \xRightarrow{\gamma^2 * d_0^2} h^2 {*} d_0^2$,

The 2-cell $s^1 {*} d_1^1 \xRightarrow{(\alpha_1^1)} x \xRightarrow{(\alpha_0^2)^{-1}} s^2 {*} d_0^2$ *equals* $s^1 {*} d_1^1 \xRightarrow{\nu^1 * d_1^1} s {*} b^1 {*} d_1^1 \xRightarrow{s * \delta} s {*} b^2 {*} d_0^2 \xRightarrow{\nu^2 * d_0^2} s^2 {*} d_0^2$.

Then we can construct a w-homotopy H from f_1 to f_3 such that $[H] = [H^2, H^1]$.

Furthermore, H can be constructed as follows. Take the w-cylinder $C = (W, Z, b^1 * d_0^1, b^2 * d_1^2, x, s, \alpha_0, \alpha_1)$*, with α_0 and α_1 defined as the compositions*

$$\alpha_0 : s * b^1 * d_0^1 \xRightarrow{\nu^1 * d_0^1} s^1 * d_0^1 \overset{\alpha_0^1}{\Rightarrow} x, \qquad \alpha_1 : s * b^2 * d_1^2 \xRightarrow{\nu^2 * d_1^2} s^2 * d_1^2 \overset{\alpha_1^2}{\Rightarrow} x$$

Then H is given by $H = (C, h, \eta, \varepsilon)$*, with η and ε defined as the compositions*

$$\eta : f_1 \overset{\eta^1}{\Rightarrow} h^1 * d_0^1 \xRightarrow{\gamma^1 * d_0^1} h * b^1 * d_0^1, \qquad \varepsilon : h * b^2 * d_1^2 \xRightarrow{\gamma^2 * d_1^2} h^2 * d_1^2 \overset{\varepsilon^2}{\Rightarrow} f_3. \qquad \square$$

We will compose w-homotopies with cylinders with the same Z and x (in particular q-homotopies) by showing that they can be made to fit in this situation, and in the case of q-homotopies, that the resulting w-homotopy is in fact a q-homotopy. This is a generalization of Quillen's construction of [27, Lemmas 3,4] to bicategories, which will be possible if we assume a further condition in the definition of model bicategory, namely the additional axioms:

MM0. $\mathcal{C}$ has Comma-objects and coComma-objects.

MM3. Fibrations (respectively cofibrations) are closed under Comma-objects (respectively coComma-objects).

MM4. If a morphism f is the Comma-object (resp. coComma-object) of a fibration (resp. cofibration) which is also a weak equivalence, then f is a weak equivalence.

Note that these axioms hold automatically if every 2-cell of $\mathcal{C}$ is invertible. Also note that, since we are working with left homotopies, we will use only the coComma case of these axioms.

Lemma 4.22. *Given $X \in \mathcal{C}$ cofibrant, arrows $X \overset{\xrightarrow{f_1}}{\underset{\xrightarrow{}}{\xrightarrow[f_3]{f_2}}} Y \in \mathcal{C}$ and w-homotopies $f_1 \overset{H^1}{\rightsquigarrow} f_2 \overset{H^2}{\rightsquigarrow} f_3$, with $Z^1 = Z^2 = Z$, $x^1 = x^2 = x$, there exists a w-homotopy H from f_1 to f_3 such that $[H] = [H^2, H^1]$. If H^1 and H^2 are q-homotopies, so is H.*

Note that this means that for any 2-functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ that maps the weak equivalences to quasiequivalences, $\widehat{FH} = \widehat{FH^2} \circ \widehat{FH^1}$.

Proof. We construct the diagram (4.21) in Lemma 4.20 as follows: Let (W, b^1, b^2, δ) be the coComma-object of d_1^1 and d_0^2. Then s, ν^1 and ν^2 are induced by the 2-cell $s^1 * d_1^1 \overset{\alpha_1^1}{\Longrightarrow} id_X \xRightarrow{(\alpha_0^2)^{-1}} s^2 * d_0^2$, and h, γ^1 and γ^2 are induced by the 2-cell $h^1 * d_1^1 \overset{\varepsilon^1}{\Rightarrow} f_2 \overset{\eta^2}{\Rightarrow} h^2 * d_0^2$. By construction all the hypothesis of the Lemma are satisfied (the arrow s is a weak equivalence since b^1 is so by axiom **MM4** and Lemma 4.5). Thus, we have a w-homotopy H such that $[H] = [H^1, H^2]$. Note that, by the construction of H, in order to conclude that H is a q-homotopy when H^1 and H^2 are so, it only remains to prove that $\binom{d_0}{d_1}$ is a cofibration. We consider the diagram

$$\begin{tikzcd}
X \amalg X \arrow[rr, "d_0^1 \amalg id_X"] \arrow[rrrr, bend left, "\binom{d_0}{d_1}"] \arrow[dr, phantom, "\Downarrow"] & & W^1 \amalg X \arrow[rr, "\binom{b^1}{d_1}"] & & W \arrow[dl, phantom, "\Downarrow"] \\
X \arrow[u, "i_0"] \arrow[r, hook, "\sim", "d_0^1"'] & W^1 \arrow[ur, "i_0"] & & X \amalg X \arrow[ul, "d_1^1 \amalg id_X"'] \arrow[r, hook, "\binom{d_0^2}{d_1^2}"'] & W^2 \arrow[u, "b^2"']
\end{tikzcd}$$

in which the upper triangle commutes (up to isomorphism) by the definition $d_0 = b^1 * d_0^1$, the left square is a Pushout and the right square is a coComma object. Using axioms **M3** and **MM3**, it follows that the top horizontal arrows in the squares are cofibrations and thus so is $\binom{d_0}{d_1}$. □

Remark 4.23. A different proof of the previous proposition, under the extra assumption that Y is fibrant but without using axiom **MM3**, is also possible as follows. Proceed as above to construct the w-homotopy H, and then use Proposition 4.37 below. However, we preferred to give this proof because we consider the vertical composition of q-homotopies to be independent of the results which lead to Proposition 4.37 (which assume Y fibrant).

4.2 Relating w-homotopies and q-homotopies

4.24. In this subsection we will show that, for arrows with fibrant codomain, there is a fibrant q-homotopy in the same class of any arbitrary w-homotopy. Our strategy for doing this will consist on finding for any w-cylinder C a fibrant q-cylinder C' which is *linked* to C by a finite sequence of cylinder morphisms, going in both directions, and using that in this case a w-homotopy H (with respect to C) determines a q-homotopy H' (with respect to C') in the same class.

Definition 4.25. *A morphism $C^1 \xrightarrow{M} C^2$ between w-cylinders (with the notation of Definition 4.1) is given by the data $M = (k,\, \ell,\, \gamma_0, \gamma_1, \mu, \nu)$, where k, ℓ are arrows fitting in the following* prism *diagram*

$$\begin{tikzcd}
 & & W^1 \arrow[rrd, "k"] \arrow[dd, "s^1", "\circ" marking] & & \\
X \arrow[urr, shift left, "d_0^1"] \arrow[urr, shift right, "d_1^1"'] \arrow[rrrr, bend right=10, shift left, "d_0^2"] \arrow[rrrr, bend right=10, shift right, "d_1^2"'] \arrow[dd, equal] & & & & W^2 \arrow[dd, "s^2", "\circ" marking] \\
 & & Z^1 \arrow[rrd, "\ell"] & & \\
X \arrow[urr, "x^1"] \arrow[rrrr, bend right=10, "x^2"'] & & & & Z^2
\end{tikzcd}$$

and γ_0, γ_1, μ and ν are invertible 2-cells filling the faces as indicated in the diagrams below, and satisfying the two underlying equalities:

$$\begin{tikzcd}
X \arrow[r, shift left, "d_0^1"] \arrow[r, shift right, "d_1^1"'] \arrow[rr, bend left, "x^1"] \arrow[rr, phantom, shift left=3, "\alpha_0^1\Uparrow\, \alpha_1^1\Uparrow"] \arrow[dr, shift left, "d_0^2"] \arrow[dr, shift right, "d_1^2"'] \arrow[dr, phantom, shift left=4, "\gamma_0\Uparrow\, \gamma_1\Uparrow"] & W^1 \arrow[r, "s^1", "\circ" marking] \arrow[d, "k"] \arrow[dr, phantom, "\mu\Uparrow"] & Z^1 \arrow[d, "\ell"] & = & X \arrow[rr, bend left, "x^1"] \arrow[rr, phantom, shift left=3, "\nu\Uparrow"] \arrow[rrd, "x^2"] \arrow[dr, shift left, "d_0^2"] \arrow[dr, shift right, "d_1^2"'] \arrow[dr, phantom, shift left=4, "\alpha_0^2\Uparrow\, \alpha_1^2\Uparrow"] & & Z^1 \arrow[d, "\ell"] \\
 & W^2 \arrow[r, "s^2", "\circ" marking] & Z^2 & & & W^2 \arrow[r, "s^2", "\circ" marking] & Z^2
\end{tikzcd} \tag{4.26}$$

That is:

$$s^2 * d_0^2 \xRightarrow{\gamma_0} s^2 * k * d_0^1 \xRightarrow{\mu * d_0^1} \ell * s^1 * d_0^1 \xRightarrow{\ell * \alpha_0^1} \ell * x^1 \quad = \quad s^2 * d_0^2 \xRightarrow{\alpha_0^2} x^2 \xRightarrow{\nu} \ell * x^1.$$

$$s^2 * d_1^2 \xRightarrow{\gamma_1} s^2 * k * d_1^1 \xRightarrow{\mu * d_1^1} \ell * s^1 * d_1^1 \xRightarrow{\ell * \alpha_1^1} \ell * x^1 \quad = \quad s^2 * d_1^2 \xRightarrow{\alpha_1^2} x^2 \xRightarrow{\nu} \ell * x^1.$$

Remark 4.27. Clearly in Definition 4.25 the 2-cells α_0^2 and α_1^2 can be obtained from the 2-cells α_0^1 and α_1^1 respectively, in such a way that the conditions in the definition hold. Also, when ℓ is a quasiequivalence, the same happens the other way around. Thus:

1. (transfer cylinder forward) Given all the data $C^1 \xrightarrow{M} C^2$ in Definition 4.25, but with C^2 lacking the invertible 2-cells α_0^2 and α_1^2, these can be uniquely defined in order to complete a cylinder C^2 in such a way that M becomes a morphism of cylinders.

2. If ℓ is a quasiequivalence,

   (transfer cylinder backwards) Given all the data $C^1 \xrightarrow{M} C^2$ in Definition 4.25, but with C^1 lacking the invertible 2-cells α_0^1 and α_1^1, these can be uniquely defined in order to complete a cylinder C^1 in such a way that M becomes a morphism of cylinders.

The following is a bicategorical version of the *germ* relation between homotopies considered in [6, Appendix A] and [12].

Definition 4.28. *Let $f \overset{H^1}{\rightsquigarrow} g$, $f \overset{H^2}{\rightsquigarrow} g$ be w-homotopies (with the notation of Definition 4.1) with w-cylinders C^1, C^2 respectively. We say that H^1 is* germ related *to H^2, denoted $H^1 \underset{g}{\sim\!>} H^2$, if there exist a morphism $C^1 \xrightarrow{M} C^2$ as in Definition 4.25 and an invertible 2-cell $h^2 * k \overset{\rho}{\Rightarrow} h^1$ such that*

1. η^1 equals the composition $f \overset{\eta^2}{\Rightarrow} h^2 * d_0^2 \xRightarrow{h^2 * \gamma_0} h^2 * k * d_0^1 \xRightarrow{\rho * d_0^1} h^1 * d_0^1$,

2. ε^1 equals the composition $h^1 * d_1^1 \xRightarrow{\rho^{-1} * d_1^1} h^2 * k * d_1^1 \xRightarrow{h^2 * \gamma_1^{-1}} h^2 * d_1^2 \overset{\varepsilon^2}{\Rightarrow} g$.

Remark 4.29. Clearly in Definition 4.28 the 2-cells η^1 and ε^1 can be obtained from the 2-cells η^2 and ε^2 respectively, in such a way that the conditions in the definition hold. Also, the same happens the other way around. Thus:

1. (transfer homotopy forward) Given all the data in Definition 4.28 but with H^2 lacking the 2-cells η^2 and ε^2, these can be uniquely defined in order to complete a w-homotopy H^2 in such a way that $H^1 \underset{g}{\sim\!>} H^2$.

2. (transfer homotopy backwards) Given all the data in Definition 4.28 but with H^1 lacking the 2-cells η^1 and ε^1, these can be uniquely defined in order to complete a w-homotopy H^1 in such a way that $H^1 \underset{g}{\sim\!>} H^2$.

Lemma 4.30. *With the notation of Definition 4.28, if $H^1 \underset{g}{\sim\!>} H^2$, then $[H^1] = [H^2]$. Thus, the same holds for the generated equivalence relation $\underset{g}{\sim}$.*

Proof. Recalling Definition 4.8, let $(\mathcal{C}, \mathcal{W}) \xrightarrow{F} (\mathcal{D}, q\Theta)$, and let $Fd_0^i \xRightarrow{\widehat{FC^i}} Fd_1^i$ be the 2-cell such that $Fs^i * \widehat{FC^i} = F\alpha^i$, $i = 1, 2$. From Definition 4.25, it follows that α^2 equals the composition

$$s^2 * d_0^2 \xRightarrow{s^2 * \gamma_0} s^2 * k * d_0^1 \xRightarrow{\mu * d_0^1} \ell * s^1 * d_0^1 \xRightarrow{\ell * \alpha^1} \ell * s^1 * d_1^1 \xRightarrow{\mu^{-1} * d_1^1} s^2 * k * d_1^1 \xRightarrow{s^2 * \gamma_1^{-1}} s^2 * d_1^2. \quad (4.31)$$

We start by showing that the 2-cell $\widehat{FC^2}$ equals the composition

$$Fd_0^2 \xRightarrow{F\gamma_0} Fk * Fd_0^1 \xRightarrow{Fk * \widehat{FC^1}} Fk * Fd_1^1 \xRightarrow{F\gamma_1^{-1}} Fd_1^2.$$

By applying $Fs^2 * (-)$ to this composition, and comparing it with the value of the 2-functor F at the composition in (4.31) above, it follows that it suffices to show that $F(s^2 * k) * \widehat{FC}^1$ equals the composition

$$F(s^2 * k * d_0^1) \xRightarrow{F(\mu * d_0^1)} F(\ell * s^1 * d_0^1) \xRightarrow{F(\ell * \alpha^1)} F(\ell * s^1 * d_1^1) \xRightarrow{F(\mu^{-1} * d_1^1)} F(s^2 * k * d_1^1).$$

This followws from the interchange law applied to the configuration

$$\begin{tikzcd} FX \arrow[r, bend left, "Fd_0^1"] \arrow[r, bend right, "Fd_1^1"'] \arrow[r, phantom, "\widehat{FC}^1 \Downarrow"] & Fd_1^1 \arrow[r, bend left, "F(\ell * s^1)"] \arrow[r, bend right, "F(s^2 * k)"'] \arrow[r, phantom, "F\mu^{-1} \Downarrow"] & FZ \end{tikzcd}$$

. Therefore, the 2-cell $\widehat{FH^2}$ is the composition

$$Ff \overset{F\eta^2}{\Rightarrow} Fh^2 * Fd_0^2 \xRightarrow{Fh^2 * F\gamma_0} Fh^2 * Fk * Fd_0^1 \xRightarrow{F(h^2 * k) * \widehat{FC}^1} Fh^2 * Fk * Fd_1^1 \xRightarrow{Fh^2 * F\gamma_1^{-1}} Fh^2 * Fd_1^2 \overset{F\varepsilon^2}{\Rightarrow} Fg.$$

By looking at the expression of the 2-cells η^1 and ε^1 in the hypothesis of the lemma, it follows that in order to show $\widehat{FH^2} = \widehat{FH^1}$ it suffices to show that $F(h^2 * k) * \widehat{FC}^1$ equals the composition

$$Fh^2 * Fk * Fd_0^1 \xRightarrow{F\rho * Fd_0^1} Fh^1 * Fd_0^1 \xRightarrow{Fh^1 * \widehat{FC}^1} Fh^1 * Fd_1^1 \xRightarrow{F\rho^{-1} * Fd_1^1} Fh^2 * Fk * Fd_1^1,$$

which follows from the interchange law applied to the configuration

$$\begin{tikzcd} FX \arrow[r, bend left, "Fd_0^1"] \arrow[r, bend right, "Fd_1^1"'] \arrow[r, phantom, "\widehat{FC}^1 \Downarrow"] & Fd_1^1 \arrow[r, bend left, "Fh^1"] \arrow[r, bend right, "F(h^2 * k)"'] \arrow[r, phantom, "F\rho^{-1} \Downarrow"] & FY \end{tikzcd}$$

. □

If H^1, H^2 in the previous lemma have the same w-cylinder $C = C^1 = C^2$, we may take M to be the evident identity morphism and we have the following

Corollary 4.32. *Let* $f \overset{H^1}{\rightsquigarrow} g$, $f \overset{H^2}{\rightsquigarrow} g$ *be w-homotopies (with the notation as in Definition 4.1) with the same w-cylinder C. Assume there exists an invertible 2-cell* $h^2 \overset{\rho}{\Rightarrow} h^1$ *such that*

1. η^1 *equals the composition* $f \overset{\eta^2}{\Rightarrow} h^2 * d^2 \xRightarrow{\rho * d^1} h^1 * d^1$.
2. ε^1 *equals the composition* $h^1 * c^1 \xRightarrow{\rho^{-1} * c^1} h^2 * c^2 \overset{\varepsilon^2}{\Rightarrow} g$.

Then $[H^1] = [H^2]$. □

We develop now the strategy envisaged in 4.24. The following lemmas lead to Proposition 4.37, which states that for arrows with fibrant codomain, each w-homotopy has an associated fibrant q-homotopy in the same class.

Lemma 4.33. *Let* $\begin{tikzcd} X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \end{tikzcd} \in \mathcal{C}$, *with Y a fibrant object. Let* $f \overset{H}{\rightsquigarrow} g$ *be a w-homotopy as in Definition 4.1. Then, there is a w-homotopy* $f \overset{H'}{\rightsquigarrow} g$ *with W' and Z' fibrant objects, such that* $H \underset{g}{\sim>} H'$.

Proof. First we factorize $\begin{tikzcd} Z \arrow[rr] \arrow[dr, hook, "\sim" description, "\ell"'] & \cong & 1 \\ & Z' \arrow[ur, two heads] & \end{tikzcd}$ and $\begin{tikzcd} W \arrow[rr, "h"] \arrow[dr, hook, "\sim" description, "k"'] & \rho \Uparrow \cong & Y \\ & W' \arrow[ur, two heads, "h'"'] & \end{tikzcd}$ using axiom **M2**.

Then we use axioms **M1** and **M5** in order to have $\begin{tikzcd} W \arrow[r, "\ell * s", "\sim"'] \arrow[d, hook, "k"', "\wr"] \arrow[dr, phantom, "\mu \Uparrow \cong"] & Z' \arrow[d, two heads] \\ W' \arrow[r] \arrow[ur, dotted, "s'"', "\sim" description] & 1 \end{tikzcd}$, and we set $x' = \ell * x$.

In this way we have a cylinder C' except for the 2-cells α'_0, α'_1, and all the data of a morphism $M = (k,\, \ell,\, id,\, id,\, \mu,\, id) : C \longrightarrow C'$ as follows (compare with diagram (4.26)):

$$\begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rr, bend left=30, "x"] \arrow[dr, bend right, shift left, "d'_0"] \arrow[dr, bend right, shift right, "d'_1"'] & W \arrow[r, "s", "\sim"'] \arrow[d, "k"] \arrow[u, phantom, "\alpha_0 \Uparrow \alpha_1 \Uparrow"] \arrow[dl, phantom, "id \Uparrow id \Uparrow"] & Z \arrow[d, "\ell"] \arrow[dl, phantom, "\mu \Uparrow"] \\ & W' \arrow[r, "s'", "\sim"'] & Z' \end{tikzcd} = \begin{tikzcd} X \arrow[rr, bend left=30, "x"] \arrow[drr, bend left, "x'"] \arrow[dr, bend right, shift left, "d'_0"] \arrow[dr, bend right, shift right, "d'_1"'] \arrow[rr, phantom, "id \Uparrow"] & & Z \arrow[d, "\ell"] \\ & W' \arrow[r, "s'", "\sim"'] & Z' \end{tikzcd}$$

We can then *transfer forward* C and H to get the desired C' and H': first, by item 1 in Remark 4.27, we complete C' with 2-cells α'_0, α'_1. In turn, since we have an invertible 2-cell $h' * k \overset{\rho}{\Rightarrow} h$, the proof finishes by item 1 in Remark 4.29. $\square$

Lemma 4.34. *Let* $f \overset{H}{\rightsquigarrow} g$ *be a w-homotopy as in Definition 4.1, with* W *a fibrant object. Then, there is a fibrant w-homotopy* $f \overset{H'}{\rightsquigarrow} g$ *such that* $H \underset{g}{\sim\!>} H'$.

Proof. The strategy of this proof is similar to the one of the previous lemma. First we factorize $\begin{tikzcd} W \arrow[rr, "s", "\sim"'] \arrow[dr, hook, "\sim" description, "k"'] & \mu \Uparrow \cong & Z \\ & W' \arrow[ur, two heads, "\sim" description, "s'"'] & \end{tikzcd}$ using axioms **M2** and **M5**, and we set $x' = x$.

In this way we have a cylinder C' except for the 2-cells α'_0, α'_1, and all the data of a morphism $M = (k,\, id_Z,\, id,\, id,\, \mu,\, id) : C \longrightarrow C'$ as follows:

$$\begin{tikzcd} X \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[rr, bend left=30, "x"] \arrow[dr, bend right, shift left, "d'_0"] \arrow[dr, bend right, shift right, "d'_1"'] & W \arrow[r, "s", "\sim"'] \arrow[d, "k"] \arrow[u, phantom, "\alpha_0 \Uparrow \alpha_1 \Uparrow"] \arrow[dl, phantom, "id \Uparrow id \Uparrow"] & Z \arrow[d, "id_Z"] \arrow[dl, phantom, "\mu \Uparrow"] \\ & W' \arrow[r, "s'", "\sim"'] & Z \end{tikzcd} = \begin{tikzcd} X \arrow[rr, bend left=30, "x"] \arrow[drr, bend left, "x'"] \arrow[dr, bend right, shift left, "d'_0"] \arrow[dr, bend right, shift right, "d'_1"'] \arrow[rr, phantom, "id \Uparrow"] & & Z \arrow[d, "id_Z"] \\ & W' \arrow[r, "s'", "\sim"'] & Z \end{tikzcd}$$

Then, by item 1 in Remark 4.27 we complete C' with 2-cells α'_0, α'_1.

Now we use axiom **M1** in order to have
$$\begin{tikzcd}
W \arrow[r, "id_W"] \arrow[d, hook, "k"', "\wr"] \arrow[dr, phantom, "\varepsilon \Downarrow \cong", pos=0.2] & W \arrow[d, two heads] \\
W' \arrow[r] \arrow[ur, dotted, "m"'] & 1
\end{tikzcd}$$
, and set $h' = h * m$, $\rho = h * \varepsilon : h' * k \Rightarrow h$. The proof finishes then by item 1 in Remark 4.29. $\square$

Remark 4.35. The previous two lemmas admit a unified proof using Remarks 4.27 and 4.29 only once. However we consider that the proof in two separate steps is easier to follow.

Lemma 4.36. *Let* $f \overset{H}{\rightsquigarrow} g$ *be a fibrant w-homotopy, then there is a fibrant q-homotopy* $f \overset{H'}{\rightsquigarrow} g$ *such that* $H' \underset{g}{\rightsquigarrow} H$.

Proof. Like it was the case for the previous two lemmas, for clarity it is convenient to proceed in two steps that have a similar strategy (we will now transfer C and H backwards instead of forward). In step 1, we will show that given H as in the hypothesis, there exists H' with $H' \underset{g}{\rightsquigarrow} H$, which is still fibrant and such that $Z' = X$ and $x' = id_X$. In step 2, we will show that given H satisfying these extra conditions, there exists a fibrant q-homotopy H' (i.e. satisfying also the condition that $\binom{d_0}{d_1}$ is a cofibration) such that $H' \underset{g}{\rightsquigarrow} H$.

1. By Remark 4.9 we can consider $C = (W, Z, d_0, d_1, s * d_1, s, \alpha, s * d_1)$ to be the cylinder of H, $H = (C, h, \eta, \varepsilon)$. Let W' in the diagram below be the Pullback of s and $s * d_1$ (recall axioms **M3** and **M4** so that s' results a trivial fibration). From the universal property of the coProduct (recall our notation from 2.9), we have a 2-cell $\binom{\alpha}{id} : s * \binom{d_0}{d_1} \Rightarrow s * d_1 * \nabla_X$, which induces by the universal property of the Pullback the diagram below

$$\begin{tikzcd}
X \amalg X \arrow[r, "\binom{d_0'}{d_1'}"] \arrow[rr, bend left=30, "\nabla_X"] \arrow[dr, bend right, "\binom{d_0}{d_1}"'] \arrow[r, phantom, bend left=40, "\alpha' \Uparrow \cong"] & W' \arrow[r, two heads, "s'", "\sim"'] \arrow[d, "k"] \arrow[dr, phantom, "\mu \Uparrow"] & X \arrow[d, "s * d_1"] & = & X \amalg X \arrow[rr, bend left=30, "\nabla_X"] \arrow[dr, bend right, "\binom{d_0}{d_1}"'] & \binom{\alpha}{id} \Uparrow & X \arrow[d, "s * d_1"] \\
\gamma \Uparrow \cong & W \arrow[r, two heads, "s", "\sim"'] & Z & & & W \arrow[r, two heads, "s", "\sim"'] & Z
\end{tikzcd}$$

The equality above is clearly equivalent to the following two equalities as in (4.26):

$$\begin{tikzcd}
X \arrow[r, shift left, "d_0'"] \arrow[r, shift right, "d_1'"'] \arrow[rr, bend left=35, "id"] \arrow[dr, bend right=15, shift left, "d_0"] \arrow[dr, bend right=15, shift right, "d_1"'] & W' \arrow[r, "s'", "\sim"'] \arrow[d, "k"] \arrow[dr, phantom, "\mu \Uparrow"] & X \arrow[d, "s * d_1"] & = & X \arrow[rr, bend left=35, "id_X"] \arrow[rrd, bend left=15, "s * d_1"] \arrow[dr, bend right=15, shift left, "d_0"] \arrow[dr, bend right=15, shift right, "d_1"'] & id \Uparrow & X \arrow[d, "s * d_1"] \\
\gamma_0 \Uparrow \gamma_1 \Uparrow & W \arrow[r, "s", "\sim"'] & Z & & & W \arrow[r, "s", "\sim"'] \arrow[u, phantom, "\alpha \Uparrow id \Uparrow"] & Z
\end{tikzcd}$$

with $\alpha_0' \Uparrow \alpha_1' \Uparrow$ between id and d_0', d_1'.

We have then a cylinder C' and a morphism $M = (k,\, s * d_1,\, \gamma_0, \gamma_1, \mu, id) : C' \longrightarrow C$, which allows to transfer H backwards: we set $h' = h * k$, $\rho = id$, and we use Remark 4.29, item 2 in order to have H' such that $H' \underset{g}{\sim\!>} H$.

2. We factorize

$$\begin{tikzcd}
X \amalg X \arrow[rr, "\binom{d_0}{d_1}"] \arrow[dr, hook, "\binom{d'_0}{d'_1}"'] & \gamma \Uparrow \cong & W \\
& W' \arrow[ur, "\sim", "k"'] &
\end{tikzcd}$$

using axiom **M2**, and we set $s' = s * k$.

In this way we have a cylinder C' except for the 2-cells α'_0, α'_1, and all the data of a morphism $M = (k,\, id_X,\, \gamma_0, \gamma_1, id, id) : C' \longrightarrow C$ as follows:

$$\begin{tikzcd}
X \arrow[r, shift left, "d'_0"] \arrow[r, shift right, "d'_1"'] \arrow[rr, bend left=40, "id_X"] \arrow[dr, bend right, shift left, "d_0"] \arrow[dr, bend right, shift right, "d_1"'] \arrow[dr, phantom, "\gamma_0 \Uparrow \gamma_1 \Uparrow"] & W' \arrow[r, "s'", "\sim"'] \arrow[d, "k"] \arrow[dr, phantom, "id \Uparrow"] & X \arrow[d, "id_X"] \\
& W \arrow[r, "s", "\sim"'] & X
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[rr, bend left=30, "id_X"] \arrow[rr, phantom, bend left=15, "id \Uparrow"] \arrow[drr, bend left=10, "id_X"] \arrow[dr, bend right, shift left, "d_0"] \arrow[dr, bend right, shift right, "d_1"'] \arrow[dr, phantom, "\alpha_0 \Uparrow \alpha_1 \Uparrow"] & & X \arrow[d, "id_X"] \\
& W \arrow[r, "s", "\sim"'] & X
\end{tikzcd}$$

We can then *transfer backwards* C and H to get the desired C' and H': first, by item 2 in Remark 4.27, we complete C' with 2-cells α'_0, α'_1. Then, just like in step 1, we set $h' = h * k$, $\rho = id$, and we use Remark 4.29, item 2 in order to have H' such that $H' \underset{g}{\sim\!>} H$. $\square$

Proposition 4.37. *Let* $X \underset{g}{\overset{f}{\rightrightarrows}} Y \in \mathcal{C}$, *with* Y *a fibrant object. Let* $f \overset{H}{\rightsquigarrow} g$ *be a w-homotopy. Then there is a fibrant q-homotopy* H' *such that* $[H'] = [H]$.

Proof. Using in turn Lemmas 4.33, 4.34, and 4.36 we get a fibrant q-homotopy H' such that $[H'] \underset{g}{\sim} [H]$, and the proof finishes by Lemma 4.30. $\square$

4.3 Some further properties of the homotopy bicategory

In the remainder of this section we will prove two results that hold for q-homotopies, namely Lemmas 4.43 and 4.47, which are bicategorical versions of two basic results in [27]. These results will be used in the following sections, and both depend on the following *double lifting property*:

Proposition 4.38.

a) Let $A \xrightarrow{i} X$ *be a trivial cofibration, and* $Y \xrightarrow{p_0} B$, $Y \xrightarrow{p_1} B$ *be morphisms in* $\mathcal{C}$, *such that* $(p_0,\, p_1) : Y \to B \times B$ *is a fibration. Then, for each pair of invertible* 2*-cells* γ_0, γ_1 *as in the two diagrams on the left, there exist a morphism* f *and invertible* 2*-cells*

λ_0, λ_1, ρ_0, ρ_1 satisfying the equations written below:

$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\gamma_0\Downarrow"] & Y \arrow[d, "p_0"] & & A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\gamma_1\Uparrow"] & Y \arrow[d, "p_1"] & & A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\lambda_0\Uparrow", pos=0.2] \arrow[dr, phantom, "\rho_0\Downarrow", pos=0.8] & Y \arrow[d, "p_0"] & & A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "\lambda_1\Downarrow", pos=0.2] \arrow[dr, phantom, "\rho_1\Uparrow", pos=0.8] & Y \arrow[d, "p_1"] \\
X \arrow[r, "b_0"'] & B & , \ X \arrow[r, "b_1"'] & B & \rightsquigarrow & X \arrow[r, "b_0"'] \arrow[ur, "f"] & B & , & X \arrow[r, "b_1"'] \arrow[ur, "f"] & B
\end{tikzcd} \tag{4.39}$$

$$\gamma_0 \circ p_0 * \lambda_0 \;=\; \rho_0 * i\,, \qquad p_1 * \lambda_1 \,\circ\, \gamma_1 \;=\; \rho_1 * i\,, \qquad \lambda_0 \circ \lambda_1 \;=\; id, \quad \lambda_1 \circ \lambda_0 \;=\; id.$$

b) Let $Y \xrightarrow{p} B$ be a trivial fibration and $A \xrightarrow{i_0} X$, $A \xrightarrow{i_1} X$ such that $\binom{i_0}{i_1} : A \amalg A \to X$ is a cofibration. Then, for each pair of invertible 2-cells γ_0, γ_1 as in the two diagrams on the left, there exist a morphism f and invertible 2-cells λ_0, λ_1, ρ_0, ρ_1 satisfying the equations written below:

$$\begin{tikzcd}
A \arrow[r, "a_0"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\gamma_0\Uparrow"] & Y \arrow[d, "p"] & & A \arrow[r, "a_1"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\gamma_1\Downarrow"] & Y \arrow[d, "p"] & & A \arrow[r, "a_0"] \arrow[d, "i_0"'] \arrow[dr, phantom, "\lambda_0\Uparrow", pos=0.2] \arrow[dr, phantom, "\rho_0\Downarrow", pos=0.8] & Y \arrow[d, "p"] & & A \arrow[r, "a_1"] \arrow[d, "i_1"'] \arrow[dr, phantom, "\lambda_1\Downarrow", pos=0.2] \arrow[dr, phantom, "\rho_1\Uparrow", pos=0.8] & Y \arrow[d, "p"] \\
X \arrow[r, "b"'] & B & , \ X \arrow[r, "b"'] & B & \rightsquigarrow & X \arrow[r, "b"'] \arrow[ur, "f"] & B & , & X \arrow[r, "b"'] \arrow[ur, "f"] & B
\end{tikzcd} \tag{4.40}$$

$$\gamma_0 \,\circ\, \rho_0 * i_0 \;=\; p * \lambda_0\,, \qquad \rho_1 * i \,\circ\, \gamma_1 \;=\; p * \lambda_1\,, \qquad \rho_0 \circ \rho_1 \;=\; id, \quad \rho_1 \circ \rho_0 \;=\; id.$$

Proof.

a) Apply axiom **M1** to the 2-cell
$$\begin{tikzcd}
A \arrow[r, "a"] \arrow[d, "i"'] \arrow[dr, phantom, "(\gamma_0{,}\gamma_1^{-1})\Downarrow"] & Y \arrow[d, "(p_0{,}p_1)"] \\
X \arrow[r, "(b_0{,}b_1)"'] & B \times B
\end{tikzcd}$$

b) is the dual statement. ☐

Remark 4.41. We think of the diagrams in (4.39) (resp. (4.40)), as *double lifting properties.* That is, we say that a triple (i, p_0, p_1) (resp. (i_0, i_1, p)) satisfies **LLs** (resp. **LLs**$^{\mathbf{op}}$) if any pair of squares as in the left admit fillers as in 3.3 **Ls**, for which we can take the same f, and the 2-cells λ_0, λ_1 (resp. ρ_0, ρ_1) mutually inverse.

Lemma 4.42. *Let $Z \overset{f}{\underset{g}{\rightrightarrows}} X \xrightarrow{p} Y$ be arrows in $\mathcal{C}$, with p a trivial fibration, and $p * f \overset{H}{\leadsto} p * g$ be a q-homotopy with invertible cells $H = (C,\, h,\, \eta,\, \varepsilon)$ (we refer to the notation in Definition 4.1). Then there exists a q-homotopy $f \overset{H'}{\leadsto} g$ such that $[H] = [p * H'] = p * [H']$, which furthermore has the same cylinder as H. Note then that if H is fibrant, so is H'. Note also that by Proposition 4.19 such an $[H']$ is unique.*

Proof. We apply proposition 4.38 b) as follows:

$$\begin{tikzcd}
Z \arrow[r, "g"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\varepsilon\Uparrow"] & X \arrow[d, "p"] & & Z \arrow[r, "f"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\eta\Downarrow"] & X \arrow[d, "p"] & & Z \arrow[r, "g"] \arrow[d, "d_1"'] \arrow[dr, phantom, "\varepsilon'\Uparrow", pos=0.2] \arrow[dr, phantom, "\rho_0\Downarrow", pos=0.8] & X \arrow[d, "p"] & & Z \arrow[r, "f"] \arrow[d, "d_0"'] \arrow[dr, phantom, "\eta'\Downarrow", pos=0.2] \arrow[dr, phantom, "\rho_1\Uparrow", pos=0.8] & X \arrow[d, "p"] \\
W \arrow[r, "h"'] & Y & \textit{and} \ W \arrow[r, "h"'] & Y & \rightsquigarrow & W \arrow[r, "h"'] \arrow[ur, "h'"] & Y & \textit{and} & W \arrow[r, "h"'] \arrow[ur, "h'"] & Y
\end{tikzcd}$$

This yields a fibrant q-homotopy $f \overset{H'}{\rightsquigarrow} g$, $H' = (C,\, h',\, \eta',\, \varepsilon')$ such that

$$p * \varepsilon' \, = \, \varepsilon \circ \rho_0 * d_1 \,, \quad p * \eta' \, = \, \rho_1 * d_0 \circ \eta \,, \quad \rho_0 \circ \rho_1 = id \,.$$

It follows

$$p * \varepsilon' \circ \rho_1 * d_1 \, = \, \varepsilon \,, \qquad \rho_0 * d_0 \circ p * \eta' \, = \, \eta \,.$$

We show that $[p * H'] = [H]$ using Corollary 4.32 with $H^1 = H$ and $H^2 = p * H'$. It remains to check the hypothesis of the Corollary, which in this case are exactly the two preceding equations. ☐

The following Lemma corresponds to Quillen's [27, §1, Lemma 7]. It doesn't seem to hold for arbitrary homotopies whose 2-cells are not necessarily invertible.

Lemma 4.43. *Given a cofibrant object Z and a trivial fibration $X \xrightarrow{p} Y$, the functor $\mathcal{H}o^{f}_{inv}(\mathcal{C}, \mathcal{W})(Z, X) \xrightarrow{p_*} \mathcal{H}o^{f}_{inv}(\mathcal{C}, \mathcal{W})(Z, Y)$ of postcomposition with p (see 4.10) is an equivalence of categories. (where the subscript "inv" and the superscript "f" indicate the sub-bicategory defined by the fibrant homotopies with invertible cells (indistinctly w- or q-), see Notation 5.1 below).*

Proof. By Proposition 4.19 we know that p_* is faithful, and by Proposition 4.44 it follows that it is essentially surjective. The fact that it is full is proved in Lemma 4.42. ☐

Our next task is to prove an analogous statement to Lemma 4.42, but for the q-homotopies I^μ given by the 2-cells μ of $\mathcal{C}$, not necessarily invertible. It is here that we will need the full strength of the lifting property **L**, and not just **Ls**.

The following proposition expresses the content of axiom **M1** when the object A is the biInitial object of the bicategory.

Proposition 4.44. *Given a cofibrant object Z and a trivial fibration $X \xrightarrow{p} Y$, the functor $\mathcal{C}(Z,\, X) \xrightarrow{p_*} \mathcal{C}(Z,\, Y)$ is essentially surjective and essentially full.*

Proof. Consider the lifting property **L** for the arrows $0 \longrightarrow Z$ and $X \xrightarrow{p} Y$. The categories $\mathcal{C}(0,\, X)$, $\mathcal{C}(0,\, Y)$ are equivalent to the singleton category $\mathbf{1}$, thus since equivalences are stable by biPullingback, the functor $\mathbf{P} \longrightarrow \mathcal{C}(Z,\, Y)$ in the diagram below is an equivalence of categories:

$$\begin{tikzcd}
\mathcal{C}(Z,\, X) \arrow[r, "h"] \arrow[rr, bend left] \arrow[dr, bend right, "p_*"'] & \mathbf{P} \arrow[r] \arrow[d, "\simeq"'] \arrow[dr, phantom, "\cong"] & \mathbf{1} \arrow[d, "\simeq"] \\
 & \mathcal{C}(Z,\, Y) \arrow[r] & \mathbf{1}
\end{tikzcd}$$

Since Z is cofibrant, by axiom **M1** the functor h is essentially surjective and essentially full, thus so is p_*. ☐

Recalling Remark 2.8 it follows:

Corollary 4.45. *Let* $Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{p} Y$*, with* p *a trivial fibration and* Z *cofibrant, and let* $p * f \overset{\mu}{\Rightarrow} p * g$ *be a 2-cell. Then:*

There exist morphisms $Z \underset{g'}{\overset{f'}{\rightrightarrows}} X$*, invertible* 2*-cells* $p * f \overset{\rho_1}{\Rightarrow} p * f'$, $p * g \overset{\rho_2}{\Rightarrow} p * g'$*, and a 2-cell* $f' \overset{\delta}{\Rightarrow} g'$ *such that* $p * \delta \circ \rho_1 = \rho_2 \circ \mu$*.* □

Lemma 4.46. *Let* $Z \underset{g}{\overset{f}{\rightrightarrows}} X \xrightarrow{p} Y$ *be arrows in* $\mathcal{C}$*, with* p *a trivial fibration, and* Z *cofibrant. Given any* 2*-cell* $p * f \overset{\mu}{\Rightarrow} p * g$ *in* $\mathcal{C}$*, there exists a 2-cell* $f \overset{\alpha}{\Rightarrow} g$ *in* $\mathcal{H}o(\mathcal{C}, \mathcal{W})$ *such that* $[I^{\mu}] = p * \alpha$*, Note that by Proposition 4.19 such an* α *is unique.*

Furthermore, α *can be taken to be the class of a sequence* $[H_2,\ H_3,\ H_1]$ *as in 4.11, with these three homotopies fibrant.*

Proof. By the corollary above we have a factorization of μ, $\mu = \rho_2^{-1} \circ p * \delta \circ \rho_1$. Since $[I^{\rho_1}]$ and $[I^{\rho_2^{-1}}]$ can be taken fibrant and with invertible cells, we can apply Proposition 4.42 to get H_1, H_2 such that $p * [H_1] = [I^{\rho_1}]$, $p * [H_2] = [I^{\rho_2^{-1}}]$. We define α as the class of the sequence $[H_2,\ I^{\delta},\ H_1]$, then we have

$$p * \alpha = p * [H_2] \circ p * [I^{\delta}] \circ p * [H_1] = [I^{\rho_2^{-1}}] \circ p * [I^{\delta}] \circ [I^{\rho_1}] = [I^{\mu}].$$

□

4.4 Switching between right and left homotopies

The following lemma is a bicategorical version of [27, I, §1, Lemma 5 (i)]. It allows to *switch* from left to right homotopies (recall Definitions 4.1 and 4.3) without changing the equivalence class. In addition, the right homotopy can be taken with respect to an arbitrary fixed q-path-object, which by the dual of Remark 4.4 can be assumed to be cofibrant.

.

Lemma 4.47. *Let* $X \underset{g}{\overset{f}{\rightrightarrows}} Y \ \in \mathcal{C}$*, with* X *a cofibrant object,* $f \overset{H}{\rightsquigarrow} g$ *a left q-homotopy, and* P *a q-path-object for* Y*. Then there exists a right q-homotopy* $f \overset{K}{\rightsquigarrow} g$ *with path object* P *such that* $[K] = [H]$*.*

Proof. Let $H = (C, h, \eta, \varepsilon)$, $C = (W, d_0, d_1, s, \alpha_0, \alpha_1)$ and $P = (V, c_0, c_1, t, \beta_0, \beta_1)$ (see Definitions 4.1 and 4.3). We define 2-cells γ_0 and γ_1 as the composites:

$$\gamma_0 \overset{(1)}{=} c_1 * t * f \xRightarrow{\beta_1 * f} f \overset{\eta}{\Rightarrow} h * d_0, \qquad \gamma_1 \overset{(2)}{=} f * s * d_0 \xRightarrow{f * \alpha_0} f \xRightarrow{\beta_0^{-1} * f} c_0 * t * f$$

We assume first that H has invertible cells, thus η is invertible, and so is γ_0. Since γ_1 was already invertible, we apply Proposition 4.38 a) and we have (note that d_0 is a trivial cofibration by Lemma 4.5, it is here where we need X to be cofibrant):

$$\begin{tikzcd}
X \arrow[r, "t*f"] \arrow[dd, "d_0"] \arrow[ddr, phantom, "\gamma_0\Downarrow"] & V \arrow[dd, "c_1"] & & X \arrow[r, "t*f"] \arrow[dd, "d_0"] \arrow[ddr, phantom, "\gamma_1\Uparrow"] & V \arrow[dd, "c_0"] & & X \arrow[r, "t*f"] \arrow[dd, "d_0"] & V \arrow[dd, "c_1"] & & X \arrow[r, "t*f"] \arrow[dd, "d_0"] & V \arrow[dd, "c_0"] \\
& & , & & & \rightsquigarrow & & & , & & \\
W \arrow[r, "h"] & Y & & W \arrow[r, "f*s"] & Y & & W \arrow[r, "h"] \arrow[uur, "\ell"] \arrow[uur, phantom, near end, "\lambda_0\Uparrow"] \arrow[uur, phantom, near end, "\rho_0\Downarrow"'] & Y & & W \arrow[r, "f*s"] \arrow[uur, "\ell"] \arrow[uur, phantom, near end, "\lambda_1\Downarrow"] \arrow[uur, phantom, near end, "\rho_1\Uparrow"'] & Y
\end{tikzcd}$$

$$\rho_0 * d_0 \overset{(3)}{=} \gamma_0 \circ c_1 * \lambda_0 \,, \quad \rho_1 * d_0 \overset{(4)}{=} c_0 * \lambda_1 \circ \gamma_1 \,, \quad \lambda_0 \circ \lambda_1 \overset{(5)}{=} id, \quad \lambda_1 \circ \lambda_0 = id.$$

We define $k = \ell * d_1 : X \xrightarrow{d_1} W \xrightarrow{\ell} V$, and the 2-cells δ and ϵ as the composites:

$$\delta \overset{(a)}{=} f \xRightarrow{f*\alpha_1^{-1}} f * s * d_1 \xRightarrow{\rho_1 * d_1} c_0 * \ell * d_1, \qquad \epsilon \overset{(6)}{=} c_1 * \ell * d_1 \xRightarrow{\rho_0 * d_1} h * d_1 \overset{\varepsilon}{\Rightarrow} g$$

We define the right homotopy K as $K = (P, k, \delta, \epsilon)$.

We prove now that $[K] = [H]$, that is, $\widehat{FK} = \widehat{FH}$, see Definition 4.8.

Since F is a 2-functor, to simplify the notation we can omit the letter F, and we do so in what follows.

Recall the 2-cells from 4.7

$$d_0 \overset{\widehat{C}}{\Rightarrow} d_1 \;\text{ and }\; c_0 \overset{\widehat{P}}{\Rightarrow} c_1 \;, \quad s * \widehat{C} \overset{(b)}{=} \alpha_1^{-1} \circ \alpha_0 \;\text{ and }\; \widehat{P} * t \overset{(7)}{=} \beta_1^{-1} \circ \beta_0$$

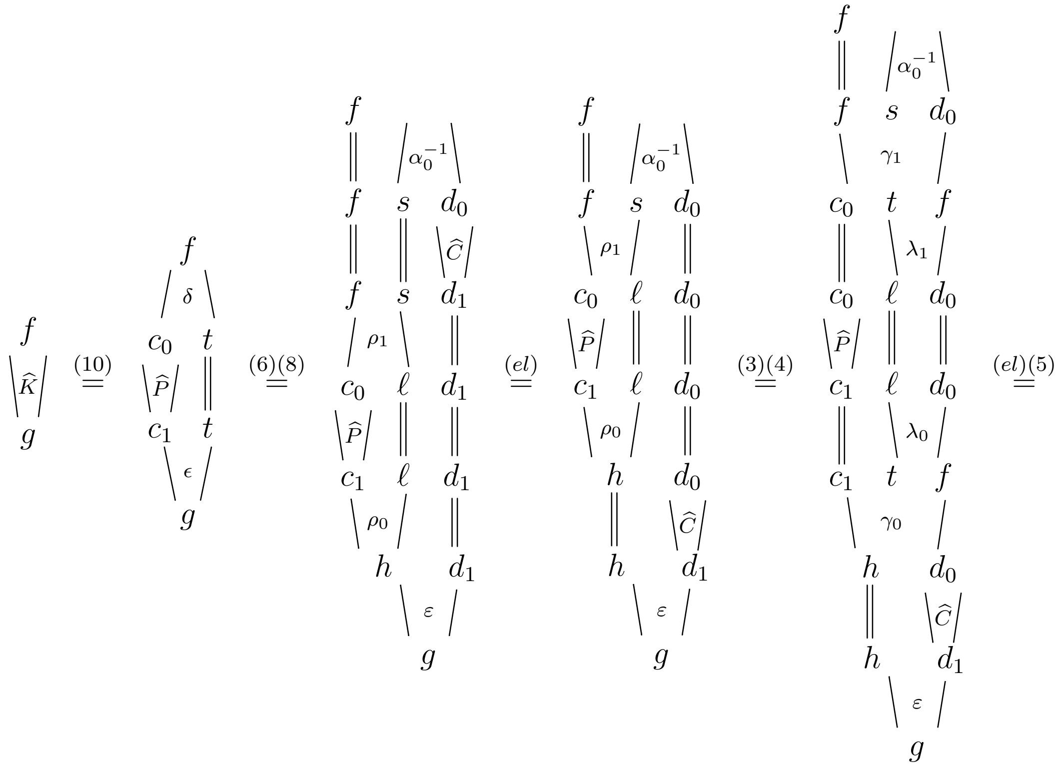
from (a) and (b) it follows that

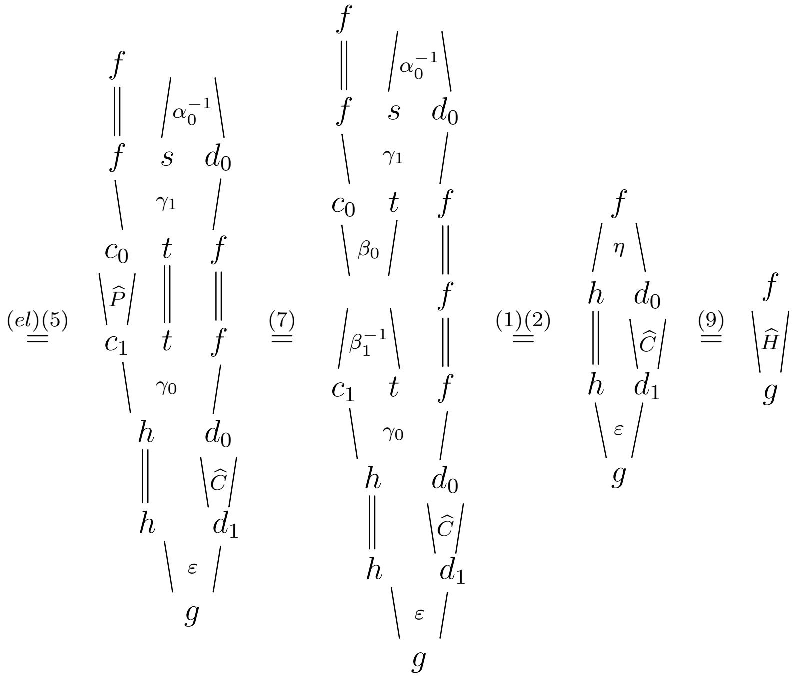
$$\delta \overset{(8)}{=} f \xRightarrow{f*\alpha_0^{-1}} f * s * d_0 \xRightarrow{f*s*\widehat{C}} f * s * d_1 \xRightarrow{\rho_1 * d_1} c_0 * \ell * d_1$$

Recall now that $\widehat{H}$ and $\widehat{K}$ are defined as the composites

$$\widehat{H} \overset{(9)}{=} f \overset{\eta}{\Rightarrow} h * d_0 \xRightarrow{h*\widehat{C}} h * d_1 \overset{\varepsilon}{\Rightarrow} g \;\text{ and }\; \widehat{K} \overset{(10)}{=} f \overset{\delta}{\Rightarrow} c_0 * t \xRightarrow{\widehat{P}*t} c_1 * t \overset{\epsilon}{\Rightarrow} g$$

We prove now that $\widehat{K} = \widehat{H}$, using the elevator calculus described in §A.1, in Appendix A:





This finishes the proof for H with invertible cells. Let now $[H]$ be with arbitrary cells.

By proposition 4.16 we have

$$[H] = [\varepsilon \circ (h * H^C) \circ \eta] = [I^\varepsilon] \circ [h * H^C] \circ [I^\eta]$$

Recalling that $h * H^C$ has invertible cells, we can apply the above and get K' such that $[K'] = [h * H^C]$. We can take the desired q-homotopy K to be $\varepsilon \circ K' \circ \eta$, defined by the dual of Remark 4.15, which still has P as path object and satisfies

$$[K] = [\varepsilon \circ K' \circ \eta] = [I^\varepsilon] \circ [K'] \circ [I^\eta] = [I^\varepsilon] \circ [h * H^C] \circ [I^\eta] = [H]$$

(recall that I^ε as well as I^η can be considered to be right or left homotopies, see Remark 4.13) ☐

5 Replacement for model bicategories

In this section we will define a fibrant-cofibrant replacement for model bicategories. Using just the axioms of a model bicategory, and mimicking the 1-dimensional construction in [27, I, proof of Th. 1], we can define an assignation on objects and arrows $\mathcal{C} \overset{RQ}{\rightsquigarrow} \mathcal{C}$, $X \xrightarrow{f} Y \rightsquigarrow RQX \xrightarrow{RQf} RQY$, such that RQX is a fibrant-cofibrant object for all $X \in \mathcal{C}$. We call such an assignation a *preassignment* of bicategories (see Definition A.5). As we mentioned in the introduction of this paper, we are considering Quillen's axioms in [27] as a base for our work. Since the appearance of [27], in the vast literature on model categories, these axioms have been strengthened and for example in [15], [16], there is the assumption of the existence of functorial factorizations which allow to define, for a model category $\mathbf{C}$, a fibrant-cofibrant replacement functor $\mathbf{C} \longrightarrow \mathbf{C}$. However, we are not assuming a (pseudo)functorial factorization in the model bicategory axioms, and this is the reason why we do not have a fibrant-cofibrant replacement pseudofunctor $\mathcal{C} \xrightarrow{RQ} \mathcal{C}$.

Our strategy for having nevertheless a fibrant-cofibrant replacement pseudofunctor, which can be used to show a localization theorem as in [27, I, proof of Th. 1] is the following. First we note that by definition $\mathcal{C}$ and $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})$ have the same objects and arrows, so that RQ above is equivalently a preassignment $\mathcal{H}o^f(\mathcal{C}, \mathcal{W}) \overset{RQ}{\rightsquigarrow} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})$. Then, by results in Appendix A on *transfer of structures*, we will extend this preassignment to a pseudofunctor $R_\ell Q$, which is the composition of two pseudofunctors R_ℓ and Q, satisfying that $R_\ell Q = RQ$ on objects and arrows. $R_\ell Q$ is a fibrant-cofibrant replacement $\mathcal{H}o^f(\mathcal{C}, \mathcal{W}) \xrightarrow{R_\ell Q} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})$ defined on the fibrant homotopies. In particular, we can consider the pseudofunctor q defined as the composition $\mathcal{C} \xrightarrow{i} \mathcal{H}o^f(\mathcal{C}, \mathcal{W}) \xrightarrow{R_\ell Q} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})$. Note that, even though there may not be structural 2-cells of $\mathcal{C}$ making $\mathcal{C} \overset{RQ}{\rightsquigarrow} \mathcal{C}$ above into a pseudofunctor, the existence of q is showing that there are structural homotopies serving that purpose, just like in the original case [27]. In Section 6, we will show that q is the pseudofunctor giving the localization of $\mathcal{C}$ at the weak equivalences.

Notation 5.1. Until now, we have usually omitted to specify that we were working with left homotopies. Since in this section we will work with both left and right homotopies, for clarity we will indicate with a superscript "ℓ" or "r" which are the 2-cells of the homotopy bicategory being considered.

The fibrant left homotopies (and cofibrant right homotopies) will play a major role in the remainder of the paper, we denote by $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ the sub-bicategory of $\mathcal{H}o(\mathcal{C}, \mathcal{W})^\ell$ with the same objects and arrows, whose 2-cells are given by the fibrant homotopies (indistinctly w- or q-, recall Lemma 4.36). By this we mean precisely that the 2-cells of $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ are the classes of finite sequences $[H^n, \ldots, H^2, H^1]$ of homotopies, as in 4.11, which are all fibrant. We define $\mathcal{H}o^f_{inv}(\mathcal{C}, \mathcal{W})^\ell \subseteq \mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ as the sub-bicategory for which, in addition, we ask all these homotopies to have invertible cells.

The reader will note that it could be possible to vertically compose the homotopies $H^1, H^2..., H^n$ above as in Lemma 4.22, so that $[H^n, \ldots, H^2, H^1]$ is also the class of a single fibrant homotopy, as is the case when X is cofibrant and Y is fibrant by Lemma 4.37. However, this fact is not relevant for the definitions of $\mathcal{H}o^f_{inv}(\mathcal{C}, \mathcal{W})^\ell$ and $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$, which also include the case of arbitrary objects X and Y.

Note that $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ and $\mathcal{H}o^f_{inv}(\mathcal{C}, \mathcal{W})^\ell$ are indeed sub-bicategories of $\mathcal{H}o(\mathcal{C}, \mathcal{W})^\ell$, this follows because the composition $r * H$ is defined to have the same cylinder as H, and $H * \ell$ with a cylinder that has the same "s", see 4.10. This is not the case if we consider arbitrary q-homotopies instead of the fibrant ones.

We will also work with the dual cofibrant right homotopies, that we denote with a c instead of an f, $\mathcal{H}o^c(\mathcal{C}, \mathcal{W})^r$.

Note that we use the same letters f and/or c but as subscripts, to indicate that the objects are supposed to be fibrant, resp. cofibrant.

As a mnemonic, it is convenient to remember that the superscripts refer to the homotopies we are considering, and the subscripts refer to the objects.

Proposition 5.2 (Cofibrant replacement). *There exist a pseudofunctor $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell \xrightarrow{Q} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ and a pseudonatural transformation $Q \overset{\rho}{\Rightarrow} id$ such that ρ_X is a trivial fibration for each X. All the QX are cofibrant objects and, when restricted to cofibrant objects, Q and ρ coincide with the identity.*

Proof. **Cofibrant replacement on objects and arrows.**

For each object $X \in \mathcal{C}$, we choose a cofibrant object QX and a trivial fibration $QX \xrightarrow{\rho_X} X$. If X is already cofibrant, we require $QX = X$ and $\rho_X = id_X$. Note that this can be obtained by factorizing $0 \longrightarrow X$ using axiom **M2**.

For each arrow $X \xrightarrow{f} Y$, we choose an arrow $QX \xrightarrow{Qf} QY$ and an invertible 2-cell

$$f * \rho_X \overset{\rho_f}{\Rightarrow} \rho_Y * Qf \,, \qquad \begin{tikzcd} QX \arrow[r, "\rho_X"'] \arrow[d, "Qf"'] \arrow[dr, phantom, "\Uparrow \rho_f"] & X \arrow[d, "f"] \\ QY \arrow[r, "\rho_Y"'] & Y \end{tikzcd}$$

.

If X and Y are cofibrant, we require $Qf = f$ and $\rho_f = id_f$. Note that this can be obtained using axiom **M1s**:

$$\begin{tikzcd} 0 \arrow[rr] \arrow[d] & \cong & QY \arrow[d, "\rho_Y"] \\ QX \arrow[r, "\rho_X"'] & X \arrow[r, "f"'] & Y \end{tikzcd} \rightsquigarrow \begin{tikzcd} 0 \arrow[rr] \arrow[d] & \cong & QY \arrow[d, "\rho_Y"] \\ QX \arrow[r, "\rho_X"'] \arrow[rru, dashed, "Qf"] & X \arrow[r, "f"'] & Y \arrow[ul, phantom, "\cong \Downarrow \rho_f"] \end{tikzcd}$$

Before finishing the proof of of Proposition 5.2, we note:

Remark 5.3. If X is fibrant, then so is QX by axiom **M3**.

Remark 5.4. If f is a weak equivalence, then so is Qf by axiom **M5**.

Cofibrant replacement on fibrant w-homotopies. Recall from Remark 4.12 that, for each invertible 2-cell ρ_f, I^{ρ_f} can be taken to be a fibrant w-homotopy with invertible cells. By Proposition 4.17 the class of such a w-homotopy is an invertible 2-cell of $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})$ that we will abuse and denote also by ρ_f.

We refer to Definition A.5 in Appendix A, with $\mathcal{B} = \mathcal{D} = \mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$.

5.5. *The constructions above determine a preassignment* $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell \overset{F=Q}{\rightsquigarrow} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ *furnished with a pretransfer data into the identity assignment* $G = id$, $\theta = \rho : Q \rightsquigarrow id$.

We begin by showing:

5.6. *The preassignment in 5.5 can be extended to an assignment Q furnished with a transfer data ρ into the identity assignment.*

Proof of 5.6. This follows by Proposition A.10. To verify that the assumption (a1) in Proposition A.10 holds, we consider Remark A.12, and Propositions 4.19 and 4.16. It suffices thus to show the assumption (a1) in Proposition A.10 considering α to be either a q-homotopy of the form I^μ or a q-homotopy with invertible 2-cells. This follows by Propositions 4.46 and 4.42 respectively. □

5.7. *The preassignment in 5.5 determines a pseudofunctor* $\mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell \xrightarrow{Q} \mathcal{H}o^f(\mathcal{C}, \mathcal{W})^\ell$ *furnished with a pseudonatural transformation into the identity pseudofunctor.*

Proof of 5.7. We apply Theorem A.13. We consider $\Omega \subset \mathcal{H}o^f(\mathcal{C}, \mathcal{W})$ to be the subcategory generated by the homotopies with invertible cells, $\Omega = \mathcal{H}o^f_{inv}(\mathcal{C}, \mathcal{W})$ as in Lemma 4.43. Note that the identity and the multiplication of $G = id$ as well as the 2-cells ρ_f of the transfer data are in Ω. The assumption (a2) in Theorem A.13 holds by Proposition 4.42. □

This finishes the proof of Proposition 5.2. □

A fibrant replacement is a cofibrant replacement in the dual bicategory. To set up the notation we write the dual of Proposition 5.2.

Proposition 5.8 (Fibrant replacement). *There exists a pseudofunctor* $\mathcal{H}o^c(\mathcal{C},\mathcal{W})^r \xrightarrow{R} \mathcal{H}o^c(\mathcal{C},\mathcal{W})^r$*, and a pseudonatural transformation* $id \overset{\lambda}{\Rightarrow} R$ *such that* λ_X *is a trivial cofibration for each* X*. All the* RX *are fibrant objects and, when restricted to fibrant objects,* R *and* λ *coincide with the identity.* □

Composing a fibrant and a cofibrant replacement

It is clear how to compose a fibrant and a cofibrant replacement on objects and arrows, but not so clear how to compose the corresponding pseudofunctors. As it could be expected, Lemma 4.47 and its dual provide the key to do this. They allow to define the "switch" 2-functors s and s^{op}, $s\,s^{op} = id$, $s^{op}\,s = id$, which are the identity on objects and arrows, and in the 2-cells switch the homotopies from left to right and viceversa, but the 2-cell remains the same. Note that Lemma 4.47 holds when all the involved objects are cofibrant, thus its dual will hold when they are fibrant. Note also that, by the dual of Remark 5.3, R and λ can be restricted to $\mathcal{H}o^c_c(\mathcal{C},\mathcal{W})^r \xrightarrow{R} \mathcal{H}o^c_c(\mathcal{C},\mathcal{W})^r$, $id \overset{\lambda}{\Rightarrow} R$.

We abuse and denote with the same letters $\mathcal{H}o^f(\mathcal{C},\mathcal{W})^\ell \xrightarrow{Q} \mathcal{H}o^f_c(\mathcal{C},\mathcal{W})^\ell$, $\mathcal{H}o^c_c(\mathcal{C},\mathcal{W})^r \xrightarrow{R} \mathcal{H}o^c_{fc}(\mathcal{C},\mathcal{W})^r$, and $\mathcal{H}o^c_{fc}(\mathcal{C},\mathcal{W})^r \xrightarrow{s^{op}} \mathcal{H}o^f_c(\mathcal{C},\mathcal{W})^\ell$ the pseudofunctors which are either co-restrictions to their images or post-compositions with inclusions, of the respective endo-pseudofunctors. This allows to define a *fibrant replacement on left homotopies*, that is the arrow R_l defined as the composition $R_\ell = s^{op}\,R\,s$, which can be composed with Q as follows:

$$\begin{tikzcd}
\mathcal{H}o^f(\mathcal{C},\mathcal{W})^\ell \arrow[r, "Q"] & \mathcal{H}o^f_c(\mathcal{C},\mathcal{W})^\ell \arrow[r, "s"] \arrow[rrr, bend left, "R_\ell"] & \mathcal{H}o^c_c(\mathcal{C},\mathcal{W})^r \arrow[r, "R"] & \mathcal{H}o^c_{fc}(\mathcal{C},\mathcal{W})^r \arrow[r, "s^{op}"] & \mathcal{H}o^f_c(\mathcal{C},\mathcal{W})^\ell
\end{tikzcd}$$

Note that $R_\ell X = RX$, $R_\ell f = Rf$. There is a pseudonatural transformation $id \overset{\lambda_\ell}{\Rightarrow} R_\ell$ defined as $\lambda_\ell = s^{op}\,\lambda\,s$. Note that the X-components of λ_ℓ are the same that those of λ, $(\lambda_\ell)_X = \lambda_X$, we also have $(\lambda_\ell)_f = \lambda_f$. We define a fibrant-cofibrant replacement pseudofunctor as the composite $R_\ell\,Q$. From Propositions 5.2 and 5.8 we have:

Proposition 5.9 (Fibrant-cofibrant replacement). *There exist a pseudofunctor* $\mathcal{H}o^f(\mathcal{C},\mathcal{W})^\ell \xrightarrow{R_\ell Q} \mathcal{H}o^f(\mathcal{C},\mathcal{W})^\ell$ *and pseudonatural transformations* $id \overset{\rho}{\Leftarrow} Q \xRightarrow{\lambda_\ell\,Q} R_\ell\,Q$ *such that* ρ_X *is a trivial fibration and* $(\lambda\,Q)_X$ *is a trivial cofibration for each* X*. All the* $(R_\ell Q)X$ *are fibrant-cofibrant objects and, when restricted to fibrant-cofibrant objects,* $R_\ell Q$*,* ρ *and* $\lambda_\ell\,Q$ *coincide with the identity.*

From Remark 5.4 and its dual it follows

Remark 5.10. Note that f is a weak equivalence, so is $(R_\ell Q)f$. □

6 The localization theorem.

Much like it is done in the 1-dimensional case, Lemma 4.47 can be used together with its dual to give *niceness* properties of homotopies between arrows in $\mathcal{C}(W,\,Z)$ when W is cofibrant and Z is fibrant. Note that, using also Proposition 4.37, it follows that in this case all notions of homotopy (w or q, right or left, right cofibrant or left fibrant), coincide, in the strong sense that the equality (given by the switch functors) is an isomorphism of categories $\mathcal{X}(W, Z) = \mathcal{Y}(W, Z)$, where $\mathcal{X}$, $\mathcal{Y}$ stand for the homotopy bicategories corresponding to any two choices of these concepts. We have

$$\mathcal{H}\mathrm{o}(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C}, \mathcal{W}) = \mathcal{H}o^{c}_{fc}(\mathcal{C}, \mathcal{W})^{r} = \mathcal{H}o^{f}_{fc}(\mathcal{C}, \mathcal{W})^{\ell}$$

As a consequence, although for the fibrant-cofibrant replacement both left and right homotopies are necessary, the latter ones become *implicit*, and do not occur explicitly. Thus in this section we will drop the superscript "ℓ" in the $\mathcal{H}o$ bicategories, that *will always have left homotopies as* 2*-cells.*

Definition 6.1. *We set $\mathcal{H}o(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C}, \mathcal{W})$, that is the full sub-bicategory of $\mathcal{H}o(\mathcal{C}, \mathcal{W})$ given by the fibrant-cofibrant objects.*

From the Lemma 4.22 we have

Remark 6.2. The 2-cells of $\mathcal{H}\mathrm{o}(\mathcal{C})$ are classes of a single q-homotopy. For any pair of q-homotopies $[H_1, H_2]$ in $\mathcal{H}\mathrm{o}(\mathcal{C})$, there is a single q-homotopy $[H]$ in $\mathcal{H}\mathrm{o}(\mathcal{C})$ such that $[H] = [H_1, H_2]$.

We consider the 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H}o^{f}(\mathcal{C}, \mathcal{W})$ in 4.11, Remark 4.12, and the fibrant-cofibrant replacement pseudofunctor $R_\ell Q$ in Proposition 5.9. Note that $R_\ell Q$ takes its values in $\mathcal{H}\mathrm{o}(\mathcal{C})$, here it is convenient to denote its co-restriction as $\mathcal{H}o^{f}(\mathcal{C}, \mathcal{W}) \xrightarrow{r} \mathcal{H}\mathrm{o}(\mathcal{C})$. We have thus the following commutative diagram , where q is defined as the composite $q = r\, i$ and j denotes the inclusion:

$$\begin{tikzcd}
\mathcal{C} \arrow[rr, "q"] \arrow[d, "i"'] & & \mathcal{H}\mathrm{o}(\mathcal{C}) \arrow[d, "j"] \\
\mathcal{H}o^{f}(\mathcal{C}, \mathcal{W}) \arrow[rru, "r"] \arrow[rr, "R_\ell Q"'] & & \mathcal{H}o^{f}(\mathcal{C}, \mathcal{W})
\end{tikzcd} \tag{6.3}$$

We will show that q is the localization of $\mathcal{C}$ with respect to $\mathcal{W}$, that is, we will prove the following theorem:

Theorem 6.4. *The pseudofunctor $q = r\, i$ is the localization of $\mathcal{C}$ with respect to $\mathcal{W}$, in the sense that it maps the arrows of $\mathcal{W}$ to equivalences, and furthermore for each bicategory $\mathcal{D}$, precomposition with q, $Hom(\mathcal{H}o(\mathcal{C}), \mathcal{D}) \xrightarrow{q^*} Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ is a biequivalence of bicategories, where $Hom_{\mathcal{W},\Theta}(\mathcal{C}, \mathcal{D})$ stands for the full sub-bicategory of $Hom(\mathcal{C}, \mathcal{D})$ given by those pseudofunctors that send weak equivalences into equivalences.*

The proof of this theorem depends on the following three results.

Proposition 6.5. *The pseudofunctor q maps the arrows of $\mathcal{W}$ to equivalences.*

Proof. If f is a weak equivalence, by Remark 5.10 $q(f)$ is also a weak equivalence, but now between fibrant-cofibrant objects. Thus by Proposition 3.17 it is a composite of split weak equivalences (see Definition 3.16), which by [6, Prop. 3.45] is an equivalence. ☐

Let $i\mathcal{W}$ be the class of arrows in $\mathcal{H}o^f(\mathcal{C},\mathcal{W})$ of the form $i\,f$ with $f \in \mathcal{W}$.

Proposition 6.6. *Consider the diagram* (6.3). *For each bicategory $\mathcal{D}$, precomposing with r, $Hom(\mathcal{H}o(\mathcal{C}),\mathcal{D}) \xrightarrow{r^*} Hom(\mathcal{H}o^f(\mathcal{C},\mathcal{W}),\mathcal{D})$ factors through the full sub-bicategory $Hom_{i\mathcal{W},\Theta}(\mathcal{H}o^f(\mathcal{C},\mathcal{W}),\mathcal{D})$ and together with j^* establishes a biequivalence of bicategories:*

$$\begin{tikzcd} Hom(\mathcal{H}o(\mathcal{C}),\mathcal{D}) \arrow[r, shift left, "r^*"] & Hom_{i\mathcal{W},\Theta}(\mathcal{H}o^f(\mathcal{C},\mathcal{W}),\mathcal{D}) \arrow[l, shift left, "j^*"] \end{tikzcd}$$

Proof. Let $F \in Hom(\mathcal{H}\mathrm{o}(\mathcal{C}),\mathcal{D})$ and $f \in \mathcal{W}$, then $(r^*F)(if) = F r\, i f = Fqf$ which is an equivalence by Proposition 6.5. This shows the factorization of r^*.

We already have $j^*\, r^* = (r\,j)^* = id^* = id$, so it only remains to establish an equivalence $id \simeq r^*\, j^* = (jr)^* = (R_\ell Q)^*$. Let $id \overset{\rho^*}{\Leftarrow} Q^* \xRightarrow{(\lambda_\ell Q)^*} (R_\ell Q)^*$ be the pseudonatural transformations induced by those in Proposition 5.9 , to conclude the proof we will show that they are equivalences. By item 3. in 2.4, it suffices to check that $((\rho^*)_F)_X = F(\rho_X)$ and $(((\lambda_\ell Q)^*)_F)_X = F(\lambda_{QX})$ are equivalences, for each $\mathcal{H}o^f(\mathcal{C},\mathcal{W}) \xrightarrow{F} \mathcal{D}$ that maps weak equivalences to equivalences, and for each X. Since ρ_X and λ_{QX} are indeed weak equivalences (see Propositions 5.2 and 5.8), this is precisely the case. ☐

We have also the following result proven in [6, Theorem 3.42, Remark 3.47]:

Theorem 6.7. *The 2-functor $\mathcal{C} \xrightarrow{i} \mathcal{H}o^f(\mathcal{C},\mathcal{W})$ is such that precomposing with i establishes a biequivalence of bicategories, which in fact is an isomorphism:*

$$Hom_{i\mathcal{W},\Theta}(\mathcal{H}o^f(\mathcal{C},\mathcal{W}),\mathcal{D}) \xrightarrow{i^*} Hom_{\mathcal{W},\Theta}(\mathcal{C},\mathcal{D})$$ ☐

Finally, since $q^* = (r\,i)^* = i^*\, r^*$, putting together Proposition 6.6 and Theorem 6.7 finishes the proof of the localization theorem, Theorem 6.4. ☐

We finish the paper by showing that the set-theoretical difficulties in the construction of localizations are resolved for the homotopy bicategory. From Proposition 4.47 (and its dual version) it follows immediately:

Proposition 6.8 (Single cylinder)**.** *Let $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ be any pair of arrows in $\mathcal{H}o(\mathcal{C}) = \mathcal{H}o_{fc}(\mathcal{C},\mathcal{W})$, and fix a cylinder C. Then all the q-homotopies $f \rightsquigarrow g$ can be chosen with C as its cylinder.* ☐

Recall that a bicategory is said to be locally small if, for each pair of objects X, Y, the category $\mathcal{C}(X,Y)$ is small, that is, has a set of objects and a set of arrows. From Proposition 6.8 it follows:

Corollary 6.9. *If $\mathcal{C}_{fc}$ is locally small (in particular when $\mathcal{C}$ is so), then so is $\mathcal{H}o(\mathcal{C})$.* ☐

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A Transfer of structure for bicategories

A.1 Preliminaries.

While the theory of bicategories is nowadays well-established, it becomes necessary to recall here the explicit definitions of some basic concepts to fix the notation that we will be using. For the rest of the notation used here see 2.1. In the proofs involving 2-cells we use the elevator calculus rather than pasting diagrams. This is justified by the Coherence result in 2.1.

Elevator calculus.[3] A 2-cell $f \overset{\alpha}{\Rightarrow} g$ is written $\begin{array}{c} f \\ \big\backslash \alpha \big/ \\ g \end{array}$. Each elevators diagram represents a composition of 2-cells in a bicategory. Objects are omitted, arrows are composed from right to left, and 2-cells from top to bottom. Using the basic move "(el)", which corresponds

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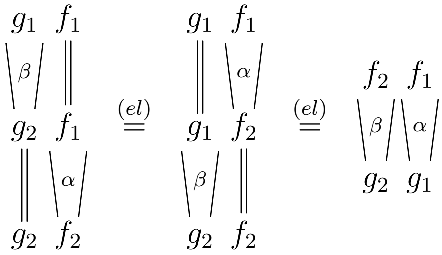
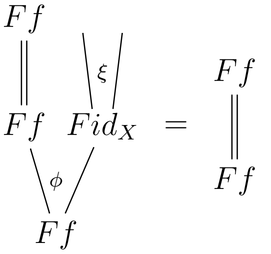
[3] Developed in 1969 by the second author for draft use.

to the middle-four interchange, we form configurations of cells that fit valid equations in order to prove new equations.

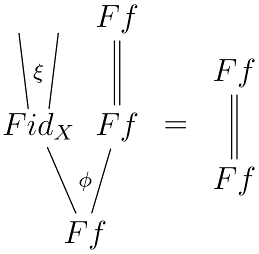


Definition A.1. *A lax functor $\mathcal{B} \xrightarrow{F} \mathcal{D}$ between bicategories is given by a family of functors $\mathcal{B}(X,Y) \xrightarrow{F} \mathcal{D}(FX,FY)$, one for each pair of objects X, Y of $\mathcal{B}$, 2-cells $id_{FX} \xRightarrow{\xi_X} F(id_X)$, one for each object X of $\mathcal{B}$, refered to as the* unit *of the lax functor, and natural transformations $* \circ (F \times F) \overset{\phi}{\Rightarrow} F \circ * : \mathcal{B}(X,Y) \times \mathcal{B}(Y,Z) \longrightarrow \mathcal{B}(X,Z)$ with components $Fg * Ff \xRightarrow{\phi_{f,g}} F(g * f)$, one for each triplet X, Y, Z of objects of $\mathcal{B}$, refered to as the* multiplication *of the lax functor. We will omit the subindices of ξ and ϕ. The following equalities are required to hold:*

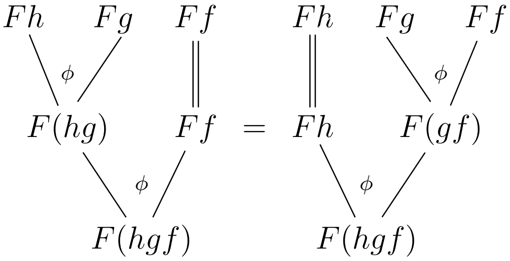
For each $X \xrightarrow{f} Y \in \mathcal{B}$, **LF1.**



LF2.

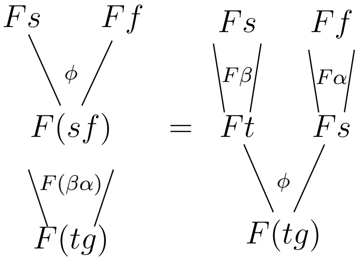


For each $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W \in \mathcal{B}$, **LF3.**



We will often use the naturality of ϕ, thus we make it explicit:

For each $$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\alpha\Downarrow"] & Y \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\beta\Downarrow"] & Z \end{tikzcd} \in \mathcal{B},$$ **Nϕ.**



A pseudofunctor (resp. 2-functor) is a lax functor such that all the 2-cells ϕ and ξ are invertible (resp. identities).

Definition A.2. *A lax natural transformation $\theta : F \Rightarrow G : \mathcal{B} \to \mathcal{D}$ between lax functors consists of a family of arrows $FX \xrightarrow{\theta_X} GX$, one for each $X \in \mathcal{B}$ and a family of 2-cells*

$$\begin{tikzcd}[column sep=small] Gf \arrow[d, dash] \arrow[dr, phantom, "\theta_f"] & \theta_X \arrow[d, dash] \\ \theta_Y & Ff \end{tikzcd}$$

, one for each $X \xrightarrow{f} Y \in \mathcal{B}$, satisfying the following axioms:

LN0. *For each $X \in \mathcal{B}$,*

$$\begin{tikzcd}[column sep=small] \theta_X \arrow[d, equal] & {} \arrow[d, dash, "\xi"] \\ \theta_X & Fid_X \end{tikzcd} = \begin{tikzcd}[column sep=small] {} \arrow[d, dash, "\xi"] & \theta_X \arrow[d, equal] \\ Gid_X \arrow[d, dash] \arrow[dr, phantom, "\theta_{id_X}"] & \theta_X \arrow[d, dash] \\ \theta_X & Fid_X \end{tikzcd}$$

LN1. *For each $X \xrightarrow{f} Y \xrightarrow{g} Z \in \mathcal{B}$,*

$$\begin{tikzcd}[column sep=small] Gg \arrow[d, equal] & Gf \arrow[d, dash] \arrow[dr, phantom, "\theta_f"] & \theta_X \arrow[d, dash] \\ Gg \arrow[d, dash] \arrow[dr, phantom, "\theta_g"] & \theta_Y \arrow[d, dash] & Ff \arrow[d, equal] \\ \theta_Z \arrow[d, equal] & Fg \arrow[dr, dash, "\phi"] & Ff \arrow[d, dash] \\ \theta_Z & & F(gf) \end{tikzcd} = \begin{tikzcd}[column sep=small] Gg \arrow[dr, dash, "\phi"] & & Gf \arrow[dl, dash] & \theta_X \arrow[d, equal] \\ & G(gf) \arrow[d, dash] \arrow[drr, phantom, "\theta_{gf}"] & & \theta_X \arrow[d, dash] \\ & \theta_Z & & F(gf) \end{tikzcd}$$

LN2. *For each $\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow \alpha"] & Y \end{tikzcd} \in \mathcal{B}$,*

$$\begin{tikzcd}[column sep=small] Gf \arrow[d, dash, "G\alpha"] & \theta_X \arrow[d, equal] \\ Gg \arrow[d, dash] \arrow[dr, phantom, "\theta_g"] & \theta_X \arrow[d, dash] \\ \theta_Y & Fg \end{tikzcd} = \begin{tikzcd}[column sep=small] Gf \arrow[d, dash] \arrow[dr, phantom, "\theta_f"] & \theta_X \arrow[d, dash] \\ \theta_Y \arrow[d, equal] & Ff \arrow[d, dash, "F\alpha"] \\ \theta_Y & Fg \end{tikzcd}$$

An oplax natural transformation is as above, only with the 2-cells θ_f reversed, satisfying dual axioms $LN0^{op}$, $LN1^{op}$ and $LN2^{op}$.

A pseudonatural (resp 2-natural) transformation is a lax natural transformation such that all the 2-cells θ_f are invertible (resp. identities). Note than in this case the lax and oplax axioms are equivalent. The collection θ_f^{-1} determines a pseudonatural (in particular oplax) transformation.

Note that we can consider pseudonatural transformations between lax functors.

Definition A.3. *A* modification *$\rho : \theta \to \eta : F \Rightarrow G : \mathcal{B} \to \mathcal{D}$ between lax, pseudo, or* 2*-natural transformations is a family of 2-cells $\theta_X \overset{\rho_X}{\Longrightarrow} \eta_X$ of $\mathcal{D}$, one for each $X \in \mathcal{B}$ such that:*

M. *For each* $X \xrightarrow{f} Y \in \mathcal{B}$,

$$\begin{array}{ccc} Gf & & \theta_X \\ \backslash & \theta_f & / \\ \theta_Y & & Ff \\ \backslash\rho_Y/ & & \| \\ \eta_Y & & Ff \end{array} \quad = \quad \begin{array}{ccc} Gf & & \theta_X \\ \| & & \backslash\rho_X/ \\ Gf & & \eta_X \\ \backslash & \eta_f & / \\ \eta_Y & & Ff \end{array}$$

Lax functors, in particular pseudofunctors, with lax or pseudo natural transformations and modifications can be composed in order to define, for each pair of bicategories $\mathcal{B}, \mathcal{D}$, bicategories $\mathcal{X}(\mathcal{B}, \mathcal{D})$ as the case may be. We omit the details as they are ubiquitous in the literature.

Notation. We denote $\mathcal{H}om(\mathcal{B}, \mathcal{D})$ the case where the objects are either lax or pseudo functors, and the arrows are pseudonatural transformations.

Recall from 4.7 that an arrow $X \xrightarrow{f} X'$ in a bicategory $\mathcal{D}$ is a *quasiequivalence* if for every object Y in $\mathcal{D}$ the functors $\mathcal{D}(Y, X) \xrightarrow{f_*} \mathcal{D}(Y, X')$ and $\mathcal{D}(X', Y) \xrightarrow{f^*} \mathcal{D}(X, Y)$ are full and faithful.

Proposition A.4. *If a pseudonatural transformation* $F \overset{\theta}{\Rightarrow} G$ *between lax functors* $\mathcal{B} \overset{F}{\underset{G}{\rightrightarrows}} \mathcal{D}$ *is such that* $FX \xrightarrow{\theta_X} GX$ *is a quasiequivalence in* $\mathcal{D}$ *for each* $X \in \mathcal{B}$, *then it is a quasiequivalence in the bicategory* $\mathcal{H}om(\mathcal{B}, \mathcal{D})$.

Proof. We will show that θ_* is full and faithful, the case of θ^* is dual. Let $\mathcal{B} \xrightarrow{H} \mathcal{D}$ be any lax functor, and $H \overset{\alpha}{\Rightarrow} F$, $H \overset{\beta}{\Rightarrow} F$ be pseudonatural transformations. Consider for each $X \in \mathcal{B}$ the following diagram:

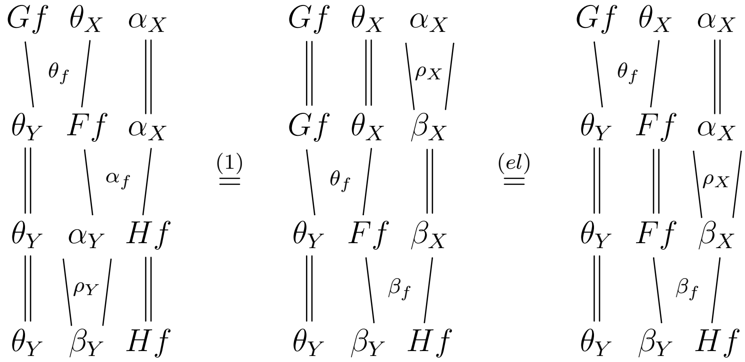
$$\begin{tikzcd}
\mathcal{H}om(\mathcal{B}, \mathcal{D})(H, F)(\alpha, \beta) \arrow[r, "\theta_*"] \arrow[d] & \mathcal{H}om(\mathcal{B}, \mathcal{D})(H, G)(\theta \circ \alpha, \theta \circ \beta) \arrow[d] \\
\mathcal{D}(HX, FX)(\alpha_X, \beta_X) \arrow[r, "(\theta_X)_*"] & \mathcal{D}(HX, GX)((\theta \circ \alpha)_X, (\theta \circ \beta)_X)
\end{tikzcd}$$

where the arrow in the bottom is a bijection by hypothesis. We will show that then the upper horizontal arrow is also a bijection.

Let $\theta \circ \alpha \xrightarrow{\xi} \theta \circ \beta$ be a modification, for each X we have a 2-cell $(\theta \circ \alpha)_X \overset{\xi_X}{\Rightarrow} (\theta \circ \beta)_X$, thus there is a unique 2-cell $\alpha_X \overset{\rho_X}{\Rightarrow} \beta_X$ such that $\theta_X * \rho_X = \xi_X$. It remains to prove the modification axiom **M.** for ρ. By the modification axiom for ξ we have

$$\begin{array}{ccc} Gf & & (\theta \circ \alpha)_X \\ \backslash & (\theta \circ \alpha)_f & / \\ (\theta \circ \alpha)_Y & & Hf \\ \backslash\xi_Y/ & & \| \\ (\theta \circ \beta)_Y & & Hf \end{array} \quad \overset{M}{=} \quad \begin{array}{ccc} Gf & & (\theta \circ \alpha)_X \\ \| & & \backslash\xi_X/ \\ Gf & & (\theta \circ \beta)_X \\ \backslash & (\theta \circ \beta)_f & / \\ (\theta \circ \beta)_Y & & Hf \end{array}$$

Recalling the definition of the vertical composition of pseudonatural transformations and that $\theta_X * \rho_X = \xi_X$, the previous equation becomes equation (1) below



Since $\theta_f * \alpha_X$ is invertible and $(\theta_Y)_*$ is faithful, it follows that equation **M.** holds for ρ. $\square$

It is worth mentioning that, unlike the case of equivalences, Proposition A.4 also holds for 2-natural transformations, as it can be seen by inspecting the proof above.

A.2 Transfer of laxfunctor structure.

Definition A.5.
1) A preassignment *of bicategories $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ consists of the following: for each object X and for each arrow $X \xrightarrow{f} Y$ of $\mathcal{B}$, an object FX and an arrow $FX \xrightarrow{Ff} FY$ of $\mathcal{D}$.*

2) A assignment *of bicategories is a preassignment, plus, for each 2-cell $f \overset{\alpha}{\Rightarrow} g$ in $\mathcal{B}$ a 2-cell $Ff \overset{F\alpha}{\Rightarrow} Fg$ in $\mathcal{D}$ such that for all X, Y, $\mathcal{B}(X,Y) \xrightarrow{F} \mathcal{D}(FX,FY)$ is a functor.*

Definition A.6. *Let $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ and $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ be preassignments of bicategories:*

1) A pretransfer data *$\theta : F \Rrightarrow G$ between preassignments consists of a family of arrows $FX \xrightarrow{\theta_X} GX$, one for each $X \in \mathcal{B}$, and a family of invertible 2-cells $\theta_f \colon Gf * \theta_X \Rightarrow \theta_Y * Ff$.*

2) If F and G are assignments, a transfer data *is a pretransfer data such that $\theta_g \circ (G\alpha * \theta_X) = (\theta_Y * F\alpha) \circ \theta_f$. Note that this equation is Axiom LN2 in Definition A.2.*

Theorem A.7. *Let $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ and $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ be assignments of bicategories furnished with a transfer data $\theta : F \Rrightarrow G$. Consider the assumption*

$$(*) \quad \textit{For each } X \in \mathcal{B},\ FX \xrightarrow{\theta_X} GX \textit{ is an equivalence in } \mathcal{D}$$

Then, G is the underlying assignment of a pseudofunctor if and only if F is so. More precisely:

(bt) Backward transfer*. Given a pseudofunctor structure on G, F can be uniquely furnished with a pseudofunctor structure in such a way that the arrows θ_X and the 2-cells θ_f form a pseudonatural equivalence $F \overset{\theta}{\Rightarrow} G$ in $\mathcal{H}om(\mathcal{B},\, \mathcal{D})$.*

(ft) Forward transfer*. Given a pseudofunctor structure on F, G can be uniquely furnished with a pseudofunctor structure in such a way that the arrows θ_X and the 2-cells θ_f form a pseudonatural equivalence $F \overset{\theta}{\Rightarrow} G$ in $\mathcal{H}om(\mathcal{B},\, \mathcal{D})$.*

(1) The same statement holds for preassignments of bicategories furnished with a pretransfer data.

(2) The same statement holds replacing all instances of equivalence *with* quasiequivalence.

(3) The same statement holds replacing all instances of pseudofunctor *with* lax functor.

Remark A.8. Note that a forward transfer is a backward transfer on the dual bicategories $\mathcal{B}^{op}$ and $\mathcal{D}^{op}$, so items (bt) and (ft) in the theorem are equivalent statements.

In the remainder of this Appendix we will prove Theorem A.7. Furthermore, we will show that the natural way in which a proof is obtained, leads to more general independent and relevant statements, which are the ones we will actually use in this paper, where only assumptions strictly weaker than (*) hold.

Definition A.9. *Let $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ and $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ be preassignments of bicategories furnished with a pretransfer data $\theta : F \Rrightarrow G$. If G is an assignment, we define a functor*

$$\mathcal{B}(X,Y) \xrightarrow{\widetilde{G}} \mathcal{D}(FX,\, GY)$$

by the formulas, for $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and $f \overset{\alpha}{\Rightarrow} g$ in $\mathcal{B}$:

$$\widetilde{G}f = \theta_Y \circ Ff,\;\; \widetilde{G}g = \theta_Y \circ Fg \quad \text{and} \quad \widetilde{G}\alpha = \theta_g \circ (G\alpha * \theta_X) \circ \theta_f^{-1}.$$

The functoriality of $\mathcal{B}(X,Y) \xrightarrow{\widetilde{G}} \mathcal{D}(FX,GY)$ follows from that of $\mathcal{B}(X,Y) \xrightarrow{G} \mathcal{D}(GX,GY)$.

Proposition A.10. *Let $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ and $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ be preassignments of bicategories furnished with a pretransfer data $\theta : F \Rrightarrow G$. Assume that*

(a1) For each $X \overset{f}{\underset{g}{\rightrightarrows}} Y$ and $f \overset{\alpha}{\Rightarrow} g$ in $\mathcal{B}$, there exists a unique 2-cell $Ff \overset{\beta}{\Rightarrow} Fg$ such that $(\theta_Y)_*(\beta) = \theta_Y * \beta = \widetilde{G}\alpha$ as in Definition A.9.

Then, if G is an assignment, so is F, in a unique way such that the pretransfer data becomes a transfer data.

Proof. Note that the equation $\theta_g \circ (G\alpha * \theta_X) = (\theta_Y * F\alpha) \circ \theta_f$ in the definition of transfer data is equivalent to the equation $(\theta_Y)_*(F\alpha) = \widetilde{G}\alpha$. Then, assumption (a1) means there is a unique factorization F, $F\alpha = \beta$ as indicated in the diagram (A.11). The functoriality of F follows by the uniqueness of β and the functoriality of $\widetilde{G}$.

$$\begin{tikzcd} & \mathcal{D}(FX, FY) \arrow[dr, "(\theta_Y)_*"] & \\ \mathcal{B}(X,Y) \arrow[ur, dashed, "F"] \arrow[rr, "\widetilde{G}"] & & \mathcal{D}(FX, GY) \end{tikzcd} \tag{A.11}$$

□

Remark A.12. If $\mathcal{B}$ has a family of generating 2-cells, such that any 2-cell α in $\mathcal{B}(X,Y)$ is a finite vertical composition of them, it suffices to check the *existence* part in $(a1)$ only for the generating 2-cells. Thus, when the *uniqueness* of β is known to hold, it suffices to check $(a1)$ only for the generating 2-cells.

Proof. If $\alpha = \alpha_1 \circ \alpha_2$ with α_1 and α_2 generating, using the *existence* part of $(a1)$ take β_1, β_2 such that $\theta_Y * \beta_1 = \widetilde{G}(\alpha_1)$, $\theta_Y * \beta_2 = \widetilde{G}(\alpha_2)$. Then:

$$\theta_Y * \beta = \theta_Y * (\beta_1 \circ \beta_2) = \theta_Y * \beta_1 \circ \theta_Y * \beta_2 = \widetilde{G}(\alpha_1) \circ \widetilde{G}(\alpha_2) = \widetilde{G}(\alpha_1 \circ \alpha_2) = \widetilde{G}(\alpha).$$

Also, note that $\theta_Y * id = id = \widetilde{G}(id)$, so if $\alpha = id$, we may take $\beta = id$. □

We consider now a sub-bicategory Ω of $\mathcal{D}$ with the same objects and arrows, $\Omega(X,Y)(f,g) \subset \mathcal{D}(X,Y)(f,g)$, and containing the 2-cells θ_f of the transfer data.

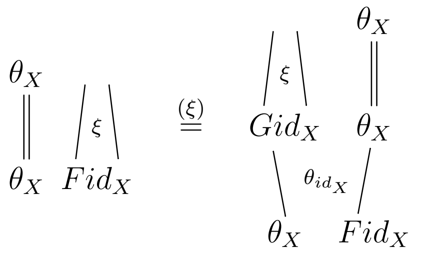
Theorem A.13. *Let* $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ *and* $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ *be assignments of bicategories furnished with a transfer data* $\theta : F \rightsquigarrow G$. *Assume that*

(a2) For each $Z \in \mathcal{B}$, *and each* W *in the image of* F, $\Omega(W, FZ) \xrightarrow{(\theta_Z)_*} \Omega(W, GZ)$ *is full and faithful. That is, for each* 2*-cell* α *in* $\Omega(W, GZ)$ *there is a unique* 2*-cell* β *in* $\Omega(W, FZ)$ *such that* $\theta_Z * \beta = \alpha$.

Then, given a lax-functor structure on G *such that its unit and multiplication are in* Ω, F *can be uniquely furnished with a lax-functor structure in such a way that the transfer data becomes a pseudonatural transformation* $\theta : F \Rightarrow G$, *and furthermore the unit and multiplication of* F *are in* Ω.

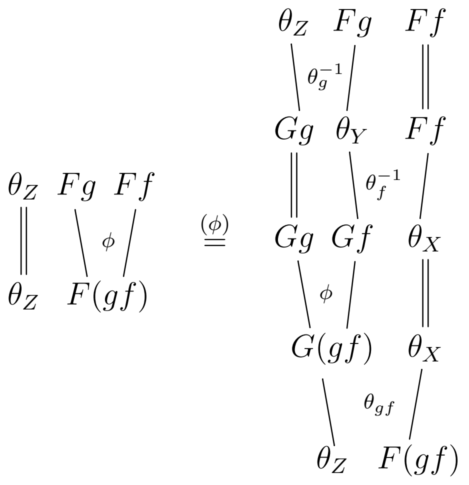
Proof. We will consider the axioms and the notation in Definitions A.1 and A.2. Axiom LN2 in A.2 holds by definition of transfer data.

Definition of the unit for F. Consider the equation



The right side elevator is in Ω, thus by assumption (a2) there exists a unique unit for F in Ω, $\xi = \beta$, such that the equation holds. This equation is clearly equivalent to axiom LN0 in Definition A.2.

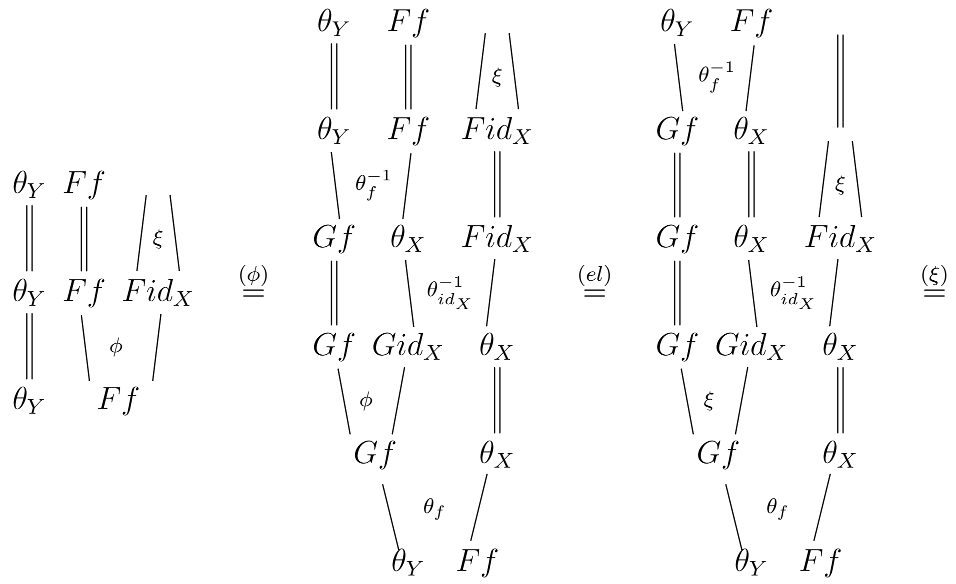
Definition of the multiplication for F. Consider the equation

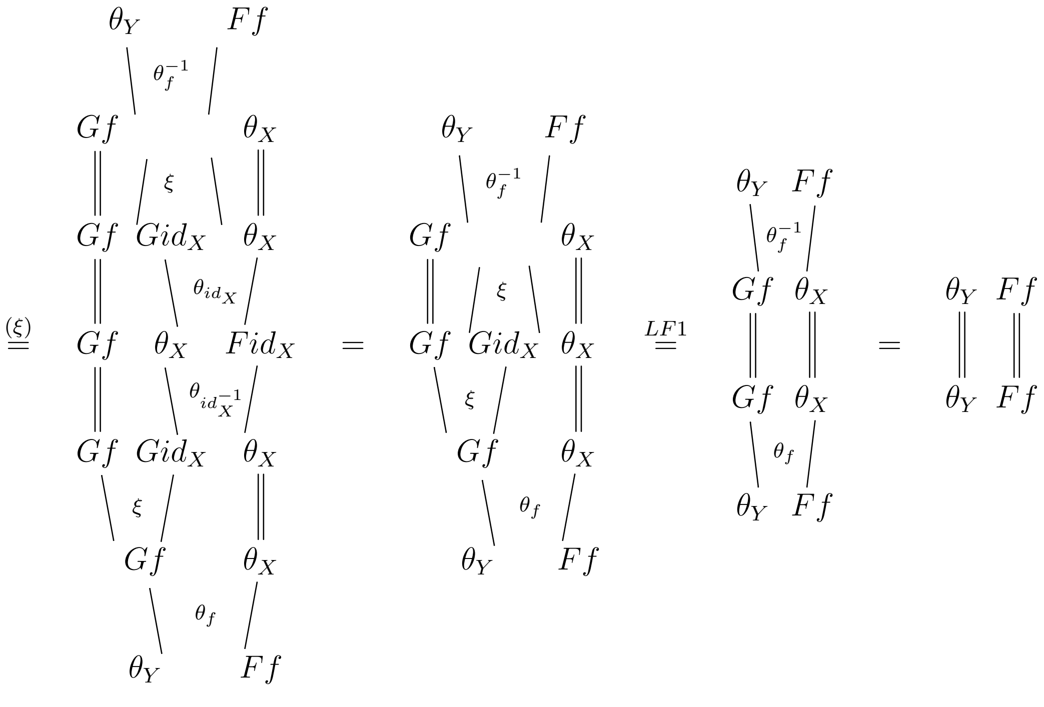


The right side elevator is in Ω, thus by assumption (a2) there exists a unique multiplication for F in Ω, $\phi = \beta$, such that the equation holds. This equation is clearly equivalent to axiom LN1 in Definition A.2.

We have shown that there is a unique possible way of furnishing F with a unit and a multiplication in Ω in such a way that the arrows θ_X and the 2-cells θ_f form a pseudonatural transformation. Now we will prove that this is indeed a lax-functor structure for F, that is, we will show the axioms in Definition A.1. Note that all the 2-cells involved are in Ω, so we can use the uniqueness in assumption (a2) to prove the equations in these axioms as follows: for each axiom, we will show that the equality of 2-cells holds because it holds after composing with an arrow of the form θ_X.

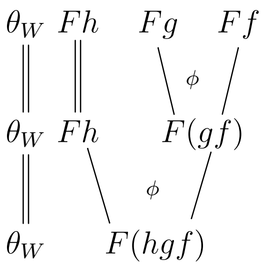
Proof of Axiom LF1.



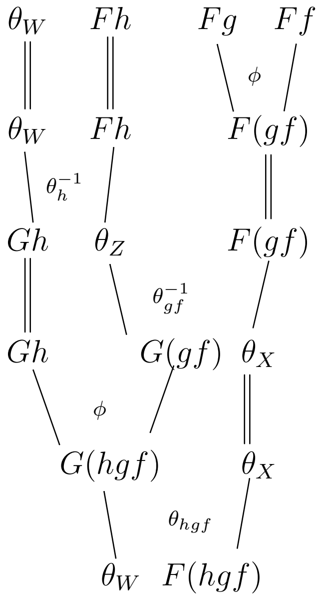


Proof of Axiom LF2. The proof is analogous to the one of LF1

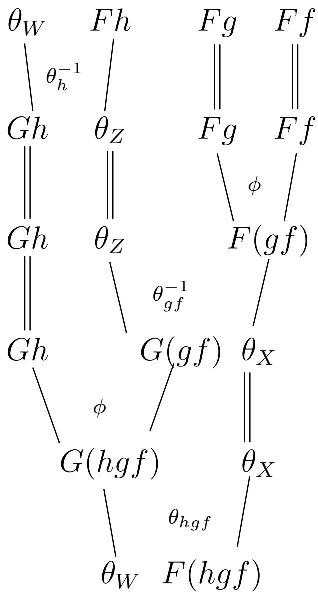
Proof of Axiom LF3.



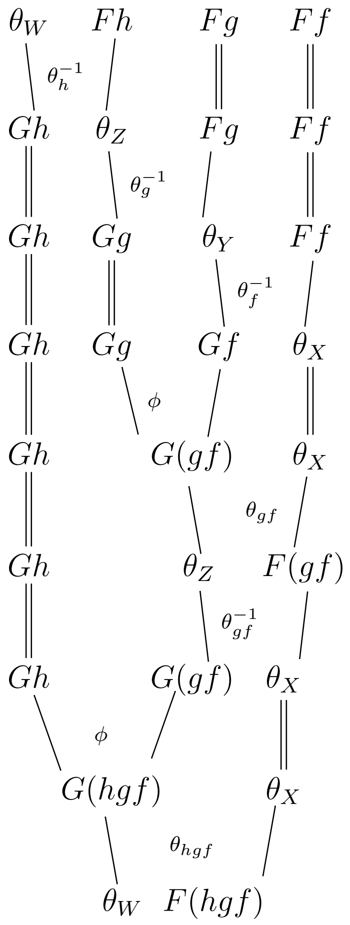
$\overset{(\phi)}{=}$



$\overset{(el)}{=}$



$\overset{(\phi)}{=}$



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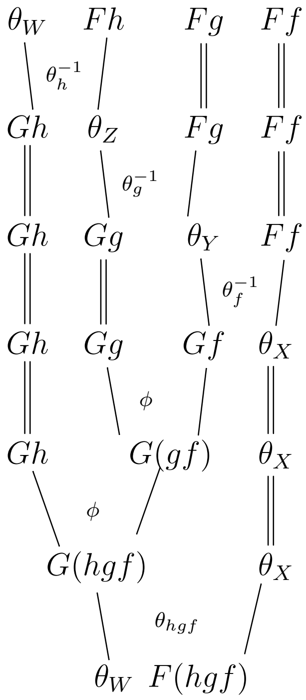
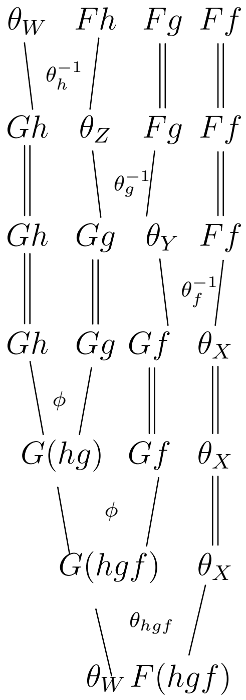
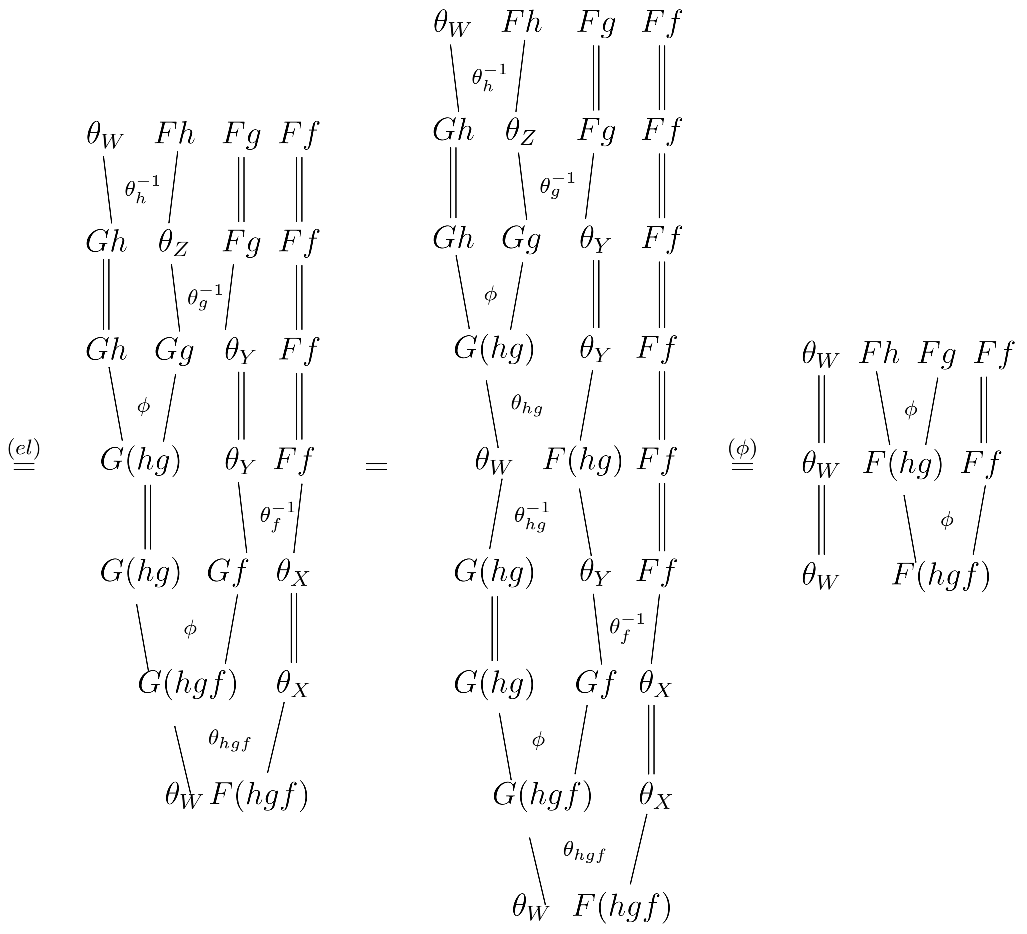


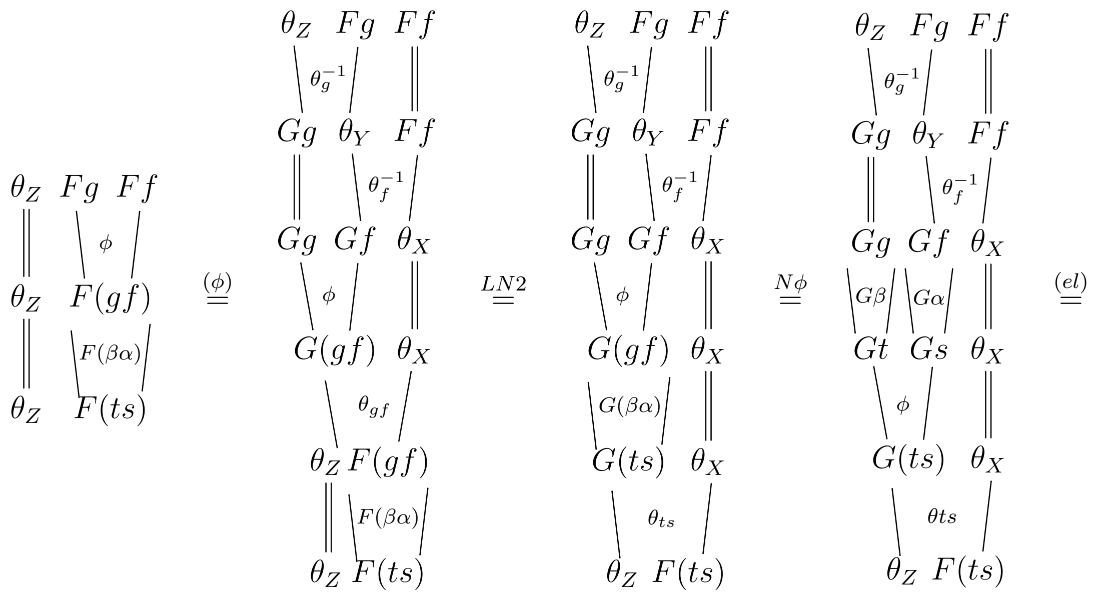
$\overset{LF3}{=}$

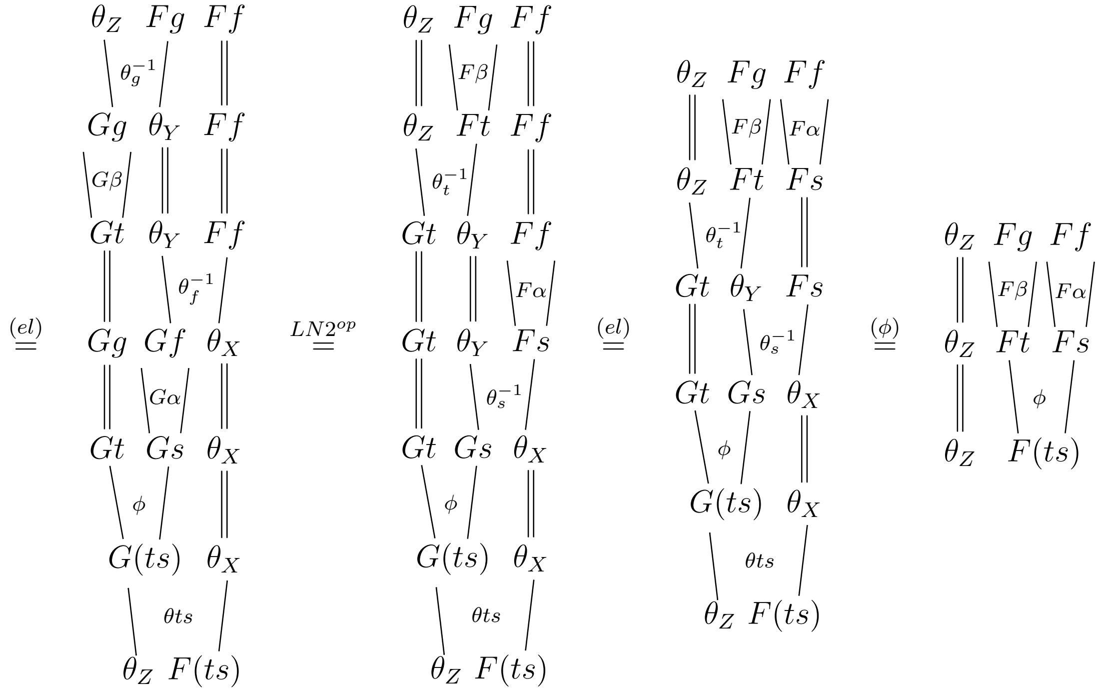


$\overset{(el)}{=}$



Proof of the naturality $N\phi$.





$\square$

We end this Appendix showing how Theorem A.7 follows from the results above. By Remark A.8 it is enough to prove item (bt). We consider then preassignments of bicategories $\mathcal{B} \overset{G}{\rightsquigarrow} \mathcal{D}$ and $\mathcal{B} \overset{F}{\rightsquigarrow} \mathcal{D}$ furnished with a pretransfer data $\theta : F \rightsquigarrow G$. If for each $X \in \mathcal{B}$, $FX \xrightarrow{\theta_X} GX$ is a quasiequivalence in $\mathcal{D}$, in particular we have

(a) $\mathcal{D}(W, FZ) \xrightarrow{(\theta_Z)_*} \mathcal{D}(W, GZ)$ is full and faithful for each W in the image of F

This implies assumption (a1) in Proposition A.10 which shows that we can assume F, G to be assignments and θ to be a transfer data (this is item (1) in Theorem A.7).

In turn, given now a lax-functor structure on G, we take $\Omega = \{\text{all 2-cells}\}$ in Theorem A.13. In this case assumption (a2) is exactly the same as (a), and in this way we have the unique lax-functor structure for F such that the transfer data forms a pseudonatural transformation $F \overset{\theta}{\Rightarrow} G$. For item (3) in Theorem A.7, note that if G is a pseudofunctor then we can take $\Omega =\{\text{invertible 2-cells}\}$ and assumption (a2) still holds.

By Proposition A.4, if for each $X \in \mathcal{B}$, $FX \xrightarrow{\theta_X} GX$ is a quasiequivalence, then so is θ as required. Finally, regarding item (2) in Theorem A.7, note that for the equivalence case we can use item 3. in 2.4 instead of Proposition A.4.