Math 2051: Problems in Geometry - Class Notes

Winter term 2022, Dalhousie University
Instructor: Martin Szyld.

This document is the recommended material for the course this term, and it is the one I will use for my lectures. It is based on the content I gave in Winter term last year. The main reference I used when preparing the course is the book [1] (one of the reasons for making these notes is that this book is out of print as I write this), as well as handwritten class notes by Professors Dorette Pronk and Bob Paré. Complementary material can also be found in the references.

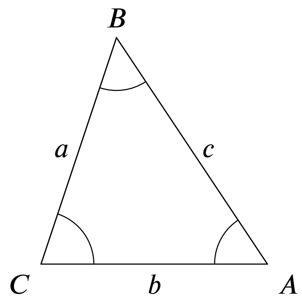
Part of the work the students did last year was to make class notes for the course out of a provided *sketch*, using the material from the lectures. I am grateful to the students Louis Bu, James Ryan, and Lareina Yang who let me use parts of their great work for this document. With that being said, mistakes and imperfections of these notes should be considered my own, and I will gratefully receive corrections and comments at `mszyld@dal.ca`.

Contents

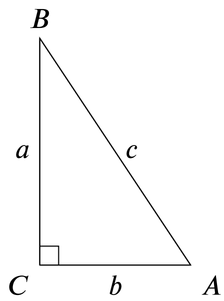
1 Basic Euclidean Geometry and Congruent Triangles

1.1 Informal and Formal Proofs, Assumptions, Converses

Notation 1.1. A triangle with vertices A, B, and C is denoted $\triangle ABC$. The length of the side opposite each vertex is denoted respectively a, b, and c. So $a = |BC|$, $b = |AC|$, and $c = |AB|$.

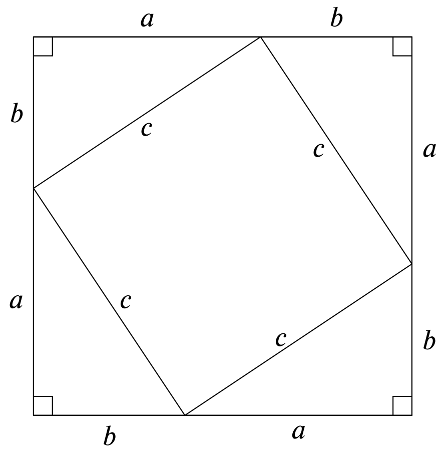


Theorem 1.2 (Pythagorean Theorem[1]). *For any triangle $\triangle ABC$, if the angle at C is a right angle, then $a^2 + b^2 = c^2$.*



Informal proof. Consider the following picture, in which we copy $\triangle ABC$ four times making a *big* square $\square$:

[1]For a nice bit of history, see for example [2]



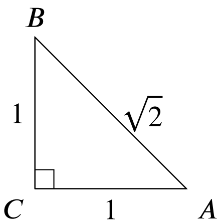
We first compute the area of the *big* square $\square$ by multiplying its sides: $Area(\square) = (a+b)^2 = a^2 + b^2 + 2ab$.

We can also compute it by adding the area of the *small* square and the four triangles: $Area(\square) = c^2 + 4(1/2)ab = c^2 + 2ab$.

We deduce: $a^2 + b^2 + 2ab = c^2 + 2ab$, and simplifying $a^2 + b^2 = c^2$ $\square$

Before moving on, here are some applications of Pythagorean Theorem. We certainly have the “real life" applications such as computing lengths in a Cartesian plane, measuring roofs, calculating lengths of ladders, which appear in elementary school problems (some of which can actually be quite funny in my opinion). There are other two which I want to look at a bit more closely here:

Corollary 1.3. *For any triangle $\triangle ABC$, if the angle at C is a right angle, and $a = b = 1$, then $c = \sqrt{2}$.*

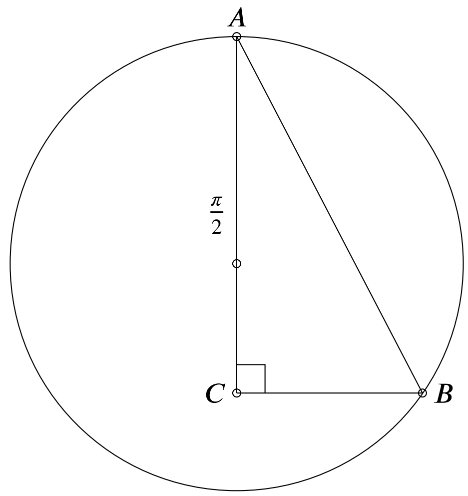


Why the *fuss* with this one? Well, history, since this is likely to have been the first discovery of an irrational number (for a nice poem accounting: [4]).

Here’s one more application of the Pythagorean Theorem, which will show what happens if we *change the rules of the game* later: squaring the circle.

Exercise 1.4. *In the figure, we have started with a circle of radius 1, and we marked in its diameter a point C such that $|AC|$ is the length of a quarter of the circumference*[2]*, that is $\frac{\pi}{2}$. Show that $|AB|^2 = \pi$.*

[2] This is much more fun to do with a piece of rope at the beach, as in [5], but unfortunately this is Winter term...



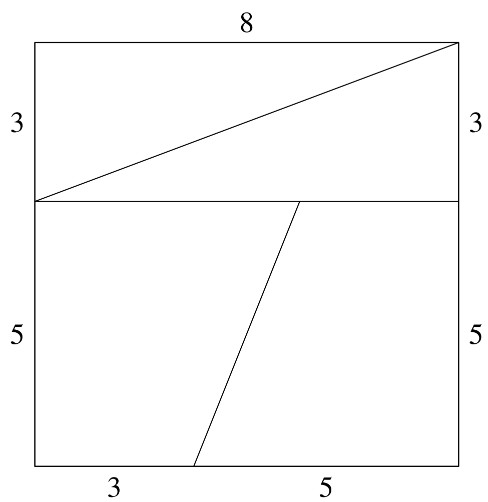
A simple hint which will be helpful throughout the course

If you're stuck, consider drawing one extra segment.

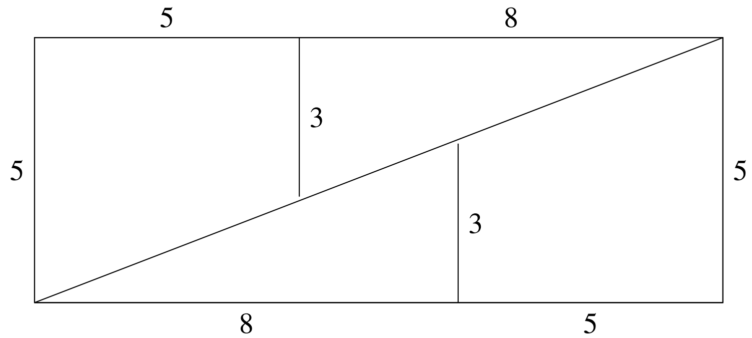
The following "false theorem" illustrates what happens when we blindly accept proofs by drawings. There were some hidden assumptions in our proof of Theorem 1.2, the main one was that the figures fitted to form that nice square. Consider this though (originally due to the genius Martin Gardner):

Theorem 1.5 (A *false* theorem)**.** $64 = 65$.

Informal proof. We consider a big square $\square$ formed as follows:



We rearrange the four pieces into a rectangle $\square$ as follows



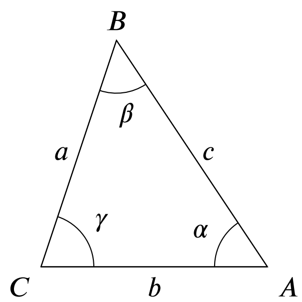
We then compute

$$64 = (3+5)^2 = Area(\square) = Area(\sqsubset\!\sqsupset) = (8+5) \times 5 = 65$$

$\square$

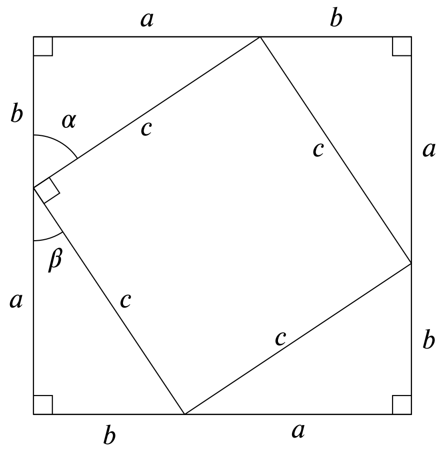
We will show that the difference between “Theorems" 1.2 and 1.5 can be observed by looking at the angles in the figures.

Notation 1.6. In a triangle $\triangle ABC$ as in Notation 1.1, the angle at each vertex is denoted respectively α, β, γ:



We also denote these angles by $\angle BAC = \angle A = \hat{A} = \alpha$, $\angle ABC = \angle B = \hat{B} = \beta$, $\angle BCA = \angle C = \hat{C} = \gamma$.

We look at the *informal proof* of Theorem 1.2 with angles marked:

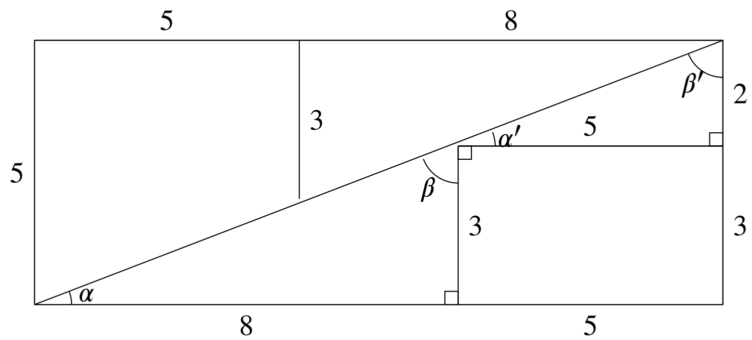


To *formalize* a proof, we need some assumptions. These are also called *axioms*. We *believe* them, or at least accept them as the *rules of the game*.

Let's **assume**[3]: right angles measure 90°, two right angles, that is 180°, form a straight line, and the three angles of any triangle add up to 180°.

We then have $\gamma = \hat{C} = 90°$, $\alpha + \beta + \gamma = 180°$. Then on the left side of the square we can verify that the three angles add up to 180°, that is a right line. The same happens for the other 3 sides of the big square. This is showing that the figures do fit to form a big square, with no overlapping and no empty spaces, so that the computation in the proof is correct.

We now look at the *informal proof* of Theorem 1.5, we draw an extra segment and we mark some angles:



For the diagonal in the big rectangle to be a straight line, we would need

$$(\star)\ \alpha' + 90° + \beta = 180°$$

[3]We will then see how these facts are true for Euclidean Geometry.

Since $\alpha + 90° + \beta = 180°$ (in the triangle on the left below), the equation ($\star$) is equivalent to $\alpha = \alpha'$. This in turn means that the two triangles with their angles marked are *equivalent*[4], so we would have $\frac{3}{8} = \frac{2}{5}$, which is just not true...

(we have shown that "the diagonal in the big rectangle is a straight line" if and only if "$\frac{3}{8} = \frac{2}{5}$", since this second statement is not true then the diagonal in the big rectangle is **not** a straight line)

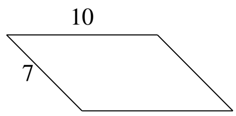
The problem with the informal proof is that the 4 pieces don't fit to form the rectangle ▭, there is an empty space in the middle that measures 1.

Takeaway

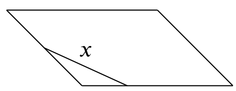
A problem with informal proofs is that *they don't distinguish what's true from what's false*. For a *formal*, or *real* proof of a theorem we need assumptions that we assume to be true, and we use them to reason and argue that the theorem is true.

We will now consider the converse of Pythagorean Theorem.

Exercise 1.7. *Imagine you have four pieces of wood, two of length* $7cm$ *and two of length* $10cm$ *that you want to build a rectangle with. You have attached the ends of these pieces of wood, and you lay them on the ground, but how can you make this parallelogram into a rectangle?*



You also have a ruler which allows to measure $1, 2, 3, 4,$ *and* $5cm$. *You're clever and you know that* $3^2 + 4^2 = 5^2$, *and Theorem 1.8. So you mark* $3cm$ *on one side,* $4cm$ *on another, and measure the* diagonal *segment:*



Do you think this segment x *will measure more or less than* $5cm$? *And if you had chosen another vertex? Rearranging the sides of the parallelogram so that* x *is* $5cm$ *will make it a rectangle (and here is where we are using Theorem 1.8!). A piece of trivia is that this is actually how carpenters make sure angles are right.*

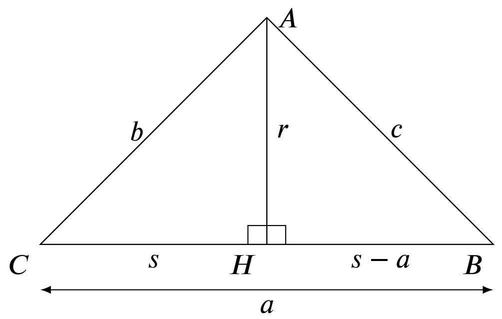
[4]This is a bit of *high-school geometry*, but we will also formalize these criteria for equivalent triangles in a few lectures...

Discussion

What is the “converse" of a theorem? Can you think of a result that is true but whose converse isn't? Can you think of a result that is true and so is its converse?

Theorem 1.8 (Converse of Pythagorean Theorem). *Let $\triangle ABC$ such that $a^2+b^2 = c^2$. Then $\angle ACB$ is a right angle.*

Proof. We draw $\overline{AH}$ such that $\overline{AH} \perp \overline{BC}$.



(1.9)

We use Theorem 1.2 (the Pythagorean Theorem) for the triangles $\triangle AHC$ and $\triangle AHB$. We get then

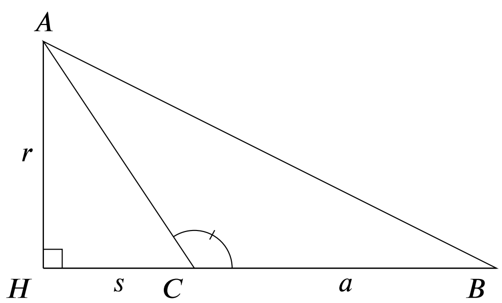
$$(A)\ r^2 + s^2 = b^2 \qquad\qquad (B)\ r^2 + (s-a)^2 = c^2$$

Equation (B) can be rewritten as $r^2 + a^2 + s^2 - 2as = c^2$. Subtracting (A) to this equation we get

$$r^2 + a^2 + s^2 - 2as - r^2 - s^2 = c^2 - b^2 = a^2$$

Simplifying, we have $a^2 - 2as = a^2$, so $as = 0$, then $s = 0$ which means that $H = C$. Since $\overline{AH} \perp \overline{BC}$, we get that $\overline{AC} \perp \overline{BC}$ so $\angle ACB$ is a right angle. $\square$

Exercise 1.10. *Note that the proof of Theorem 1.8 isn't entirely complete. The drawing in* (1.9) *assumes that $\angle C \leq 90°$ and $\angle B \leq 90°$. Write the proof in the case in which $\angle C > 90°$, using the following drawing*

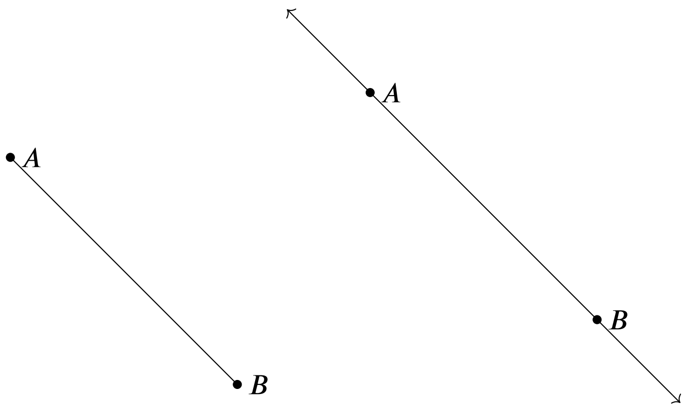


1.2 Axioms for Euclidean Geometry, Distances

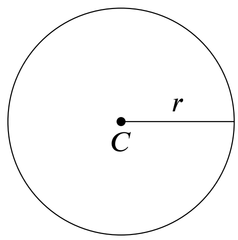
Recall that for proving results (theorems) we needed *assumptions* (can't get something from nothing right?). So if we assume nothing, we will not be able to prove anything. It then makes sense to start with some assumptions that we don't prove. I like to think of these as the *rules of the game*, in this case, the *rules of the Euclidean Geometry game*. These rules are called **axioms**. These are a (modern version of) Euclid's proposed axioms for his Geometry:

E1 We can draw a unique line segment between any two points.

E2 Any line segment can be continued indefinitely.



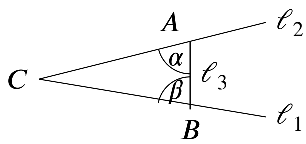
E3 A circle of any radius and any center can be drawn.



E4 Any two right angles are congruent.

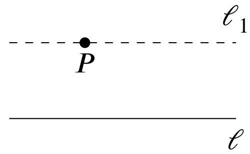
E5 Suppose a line ℓ meets two other lines, ℓ_1, ℓ_2, so that the sum of the two angles on one side is less than 2 right angles, then the two other lines meet at a point on that side.

$\alpha + \beta < 180^\circ \quad \rightsquigarrow$



We will also consider *Playfair's version* of axiom E5, that we will show later to be equivalent to it:

P5 Given a line ℓ and a point P not on ℓ, there exists a **unique** line ℓ_1 through P which does not intersect ℓ.



There is *quite a lot* we can do with just these axioms. We will assume them to be true, and we will call the set of points the Euclidean plane, that we will denote by $\mathcal{E}$.

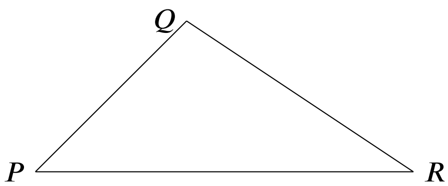
Discussion

Before starting though, can you think of a difference between the first 4 axioms and the fifth one? Do all these axioms hold in the 3-dimensional space we (think we) live in? Do they hold for the points inside a rectangle? And for the rational points of the plane?

We will now look at some notions which are implicitly part of the axioms. For example, what's the radius of a circle? For this to make sense, we could interpret it as the **distance** of all the points of the circle to its center.

Definition 1.11. A *distance* is a function that assigns a non-negative real number to any pair of points on the Euclidean plane (that is, $d : \mathcal{E} \times \mathcal{E} \to \mathbb{R}_{\geq 0}$), satisfying:

- for all points P and Q, $d(P,Q) = d(Q,P)$;
- for all points P and Q, $d(P,Q) = 0 \iff P = Q$; and
- for all points P, Q, and R, $d(P,R) \leq d(P,Q) + d(Q,R)$

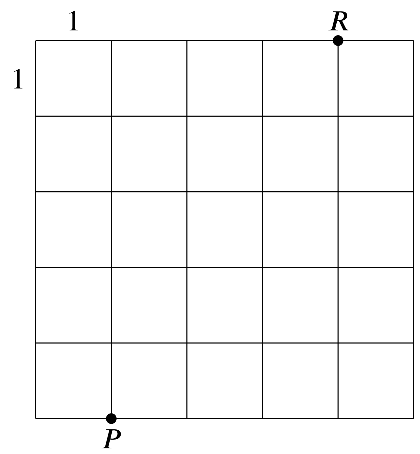


We will usually think of the Cartesian plane $\mathbb{R}^2$ and the usual distance

$$d(a,b)(c,d) = \sqrt{(c-a)^2 + (d-b)^2}$$

(once again, Pythagoras!). This is a *model* for Euclidean geometry (that is, roughly, a construction of a situation where all the axioms hold).

Example 1.12. There are many examples of *distances*, that is ways to assign non-negative real numbers to pairs of things. For example, if we were in a city whose streets form a grid:

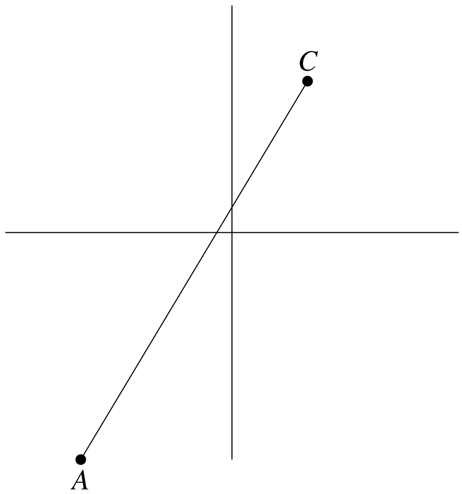


How would you define the distance between two points? What's the distance between P and R?

The equation $d(P, R) \leq d(P, Q) + d(Q, R)$ is called the *triangular inequality*. This allows us to think of a segment as the shortest path between two points.

Notation 1.13. We denote $d(P, Q) = |PQ|$, that is the length of the segment PQ.

Exercise 1.14. **1.** *In the Cartesian plane* $\mathbb{R}^2$*, show that for all points* $A = (a_1, a_2)$*,* $B = (b_1, b_2)$*, and* $C = (c_1, c_2)$*,*
$d(A, C) = d(A, B) + d(B, C)$ *if and only if* B *is in the segment from* A *to* C*:*



2. *In the* grid *of Example 1.12, what is the set of points* Q *such that* $d(P, R) = d(P, Q) + d(Q, R)$ *?*

1.3 Congruence, Isometries, and the SSS, SAS, ASA Criteria

Another notion appearing in the axioms is *congruent* angles. We want to consider when two angles, two segments, two shapes in the plane are congruent. We will define this using a notion of **isometry**, a function that preserves distance:

Definition 1.15. An *isometry* is a function $f : \mathcal{E} \to \mathcal{E}$, that is a way of assigning a new point $f(P)$ to each point P, such that for every pair of points P, Q we have

$$d(f(P), f(Q)) = f(P, Q)$$

We will define shapes and angles to be congruent when there is an isometry mapping one to the other (see Definitions 1.19, 1.20).

Exercise 1.16. *In the Cartesian plane $\mathbb{R}^2$, with the usual distance,*

1. *Let $A = (2, 4)$. show that $f : \mathbb{R}^2 \to \mathbb{R}^2$, defined by*

$$f(x_1, x_2) = (x_1, x_2) + (2, 4) = (x_1 + 2, x_2 + 4)$$

is an isometry. Compute $f(0, 0)$ and $f(A)$. What is this isometry doing*?*

2. *Is $F : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $F(x_1, x_2) = (2x_1 + 2, x_2 + 4)$, an isometry?*

3. *Show that $g : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $g(x_1, x_2) = (-x_1, x_2)$ is an isometry. Compute $g(0, 0)$, $g(1, 1)$, and $g(1, 0)$. What is this isometry* doing*?*

4. *Show that $h : \mathbb{R}^2 \to \mathbb{R}^2$, defined by $h(x_1, x_2) = (-x_2, x_1)$ is an isometry. Compute $h(0, 0)$, $h(1, 1)$, and $h(1, 0)$. What is this isometry* doing*?*

Proposition 1.17.

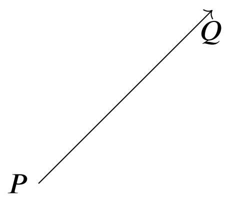
- *$id : \mathcal{E} \to \mathcal{E}$, $id(P) = P$ is an isometry.*
- *If $f, g : \mathcal{E} \to \mathcal{E}$ are isometries then their composition, $g \circ f : \mathcal{E} \to \mathcal{E}$, $g \circ f(P) = g(f(P))$ is an isometry.*

Proof. Exercise (that we may do in class). □

We will now consider some axioms on the existence of isometries. As motivation, for each axiom, we will show that this is satisfied in the Cartesian plane.

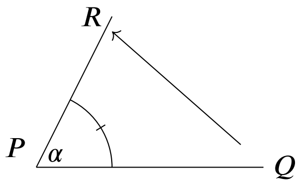
IM1 For each pair of points P, Q, there exists an isometry $f : \mathcal{E} \to \mathcal{E}$ such that $f(P) = Q$.

- Example: a translation as in item 1 of Exercise 1.16, τ: $\mathbb{R}^2 \to \mathbb{R}^2$ defined by $\tau(X) = X + (Q - P)$ for each $X \in \mathbb{R}^2$



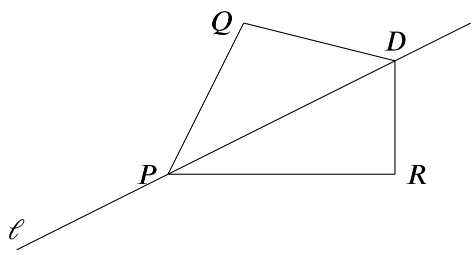
$$\tau(P) = P + Q - P = Q$$

IM2 For all points P, Q, R, if $|PQ| = |PQ|$ then there exists an isometry $f : \mathcal{E} \to \mathcal{E}$ such that $f(P) = P$, $f(Q) = R$

- Example: rotation, symmetry as as in items 3 and 4 of Exercise 1.16. Rotate $Q \curvearrowright R$ around the point P: 

IM3 For each line ℓ, there exists an isometry $f : \mathcal{E} \to \mathcal{E}$ such that $f(P) = P$ for each $P \in \ell$ but $f(Q) \neq Q$ for each $Q \notin \ell$.

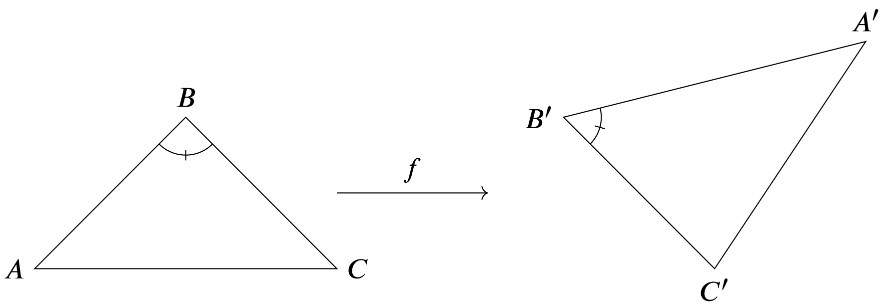
- Example: symmetry as in item 3 of Exercise 1.16. Note that given ℓ, $P \in \ell$ and $Q \notin \ell$, if we denote $f(Q) = R$ and D is another point in ℓ then $\measuredangle QPD = \measuredangle RPD$ and $|PR| = |PQ|$. Also, note that these two conditions *determine* R, that is, R is the only point different to Q such that those two equations hold. This is showing that the isometry f in axiom IM3 is the *reflection* with respect to ℓ, mapping $\triangle$PDQ to its reflection $\triangle$PDR.



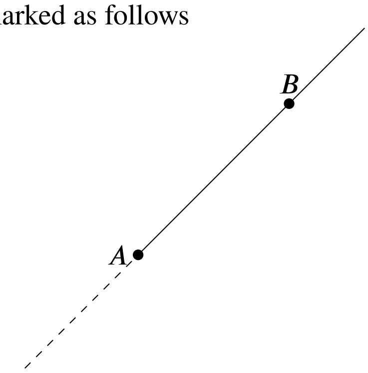
(1.18)

These axioms are *enough* to prove the SSS, SAS, and ASA criteria for congruence of triangles.

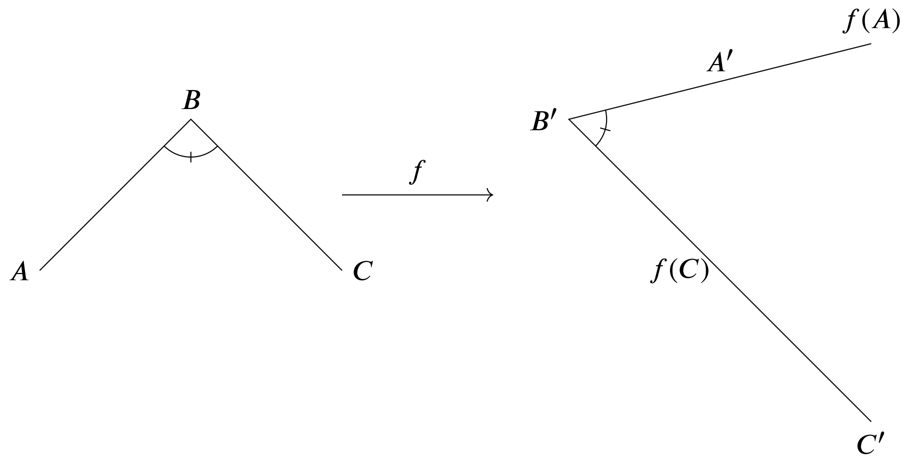
Definition 1.19. We say that two triangles $\triangle ABC$, $\triangle A'B'C'$ are congruent if there exists an isometry: $f : \mathcal{E} \to \mathcal{E}$ such that $f(A) = A'$, $f(B) = B'$, $f(C) = C'$. We write $\triangle ABC \cong \triangle A'B'C'$.



Given two points A, B, we have the *half-line* $\overrightarrow{AB}$, the *solid part* of the line $\overleftrightarrow{AB}$ marked as follows



Definition 1.20. We say that two angles $\angle ABC$, $\angle A'B'C'$ are congruent if there exists an isometry: $f : \mathcal{E} \to \mathcal{E}$ such that $f(B) = B'$, $f(A)$ is in $\overrightarrow{B'A'}$, and $f(C)$ is in $\overrightarrow{B'C'}$. We write $\angle ABC \cong \angle A'B'C'$ or $\measuredangle ABC \cong \measuredangle A'B'C'$.



Note that, by definition, we have $\angle ABC \cong \angle(f(A))(f(B))(f(C))$.

Remark 1.21. If $\triangle ABC \cong \triangle A'B'C'$, then we have $|AB| = |A'B'|$, $|BC| = |B'C'|$, $|CA| = |C'A'|$, $\angle ABC \cong \angle A'B'C'$, $\angle BCA \cong \angle B'C'A'$, and $\angle CAB \cong \angle C'A'B'$. The following theorems show a sort of converse, that some of these six conditions are enough to show that $\triangle ABC \cong \triangle A'B'C'$.

Theorem 1.22 (SAS)**.** *If two sides and the contained angle of a triangle are congruent to two sides and the contained angle of the other, then these triangles are congruent.*

A lemma is a result that is convenient to show separately from a theorem. To show Theorem 1.22, we will use a basic construction that we show as a lemma:

Lemma 1.23. *[Basic construction] Given points $A, A', B,$ and B', if $d(A, B) = d(A', B')$ then exists an isometry f such that $f(A) = A'$, $f(B) = B'$.*

Discussion 1

As in Definitions 1.19 and 1.20, we could define $\overline{AB} \cong \overline{A'B'}$ when there exists an isometry f such that $f(A) = A', f(B) = B'$. Note that then $AB \cong A'B'$ implies $|AB| = |A'B'|$, and this Lemma is showing the converse implication.

Discussion 2

Before proving this lemma, let's think how we would construct such an isometry. How would this go in $\mathbb{R}^2$? Can we do this *only using the axioms* IM1, IM2, and IM3?

Proof. Step 1: Using Axiom IM1, we get an isometry f_1 such that $f_1(A) = A'$. Note that

$$d(f_1(B), A') = d(f_1(B), f_1(A)) = d(B, A) = d(B', A'),$$

then we can do

Step 2: Using Axiom IM2, we get an isometry f_2 such that $f_2(A') = A'$ and $f_2(f_1(B)) = B'$.

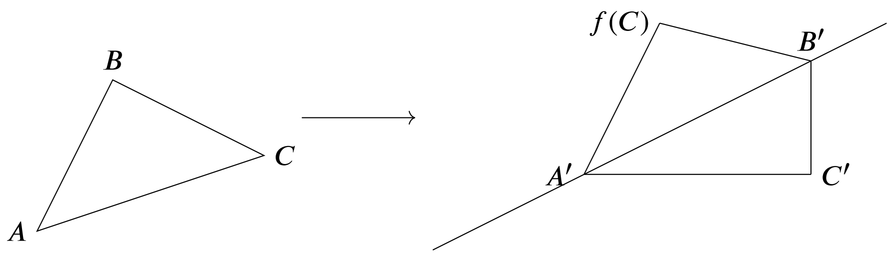
We can now define $f = f_2 \circ f_1$. Then $f(A) = f_2(f_1(A)) = f_2(A') = A'$ and $f(B) = f_2(f_1(B)) = B'$ as desired. $\square$

We will use this basic construction to prove Theorem 1.22:

Proof of Theorem 1.22. Let A, B, C, A', B', C' such that $|AB| = |A'B'|$, $|AC| = |A'C'|$, and $\measuredangle CAB = \measuredangle C'A'B'$. We have to construct an isometry g such that $g(A) = A'$, $g(B) = B'$, and $g(C) = C'$. By Lemma 1.23, there exists an isometry f such that $f(A) = A'$ and $f(B) = B'$. If $f(C) = C'$, then we are done (we just take $g = f$). Otherwise, we observe:

- $\measuredangle(f(C))A'B' = \measuredangle(f(C))(f(A))(f(B)) = \measuredangle CAB = \measuredangle C'A'B'$.
- $d(f(C), A') = d(f(C), f(A)) = d(C, A) = d(C', A')$.

This is showing that the points are set as in the diagram (1.18):



So using axiom IM3, with respect to the line $\overleftrightarrow{A'B'}$ we get h such that $h(A') = A'$, $h(B') = B'$, and $h(f(C)) = C'$. We then define $g = h \circ f$, and we have as desired

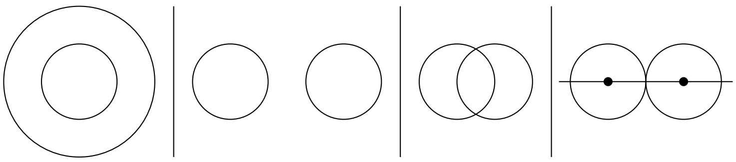
$$\begin{aligned} g(A) &= h(f(A)) = h(A') = A' \\ g(B) &= h(f(B)) = h(B') = B' \\ g(C) &= h(f(C)) = C' \end{aligned}$$

□

Theorem 1.24. *[SSS] If two triangles satisfy that their three sides have equal lengths, then they are congruent.*

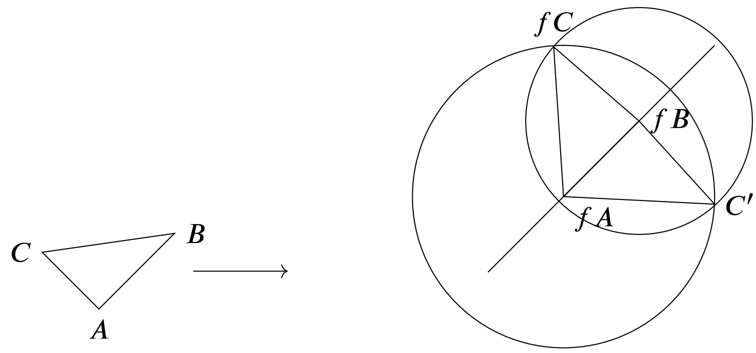
Lemma 1.25. *Two different circles cannot have more than two intersections. Furthermore if they have just one intersection point, then it lies in the line joining the 2 centres.*

This lemma is, in the words of A. Baragar [1], "a very believable lemma with a rather difficult proof". Here's a visual explanation of how it's very believable, we won't consider the proof for now:



The proof of SSS is similar to the proof of SAS, using Lemma 1.25 instead of (1.18) for the final part. We give it in full details below. There is also a similar proof of the third criteria, ASA, that we leave as an exercise.

Proof of Theorem 1.24. Let A, B, C, A', B', C' such that $|AB| = |A'B'|$, $|AC| = |A'C'|$, and $|BC| = |B'C'|$. We have to construct an isometry g such that $g(A) = A'$, $g(B) = B'$, and $g(C) = C'$. We first use the basic construction in Lemma 1.23: since $|AB| = |A'B'|$, there exists an isometry f such that $f(A) = A'$ and $f(B) = B'$. If $f(C) = C'$, then we are done (we just take $g = f$). Otherwise, like in the proof of Theorem 1.22, we will find h such that $h(A') = A'$, $h(B') = B'$, and $h(f(C)) = C'$ and then we can define $g = h \circ f$. Here too h will be obtained by applying axiom IM3 with respect to the line $\overleftrightarrow{A'B'}$, but why will $h(f(C)) = C'$? This is the only remaining thing to show:



We draw a circle C_1 with center $A' = f(A)$ and radius $|A'C'| = |A'f(C)|$, and we draw a circle C_2 with center $B' = f(B)$ and radius $|B'C'| = |B'f(C)|$.

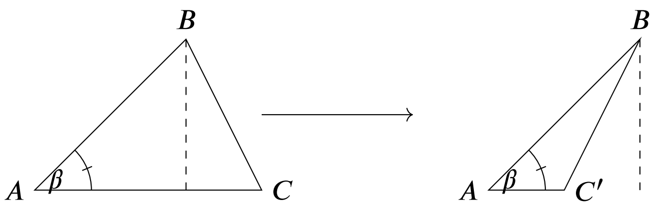
$C_1 = \{\delta \text{ such that } d(A', \delta) = d(A', C')\}, \quad C_2 = \{\delta \text{ s.t. } d(B', \delta) = d(B', C')\}.$

Note that both $f(C)$ and C' are points in $C_1 \cap C_2$. Now, for each point $\delta \in C_1 \cap C_2$, let us show that $h(\delta)$ is still in $C_1 \cap C_2$. This is so because:
d(h(δ), A') = d(h(δ), h(A')) = d(δ, A') = d(A', C') and similarly
d(h(δ), B') = d(h(δ), h(B')) = d(δ, B') = d(B', C').

Since $f(C)$ and C' are two different points in $C_1 \cap C_2$, $h(f(C))$ is also a point in $C_1 \cap C_2$ by the computation above, and $h(f(C)) \neq f(C)$ by IM3, then by Lemma 1.25 the only possibility remaining is $h(f(C)) = C'$. $\square$

Exercise 1.26. *Prove the ASA-congruence result for triangles: if $\triangle ABC$ and $\triangle A'B'C'$ satisfy that $\measuredangle ABC = \measuredangle A'B'C'$, $|AB| = |A'B'|$, and $\measuredangle BAC = \measuredangle B'A'C'$, then there is an isometry f such that $f(A) = A'$, $f(B) = B'$, $f(C) = C'$.*

Remark 1.27. Is there a theorem SSA? No, there may be two possible different configurations. For example, take the following triangle



Reflecting with respect to the height of the triangle maps C to C', so $\triangle ABC$ and $\triangle ABC'$ are two non-congruent triangles with two congruent sides and the same angle $\angle BAC$.

1.4 Axioms E5 and P5

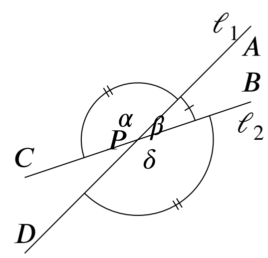
In this section we will show that axioms E5 and P5 are equivalent.

Discussion

What do you think of these two axioms? Here we will introduce the problem of infinity and we will meet Gracie, the infinite cow [3]. We will also mention the relevance of this axiom and the fact that you can also work without assuming it (non-Euclidean geometries).

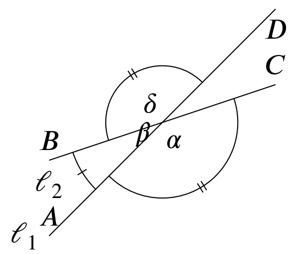
What does it mean to show that two axioms are equivalent? How do we show this? Why is this useful?

Definition 1.28. Two lines that meet at a point determine four angles:



The angles α and β are called adjacent, and they add up to $180°$ by definition (we define $180°$ as the measure of a straight angle formed by any point in any line) the angles α and δ are called opposite.

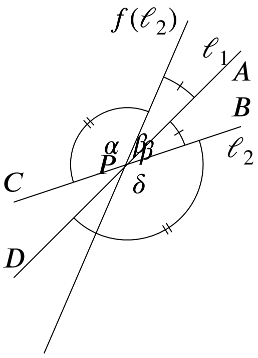
Remark 1.29. Opposite angles are congruent. We can give two proofs: one is observing that $\alpha + \beta = 180° = \delta + \beta$, so $\alpha = \delta$. For the other, we rotate the drawing $180°$:



Definition 1.30. ℓ_1 is perpendicular to ℓ_2 when two adjacent angles are equal. We denote $\ell_1 \perp \ell_2$

(note that in this case the four angles are equal to $180°/2 = 90°$, that we may define as *right angles*).

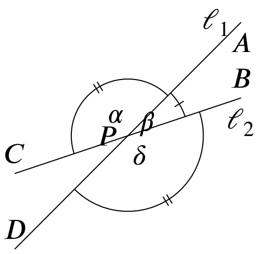
Remark 1.31. What happens above if instead of rotating we use the reflection f from axiom IM3 with respect to ℓ_1?



In general, $f(\ell_2)$ will not be ℓ_2. $f(\ell_2)$ will be ℓ_2 precisely when $\ell_1 \perp \ell_2$.

Proposition 1.32. *ℓ_1 is perpendicular to ℓ_2 if and only if the reflection with respect to ℓ_1 maps the points of ℓ_2 to ℓ_2. For this to happen, it is enough to check that a point of ℓ_2 that in not in ℓ_1 is mapped to ℓ_2.*

Proof. Let ℓ_1 and ℓ_2 meet and determine adjacent angles α and β, and let f be the reflection with respect to ℓ_1



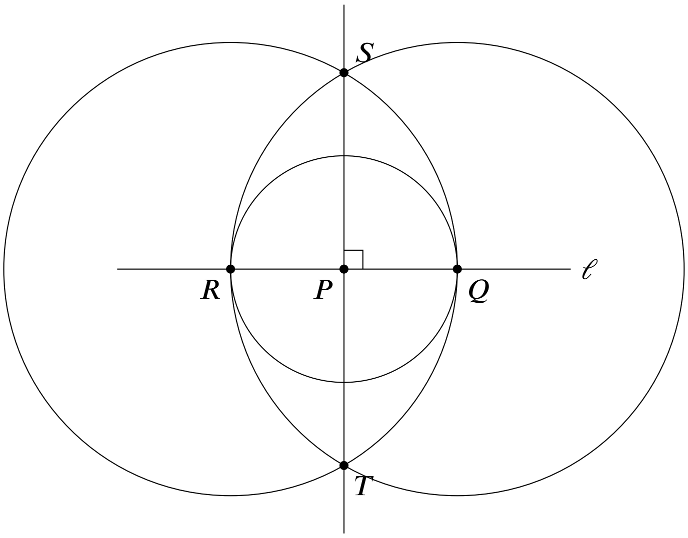
$\Rightarrow$: For any point C in ℓ_2, $\angle f(C)PA = \angle f(C)f(P)f(A) = \angle CPA = 90°$, so $f(C)$ is also in ℓ_2.

$\Leftarrow$: If a point C in ℓ_2 is mapped to a point B in ℓ_2, then $\alpha = \angle CPA = \angle f(C)f(P)f(A) = \angle BPA = \beta$.

□

Construction 1.33. *Given a line ℓ and a point P in it, we can construct the perpendicular line to ℓ through P.*

Proof. We will show **how** to construct this line and **why** it is perpendicular.

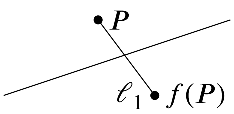


We first mark Q$\in \ell$, Q $\neq$ P. We then draw a circle C with centre P and radius $|PQ|$; and denote by R the point in $\ell \cap$ C. Finally we draw circles C_1 with centre R and radius $|RQ|$ and circle C_2 with centre Q and radius also $|RQ|$. We denote by S and T the two points in $C_1 \cap C_2$.

Now, in the proof of Theorem 1.24 we have shown that in this case the reflection of S with respect to ℓ is T. By Proposition 1.32, we get that $\overleftrightarrow{SP} \perp \ell$ as desired. $\square$

Construction 1.34. *Given a line ℓ and a point P* **not** *in it, we can construct the perpendicular line to ℓ through P.*

Proof. Given ℓ a line and $P \notin \ell$, we take f, the reflection with respect to ℓ, and define ℓ_1 as the line through P and $f(P)$:



By Proposition 1.32, ℓ_1 is perpendicular to ℓ. $\square$

Which of the two implications of Proposition 1.32 are we using in this proof?

Recall axiom P5:

P5 Given a line ℓ and a point P not on ℓ, there exists a **unique** line ℓ_1 through P which does not intersect ℓ.

We will now show how Constructions 1.33 and 1.34 can be *combined* to construct the line ℓ_1 appearing in this axiom. Note that we will not use axiom E5 for this construction (it will be used for showing the *unicity* in the statement of P5.

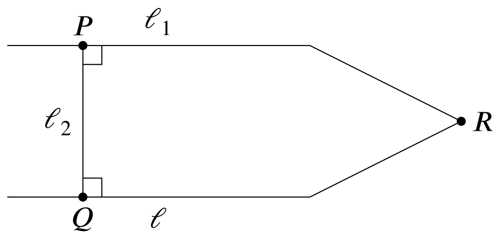
Definition 1.35. Two lines are called parallel if they are the same or if they have no intersection.

Lemma 1.36. *Given lines ℓ, ℓ_1, and ℓ_2, if $\ell_1 \perp \ell_2 \perp \ell$ then ℓ_1 is parallel to ℓ*

Proof. We will show that if ℓ and ℓ_1 meet at a point, then they are the same line.

Why is this enough to prove the Lemma?

So let us assume that $\ell \cap \ell_1$ is a point that we denote R.



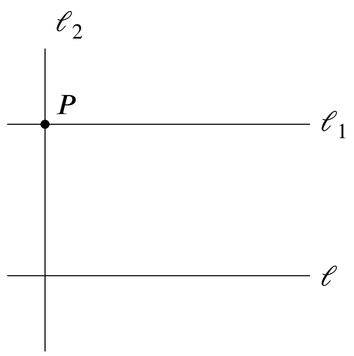
We consider now the reflection f with respect to ℓ_2. Since $\ell \perp \ell_2$ and $\ell_1 \perp \ell_2$, then f maps the points of ℓ to points of ℓ and maps the points of ℓ_1 to points of ℓ_1.

Which of the two implications of Proposition 1.32 are we using in this proof?

In particular, f(R) still $\in \ell \cap \ell_1$. so f(R) = R, this means that R $\in \ell_2$ (f is the reflection with respect to ℓ_2, so if $f(R) = R$, that means that $R \in \ell_2$). So we have R $\in \ell_2 \cap \ell_1 \cap \ell$, $\ell_1 \perp \ell_2$, $\ell \perp \ell_2$, but perpendicular lines to ℓ_2 through R $\in \ell_2$ are unique! Then $\ell = \ell_1$. $\square$

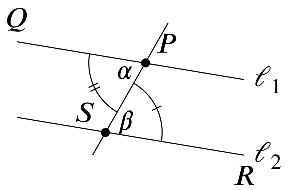
We can now construct the line in axiom P5:

Remark 1.37. Given a point P which is not in a line ℓ we can apply first Construction 1.34 and then Construction 1.33 to get $\ell_1 \perp \ell_2 \perp \ell$, with P in ℓ_1 and P in ℓ_2. By Lemma 1.36, ℓ_1 is parallel to ℓ.



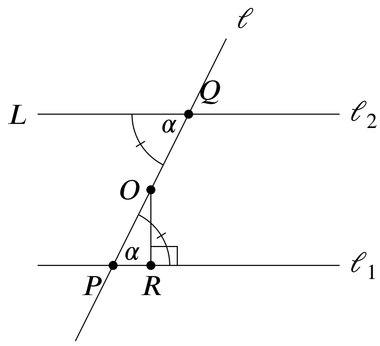
We will now show the equivalence E5 $\Longleftrightarrow$ Unicity in P5. For this we will use the notion of opposite interior angles. Warning: Sometimes these are called alternate interior angles

Definition 1.38. Given lines ℓ, ℓ_1, and ℓ_2 such that ℓ meets ℓ_1 and ℓ_2 the angles α and β in the figure below are called opposite interior angles.

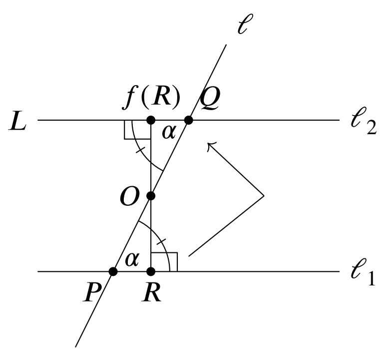


Corollary 1.39 (of Lemma 1.36)**.** *If ℓ meets ℓ_1, ℓ_2 in a way such that opposite interior angles are congruent, then ℓ_1 is parallel to ℓ_2.*

Proof. Let $P = \ell_1 \cap \ell$, $Q = \ell_2 \cap \ell$, and mark O the midpoint of the segment PQ. We use Construction 1.34, we can then mark $R \in \ell_1$ such that $\overline{OR} \perp \ell_1$.



We rotate 180° with center O, let's call this rotation f. Then f(P)=Q, and R goes to a point $f(R)$ that we will show is in ℓ_2.



Indeed, since

$$\measuredangle(f(R), Q, O) = \measuredangle(f(R), f(P), f(O)) = \measuredangle(R, P, O) = \alpha = \measuredangle(\ell, \ell_2),$$

$f(R) \in \ell_2$. Also $\measuredangle$(O,f(R),Q) = $\measuredangle$(f(O),f(R),f(P)) = $\measuredangle$(O,R,P) = 90°. So, if we take ℓ_3 the line containing R and $f(R)$, we then have $\ell_1 \perp \ell_3 \perp \ell_2$, and by Lemma 1.36 it follows that $\ell_1 \parallel \ell_2$. □

Theorem 1.40. *Axioms P5 and E5 are equivalent.*

Remember...

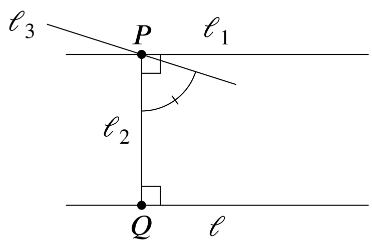
Anytime that we have to show that two statements are equivalent (an "if and only if"), we have to show two implications. In fact, when doing exercises and assignments, I suggest you write two arrows ⇒ and ⇐ so you don't forget one.

Proof. **P5 ⇐ E5:** Let ℓ and $P \notin \ell$, in Remark 1.37 we have shown that there exists an arrow ℓ_1 parallel to ℓ.

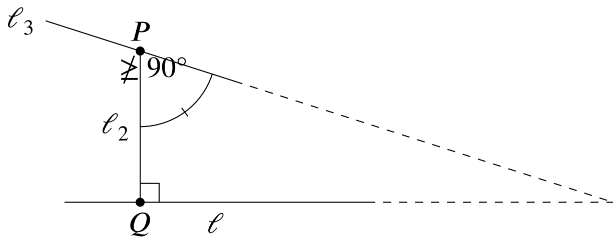
Discussion

It *only* remains to check the unicity of ℓ_1. What does unicity mean? How do we show unicity? One way is: "Give me a different line (ℓ_3) satisfying these properties (P $\in \ell_3$, $\ell_3 \parallel \ell$), and I'll show you a contradiction"

Recall that in Remark 1.37 we have constructed $\ell \perp \ell_2 \perp \ell_1$, $P \in \ell_2 \cap \ell_1$. Now, if ℓ_3 is a line that is different to ℓ_1 and satisfies P $\in \ell_3$, then $\measuredangle(\ell_3, \ell_2) \neq 90°$ (because there is only one perpendicular line to ℓ_2 through a point $P \in \ell_2$, and that line is ℓ_1).

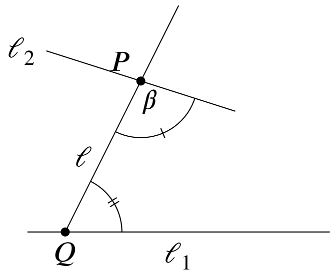


Then one of the angles formed by ℓ and ℓ_3 is less then 90°, so we can apply axiom E5 to these 3 lines and we get that ℓ_3 and ℓ meet:

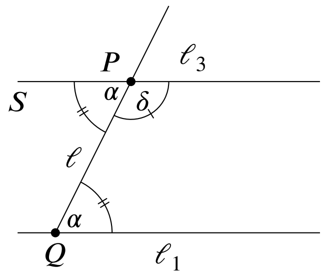


But if $\ell_3 \parallel \ell$, $P \in \ell_3$, and $P \notin \ell$, these two lines can't meet (why?). So this is a contradiction.

P5 ⇒ E5: We now have to show axiom E5, so we let ℓ, ℓ_1, and ℓ_2 meet such that the angles between them satisfy $\alpha + \beta < 180°$, and we have to show that ℓ_1 and ℓ_2 also meet (then they will certainly meet on that side):



We draw ℓ_3 such that $\measuredangle(\ell, \ell_3) = \alpha$, $P \in \ell_3$. Note that $\delta = 180° - \alpha \neq \beta$, so ℓ_2 and ℓ_3 are different lines.



(1.41)

We now use Corollary 1.39: since opposite interior angles are equal then ℓ_1 is parallel to ℓ_3. We can then use axiom P5 (more precisely, the *unicity*) to conclude that ℓ_2 is not parallel to ℓ_1, and so they meet. □

Theorem 1.40 is showing that, if we accept axioms E1 to E4, then E5 and P5 are two different *versions* of the same axiom. In what comes next, we will always assume these axioms. This final axiom is the one that distinguishes *Euclidean* geometry from the others. For more, see [1], and we may revisit this too during the course.

Remark 1.42. Note that in (1.41) we constructed a parallel line (ℓ_3 parallel to ℓ_1) such that the opposite interior angles are congruent, and we didn't use ℓ_2 for this. Since parallels through P are unique, then the converse Corollary 1.39 also holds:

Proposition 1.43. *If two lines are parallel then the opposite interior angles that are formed with a third line are congruent.*

Proof. Exercise to practice the same ideas appearing above. $\square$

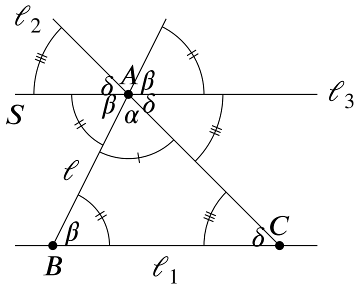
Putting Corollary 1.39 and Proposition 1.43 together:

Theorem 1.44. *Let a line ℓ meet ℓ_1 and ℓ_2. The lines ℓ_1 and ℓ_2 are parallel if and only if the opposite interior angles are congruent.* $\square$

We finish this chapter with two classical results of Euclidean geometry

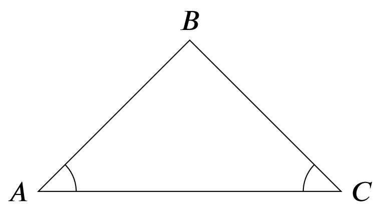
Theorem 1.45. *The sum of the angles of a triangle is 180 degrees.*

Proof. Let ΔABC be a triangle, let ℓ be the parallel line to $\overline{BC}$ through A.



By Proposition 1.43 and Remark 1.29, the three angles marked β are congruent, and so are the three angles marked δ. Since they form a straight angle, we then have that $\gamma, \alpha + \beta + \gamma = 180°$ as desired. $\square$

Definition 1.46. A triangle is isosceles if two of its sides are congruent. In such a triangle $\triangle ABC$, if $|AB| = |BC|$, then the angles at A and at C are called the *base angles*.



Theorem 1.47 (Pons asinorum)**.** *In an isosceles triangle, the base angles are equal.*

Proof. Let A, B, C such that $d(A, B) = d(A, C)$, then from the three equalities

$$|\overline{AB}| = |\overline{AC}|, \qquad |\overline{AC}| = |\overline{AB}|, \qquad |\overline{BC}| = |\overline{CB}|$$

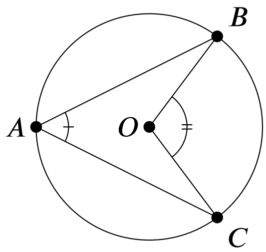
by Theorem 1.24 (SSS) we have $\Delta ABC \cong \Delta ACB$. By Remark 1.21, we have in particular $\angle ABC = \angle ACB$. $\square$

2 Geometry with Circles and Similar Triangles

2.1 Star Trek and Bow Tie

Notation 2.1. Given a point O and a length r, we denote by $\mathscr{C}(0, r)$ the circle of center O and radius r, that is the set of points P such that $d(P, O) = r$.

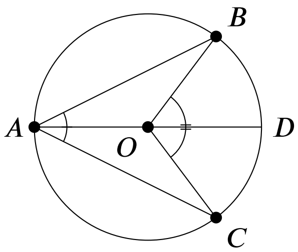
Definition 2.2. Let $\mathscr{C} = \mathscr{C}(0, r)$. Three points $A, B, C \in \mathscr{C}$ determine an *inscribed angle* $\measuredangle$ BAC. This angle *subtends* an arc $\widehat{BC}$: this means that these 3 points also determine the arc $\widehat{BC}$ (the part of the circle going from B to C, with A not in it).



The arc $\widehat{BC}$ has an *angular measure* defined as the measure of $\measuredangle$ BOC.

Lemma 2.3 (Star Trek Lemma)**.** *The measure of an inscribed angle is half of the angular measure of the arc it subtends.*

Proof. This is the case in which O is *inside* the angle:



Let A, B, C form an inscribed angle $\measuredangle BAC$ in a circle $\mathscr{C} = \mathscr{C}(O, r)$. We have to show that $2\measuredangle$ BAC $= \measuredangle$ BOC. We draw $\overline{AO}$, extend to a line ℓ , and denote $D = \ell \cap \mathscr{C}$.

Note that $|AO| = |BO| = |DO| = |CO|$ since they are all radius of $\mathscr{C}$. Then, by Theorem 1.47 (Pons Asinorum) in $\triangle$ AOB, $\measuredangle$ OAB $= \measuredangle$ OBA. We then have

$$\measuredangle BOD = 180^\circ - \measuredangle BOA = \measuredangle OBA + \measuredangle OAB = 2\measuredangle OAB$$

(in the equality in the middle we are using Theorem 1.45, i.e. that the angles in the triangle $\triangle AOB$ add up to 180°).

A similar reasoning with $\triangle AOC$ instead of $\triangle AOB$ shows that

$$\measuredangle COD = \qquad\qquad\qquad\qquad = 2\measuredangle OAC$$

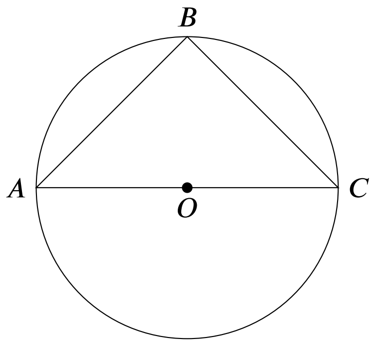
Adding the two equations, we get

$$\measuredangle BOC = \measuredangle BOD + \measuredangle COD = 2\measuredangle OAB + 2\measuredangle OAC = 2\measuredangle BAC$$

so $\measuredangle BAC = \dfrac{1}{2}\measuredangle BOC$ as desired. □

Exercise 2.4. *Write the proof of Lemma 2.3 case when O is* outside *the angle.*

Remark 2.5 (Star Trek for diameters)**.** If A, B, C are in a circle $\mathscr{C}$, then $\measuredangle ABC = 90° \iff \measuredangle AOC = 180° \iff AC$ is a diameter.



Consequence of the Star Trek Lemma:

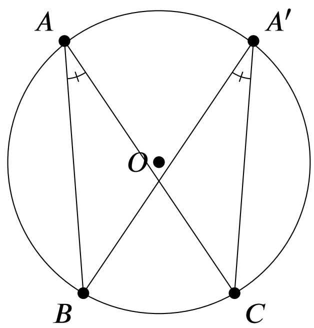
The measure of $\measuredangle BAC$ does not depend on A! That is, for two different choices A and A', we have $\measuredangle BAC = \measuredangle BA'C$:

Lemma 2.6 (Bow Tie Lemma)**.** *If two inscribed angles in a circle subtend the same arc, then they are congruent.*

More explicitly: if A, A', B, C are in a circle $\mathscr{C}$ as in the figure below, then $\measuredangle BA'C = \measuredangle BAC$

Proof. The angles are congruent because by Lemma 2.3 the measure of each of them is half of the angular measure of the arc it subtends.

More explicitly, if we have the figure



$$\measuredangle BA'C = \frac{1}{2}\measuredangle BOC = \measuredangle BAC$$

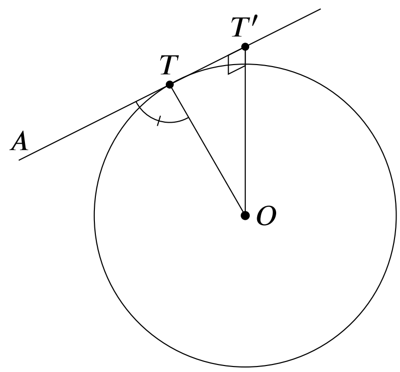
$\square$

Here we will look at some exercises where we can apply the previous results.

Definition 2.7. A line ℓ is tangent to a circle $\mathscr{C}$ when $\ell \cap \mathscr{C}$ consists of a single point

Lemma 2.8 (Tangents are perpendicular to radius)**.** *If ℓ is tangent to $\mathscr{C}(O, r)$, $\ell \cap \mathscr{C} = \{T\}$, and A is another point in ℓ, then $\angle ATO$ is right.*

Proof. This is a proof by contradiction. If $\measuredangle ATO \neq 90°$, then we can draw the perpendicular line to AT passing through O, intersecting AT at T'.



Since ℓ is tangent to $\mathscr{C}$, then T' is outside the circle $\mathscr{C}$, so $|OT'| > |OT|$. However, by the Pythagorean Theorem, $|OT'|^2 + |TT'|^2 = |OT|^2$, so $|OT| > |OT'|$ and we have a contradiction.

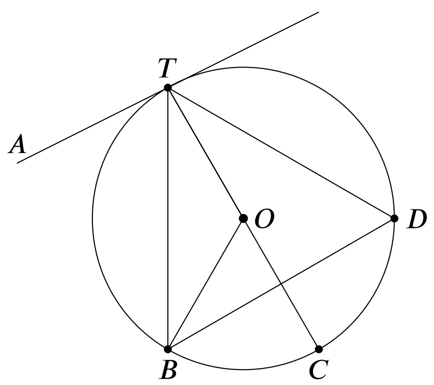
$\square$

Proposition 2.9 (Tangential Star Trek)**.** *If AT is tangent to a circle, B another point in the circle, then $\measuredangle ATB = \frac{1}{2}\measuredangle TOB$.*

Discussion before the proof

Why is this proposition called Tangential Star Trek?

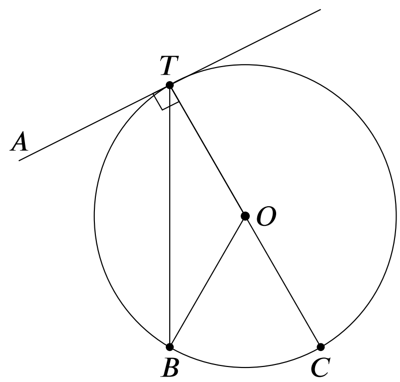
Consider any other point D in the circle.



Recall that Star Trek states that $\measuredangle TDB = \frac{1}{2}\measuredangle TOB$ ($\measuredangle TDB$ is *half the measure of the arc $\overset{\frown}{TB}$ which it subtends*). In Tangential Star Trek, we say that A, T, and B also subtend the arc $\overset{\frown}{TB}$, and just like for Star Trek, $\measuredangle ATB$ is *half the measure of the arc $\overset{\frown}{TB}$ which it subtends*.

Also, note that we are trying to prove $\measuredangle ATB = \frac{1}{2}\measuredangle TOB$, but we know that $\frac{1}{2}\measuredangle TOB = \measuredangle TDB$ for any choice of D (remember the Important consequence of Star Trek from last lecture?). So, to prove Proposition 2.9 it will be enough to show that $\measuredangle ATB = \measuredangle TDB$ for any choice of D. Since we can consider the D which is the most convenient point for us, we will extend TO, and consider C its intersection with $\mathscr{C}$ to play this role.

Proof. Recall from Lemma 2.8 that $\measuredangle ATO = 90°$, extend TO, and consider C its intersection with $\mathscr{C}$.



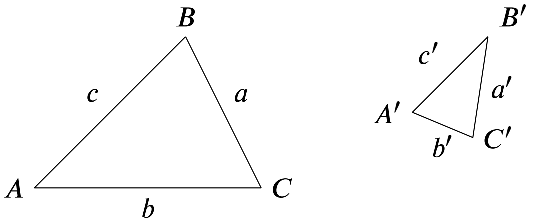
By Remark 2.5 (Star Trek for diameters), $\measuredangle TBC$ is right, so since the angles of $\triangle TBC$ add up to $180°$ we have $\measuredangle BTC + \measuredangle TCB = 90°$. We conclude that

$$\measuredangle ATB = 90° - \measuredangle BTC = \measuredangle TCB = \frac{1}{2}\measuredangle TOB$$

(the last equality holds by the Star Trek Lemma for T, B, and C; this is the same reasoning we did before the proof but with $D = C$) $\square$

2.2 Similar Triangles

Informally, similar triangles can be mapped one to the other by using not only an isometry (like congruent triangles) but with a more general "widening" (aka homeothetic function). Note how this doesn't preserve lengths, but preserves the *ratios* between them.



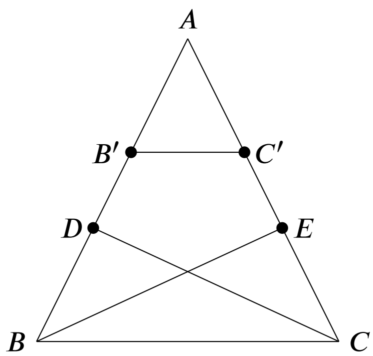
There ratios ($\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$) are telling us how bigger or smaller $\triangle ABC$ is with respect to $\triangle A'B'C'$.

The following theorem is fundamental for dealing with ratios like the ones above. We will define triangles to be similar when their angles are equal. In this case, we will be able to use an isometry to make them fit the situation in this theorem.

Theorem 2.10 (internal name given in 2021: "Onion Theorem"[5])**.** *For any triangle $\triangle ABC$, given points B' on the side AB and C' on the side AC such that $B'C'$ is parallel to BC, then*

$$\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$$

Proof. We will consider areas of various triangles computed using different heights.



[5]The different equations that we encounter and that we have to show to prove the theorem appear one after the other like layers of an onion. Also, I guess anything that made online teaching a bit more amusing was welcome!

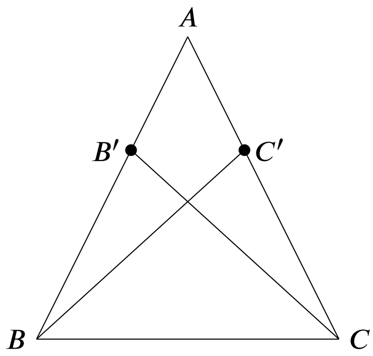
We consider $D \in AB$ such that $CD \perp AB$, and $E \in AC$ such that $BE \perp AC$. Therefore we have:

$$\frac{|\Delta AB'C|}{|\Delta ABC|} = \frac{\frac{1}{2}|AB'| \cdot |DC|}{\frac{1}{2}|AB| \cdot |DC|} = \frac{|AB'|}{|AB|} = (1)$$

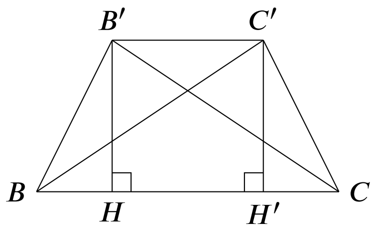
$$\frac{|\Delta ABC'|}{|\Delta ABC|} = \frac{\frac{1}{2}|AC'| \cdot |EB|}{\frac{1}{2}|AC| \cdot |EB|} = \frac{|AC'|}{|AC|} = (2)$$

We *just* have to show now that $|\Delta AB'C| \overset{\star}{=} |\Delta ABC'|$ and then we will have (1) = (2) as desired. We note first that

$$|\Delta AB'C| = |\Delta ABC| - |\Delta BB'C| \qquad \text{and} \qquad |\Delta ABC'| = |\Delta ABC| - |\Delta BC'C|,$$



so we may show that $|\Delta BB'C| = |\Delta BC'C|$ and we will have $\star$. To show this, note that both triangles have BC as a side (*base*), and since $BC \| B'C'$, both triangles have heights with the same length:



We deduce $|\Delta BB'C| = \frac{1}{2}|BC| \cdot |B'H| = \frac{1}{2}|BC| \cdot |C'H'| = |\Delta BC'C|$, so the equality $\star$ holds and then (1) = (2) as desired, proving the *Onion Theorem.* □

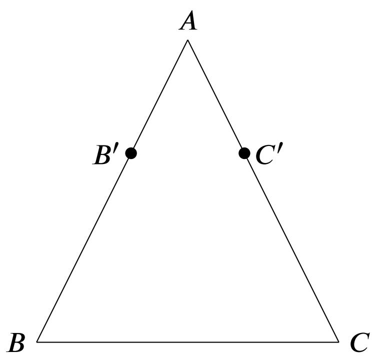
The other *ingredient* that we will use for showing the criteria for similar triangles is the converse of the *Onion Theorem.* This will allow us to have a statement of the form “two triangles are congruent **if and only if**..."

Theorem 2.11 (converse of Onion Theorem)**.** *For any triangle $\triangle ABC$, given points B' on the side AB and C' on the side AC such that $\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$, we have that $B'C'$ is parallel to BC.*

A trick for proving converses: "construct the point"

This is a trick that is usual in geometry for proving the converse $B \Rightarrow A$ of a result $A \Rightarrow B$: we construct the point satisfying A and we show that (since $A \Rightarrow B$) it's equal to the point satisfying B.

Proof. We draw the parallel line to BC at B', say that it intersects AC at C'', then $B'C''||BC$ and we want to show that $C' = C''$.



We know that $\frac{|AB'|}{|AB|} = \frac{|AC'|}{|AC|}$, and by the onion theorem (applied to C''), we know that $\frac{|AB'|}{|AB|} = \frac{|AC''|}{|AC|}$.

Therefore, we conclude that $|AC'| = |AC''|$ and since $C', C'' \in AC$, then $C' = C''$ and $B'C'$ is parallel to BC. □

Definition 2.12. Two triangles are similar if their angles are congruent. We write $\triangle ABC \sim \triangle A'B'C'$, this means precisely that $\angle ABC \cong \angle A'B'C'$, $\angle BCA \cong \angle B'C'A'$, and $\angle CAB \cong \angle C'A'B'$.

Remark 2.13. Since the angles of a triangle add up to 180°, it is always enough to show that 2 of the angles of the triangles are congruent (if 2 of the angles are congruent, the 3rd is automatically congruent and the triangles are similar).

Recall Remark 1.21, two triangles are congruent when in addition of having congruent angles they have congruent lengths. So we have:

Remark 2.14.
- $\triangle ABC \cong \triangle A'B'C' \implies \triangle ABC \sim \triangle A'B'C'$ (**congruent implies similar**).
- For any isometry f, we have $\triangle ABC \cong \triangle f(A)f(B)f(C)$, then in particular $\triangle ABC \sim \triangle f(A)f(B)f(C)$
- being congruent and being similar are both "transitive relations" between triangles: $\triangle ABC \cong \triangle A'B'C' \cong \triangle A''B''C'' \implies \triangle ABC \cong \triangle A''B''C''$, and the same happens replacing the three $\cong$ by $\sim$.

The importance of the Onion Theorem and its converse is that they allow to show the following “Basic similarity trick" or “basic trick" that we will use for proving the criteria for similar triangles (and more!).

As we mentioned before, having equivalences is useful because we can show any of the statements and get any of the others. We will do this many times using this basic trick.

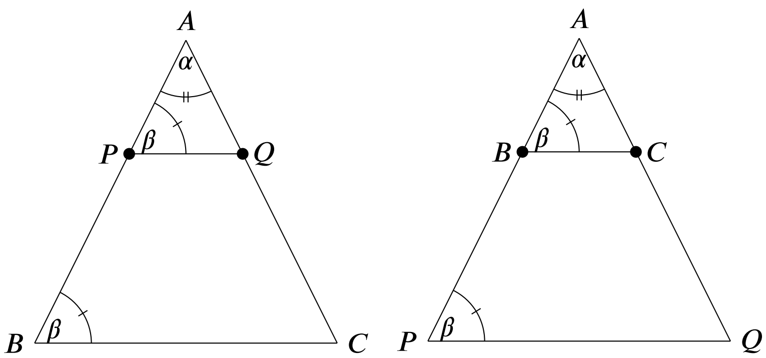
Lemma 2.15 (“Basic similarity trick" or “basic trick"). *Let $\triangle ABC$, $P \in \overrightarrow{AB}$, $Q \in \overrightarrow{AC}$, then the following statements are all equivalent:*

$$\frac{|AP|}{|AB|} = \frac{|AQ|}{|AC|} \iff PQ \| BC \iff \measuredangle APQ = \measuredangle ABC \iff \triangle APQ \sim \triangle ABC$$

Discussion

Is having two points P $\in \overrightarrow{AB}$, Q $\in \overrightarrow{AC}$ here the same as having two points B' on the side AB and C' on the side AC like in the Onion Theorem? Which situation is *more general*?

Proof. Note that if any of the four conditions hold then the points have to be set as in one of the two figures below (we can't have for example P betweeen A and B, and C between A and Q simultaneously, but you should check and convince yourselves of this).



The first $\iff$ is then precisely the onion theorem and its converse (this should be quite clear for the diagram on the left, do you see it for the one on the right too?). The second $\iff$ is given by Theorem 1.44 and Remark 1.29, and the third is given by Remark 2.13. $\square$

Note that, using basically just the Onion Theorem, we have shown a *more general* result.

The first criteria for similar triangles is given by the following proposition and its converse:

Proposition 2.16. *If* $\triangle ABC \sim \triangle A'B'C'$, *then*

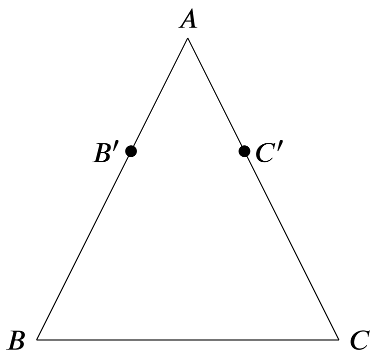
$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|B'C'|}{|BC|}$$

Some applications

Simple as it may seem, the previous proposition is of fundamental importance in geometry. For its applications, what matters is that checking $\triangle ABC \sim \triangle A'B'C'$ is *just* a matter of checking angles (by definition), and using one of the equalities above we may compute one of the lengths if we know the other two. Equally importantly, this allows to define the trigonometric functions (we will come back to this).

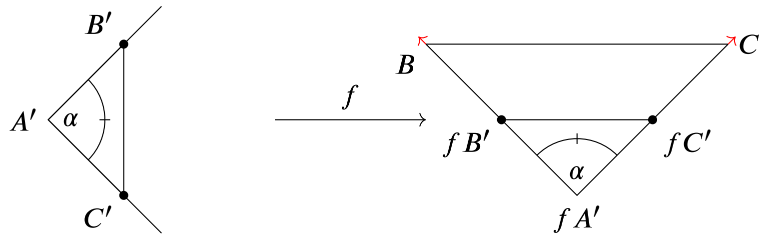
Example 2.17. This one is quite well-known but we will revisit it: “Measuring the height of the Great Pyramid with a stick". Depending on the source, this was done by waiting until the shadow was as long as the stick, or without waiting.

Example 2.18. This one is also a classic. Since the sun and the moon have similar sizes when we look at them from the point A on Earth where we are, we can imagine two other points on the moon giving a diameter and two other points on the sun giving a diameter also fitting the situation of the basic trick and giving two similar triangles, so $\frac{D_M}{R_M} = \frac{D_S}{R_S}$.



We will later (maybe as an assignment) use this equation and others to compute these numbers, much like (we believe) the Ancient Greeks did.

Proof of Proposition 2.16. Let $\triangle ABC \sim \triangle A'B'C'$, so by definition their three angles are congruent. Recall the Definition 1.20 of congruent angles: $\measuredangle BAC = \measuredangle B'A'C'$ meant that there is an isometry f such that $f(A') = A$, $f(B')$ is in the ray $\overrightarrow{AB}$ and $f(C')$ is in the ray $\overrightarrow{AC}$



Note that the 5 points on the right fit the situation of the "basic trick" Lemma 2.15. By the second bullet point in Remark 2.14, we have $\triangle ABC \sim \triangle f(A')f(B')f(C')$, which is the fourth equivalence in the "basic trick" Lemma, so we also have the first:

$$\frac{|f(A')f(B')|}{|AB|} = \frac{|f(A')f(C')|}{|AC|}$$

and since f is an isometry:

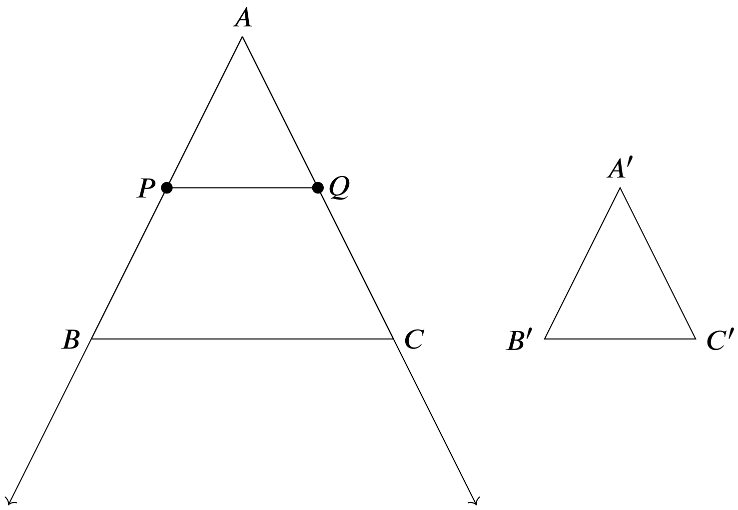
$$\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}$$

This is the first equality we had to show. For the other equality, we do the same using C instead of A and the same reasoning as above (Exercise: write this explicitly if you're not convinced). $\square$

We will now how the converse of Proposition 2.16. Remember the trick for showing converses?

Proposition 2.19. *If* $\dfrac{|A'B'|}{|AB|} = \dfrac{|A'C'|}{|AC|} = \dfrac{|B'C'|}{|BC|}$ *, then* $\triangle ABC \sim \triangle A'B'C'$.

Proof. Let six points satisfy $\dfrac{|A'B'|}{|AB|} = \dfrac{|A'C'|}{|AC|} = \dfrac{|B'C'|}{|BC|}$. We will construct the points P, Q to fit the "basic trick" Lemma and use Proposition 2.16.



We mark $P \in \overrightarrow{AB}$ such that $d(A', B') = d(A, P)$ (note that P may or may not be in $\overline{AB}$ as drawn in the left, it is possible for $P \in \overrightarrow{AB}$, $P \notin \overline{AB}$ but the basic trick Lemma still applies). And we mark $Q \in \overrightarrow{AC}$ such that $d(A', C') = d(A, Q)$ (again, Q may or may not be in $\overline{AC}$). Now, since

$$\frac{|AP|}{|AB|} = \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|AQ|}{|AC|},$$

by the basic trick we know that $\triangle APQ \sim \triangle ABC$. Then, by Proposition 2.16 we have

$$\frac{|PQ|}{|BC|} = \frac{|AP|}{|AB|} = \frac{|A'B'|}{|AB|} = \frac{|B'C'|}{|BC|}$$

Therefore $|PQ| = |B'C'|$, and we can apply SSS (for congruent triangles, Theorem 1.24): we get that $\triangle A'B'C' \cong \triangle APQ$. Since as explained in Remark 2.14 congruence implies similarity, and similarity is transitive:

$$\triangle A'B'C' \sim \triangle APQ \sim \triangle ABC \implies \triangle ABC \sim \triangle A'B'C'.$$

□

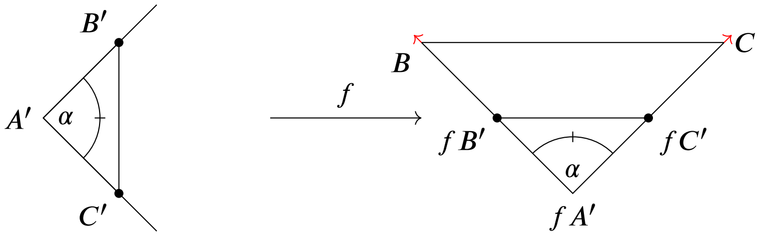
Putting both propositions together, we get an equivalence known as *SSS for similar triangles*:

Theorem 2.20 (SSS for similar triangles)**.** *For two triangles $\triangle ABC$ and $\triangle A'B'C'$, $\dfrac{|A'B'|}{|AB|} = \dfrac{|A'C'|}{|AC|} = \dfrac{|B'C'|}{|BC|}$ if and only if $\triangle ABC \sim \triangle A'B'C'$.*

Note that this theorem is telling us that similar triangles not only have congruent angles, as defined, but also proportional 3 sides. We also have a *SAS for similar triangles* stating that it's enough to check "proportional 2 sides and the angle between them". The proof is (again!) using the basic trick:

Theorem 2.21. *For two triangles $\triangle ABC$ and $\triangle A'B'C'$, if $\angle BAC = \angle B'A'C'$ and $\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}$, then $\triangle ABC \sim \triangle A'B'C'$*

Proof. We start just like the proof of Proposition 2.16 (this is really a copy and paste!): $\measuredangle BAC = \measuredangle B'A'C'$ means that there is an isometry f such that $f(A') = A$, $f(B')$ is in the ray $\overrightarrow{AB}$ and $f(C')$ is in the ray $\overrightarrow{AC}$



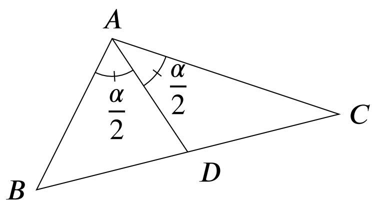
Note that the 5 points on the right fit the situation of the "basic trick" Lemma 2.15. But this time our hypothesis $\frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|}$ gives us the first of the equivalent conditions in the lemma:

$$\frac{|Af(B')|}{|AB|} = \frac{|f(A')f(B')|}{|AB|} = \frac{|A'B'|}{|AB|} = \frac{|A'C'|}{|AC|} = \frac{|f(A')f(C')|}{|AC|} = \frac{|Af(C')|}{|AC|}$$

We conclude that $\triangle ABC \sim \triangle f(A')f(B')f(C')$, and since $\triangle f(A')f(B')f(C') \sim \triangle A'B'C'$ we have $\triangle ABC \sim \triangle A'B'C'$ as desired. $\square$

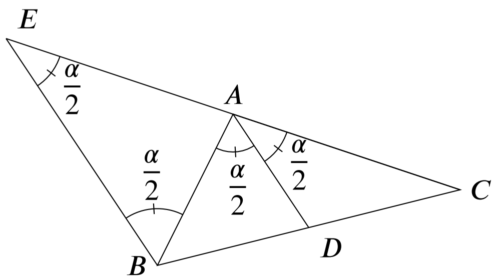
Sometimes we don't have the situation of a known result, but we can construct new lines to make it fit. In the proof of the following theorem we will do this, we will be constructing new lines to make it fit the "basic trick" Lemma 2.15.

Theorem 2.22 (Angle bisector theorem)**.** *For any triangle $\triangle ABC$, if the interior angle bisector at A intersects the side BC at D, then $\frac{|BD|}{|DC|} = \frac{|AB|}{|AC|}$*



Proof. $\measuredangle$ BAC = α, $\measuredangle$ BAD = $\measuredangle$ BAC = $\frac{\alpha}{2}$ (because AD bisects $\measuredangle$ BAC)

First we extend $\overrightarrow{CA}$, draw the parallel to DA through B, the two lines will meet at E.



Since BE || DA, $\measuredangle$ EBA = $\measuredangle$ BAD = $\frac{\alpha}{2}$ (opposite interior angles), and $\measuredangle$ BEA = $\measuredangle$ DAC = $\frac{\alpha}{2}$ (corresponding angles).

Also since BE || DA, by "basic trick" Lemma 2.15 $\frac{|DC|}{|BC|} = \frac{|AC|}{|EC|}$, which is the same as $\frac{|DC|}{|AC|} = \frac{|BC|}{|EC|}$. Now it's *just algebra*:

$$\frac{|DC|}{|AC|} = \frac{|BC|}{|EC|} = \frac{|BD| + |DC|}{|EA| + |AC|} = \frac{|BD| + |DC|}{|AB| + |AC|},$$

where the last equality holds by Pons Asinorum Theorem 1.47.

So $\frac{|AB| + |AC|}{|AC|} = \frac{|BD| + |DC|}{|DC|}$, that is, $\frac{|AB|}{|AC|} + 1 = \frac{|BD|}{|DC|} + 1$, so $\frac{|AB|}{|AC|} = \frac{|BD|}{|DC|}$ as desired. □

2.3 Power of a Point

This is the last result involving a point and a circle that we will need before starting with *geometry of triangles*.

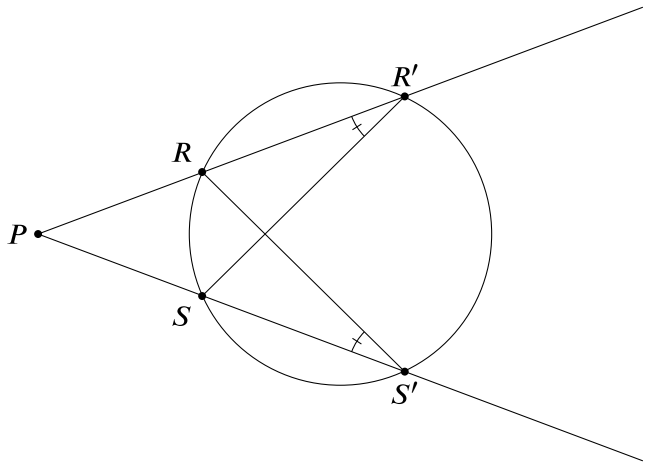
Theorem 2.23. *for P a point and a circle, given two lines that intersect at P and that each intersect the circle at R, R', and S, S' respectively ,*

$$|PR||PR'| = |PS||PS'|.$$

A typed proof is below, but I think we may first try to do this all together in class.

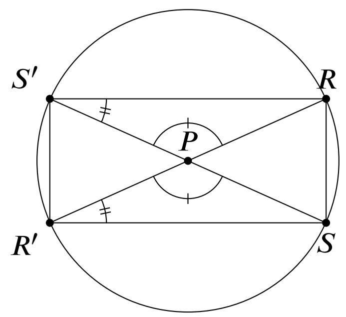
Proof. **Case 1:**, P outside the circle, $R \neq R', S \neq S'$.

Let's show $\triangle PR'S \sim \triangle PS'R$:



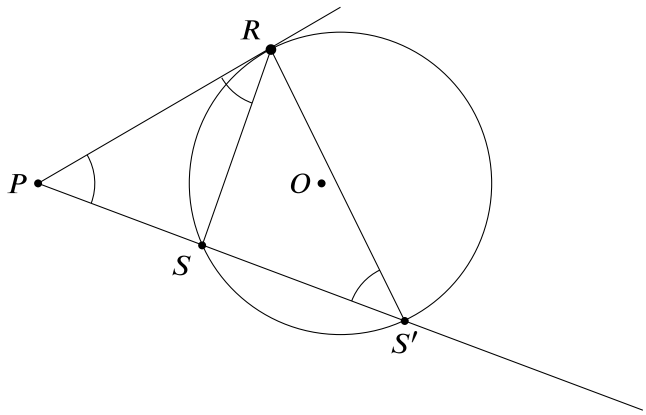
We have that $\measuredangle R'PS = \measuredangle S'PR$, and $\measuredangle PR'S = \measuredangle PS'R$ (by the Bow Tie Lemma 2.6). By Remark 2.13 (it suffices to show 2 congruences of angles to prove similarity), we have proven $\triangle PR'S \sim \triangle PS'R$ which by Theorem 2.20 implies $|PR'|/|PS'| = |PS|/|PR|$ and then $|PR||PR'| = |PS||PS'|$.

Case 2: P inside the circle.



Once again, we will show $\triangle PR'S \sim \triangle PS'R$: we have $\measuredangle R'PS = \measuredangle S'PR$ (by Remark 1.29, opposite angles are congruent) and $\measuredangle PR'S = \measuredangle PS'R$ (again by Bow Tie Lemma). Just like in case 1, we have then $\triangle PR'S \sim \triangle PS'R$ which implies $|PR||PR'| = |PS||PS'|$.

Case 3: P outside the circle, $S \neq S'$ but $R = R'$ (that is, PR is tangent to the circle):



In this third case, we will also show $\triangle PR'S \sim \triangle PS'R$, except that since $R' = R$ we will show $\triangle PRS \sim \triangle PS'R$.

We consider O the center of the circle. We have $\measuredangle RPS = \measuredangle S'PR$, and also

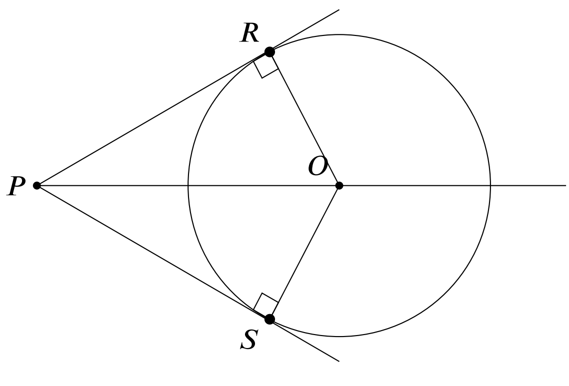
$$\measuredangle PRS = \frac{1}{2}\measuredangle ROS = \measuredangle RS'S = \measuredangle PS'R \tag{2.24}$$

where the first equality holds by tangential Star Trek Proposition 2.9, and the second one by Star Trek Lemma 2.3 [6]. We have then $\triangle PRS \sim \triangle PS'R$ by Remark 2.13.

In this case, just as in the other two cases but taking $R = R'$, $\triangle PRS \sim \triangle PS'R$ implies $|PR||PR| = |PS||PS'|$ as desired.

Case 4: P outside the circle, $R = R'$, and $S = S'$. We will give two proofs:

1st Proof:



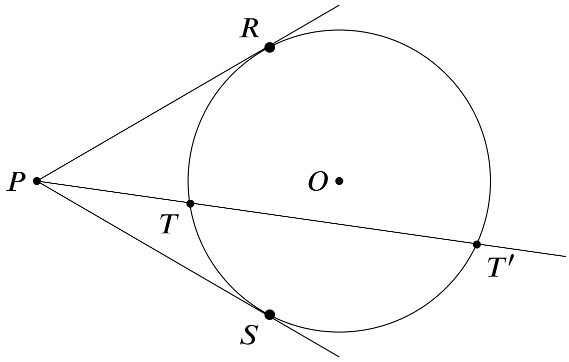
We know by Lemma 2.8 that tangents are perpendicular to the radius of a circle, so $\measuredangle PSO = \measuredangle PRO = 90°$. Also $|RO| = |SO|$ since they are both radius. Then by the Pythagorean Theorem

$$|PR|^2 = |PO|^2 - |RO|^2 = |PO|^2 - |SO|^2 = |PS|^2$$

[6] this is the same reasoning that we did before, and that can be found in these notes below Proposition 2.9. The equalities there are $\measuredangle ATB = \frac{1}{2}\measuredangle TOB = \measuredangle TDB$, this is just like the first two equalities in (2.24) but with different letters.

2nd Proof: In this proof we *reduce the 4th case to the 3rd case*.

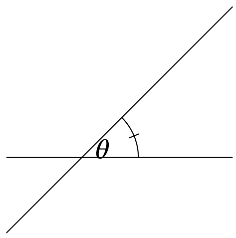
We draw another line such that it intersects the circle at two points, T, T' (it can pass through the center O or not, that's not relevant).



By the 3rd case (twice), $|PR|^2 = |PT||PT'| = |PS|^2$ as desired. $\square$

2.4 Circumcircle and Law of Sines

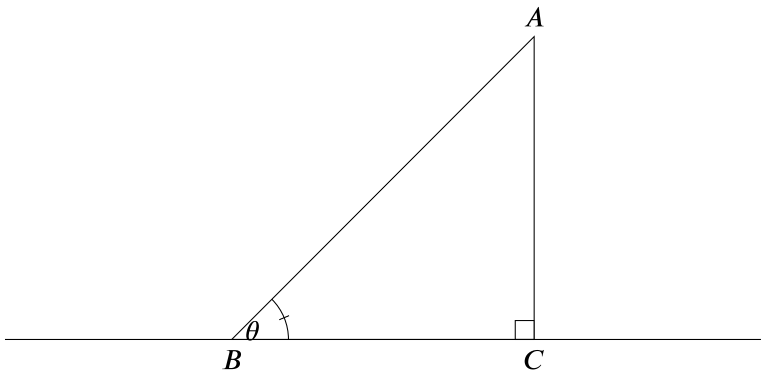
Consider θ an angle given by 2 lines.



Discussion

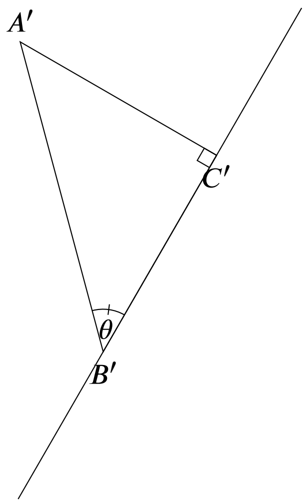
What is $\sin\theta$? We want to associate a number to θ that depends only on its *measure*. For example, congruent angles should have the same value of sin. Also, we want to have as usual $sin\theta$ as $\dfrac{opposite}{hypotenuse}$...

Definition 2.25. To define $\sin\theta$, we denote B the intersection of the lines, we choose a point A in one of the lines, and we construct C such that $\triangle ACB$ right at C. We then define $\sin\theta = \frac{|AC|}{|AB|}$



We now have to check that our definition $sin\theta = \frac{|AC|}{|AB|}$ depends only on the measure of θ, in other words that...

If we choose different A', B', C' such that $\angle A'B'C' = \theta$ and $\triangle A'C'B'$ right at C',



then we get the same value of $\sin\theta = \dfrac{|A'C'|}{|A'B'|}$. This is because the two triangles are similar by Remark 2.13, so $\dfrac{|A'C'|}{|A'B'|} = \dfrac{|AC|}{|AB|}$ by Proposition 2.16.

Since it doesn't depend on these choices, it's usual to consider B as a *center* point that we denote $0 = (0,0)$ and to choose A such that $|AB| = 1$. We get then the usual *trigonometric circle* of radius 1, which can be useful to remember what's what, since the points of the circle are then $(\cos\theta, \sin\theta)$.

Cosine and Tangent are defined similarly. Another useful mnemonic is "SOHCAHTOA". SOH is an acronym for Sine= $\dfrac{Opposite}{Hypotenuse}$ CAH: Cosine = $\dfrac{Adjacent}{Hypotenuse}$

and TOA: Tangent $= \dfrac{Opposite}{Adjacent}$

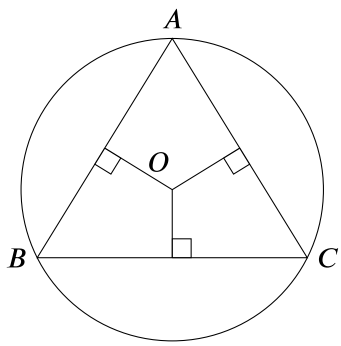
Remark 2.26. In the triangle above, $\cos\theta = \dfrac{adjacent}{hypotenuse} = \dfrac{|BC|}{|AB|}$. Note that

$$sin^2\theta + cos^2\theta = \frac{|AC|^2}{|AB|^2} + \frac{|BC|^2}{|AB|^2} = \frac{|AC|^2 + |BC|^2}{|AB|^2} = \frac{|AB|^2}{|AB|^2} = 1$$

for any value of θ, by the Pythagorean Theorem [7].

Definition 2.27. The circumcircle of a triangle $\triangle ABC$ is a circle $\mathscr{C}$ drawn *around it*: that is, such that $A, B, C \in \mathscr{C}$. In Assignment 3 you show how to construct it as the point where the perpendicular bisectors of the 3 sides meet.

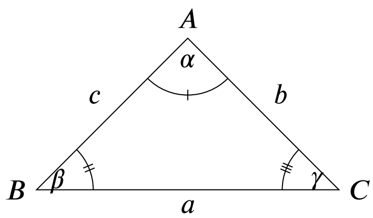
The circumradius R is the radius of circumcircle $\mathscr{C}$. Since the vertices of the triangle lay on the circumcircle, R is the distance from the circumcenter 0 to any vertex of the triangle.



[7] or if you prefer, since $(\cos\theta, \sin\theta)$ is a point in the trigonometric circle.

Or a less equilateral one:

Recall our notation: For $\triangle ABC$, A or α denotes the angle $\angle BAC$ at A, a denotes the length of the segment BC opposite to A. Similarly we use the letters B, β, C, γ and b, c.

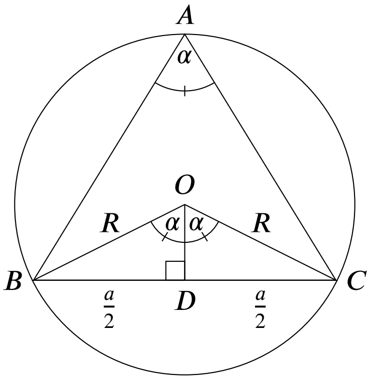


Theorem 2.28 (Extended Law of Sines)**.** *For any triangle $\triangle ABC$,*

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

Like for Power of a Point, there is a typed proof below but I want us to look at this together first and try to show this result.

Proof.



By the Star Trek Lemma, $\measuredangle$ BOC $= 2\alpha$. By SSS in $\triangle BOD$ and $\triangle COD$, $|OD| = |OD|$, $|OB| = |OC| = R$, $|BD| = |DC| = a/2 \Rightarrow \triangle BOD \cong \triangle COD \Rightarrow \angle BOD = \angle COD$.

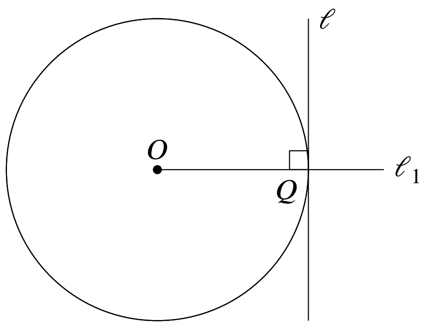
Then $2\alpha = \measuredangle$ BOC $= \measuredangle$ BOD $+ \measuredangle$ COD $= 2\measuredangle$ BOD, so $\measuredangle$ BOD $= \alpha$.

We get that $\sin\alpha = \dfrac{a}{2R}$; that is $2R = \dfrac{a}{sin\alpha}$. Doing the same with B and C instead of A finishes the proof. ☐

2.5 The Incircle and the Law of Cosines

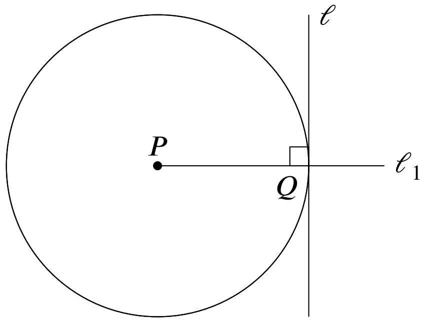
Recall that we have shown Lemma 2.8: "tangents are perpendicular to radius". The converse also holds:

Lemma 2.29. *If two lines ℓ, ℓ_1 are perpendicular and meet at Q, then ℓ is tangent to any circle $\mathscr{C}$ of center O in ℓ_1 and radius OQ.*



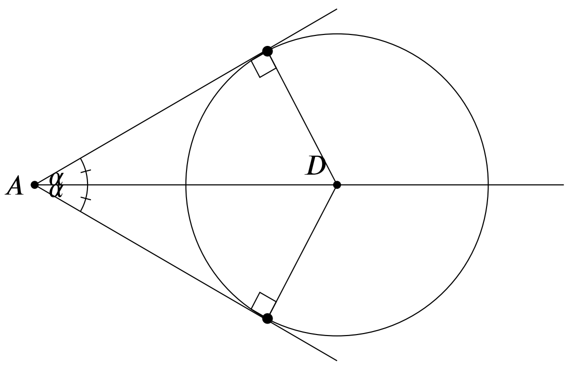
Proof. Exercise (this is a short proof, and you can look at the proof of 2.8 for an idea). ☐

Remark 2.30. Given a point P that is not on a line ℓ, let's denote $d(P,\ell)$ the distance from P to the closest point $Q \in \ell$, note that Q is the point where the line perpendicular to ℓ at P meets ℓ. Then the previous lemma tells us that the circle with center P and radius $|PQ| = d(P,\ell)$ is tangent to ℓ.



Note, we don't really need to *give* ℓ_1! So if we have just $P \notin \ell$ then we know that the circle with center P and radius $|PQ| = d(P, \ell)$ is tangent to ℓ.

Recall the angle bisector of an angle, that is the ray that cuts it in two congruent angles.



Lemma 2.31. *Consider an angle $\angle BAC$ and a point D. The following statements are equivalent:*

1. *D is a point in the angle bisector of $\angle BAC$*
2. *the perpendiculars from D to the lines ℓ_{AB} and ℓ_{AC} extending AB and AC have the same length (that is the same as having $d(D, \ell_{AB}) = d(D, \ell_{BC})$).*
3. *the circle centered at D with length $d(D, \ell_{AB})$ is tangent to both lines ℓ_{AB} and ℓ_{AC}.*

Proof. (1 => 2) Let the perpendiculars from D to the lines extending AB and AC meet the lines at P and Q. Since $\triangle$ APD and $\triangle$ AQD have the 3 corresponding angles congruent and they share AD, they are congruent triangles and then $|DP|$ = $|DQ|$.

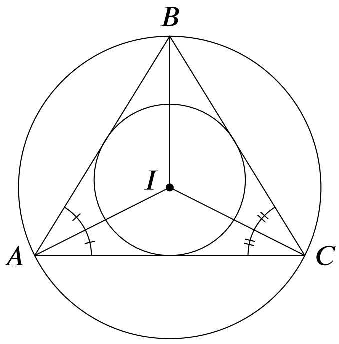
(2 => 3) We have shown above in general that "the circle with center P and radius $|PQ| = d(P, \ell)$ is tangent to ℓ." In this case, we have then that the circle with

center D and radius $d(D, \ell_{AB})$ is tangent to ℓ_{AB}. Since $d(D, \ell AB) = d(D, \ell_{BC})$, it is also tangent to ℓ_{BC}.

(3 => 1): Exercise. □

Theorem 2.32. *The angle bisectors of a triangle intersect at a common point I called the incenter, which is the center of the incircle, that is the unique circle inscribed in the triangle (that is, tangent to the three sides).*

Proof. We will use Lemma 2.31

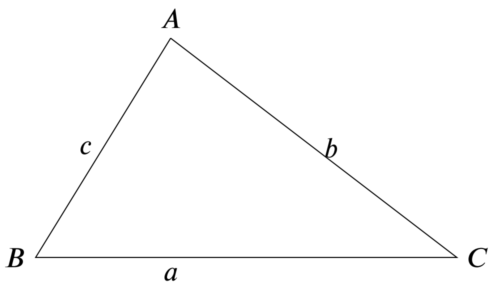


We consider the angle bisectors of ∡ BAC and ∡ BCA, they meet at a point that we call I. By 1 ⇒ 3, we know that I is the center of the circle tangent to AB, AC, and BC. And by 3 ⇒ 1, I is also in the angle bisector of ∡ ABC. □

Definition 2.33. The radius r of the incircle is called the inradius. Half of the perimeter of a triangle $\triangle ABC$, that is $s = \frac{1}{2}(a+b+c)$, is called its semiperimeter.

Theorem 2.34. *For any $\triangle ABC$, the area of $\triangle ABC$ equals rs*

Proof. We draw the incircle and recall that tangents are perpendicular to radius.



$$|\triangle ABC| = |\triangle ABI| + |\triangle ACI| + |\triangle BCI| = \frac{1}{2}cr + \frac{1}{2}br + \frac{1}{2}ar = \frac{1}{2}(a+b+c)r = sr$$

□

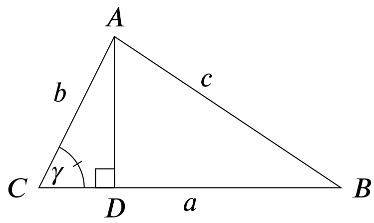
Note that $cos 90°$ equals 0. The following theorem can then be seen as a generalization of the Pythagorean Theorem to triangles which are not necessarily right.

Theorem 2.35. *[Law of Cosines] For any* $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos\gamma$.

"We do only one case"

Like in the Proof of Theorem 1.8, we will construct D s.t AD $\perp$ BC. The following proof does the case in which D is in BC, that is when the angles $\angle B$ and $\angle C$ are acute. The other case, when one of the angles is more than $90°$, is left as an exercise (it involves changing $+$ to $-$, and using the negative values of sin and cos, but no *new* ideas). This applies to all the results below as well. I think this way I can better transmit the geometric ideas in the proofs without losing ourselves in the algebraic computations.

Proof. We construct D s.t AD $\perp$ BC.



By the Pythagorean Theorem, we have the first equality, and then we use the definitions of sin and cos and the fact that $\sin^2 + \cos^2 = 1$ shown above.

$$\begin{aligned} c^2 &= BD^2 + AD^2 \\ &= (a - CD)^2 + AD^2 \\ &\overset{\star}{=} (a - b\cos\gamma)^2 + (b\sin\gamma)^2 \\ &= a^2 + b^2\cos^2\gamma - 2ab\cos\gamma + b^2\sin^2\gamma \\ &= a^2 + b^2(\sin^2\gamma + \cos^2\gamma) - 2ab\cos\gamma \\ &= a^2 + b^2 - 2ab\cos\gamma \end{aligned}$$

□

A *trick* for trigonometry; or *magic explained*

Look at the equality marked with $\star$ above. It may seem like *magic* when you fist look at it, but if you know what you have to show ($c^2 = a^2+b^2-2ab\cos\gamma$) then a useful trick is to try to write everything in terms of γ. We do that by finding a right triangle that contains γ, and then use the definitions of $\sin\gamma$ and $\cos\gamma$. I don't want to say always, but this *works* many times...

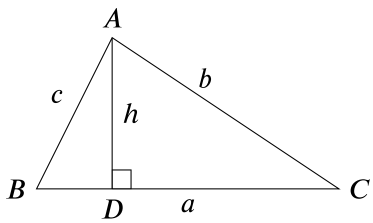
Here's a surprising formula that follows from the Law of Cosines with some work: we can compute the area of any triangle only using the lengths of its sides! Recall $s = \frac{1}{2}(a+b+c)$ is called the semiperimeter...

Theorem 2.36 (Heron's formula)**.** *For any* $\triangle ABC$*,*

$$|\triangle ABC| = \sqrt{s(s-a)(s-b)(s-c)}$$

I like to think of the proof of this result as having two parts. In the first, we show that we can compute $|\triangle ABC|$ by a formula $\star$ that only involves a, b, and c. This is *good geometry* that we will try to do. In the second part, we show that this formula $\star$ admits this *neat* expression as $\sqrt{s(s-a)(s-b)(s-c)}$. This is *algebra* (heroic algebra, if I'm allowed the pun).

Proof.



$$|\triangle ABC| = \frac{1}{2}ah =$$

A list of ingredients:

- $c^2 = a^2 + b^2 - 2ab\cos\gamma$. Note that this means that we can express $\cos\gamma$ only in terms of a, b, and c.
- $\sin^2 + \cos^2 = 1$
- And the *trick* above: finding the *right* triangle (another pun!?)

Using the definition of $\sin\gamma$, the fact that $\sin^2\gamma + \cos^2\gamma = 1$ and Theorem 2.35 we get

$$|\triangle ABC| = \frac{1}{2}ah = \frac{1}{2}ab\sin\gamma = \frac{1}{2}ab\sqrt{1-\cos^2\gamma} = \frac{1}{2}ab\sqrt{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2} = \star$$

We now work with the formula $\star$. We will use a few times the *difference of squares* property, $x^2 - y^2 = (x+y)(x-y)$.

$$\begin{aligned}
\star &= \frac{1}{2}ab\sqrt{1-\frac{(a^2+b^2-c^2)^2}{(2ab)^2}} \\
&= \frac{1}{2}ab\sqrt{\frac{(2ab)^2-(a^2+b^2-c^2)^2}{(2ab)^2}} \\
&= \frac{1}{2}ab\frac{\sqrt{(2ab)^2-(a^2+b^2-c^2)^2}}{\sqrt{(2ab)^2}} \\
&= \frac{1}{4}\sqrt{(2ab)^2-(a^2+b^2-c^2)^2} \\
&= \frac{1}{4}\sqrt{(2ab+a^2+b^2-c^2)(2ab-a^2-b^2+c^2)} \\
&= \frac{1}{4}\sqrt{((a+b)^2-c^2)(c^2-(a-b)^2)} \\
&= \frac{1}{\sqrt{(2)(2)(2)(2)}}\sqrt{(a+b+c)(a+b-c)(c+a-b)(c-a+b)} \\
&= \sqrt{(\frac{a+b+c}{2})(\frac{a+b-c}{2})(\frac{c+a-b}{2})(\frac{c-a+b}{2})} \\
&= \sqrt{s(s-c)(s-b)(s-a)}
\end{aligned}$$

□

2.6 Two more Centers and the Euler Line

Here's a little table that may be helpful before we do two more centers:

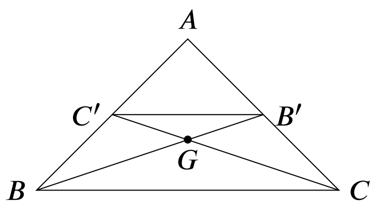
lines	**meet at**	**denoted**
Perpendicular bisectors	circumcenter	O
angle bisectors	incenter	I
medians	centroid	G
heights	orthocenter	H

Keep in mind the basic strategy we used for O and I, first intersect two lines and then show that the third one also passes through that point.

Definition 2.37. For a triangle $\triangle ABC$, the *median* at A is the segment from A to the midpoint A' of BC.

Theorem 2.38. *The three medians of a triangle $\triangle ABC$ intersect at a common point G called the centroid or center of mass. Furthermore, $|AG| = 2|A'G|$, $|BG| = 2|B'G|$, and $|CG| = 2|C'G|$.*

Proof. Let CC' be a median, so $|AC'| = |BC'|$, and let BB' be a median, so $AB' = CB'$.

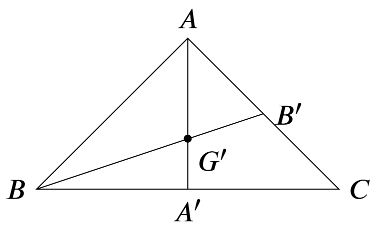


We will use the “basic trick" Lemma 2.15 that we have used before for similar triangles. Since $\dfrac{|AC'|}{|AB|} = \dfrac{1}{2} = \dfrac{|AB'|}{|AC|}$, then $B'C' \parallel BC$ and $\triangle AB'C' \sim \triangle ABC$, so by Theorem 2.20 (SSS for similar triangles) $\dfrac{|B'C'|}{|BC|} = \dfrac{|AC'|}{|AB|} = \dfrac{1}{2}$.

We can now show that $\triangle C'B'G \sim \triangle CBG$: we have $\measuredangle C'GB' = \measuredangle CGB$ since they are opposite angles, and $\measuredangle C'B'G = \measuredangle CBG$ since they are alternate interior angles between parallels.

Then, by Theorem 2.20 again, $\dfrac{|C'G|}{|CG|} = \dfrac{|B'G|}{|BG|} = \dfrac{|B'C'|}{|BC|} = \dfrac{1}{2}$. So $|BG| = 2|B'G|$ and $|CG| = 2|C'G|$ like we had to show.

Now for the 3rd median, we do the same but with A and B instead of B and C:



So we have $|BG'| = 2|B'G'|$ and $|AG'| = 2|A'G'|$. It *only* remains to show that $G = G'$!

We will be able to show that $G = G'$ because they are both in the segment BB' and satisfy that $|BG| = \dfrac{2}{3}|BB'| = |BG'|$

Indeed, looking at the first drawing, since $|BG| = 2|B'G|$ then

$$|BB'| = |BG| + |B'G| = |BG| + \frac{|BG|}{2} = \frac{3}{2}|BG|, \text{ so } |BG| = \frac{2}{3}|BB'|.$$

But the exact same happens with G' in the second drawing! $|BG'| = 2|B'G'|$ and then

$$|BB'| = |BG'| + |B'G'| = |BG'| + \frac{|BG'|}{2} = \frac{3}{2}|BG'|, \text{ so } |BG'| = \frac{2}{3}|BB'|.$$

Then $G = G'$, so the three medians meet at G and also $|AG| = 2|A'G|$ like we had to show. □

Just one more center and we're done...

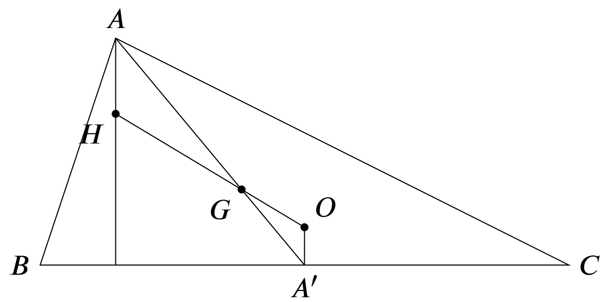
Theorem 2.39. *The three heights or altitudes of a triangle $\triangle ABC$ intersect at a common point H called the orthocenter.*

We will show this theorem *together* with the following one:

Theorem 2.40. *For any triangle $\triangle ABC$, O, G, and H are in a line that is called the* Euler line. *G is between O and H, and $2|OG| = |GH|$.*

Proof. We consider A' the midpoint of BC, O the circumcenter and we recall that it lies in the perpendicular bisector of BC, G the centroid, and we mark a point that we conveniently call H such that $|GH| = 2|OG|$.

The trick here is to construct the point H not as a intersection of two heights but with the property $2|OG| = |GH|$ that we want to show (Euler was clever!).



We now draw the line AH and we will show that it is the height of $\triangle$ ABC at A; it suffices to show that AH ⊥ BC.

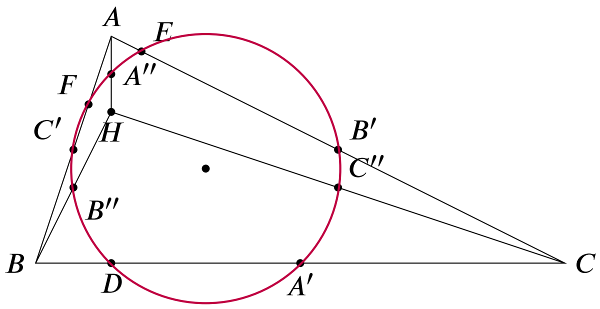
For this, we note that $\triangle AGH \sim \triangle A'GO$ by Theorem 2.21 (SAS for similar triangles); indeed $\frac{|A'G|}{|AG|} = \frac{1}{2} = \frac{|OG|}{|HG|}$, and $\measuredangle AGH = \measuredangle A'GO$ since they are opposite angles.

So we have that $\measuredangle HAG = \measuredangle GA'O$, then by Lemma 1.44 (parallel if and only if alternate angles) $AH \parallel A'O$. Since $AH \parallel A'O \perp BC$, then $AH \perp BC$.

We have shown that H is in the height at A, doing the same with B' instead of A' will show that H is also in the height at B, and the same holds for C. So H is indeed the orthocenter, and we have showed not only Theorem 2.40 but also Theorem 2.39. □

2.7 A *bonus track*: the Nine Point Circle

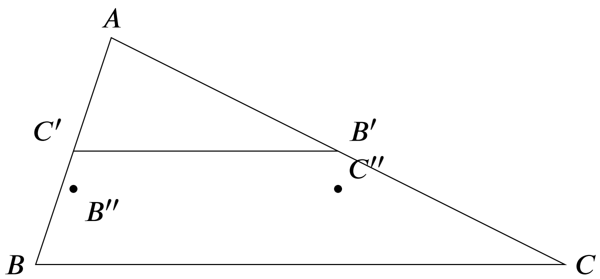
Theorem 2.41. *For a triangle $\triangle ABC$, we denote A', B', C' the midpoints of its sides as usual, we denote D, E, F the basis of the altitudes, and A'', B'', C'' the midpoints of AH, BH, CH respectively. The nine points A', B', C', D, E, F, A'', B'', and C'' all lie in a circle.*



Proof.

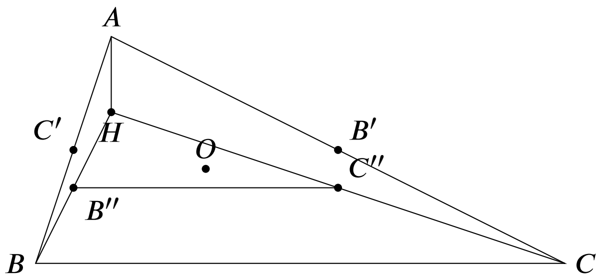
Idea: we notice that $C'B'C''B''$ is a rectangle, then the center of the circle containing these points is the intersection of the diagonals.

We first look only at B' and C':



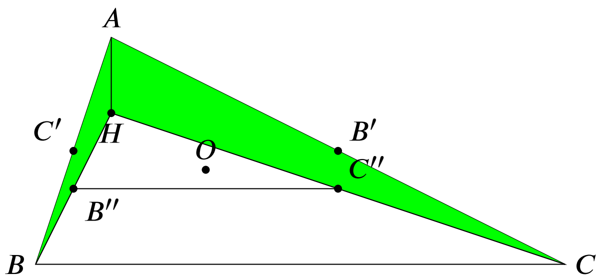
Since B' and C' are midpoints of the sides, then $\dfrac{|AC'|}{|AB|} = \dfrac{1}{2} = \dfrac{|AB'|}{|AC|}$, so by the basic trick Lemma 2.15 $B'C' \parallel BC$.

We now do a similar thing with $\triangle HBC$



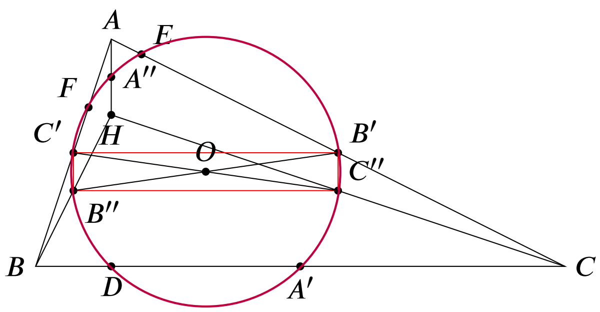
$\dfrac{|HB''|}{|HB|} = \dfrac{1}{2} = \dfrac{|HC''|}{|HC|}$, so $B''C'' \parallel BC$, then $B'C' \parallel B''C''$ since they are both parallel to BC

We now do the same with the triangles $\triangle ABH$ and $\triangle ACH$ filled in green:

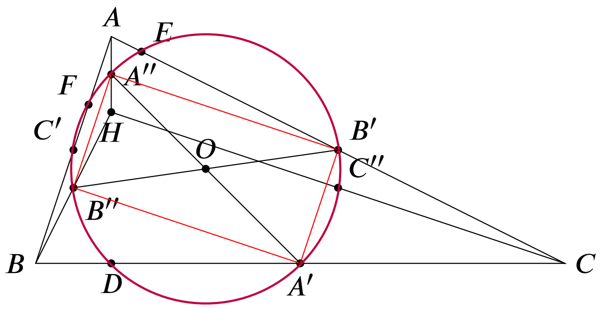


We get then that $C'B'' \parallel AH \parallel B'C''$, and since AH is a height then $AH \perp BC$ and $C'B'B''C''$ is a rectangle.

So drawing the diagonals $B'B''$, $C'C''$, they meet at their midpoint O, which is the center of the circle $\mathscr{C}$ containing B', C', B'' and C''.

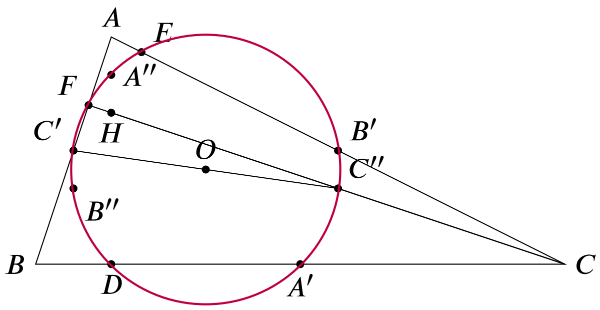


We first show that A' and A'' are also in $\mathscr{C}$. We do this by using A, B instead of B, C above, so we can show that $A''B'A'B''$ is also a rectangle.



Then the center of the circle containing these four points will be the intersection of the diagonals $B'B''$ and $A'A''$, but this is no other than the midpoint of $B'B''$, so it is the same point O!

Now we show that D, E, and F are in $\mathscr{C}$, finishing the proof.



We do this for F: note that $\measuredangle C'FC'' = \measuredangle BFC = 90°$ because CF is the height at C, and remember that $C'C''$ is a diameter of $\mathscr{C}$. By Remark 2.5 (Star Trek for diameters), this implies that F is in $\mathscr{C}$. The same reasoning goes for D and E, so we have proven that all 9 points are in $\mathscr{C}$. $\square$

3 Constructible Figures and Numbers

3.1 Constructions Using a Compass and Straightedge

In this section, we will do constructions that follow some *rules of construction*. This is closer to how the ancient Greeks did geometry.

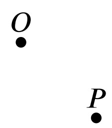
We start with two points O,P and construct new points using these rules. The rules allow us **only** to draw lines (rule 1, using a straightedge or unmarked ruler), and circles (rule 2, using a compass) **using points that were already constructed**. New points can only be obtained as intersections of these lines and circles, and can then be used to draw new ones.

How and Why

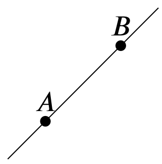
The focus here is not only in proofs but in constructing. However, you still have to prove that your construction works! And for this we will use many of the results seen so far. We will refer to this as "how and why", and both parts will usually be required.

Definition 3.1 (The Rules for constructions)**.**

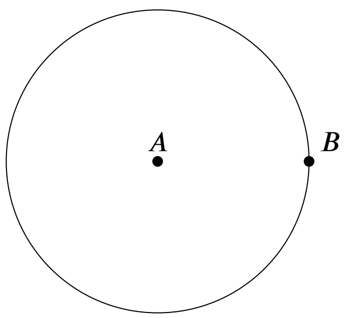
0 We start with two base constructed points.



1 We can draw the line through any two **constructed** points A and B



2 We can draw the circle $\mathscr{C}_A(B)$ with a **constructed center** A through a **constructed point** B



3 The intersections of constructed lines and circles become constructed points.

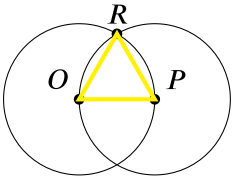
Definition 3.2. We say that a point is *constructible* if it is possible to construct it from O and P by applying rules 1-3 a finite number of times.

We will now do several basic constructions, showing in each case *how and why*. When we say *We can...* we mean that we can do that using the rules of construction.

Proposition 3.3. *We can construct an equilateral triangle.*

How can we do this?

Proof. $\underline{How}$: We construct $\mathscr{C}_O(P)$ and $\mathscr{C}_P(O)$, we choose a point in their intersection that we denote R. (this is how R is **constructed** using rules 2 and 3).



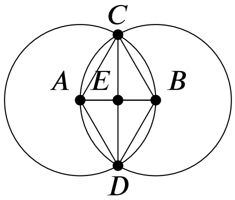
$\underline{Why}$: We show that $\triangle$ OPR is equilateral: since R is in $\mathscr{C}_O(P)$ and $\mathscr{C}_P(O)$, then $|RO| = |PO| = |PR|$.

$\square$

Proposition 3.4. *We can construct perpendicular bisectors.*

How can we do this?

Proof. $\underline{How}$: Let A, B be constructed points. $\mathscr{C}_A(B)$ and $\mathscr{C}_B(A)$ meet at C,D. Then the line through C and D (constructed using rule 1) is the perpendicular bisector of AB.



$\underline{Why}$: As in Proposition 3.3, we get five congruent segments, and six 60° angles. By SSS, $\triangle ACD \cong \triangle BCD$, then the 4 angles (∡ ACE, ∡ BCE, ∡ ADE, ∡ BDE) are congruent, then they are 30° each and AB ⊥ CD. By SAS, △ CAE ≅ △ CBE and then |AE| = |BE|... (or just note that both AE and BE are heights of congruent triangles △ ACD and △ BCD.) □

We get two *extra things for free* from this construction:

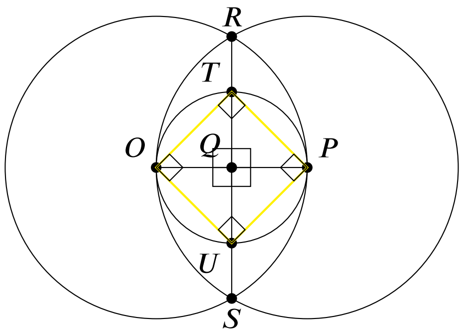
Remark 3.5. Note that in particular, we can construct midpoints of segments.

Remark 3.6. Note that we have shown that $AB \perp CD$, and $|CE| = |ED|$, so D is the reflection of C with respect to the line extending AB.

Proposition 3.7. *We can construct a square with OP as its diagonal.*

How can we do this?

Proof. <u>How</u>: We first construct the perpendicular bisector RS of OP as in Proposition 3.4. $\mathscr{C}_O(P)$, $\mathscr{C}_P(O)$ meet at two points R, S, we draw the line through them and we denote Q its intersection with OP. The circle $\mathscr{C}_Q(P) = \mathscr{C}_Q(O)$ meets RS at the points T and U. Then OTPU is a square.



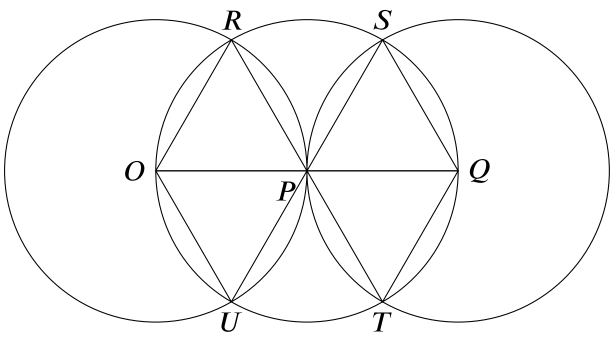
<u>Why</u>: Since O, P, T, and U are in $\mathscr{C}_Q(P)$, then $|OP| = |TQ| = |PQ| = |UP|$. Since RS is the perpendicular bisector of OP, then the 4 angles (∡OQT, ∡TQP, ∡PQU, ∡UQO) are right angles.

Then by SAS (or by the Pythagorean Theorem), we get that the 4 triangles forming OTPU are congruent, then so are the segments OT, TP, PU, UO. The 4 angles of OTPU are each 90° by Star Trek for diameters (Remark 2.5), or because each is split into two 45° angles. □

Proposition 3.8. *We can construct a regular hexagon.*

How can we do this?

Proof. <u>How</u>: We draw the line through O and P and the circle $\mathscr{C}_P(O)$, they meet at Q. We then draw $\mathscr{C}_O(P)$ and $\mathscr{C}_Q(P)$, and denote the intersections of these two circles with $\mathscr{C}_P(O)$ by R,S,T, and U. We can connect RS and UT by rule 1 of constructions, and then ORSQTU is a regular hexagon.

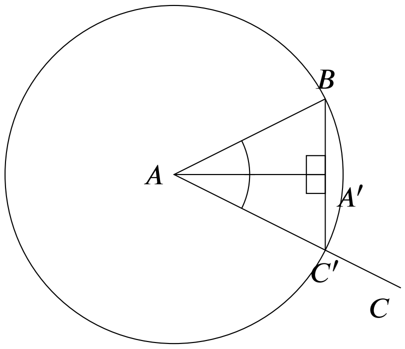


<u>Why</u>: Since we did the construction in Proposition 3.3 4 times, we have 4 equilateral triangles. The remaining two triangles ($\triangle PRS$ and $\triangle PUT$) are equilateral since they have two congruent sides and a 60° angle between them. We have then a regular hexagon formed by 6 equilateral triangles. $\square$

Proposition 3.9. *We can bisect any angle*

How can we do this?

Proof. $\underline{How}$: Given $\measuredangle BAC$, first we draw $\mathscr{C}_A(B)$, which meets the ray $\overrightarrow{AC}$ at C'. The perpendicular bisector of BC' (that we can construct by Proposition 3.4) is the angle bisector of $\measuredangle BAC$.



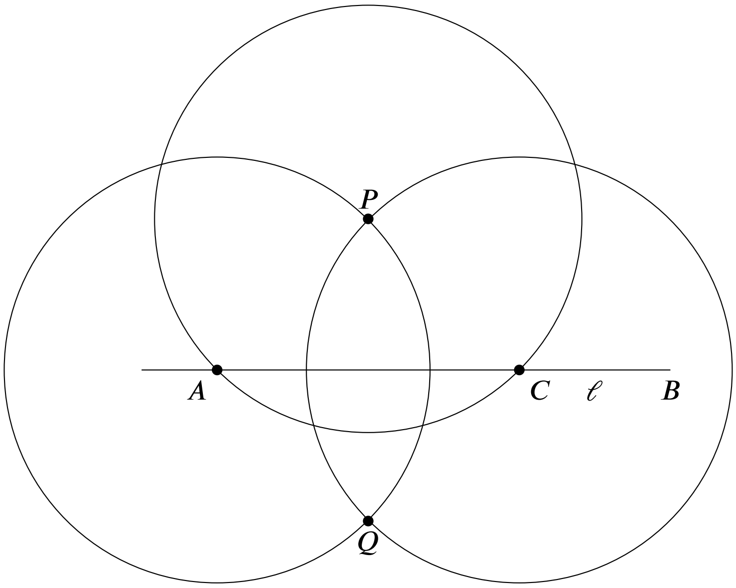
$\underline{Why}$: Since C' is in $\mathscr{C}_A(B)$, $|AB| = |AC'|$. So $\triangle ABC'$ is isosceles, and in exercise 2a) of Assignment 3 you have shown that then the median AA' is the perpendicular bisector of BC'. By SSS we get then $\triangle BAA'$ and $\triangle C'AA'$, so $\measuredangle BAA' = \measuredangle C'AA'$ as desired. □

> Can we cut any angle into 4 equal parts? Yes, we just bisect the angle and then bisect the bisected angles. Can we cut any angle into 3 equal parts? This question is *way deeper* than it may look at first sight...

> We will show now that the constructions we did when we showed how to construct the parallel line through a point (as in Playfair's version of the 5th axiom) can be done with a straightedge and a compass.

Proposition 3.10. *We can construct the reflection of any constructed point with respect to any constructed line.*

Proof. $\underline{How}$: Given constructed points P, A, and B, we show how to construct the reflection of P with respect to the line ℓ through A. and B. Draw $\mathscr{C}_P(A)$ which meets ℓ at another point C. Then do the same construction we did for constructing the perpendicular bisector of $\overline{AC}$ in Proposition 3.4 (drawing the circles $\mathscr{C}_A(P)$ and $\mathscr{C}_C(P)$, that meet at P and Q). Then Q is the reflection of P with respect to ℓ.



$\underline{Why}$: We showed in Remark 3.6 that this is so. □

Recalling Playfair's version of the 5th axiom, Remark 1.37 and Constructions 1.33 and 1.34, we have as a consequence:

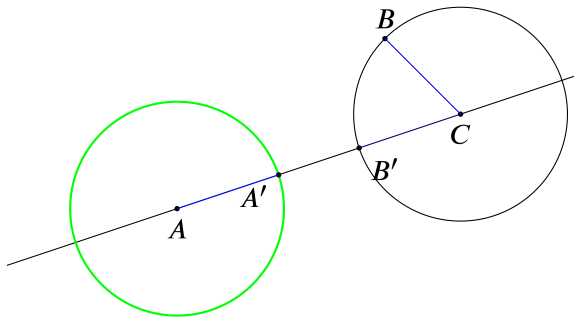
Corollary 3.11. *We can construct the parallel line to a constructed line, through any constructed point.*

Proof. We can construct is as in Remark 1.37, noting that all the steps in Constructions 1.33 and 1.34 can be done using the rules of construction and Proposition 3.10. □

We will now show a lemma that is useful to have, because it allows us to informally *move* the length r to A. This will be important when we consider *constructible lengths*.

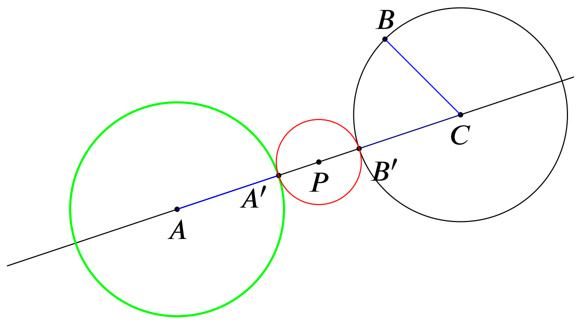
Lemma 3.12. *If A, B, C are constructed points, and $|BC| = r$, then we can construct the circle $C_A(r)$ with center A and radius r.*

The proof uses a *trick*: we first draw "informally" what we want to construct (the green circle, with the radius r in blue), and then work backwards to find the steps we need.



How do we link what we want with what we got? In the proof below we draw a red circle which connects what we want (the green circle) with what we got (the black circle).

Proof. $\underline{How}$: We draw AC and $\mathscr{C}_C(B)$, they meet at a point B' between A and C. We then construct the midpoint P of AC (which we can by Remark 3.5), and $\mathscr{C}_P(B')$, which intersects AC at a second point A'. Then $\mathscr{C}_A(A') = \mathscr{C}_A(r)$.



$\underline{Why}$: By construction,

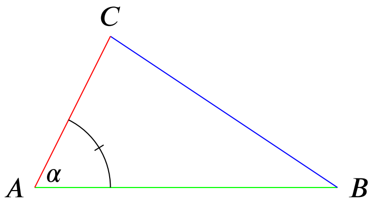
$$|AA'| = |AP| - |A'P| = |CP| - |B'P| = |B'C| = |BC| = r.$$

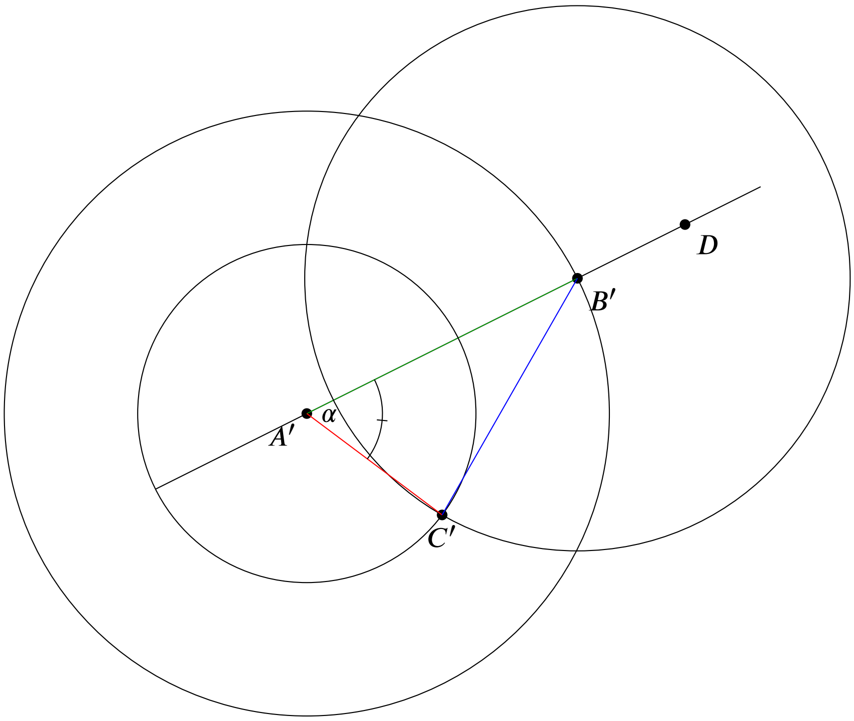
□

Since we can *move lengths*, by SSS we can also *move triangles*, as follows:

Proposition 3.13. *Given a constructed triangle $\triangle ABC$ and two constructed points A', D, we can construct a triangle $\triangle A'B'C'$ such that $\triangle ABC \cong \triangle A'B'C'$ and B' is on the line $A'D$.*

Proof. *How*: We draw the line extending $A'D$ and, using Lemma 3.12, we draw $\mathscr{C}_{A'}, |AB|)$, that intersects the line extending $A'D$ at a point B'. Using again Lemma 3.12, we draw $\mathscr{C}_{A'}, |AC|)$ and $\mathscr{C}_{B'}, |BC|)$, that meet at a point C'.





Why: By construction, $|A'B'| = |AB|$, $|A'C'| = |AC|$, and $|B'C'| = |BC|$. By SSS, $\triangle ABC \cong \triangle A'B'C'$. □

Corollary 3.14. *We can reproduce a constructed angle on any constructed line.*

Proof. Given an angle $\angle BAC$ and a line ℓ through constructed points A' and D, just repeat the construction above, and $\angle B'A'C' = \angle DA'C$ is on the line ℓ. □

Definition 3.15. We say that α (a real number between 0 and 2π, or between 0 and 360 if we prefer to think in degrees) is a **constructible angle measure** if we can construct points A, B, and C such that $\measuredangle BAC = \alpha$.

We abbreviate by saying that a number is a **constructible angle** instead of constructible angle measure. Corollary 3.14 says that constructible angles can be *copied* to any constructed point on a constructed line.

Example 3.16. $0, 90, 60$ are constructible angles (why?). If α is a constructible angle, then so is $\frac{\alpha}{2}$ (why?).

3.2 Doing Algebra with Constructible Lengths

Introduction / A bit of history

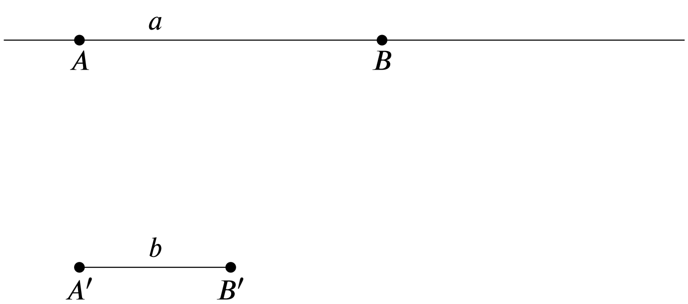
For a lot of time, people tried to show that constructions that can more or less *be done in reality*, such as doubling the volume of a cube, constructing a square with the same area of a given circle, and trisecting an arbitrary angle, could also be done only using the rules of construction. In fact, they can't, but it's usually very hard to show that something *can't be done*, and this took many years to prove.
A first step for showing that some things "X" are not constructible is to understand the structure of the things that you can construct, so you can show that X is *outside* this structure. In this case, we will show that the *constructible lengths* can be added, subtracted, multiplied, and divided, in other words they form a *field*.
We will also show that square roots can be taken. A bit informally, we will argue that this is *all that can be done* to get new construcible lengths starting with 0 and 1. The reason why two of the things mentioned above can't be done using the rules of construction is that they involve in one form or another taking a cubical root, and we will mention why this isn't possible.

Definition 3.17. We assume that $|OP| = 1$, in other words we define 1 to be distance between the two constructed points we start with.

We say that a positive number a is a constructible length if there are constructible points A, B such that $|AB| = a$

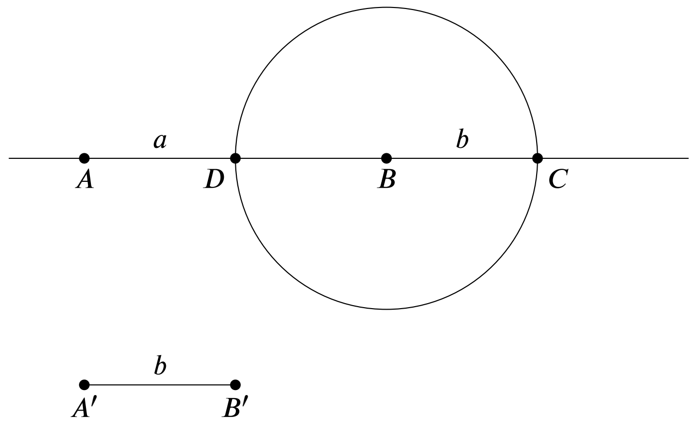
Proposition 3.18. *If a, b are constructible lengths, then so are $a + b$ and $|a - b|$.*

Proof. I'll show you how to start. Let A, B, A', B' be constructed points such that $|AB| = a$, $|A'B'| = b$.



We need first two points at distance $a + b$.

Using Lemma 3.12, we can construct the circle $\mathscr{C}_B(|A'B'|) = \mathscr{C}_B(b)$.

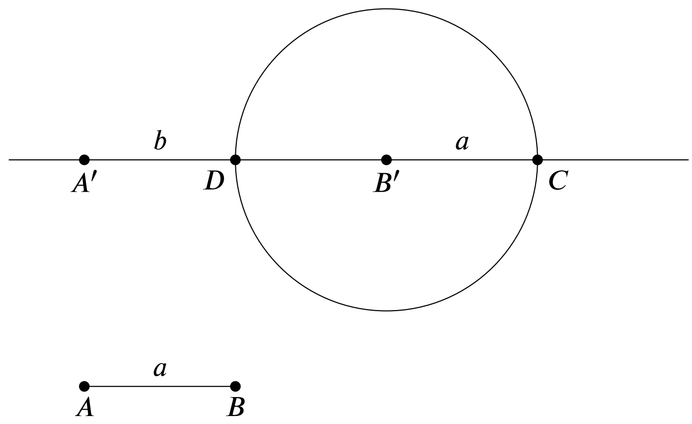


The circle will intersect the line AB at two points, one of them such that B is between A and C. Then $|AC| = |AB| + |BC| = a + b$.

For $|a - b|$, if $b < a$, then the circle will intersect the line AB at another point D, such that D is between A and B. Then $|AD| = |AB| - |DB| = a - b = |a - b|$.

If $b = a$, then $|a - b| = 0$ that certainly is a constructible length ($|AA| = 0$)

If $b > a$, then we do the same but we interchange a and b:

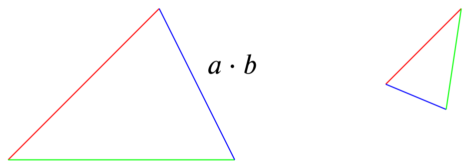


$|AD| = |A'B'| - |DB'| = b - a = |a - b|$. □

Corollary 3.19. *All natural numbers are constructible lengths.*

Proposition 3.20. *If a, b are constructible lengths, then so is $a \cdot b$.*

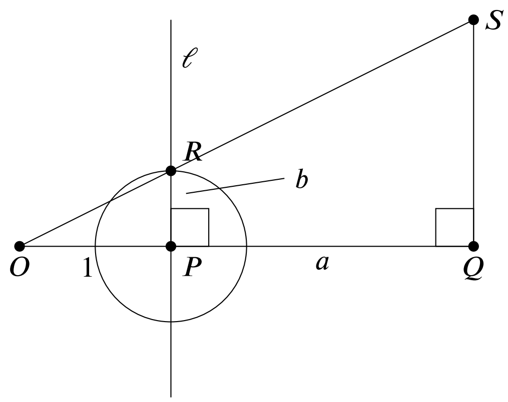
Proof. Given 2 line segments with lengths a and b respectively, how can we construct a line segment whose length is a · b? I'll give you a hint: ratio between similar triangles.



$$\frac{a \cdot b}{\quad} = \frac{\quad}{\quad}$$

How can we fill these 3 spaces using constructible lengths? That's what's behind this construction (we just construct our triangles with a right angle because that's simpler).

Recall that $|OP| = 1$, draw the line extending OP and mark Q, the point in the intersection of $\mathscr{C}_O(a)$ and the line on the same side of P, so $|OQ| = a$.



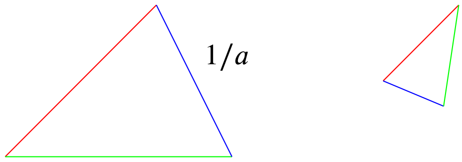
Then draw the perpendicular line ℓ to OP at P (you will show how to do this in Exercise 1 of Assignment 5) and $\mathscr{C}_P(b)$, and mark R a point in the intersection of this line and this circle.

Finally, extend OR and draw the perpendicular line to OP at Q, they meet at a point S.

Then by construction $\triangle OQS \sim \triangle OPR$. So we have $\dfrac{|QS|}{|PR|} = \dfrac{|OQ|}{|OP|}$; that is $\dfrac{|QS|}{b} = \dfrac{a}{1}$, and $|QS| = a \cdot b$. $\square$

Proposition 3.21. *If a is a constructible length, then so is $\dfrac{1}{a}$.*

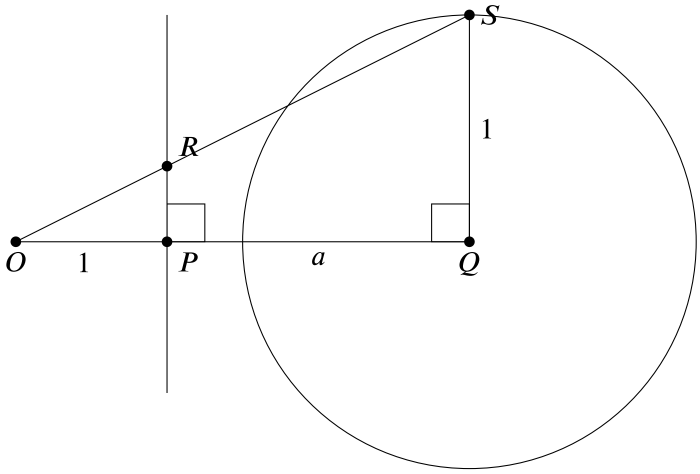
Proof. Given a line segments with length a, how can we construct a line segment whose length is $\frac{1}{a}$? Can we use the same idea from before?



$$\frac{1/a}{\_\_\_} = \_\_\_$$

How can we fill these 3 spaces using constructible lengths?

Recall that $|OP| = 1$, draw the line extending OP and mark Q, the point in the intersection of $\mathscr{C}_O(a)$ and the line on the same side of P, so $|OQ| = a$.



Then draw the perpendicular line to OQ at Q and $\mathscr{C}_Q(1)$, and mark S a point in the intersection of this line and this circle.

Finally, draw OS and draw the perpendicular line to OP at P, they meet at a point R.

Then by construction $\triangle OPR \sim \triangle OQS$. So we have $\dfrac{|PR|}{|QS|} = \dfrac{|OP|}{|OQ|}$; that is $\dfrac{|PR|}{1} = \dfrac{1}{a}$, and $|PR| = \dfrac{1}{a}$ as desired. $\square$

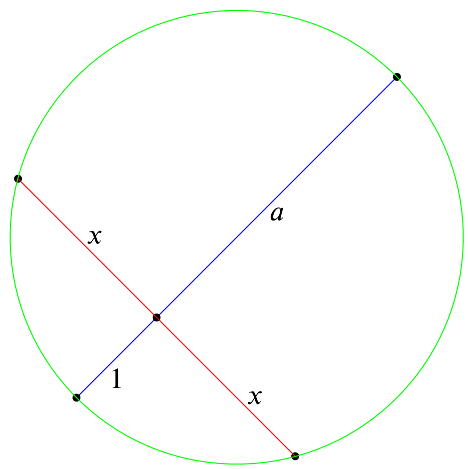
Since all positive rational numbers p/q can be obtained by adding p times $1/q$, we have

Corollary 3.22. *All positive rational numbers are constructible lengths.*

Just one more thing that *we can do:*

Proposition 3.23. *If a is a constructible length, then so is $\sqrt{a}$.*

Proof. This time we're after the length x such that $x^2 = a$. I will give you a hint: $x \cdot x = 1 \cdot a$ is an equality like the ones we had for power of a point. Could we make a drawing that fits that situation?

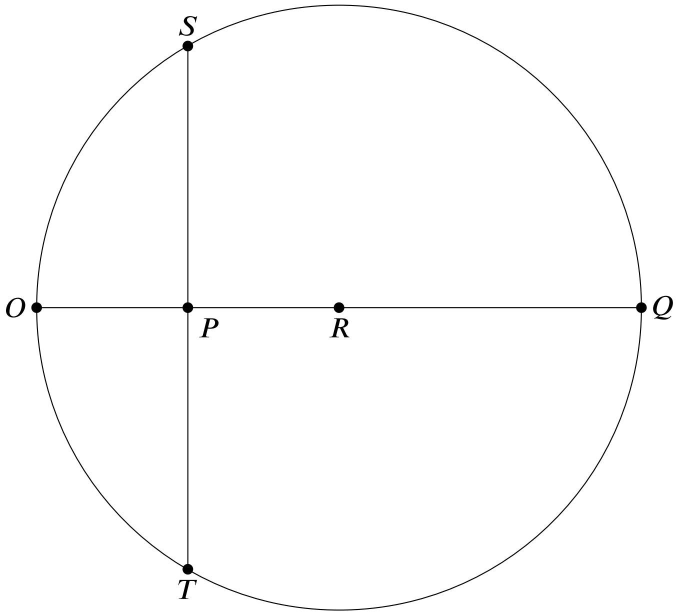


In class we did this drawing with two blue and red lines, and a green circle. We draw first the blue line, then the red circle, and get x as half the length of the red chord. We will use

Exercise 3.24. *Given two perpendicular chords Ch_1 and Ch_2 of a circle, Ch_1 is bisected by Ch_2 if and only if Ch_2 is a diameter*

How:

- Blue line: Start with O, P, $|OP| = 1$. Draw $\mathscr{C}_P(a)$, that meeets the line extending OP at a point Q such that P is between O and P.
- Green Circle: Draw the midpoint R of OQ and $\mathscr{C}_R(O) = \mathscr{C}_R(Q)$.
- Red line: Draw the perpendicular to OQ at P, that intersects $\mathscr{C}_R(O)$ at S,T. Then $|PS| = \sqrt{a}$.



Why: By Exercise 3.24, we have $|PS| = |PT| = x$. Then, by Power of a Point $x \cdot x = 1 \cdot a$, so $x = \sqrt{a}$. □

By the known formula for the solutions of a quadratic equation, combining the previous results we have:

Corollary 3.25. *Any real, positive solution of a quadratic equation with constructible coefficients is a constructible length.*

3.3 Constructible Complex Numbers and Regular Polygons

We will now generalize the notion of being constructible to the numbers in the complex plane. Let $O = 0 = (0,0) = 0 + 0i$, $P = 1 = (1,0) = 1 + 0i$ be the points in the complex plane.

Definition 3.26. We say that $R = a + bi$ is a constructible (complex) number if it can be constructed in the complex plane from 0 and 1 using the rules of construction 1,2,3.

Here is the link between being a constructible number and the work we did in the previous section. We do this first for the real numbers:

Lemma 3.27. *A real number a is a constructible number if and only if $|a|$ is a constructible length.*

Proof. ($\Rightarrow$) If a is a constructible real number, then we know that we can construct a from 0 and 1 using the rules of construction. Since $d(0, a) = |a|$, then $|a|$ is a constructible length.

($\Leftarrow$) We now assume that there are two points R, S in the complex plane such that $d(R, S) = |a|$. By Lemma 3.12, we can draw $\mathscr{C}_0(|a|)$, which will intersect the real line (the line through 0 and 1) at two points $b \geq 0$ and $c < 0$. If $a \geq 0$ then $b = a$, if $a < 0$ then $c = a$. This shows that a is a constructible real number. $\square$

When we allow non-positive numbers, we no longer have the problem we had in Proposition 3.18 regarding subtraction:

Corollary 3.28. *If a, b are constructible real numbers, then so are $a + b$ and $a - b$.*

Proof. Exercise. $\square$

Also, Propositions 3.20 and 3.21 can immediately be seen to hold for arbitrary real numbers instead of just positive, and combining them we have that

Corollary 3.29. *If a, b are constructible real numbers, and $b \neq 0$, then $\frac{a}{b}$ is constructible.*

Proof. Just because $\frac{a}{b} = a \cdot \frac{1}{b}$, so Propositions 3.20 and 3.21 give the result. $\square$

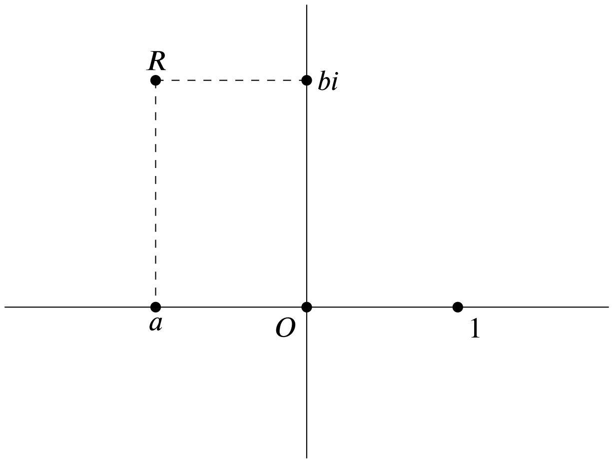
Note that then, for constructible real numbers, we can add them, subtract them, multiply them and divide them. This means that they form a *field*[a].

[a] I gave a quick definition of the notion of field during the lecture. Now, this isn't always the case, but the Wikipedia article on Fields `https://en.wikipedia.org/wiki/Field_(mathematics)` is actually good so you can check there for the definition and basic idea. Of course, any reasonable book on Algebra should have this too.

Proposition 3.30. *$R = a + bi$ is a constructible number if and only if so are a, b (so if and only if $|a|$, $|b|$ are constructible lengths).*

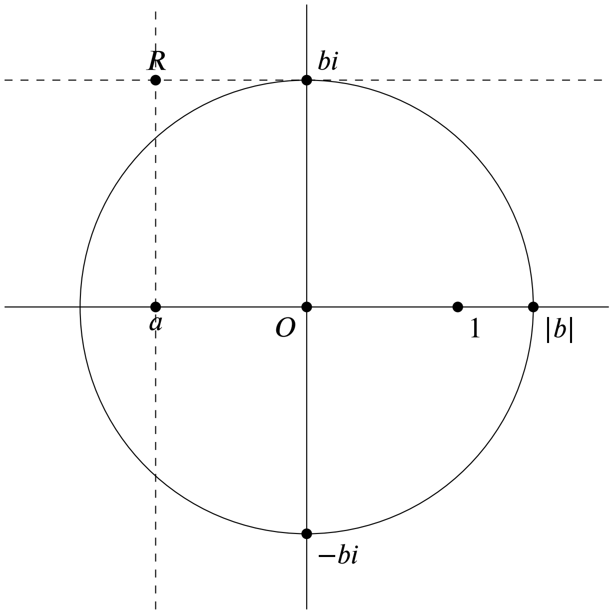
Proof. ($\Rightarrow$) First, construct the real axis, that is the line through 0 and 1, and the imaginary axis, that is its perpendicular through 0.

If $R = a+bi$ is a constructible number, then the intersection of the perpendicular line to the real axis through R (that we know can be constructed by Construction 1.33) and the real axis is a.



Also, the intersection of the perpendicular line to the imaginary axis through R and the imaginary axis is bi. Since $d(0, bi) = |b|$, then b is constructible by Lemma 3.27

($\Leftarrow$) Again, we construct first the real and the imaginary axes. Since $|b|$ is a constructible length, we can draw $\mathscr{C}_0(|b|)$ that meets the imaginary axis at the points bi and $-bi$.



Then, the parallel to the real axis through bi and the parallel to the imaginary axis through a (that we can both construct by Corollary 3.11) meet at $R = a + bi$ which is then constructible. $\square$

Corollary 3.31. *The constructible complex numbers also satisfy Propositions 3.18, 3.20, and 3.21. That is,*

1. *If R, S are constructible complex numbers, then so are $R+S$ and $R-S$.*

2. *If R, S are constructible complex numbers, then so is $R \cdot S$.*

3. *If R is a constructible complex number different to 0, then so is $1/R$.*

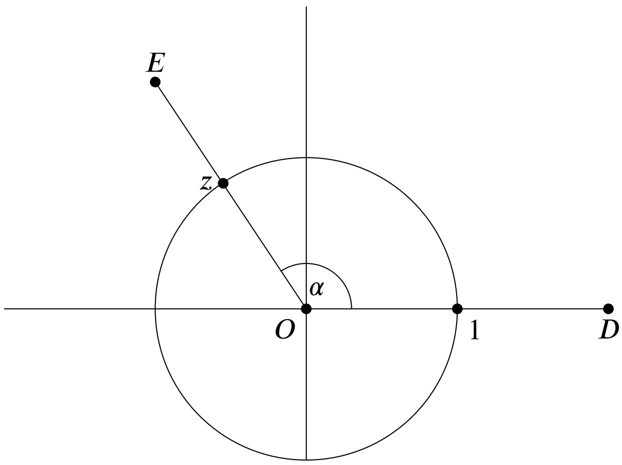
Proof. This is just because we have formulas for the real and the imaginary parts of $R+S$, $R-S$, $R \cdot S$, and $1/R$, that are built from those of R (and S) doing the operations in Propositions 3.18, 3.20, and 3.21, and Corollary 3.31. □

Note that this means that the constructible complex numbers also form a *field*.

We can now *link* constructible numbers and constructible angles as in Definition 3.15. We will be using the polar representation of any complex number $z = |z|e^{i\alpha} = |z|(\cos\alpha + i\sin\alpha)$.

Proposition 3.32. *α is the measure of a constructible angle if and only if $\cos\alpha + i\sin\alpha$ is a constructible complex number (which happens if and only if $\cos\alpha$ and $\sin\alpha$ are constructible real numbers).*

Proof. ($\Rightarrow$) By Definition 3.15 there are constructible points A, B, C such that $\angle BAC = \alpha$, and by Corollary 3.14 we can reproduce α at the real line. This means that we can construct two points D, E such that D is a real positive number and $\angle E0D = \alpha$.



We then draw $\mathscr{C}_0(1)$, that meets the line EO at a complex number z with modulus 1 and argument α, that is $z = e^{i\alpha} = \cos\alpha + i\sin\alpha$.
($\Leftarrow$) If $z = \cos\alpha + i\sin\alpha$ is a constructible complex number, then $\angle z01 = \alpha$ and so α is a constructible angle. □

Note that since $sin^2+cos^2 = 1$, it is also equivalent to ask either just for $cos\alpha$ or just for $sin\alpha$ to be constructible (because we can solve quadratic equations, remember? So in particular $\sin\alpha = \pm\sqrt{1-\cos^2\alpha}$ is a constructible real number when so is $\cos\alpha$).

We will apply this to show that the regular pentagon is a constructible figure by showing that $cos\frac{2\pi}{5} = cos72$ is a constructible number.

Proposition 3.33. *$cos72$ is a constructible number, and $cos72 = \frac{-1+\sqrt{5}}{4}$.*

Proof. We write $w = cos72 + isin72 = e^{\frac{2\pi i}{5}}$ (the complex number with $|w| = 1$ and argument 72). We know that when two complex numbers are multiplied then the modulus are multiplied and the arguments are added. We have then $w^5 = e^{\frac{2\pi i}{=}}1$, that is $w^5 - 1 = 0$, and we can factor

$$(w-1)(w^4 + w^3 + w^2 + w + 1) = 0$$

Since $w \neq 1$, then $w^4 + w^3 + w^2 + w + 1 = 0$.

How can we *make* this 4-degree polinomial into one of degree 2? This trick works only because $|w| = 1$, then $w\overline{w} = |w|^2 = 1$ and so $\overline{w} = w^{-1}$.

First we divide by w^2: $w^2 + w^1 + w^0 + w^{-1} + w^{-2} = 0$. And then we note that $c = w^1 + w^{-1} = w + \overline{w} = 2Re(w) = 2cos72$.

Then we just compute $c^2 = (w^1+w^{-1})^2 = w^2+w^{-2}+2w^1w^{-1} = w^2+w^{-2}+2$. So the equation $w^2 + w^1 + 1 + w^{-1} + w^{-2} = 0$ can be written as

$$w^2 + w^{-2} + 2 - 2 + w^1 + w^{-1} + 1 = 0,$$

that is $c^2-2+c+1 = 0$, or $c^2+c-1 = 0$. Then $c = \frac{-1+\sqrt{5}}{2}$, and $cos72 = \frac{c}{2} = \frac{-1+\sqrt{5}}{4}$.

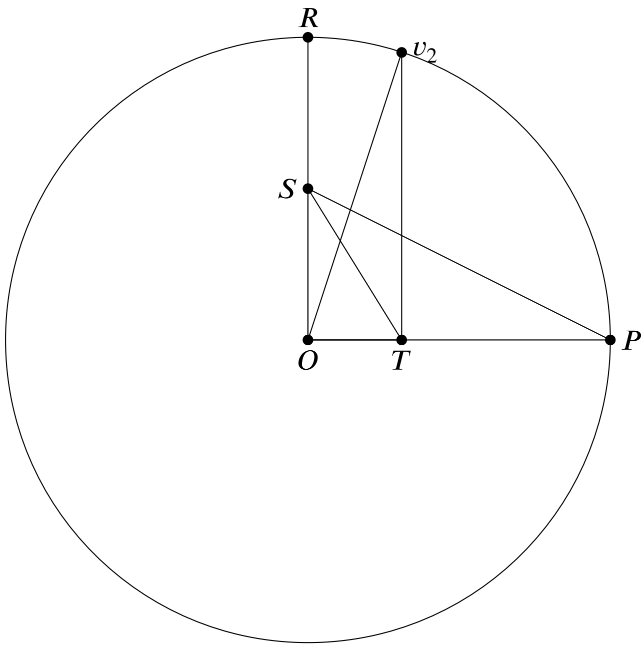
We can construct this number using the previous propositions: first 5, then $\sqrt{5}$, then $-1+\sqrt{5}$, and finally $\frac{-1+\sqrt{5}}{4}$. $\square$

Recall the regular n-gons. Note that their angle between two consecutive vertices and the center is $\frac{360}{n} = \frac{2\pi}{n}$. We consider now the case $n = 5$:

Since by Proposition 3.33 the point $e^{\frac{2\pi i}{5}}$ is constructible, then so is the regular pentagon. But writing the steps for this construction would be very very long! The following construction is much shorter (though it may seem somewhat magical...)

Construction 3.34. *The following is the construction of a regular pentagon with center a constructed point O and vertex a constructed point P. (We assume that $OP = 1$ to simplify the computations, though this isn't really needed.)*

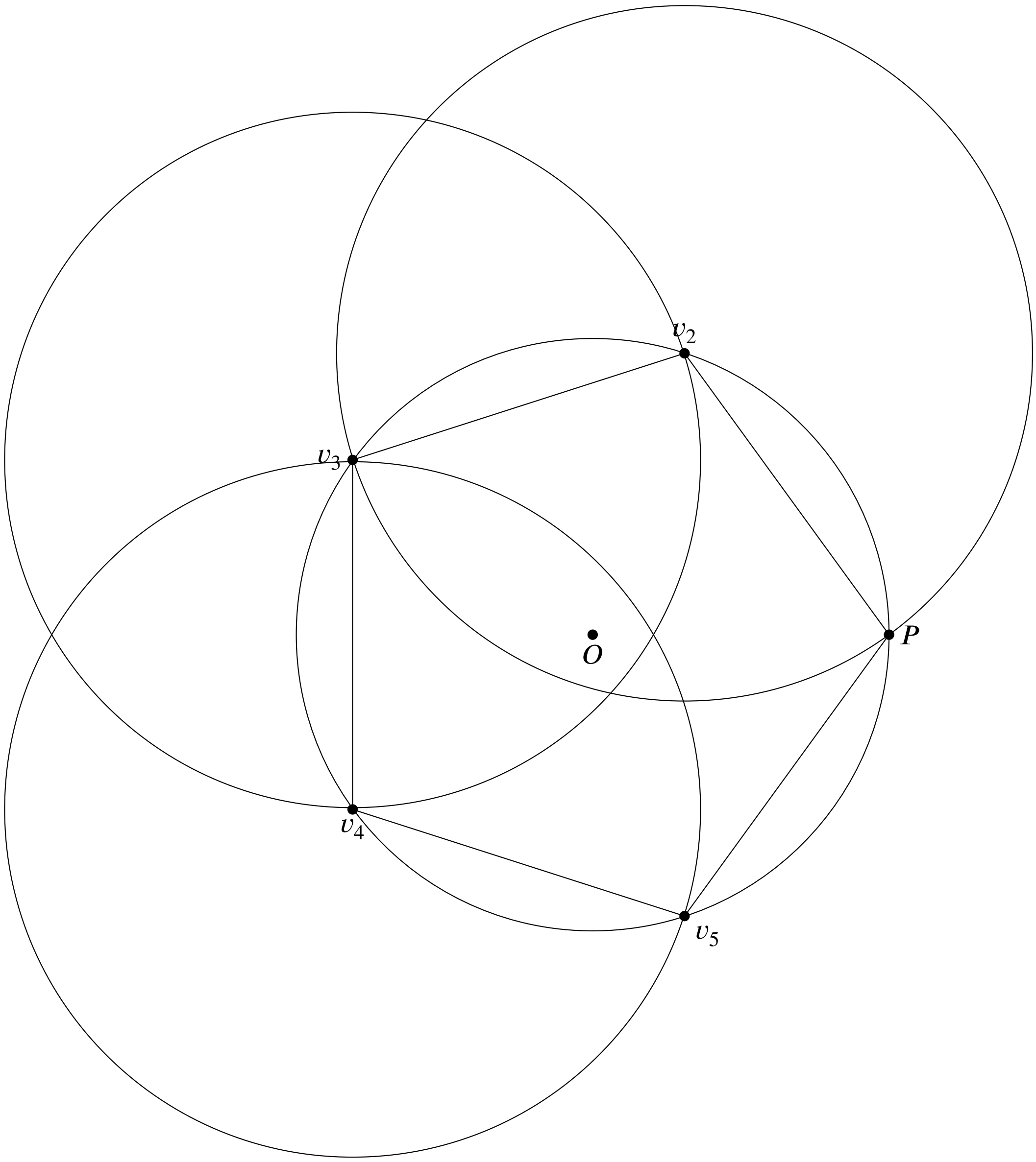
How: Draw the line extending OP and $\mathscr{C}_O(P)$. Draw the perpendicular to OP at O, that meets $\mathscr{C}_O(P)$ at R. Construct the midpoint S of OR, and the angle bisector of $\angle OSP$, that meets OP at T. Draw the perpendicular to OP at T, that meets $\mathscr{C}_O(P)$ at v_2. This is a second vertex for the regular pentagon ($P = v_1$ is the first one).



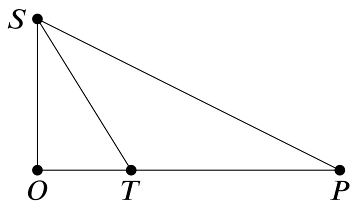
We then define

- *v_3 as the intersection of $\mathscr{C}_{v_2}(v_1)$ and $\mathscr{C}_O(P)$,*
- *v_4 as the intersection of $\mathscr{C}_{v_3}(v_2)$ and $\mathscr{C}_O(P)$,*
- *v_5 as the intersection of $\mathscr{C}_{v_4}(v_3)$ and $\mathscr{C}_O(P)$,*

and $v_1v_2v_3v_4v_5$ is a regular pentagon.



$\overline{Why}$:
We show first that $|OT| = cos72$. This is so because we have an angle bisector in the triangle



So by the Angle Bisector Theorem 2.22

$$\frac{|OT|}{|TP|} = \frac{|SO|}{|SP|}$$

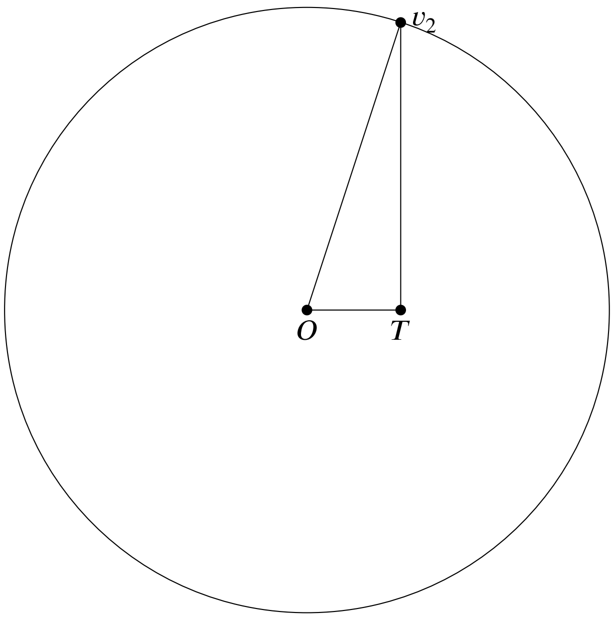
Denote $|OT| = x$*, then* $|TP| = 1-x$*,* $|SO| = \frac{1}{2}$ *and* $|SP| = \frac{\sqrt{5}}{2}$ *by Pythagoras applied to the right triangle* $\triangle SOP$*. Then we get*

$$\frac{x}{1-x} = \frac{1}{\sqrt{5}}$$

That is $\sqrt{5}x = 1 - x$*, or* $(\sqrt{5}+1)x = 1$*. So*

$$|OT| = x = \frac{1}{1+\sqrt{5}} = \frac{1}{1+\sqrt{5}}\frac{1-\sqrt{5}}{1-\sqrt{5}} = \frac{1-\sqrt{5}}{-4} = \frac{-1+\sqrt{5}}{4} = cos72$$

by Proposition 3.33. Looking now at



We have that $v_2 = (\cos 72, \sin 72)$*, so* $\angle TOv_2 = 72°$

(or, if you prefer, $\cos(\angle TOv_2) = \frac{|OT|}{1} = \cos 72$*, so* $< TOv_2 = 72°$*)*

Now that we have shown that $\angle v_1Ov_2 = 72°$*, we go back to the last drawing. When we draw* v_3*,* v_4*, and* v_5*, we have 4 congruent triangles by SSS, and since* $< v_5OP = 360° - 4 \cdot 72° = 72°$*, by SAS the 5th triangle is also congruent, and we get a regular pentagon.* $\square$

3.4 What We Can and Cannot Do

We denote C the set of constructible complex numbers. Recall Corollary 3.31: these numbers can be added, subtracted, multiplied, divided and are still constructible. This means that C is a *subfield* of the complex numbers $\mathbb{C}$:

Definition 3.35. A *subfield* of the complex numbers $\mathbb{C}$ is any subset $\mathbb{F}$ of $\mathbb{C}$ such that contains 0 and 1 and satisfies that:

$$\text{if } z, w \text{ are in } \mathbb{F} \text{ then } z + w,\ -w,\ z \cdot w \text{ and } \frac{1}{w} \text{ are in } \mathbb{F} \text{ too.}$$

(of course, $\frac{1}{w}$ only when $w \neq 0$).

Example 3.36. Examples: $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and C. We will see more.

Discussion

What numbers are in $\mathbb{Q}$? Perhaps more importantly, which are not?
What numbers are in $\mathbb{R}$? Perhaps more importantly, which are not?
What numbers are in C? Perhaps more importantly, which are not?

(Text in blue describes what we did in the lecture)

We discussed $\sqrt{2}$, i, $\sqrt{2}i$, we wrote $\sqrt{-1}$ and $\sqrt{1+3i}$ and answered the questions:

Is there a real number x such that $x^2 = -1$?

Is there a complex number z such that $z^2 = -1$?

Is there a complex number z such that $z^2 = 1 + 3i$? Yes, remember the polar representation of complex numbers and how they are multiplied.

We introduced the notation

$$\begin{tikzcd}[column sep=small] & \mathbb{C} \arrow[d, dash] \arrow[dr, dash] & \\ & \mathbb{R} \arrow[d, dash] & C \arrow[dl, dash] \\ & \mathbb{Q} & \end{tikzcd}$$

which means that we have subfields $\mathbb{Q} \subseteq C \subseteq \mathbb{C}$ and $\mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$, one *inside the other*.

Example 3.37. We have $\mathbb{Q}$ and $\sqrt{2}$, $\sqrt{2} \notin \mathbb{Q}$, $\sqrt{2}^2 = 2 \in \mathbb{Q}$. This allows to construct a new subset of the complex numbers, $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$. Let's show that $\mathbb{Q}[\sqrt{2}]$ is a subfield of $\mathbb{C}$.

We discussed why $\mathbb{Q} \cup \{\sqrt{2}\}$ is **not** a subfield of $\mathbb{C}$.

We discussed why $\mathbb{Q} \cup \{b\sqrt{2} : b \in \mathbb{Q}\}$ is also **not** a subfield of $\mathbb{C}$.

We briefly discussed why $\mathbb{Q} \cup \{b\sqrt{2} : b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$, but skipped the details, so it became an:

Exercise 3.38. *1. Show that $\mathbb{Q} \cup \{\sqrt{2}\}$ is* **not** *a subfield of $\mathbb{C}$.*

2. *Show that $\mathbb{Q} \cup \{b\sqrt{2} : b \in \mathbb{Q}\}$ is also* **not** *a subfield of $\mathbb{C}$.*

3. *Show that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{C}$.*

Example 3.39. Could we do $\mathbb{Q}[\sqrt{3}]$ similarly? **Yes** And $\mathbb{Q}[\sqrt{4}]$? **Yes, but you get $\mathbb{Q}$ again!**

Example 3.40. What is $\mathbb{R}[i]$? $\mathbb{C}$!

Example 3.41. What are the numbers in $\mathbb{Q}[i]$? For example, $\frac{17}{3} + \frac{2}{3}i$

Square roots in $\mathbb{C}$: If $z = |z|e^{i\theta}$, then $x = \sqrt{|z|}e^{i\frac{\theta}{2}}$ satisfies $x^2 = z$. We may denote $x = \sqrt{z}$.

Proposition 3.42. *If $z \in C$ (that is, if z is a constructible complex number), then so is $x = \sqrt{z}$.*

The easiest way to show this proposition is actually to first show this result, that I said during the lecture but that it's best to have as a separate lemma:

Lemma 3.43. *For any $z \in C$, $z = |z|e^{i\theta}$ is a constructible complex number if and only if $|z|$ is a constructible length* and θ is a constructible angle.*

** Recall from Lemma 3.27: a real positive number is a constructible length if and only if it is a constructible (complex) number.*

Proof. ($\Rightarrow$) Assume that $z = |z|e^{i\theta}$ is a constructible complex number. Then we can construct $|z| \in \mathscr{C}_0(z) \cap \mathbb{R}$, and then $\cos\theta + i\sin\theta = e^{i\theta} = \frac{z}{|z|}$ is a constructible complex number (by Corollary 3.31, because it's a quotient of two constructible complex numbers). By Lemma 3.32, θ is then a constructible angle.
($\Leftarrow$) Assume now that $|z|$ is a constructible length and θ is a constructible angle. Since θ is a constructible angle, then by Lemma 3.32 the complex number $\cos\theta + i\sin\theta = e^{i\theta}$ is constructible. Since it's the product of two constructible numbers, by Corollary 3.31 $z = |z|e^{i\theta}$ is a constructible complex number. □

Proof of Proposition 3.42. Assume that $z = |z|e^{i\theta}$ is a constructible complex number.

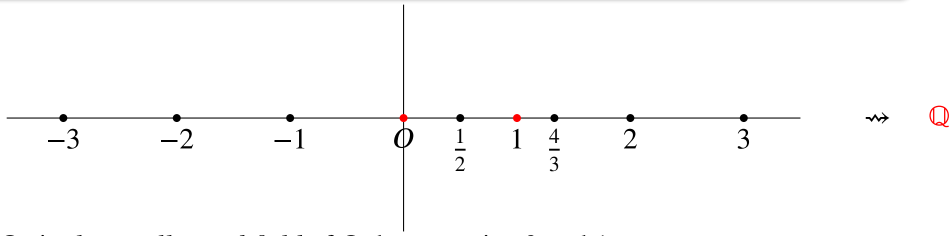
1. By Lemma 3.43, $|z|$ is a constructible length, then by Proposition 3.23 so is $\sqrt{|z|}$.

2. Also by Lemma 3.43, θ is a constructible angle, then by Proposition 3.9 so is $\frac{\theta}{2}$ (see also Example 3.16).

Finally, by Lemma 3.43, $x = \sqrt{z} = \sqrt{|z|}e^{i\frac{\theta}{2}}$ is a constructible complex number. □

Remark 3.44. Combining the results so far, we can already show that complex numbers like $\sqrt{1 + \sqrt{5 + i\sqrt{-2 + \sqrt{2 + i}}}}$ are constructible.

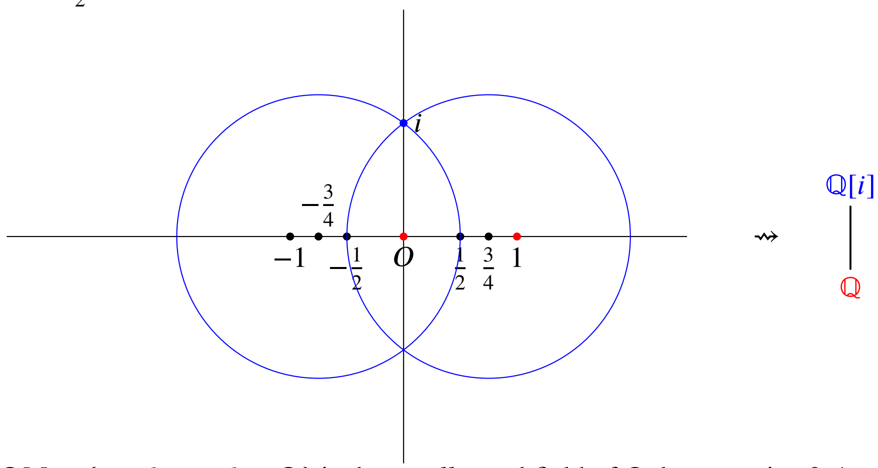
A first example of how to link constructions and subfields of $\mathbb{C}$:



$\mathbb{Q}$: is *the smallest subfield* of $\mathbb{C}$ that contains 0 and 1.

We have shown that all the numbers in $\mathbb{Q}$ are constructible, so $\mathbb{Q}$ is a subfield of C.

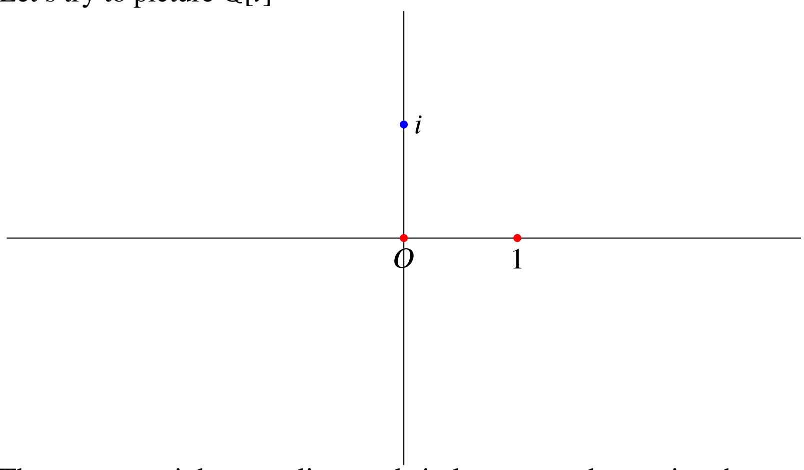
We now draw the circle with center $-\frac{3}{4}$ through $\frac{1}{2}$ and the circle with center $\frac{3}{4}$ through $-\frac{1}{2}$, they meet at i and $-i$:



$\mathbb{Q}[i] = \{a + bi \ : \ a, b \in \mathbb{Q}\}$ is the *smallest* subfield of $\mathbb{C}$ that contains $0, 1, i$.

Some numbers that are in $\mathbb{Q}[i]$: 3, $5i$, $1 + 2i$, $\frac{3}{35} - \frac{2}{97}i$, ...

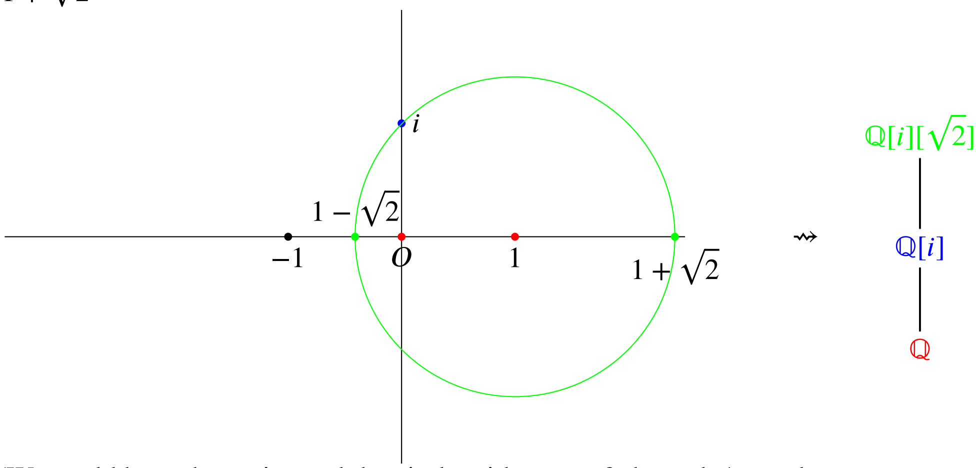
Let's try to picture $\mathbb{Q}[i]$



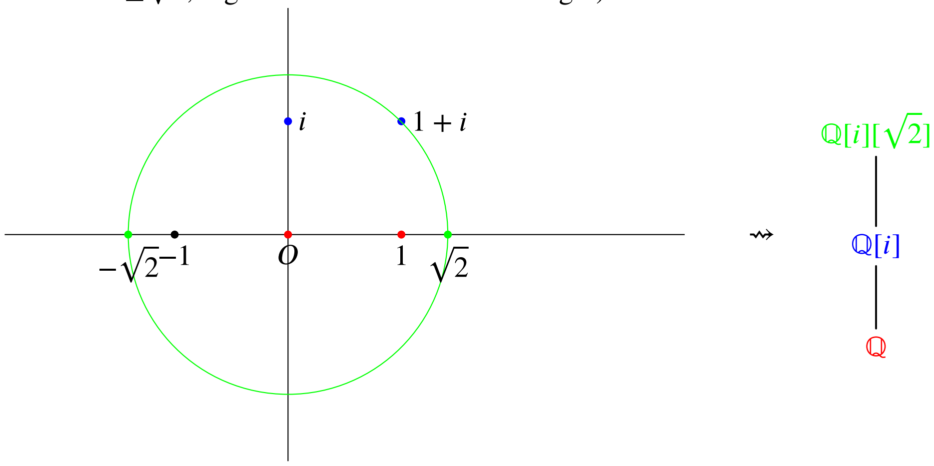
There are certainly many lines and circles we can draw using these points...

Is $\sqrt{2}$ in $\mathbb{Q}[i]$? No, though we can add it, so in a sense we don't *really* care if it was already there or not:

We now draw the circle with center 1 through i, it meets the real line at $1-\sqrt{2}$ and $1+\sqrt{2}$



(We could have drawn instead the circle with center 0 through $1+i$, that meets the real line at $\pm\sqrt{2}$, it gives the same field on the right):



The top subfield of $\mathbb{C}$ that appears on the right, in this case $\mathbb{Q}[i][\sqrt{2}]$, contains all the points that we constructed on the left.

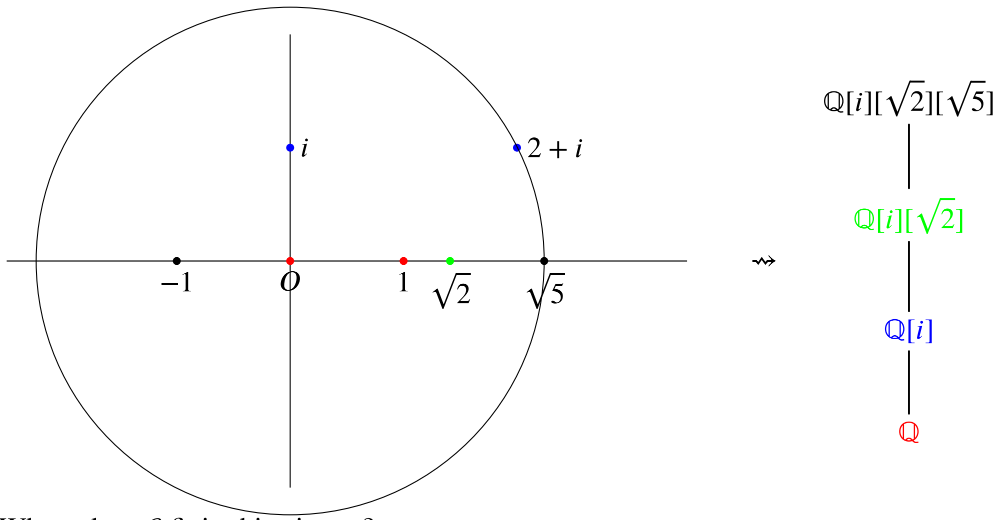
Let's try to develop some intuition about the numbers in $\mathbb{Q}[i][\sqrt{2}]$.

By definition, $\mathbb{Q}[i][\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Q}[i]\}$: is *the* subfield of $\mathbb{C}$ that contains $0, 1, i$, and $\sqrt{2}$.

Since $a, b \in \mathbb{Q}[i]$, then $a = c + di$, $b = e + fi$, $c, d, e, f \in \mathbb{Q}$. So a number of $\mathbb{Q}[i][\sqrt{2}]$ is of the form $c + di + (e + fi)\sqrt{2} = c + di + e\sqrt{2} + fi\sqrt{2}$.

Is $\sqrt{5}$ in $\mathbb{Q}[i]$? No, though once again we can add it, so in a sense we don't *really* care, because even if it was there we could still do this:

We draw the circle with center 0 throught $2+i$, that meets the real line at $\pm\sqrt{5}$:



Where does C fit in this picture?

Let's try to develop some intuition about the numbers in $\mathbb{Q}[i][\sqrt{2}][\sqrt{5}]$.

For each of the vertical lines on the right, the situation is like that of Example 3.37, we are *just adding square roots*. We have $z \in \mathbb{C}$, $z^2 \in \mathbb{F}$, and we build a new field $\mathbb{F}[z]$ (you may denote $x = z^2$, $z = \sqrt{x}$). The same thing we did for Example 3.37 works in general:

Proposition 3.45. *If $\mathbb{F} \subseteq \mathbb{C}$ is a subfield, $z \in \mathbb{C}$, $z^2 \in \mathbb{F}$, then $\mathbb{F}[z] = \{a + bz : a, b \in \mathbb{F}\}$ is also a subfield of $\mathbb{C}$.*

Proof. Can be an exercise, we just repeat what we did for $\mathbb{Q}[\sqrt{2}]$ in Example 3.37. $\square$

In the examples, we have seen how we can construct the numbers in these fields by choosing appropriate constructions of lines and circles. These numbers can certainly get much more *nasty* than $\mathbb{Q}[i][\sqrt{2}][\sqrt{5}]$. For example, since the constructible numbers are a subfield, using Proposition 3.42 we know that it is possible to construct $\sqrt{1+\sqrt{5+i\sqrt{-2+\sqrt{2+i}}}}$, and it could be worse... C is in a sense really big (it is infinite-dimensional over $\mathbb{Q}$, if that makes sense to you).

What we will do now is show that, no matter how we choose to draw lines and circles, in a sense constructing numbers like the one above is really *all* that we can do. Let's start by recognizing a property that these subsets of $\mathbb{C}$ (C, $\mathbb{Q}[i]$, $\mathbb{Q}[i][\sqrt{2}]$, $\mathbb{Q}[i][\sqrt{2}][\sqrt{5}]$) all have:

Definition 3.46. A subfield $\mathbb{F}$ of $\mathbb{C}$ such that $i \in \mathbb{F}$ is called *happy-sad* if it satisfies that, for every $z \in \mathbb{C}$, $z \in \mathbb{F}$ if and only if $\overline{z} \in \mathbb{F}$

In other words, *happy-sad* subfields are symmetric with respect to the real axis. Hence the name (though I accept improvements!):

Remark 3.47. If $\mathbb{F}$ is happy-sad, and a **real number** x satisfies that $x^2 \in \mathbb{F}$, then $\mathbb{F}[x]$ is still happy-sad.

Proof. Remember that $\mathbb{F}[x] = \{a + bx : a, b \in \mathbb{F}\}$. Then $\overline{i} = i + 0x$ is also in $\mathbb{F}[x]$. Also, if $a + bx$ is any number in $\mathbb{F}[x]$, then $\overline{a + bx} = \overline{a} + \overline{b}x$ is still in $\mathbb{F}[x]$. $\square$

If it's more clear, you can also think like this:

Remark 3.48. If $\mathbb{F}$ is happy-sad, and a **real positive number** z is in $\mathbb{F}$, then $\mathbb{F}[\sqrt{z}]$ is still happy-sad.

Proof. Remember that $\mathbb{F}[\sqrt{z}] = \{a + b\sqrt{z} : a, b \in \mathbb{F}\}$. Then $\overline{i} = i + 0\sqrt{z}$ is also in $\mathbb{F}[\sqrt{z}]$. Also, if $a + b\sqrt{z}$ is any number in $\mathbb{F}[\sqrt{z}]$, then $\overline{a + b\sqrt{z}} = \overline{a} + \overline{b}\sqrt{z}$ is still in $\mathbb{F}[\sqrt{z}]$. $\square$

Since $\mathbb{Q}[i]$ is happy-sad, this is why all the subfields $\mathbb{Q}[i][\sqrt{2}]$, $\mathbb{Q}[i][\sqrt{2}][\sqrt{5}]$ (and we can keep adding real square roots) are also happy-sad!

It's a bit easier to do geometry with coefficients (basic analytical geometry), like in the Cartesian plane $\mathbb{R}^2$, when a subfield is *happy-sad*. This is because we have:

Lemma 3.49. *If $\mathbb{F} \subseteq \mathbb{C}$ is a happy-sad subfield, then for every $z \in \mathbb{C}$, $z \in \mathbb{F}$ if and only if $Re(z) \in \mathbb{F}$ and $Im(z) \in \mathbb{F}$*

Proof. ($\Rightarrow$) We have $z \in \mathbb{F}$ and since $\mathbb{F}$ is happy-sad also $\overline{z} \in \mathbb{F}$. Since $Re(z) = \frac{1}{2}(z + \overline{z})$, then $Re(z) \in \mathbb{F}$, and then since $iIm(z) = z - Re(z)$, dividing by i shows that $Im(z) \in \mathbb{F}$ too.

($\Leftarrow$) Because $z = Re(z) + iIm(z)$. □

We will now show that something interesting happens for each of the <u>three</u> possible intersections (lines with lines, lines with circles, and circles with circles) when the defining points are in a happy-sad subfield:

Proposition 3.50. *If $z \in \mathbb{C}$ is a point in the intersection of two lines, each given by two points of a happy-sad subfield $\mathbb{F} \subseteq \mathbb{C}$, then $z \in \mathbb{F}$.*

Example 3.51. An example may help before the proof. Take $\mathbb{F} = \mathbb{Q}[i]$, and the lines ℓ_1 through 0 and $1 + i$ and ℓ_2 through i and 2. They meet at $\frac{2}{3} + \frac{2}{3}i$. How is this point computed?

Proof. We have the complex numbers A, B, C, D giving the two lines, that we think of as points $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$ in the Cartesian plane $\mathbb{R}^2$ where we can do basic analytical geometry. By Lemma 3.49, we have that the **real** numbers $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are all also in $\mathbb{F}$.

Do we remember how we intersect two lines in $\mathbb{R}^2$?

$\square$

Proposition 3.52. *If $z \in \mathbb{C}$ is a point in the intersection of a line and a circle, each given by two points of a happy-sad subfield $\mathbb{F} \subseteq \mathbb{C}$, then there exists $x \in \mathbb{R}$ such that $x^2 \in \mathbb{F}$ and $z \in \mathbb{F}[x]$.*

Note that by Remark 3.48 $\mathbb{F}[x]$ is also happy-sad.

Example 3.53. Again, let's do an example before the proof. Take $\mathbb{F} = \mathbb{Q}[i]$, and the lines ℓ_1 through 0 and $1 + i$ and $\mathscr{C}_0(1 + i)$. The point $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i \in \mathbb{Q}[i][\sqrt{2}]$ is in the intersection. How is this point computed?

Proof. We now have the complex numbers A, B, C, D giving the line and the circle, that we think of as points $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$ in the Cartesian plane $\mathbb{R}^2$ where we can do basic analytical geometry. By Lemma 3.49, we have that $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are **real** numbers in $\mathbb{F}$.

Do we remember how we intersect a line and a circle in $\mathbb{R}^2$?

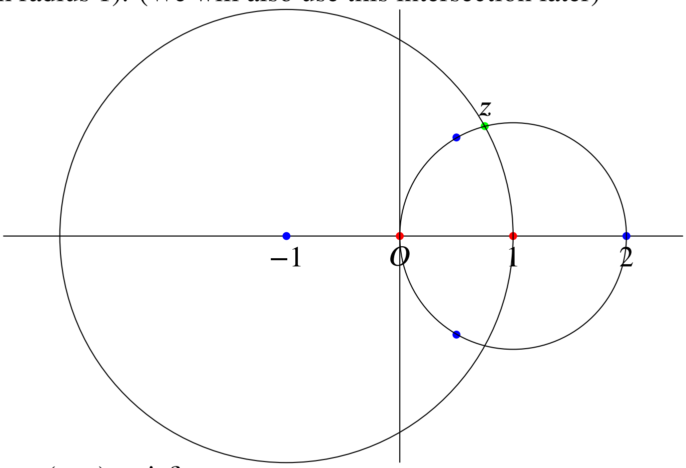
$\square$

And finally

Proposition 3.54. *If $z \in \mathbb{C}$ is a point in the intersection of two circles, each given by two points of a happy-sad subfield $\mathbb{F} \subseteq \mathbb{C}$, then there exists $x \in \mathbb{R}$ such that $x^2 \in \mathbb{F}$ and $z \in \mathbb{F}[x]$.*

Note that by Remark 3.48 $\mathbb{F}[x]$ is also happy-sad.

Example 3.55. Let's compute the intersection of $\mathscr{C}_{-1}(1)$ (with radius 2) and $\mathscr{C}_1(0)$ (with radius 1). (We will also use this intersection later)



$z = (x, y)$ satisfies

$$(x+1)^2 + y^2 = 4; \qquad (x-1)^2 + y^2 = 1$$

Subtracting $x^2 + 2x + 1 - x^2 + 2x - 1 = 3$, so $x = \frac{3}{4}$, and $y^2 = 1 - \frac{1}{16}$, so $y = \frac{\sqrt{15}}{4}$, and $z = \frac{3}{4} + \frac{\sqrt{15}}{4}i$.

Note that z is also in the intersection of the vertical line $\{x = \frac{3}{4}\}$ and any of the circles. (By the way, this line is called the radical axis of the two circles).

Proof. Once again, we have four points $(a_1, a_2), (b_1, b_2), (c_1, c_2), (d_1, d_2)$ in the Cartesian plane $\mathbb{R}^2$ giving the circles, such that $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are all in $\mathbb{F}$.

Do we remember how we intersect circles in $\mathbb{R}^2$? We can *reduce* it to the line-circle case in Proposition 3.52 as above.

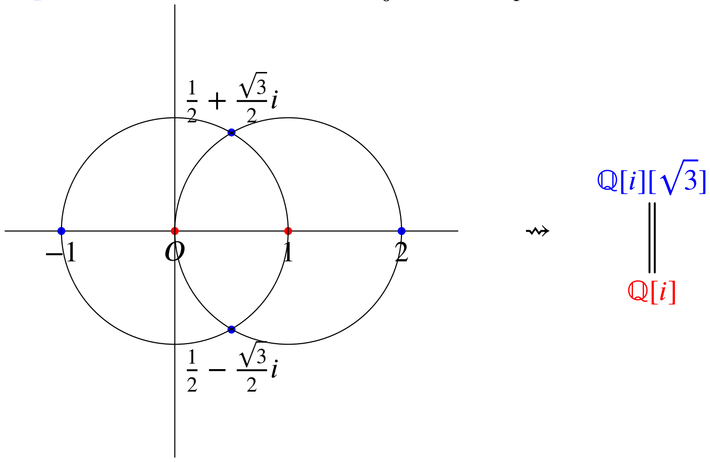
□

Let's see now what we can do by combining Propositions 3.50, 3.52, 3.54. Assume that a complex number z is constructible. This means that there is a finite sequence of steps, using the rules of construction, that will construct z from 0 and 1. We look at these steps one at a time, and at the corresponding happy-sad field on the right.

This can be easier to understand with an example:

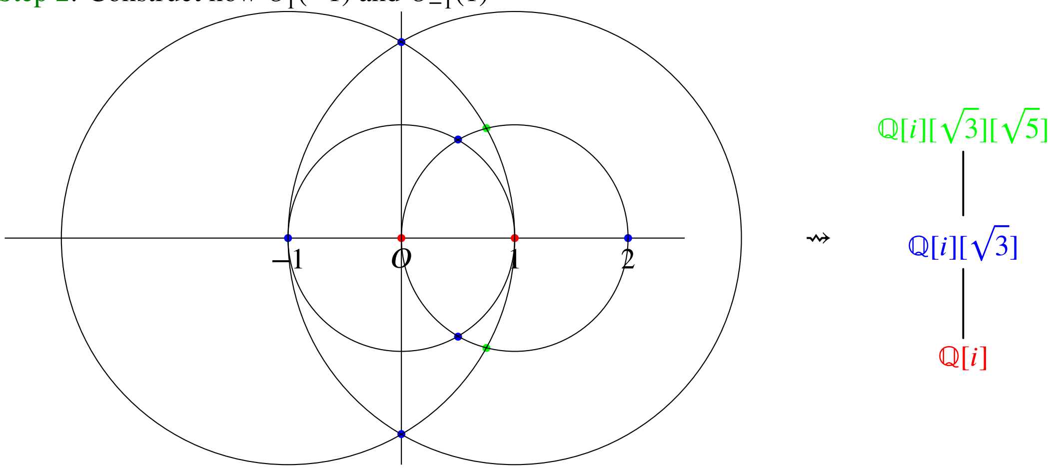
Example 3.56. Construct $3\sqrt{15}+12$ using straightedge and compass. In each step of the construction, give a happy-sad subfield of the form $\mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]...[\sqrt{x_n}]$, (with $x_i \in \mathbb{R}$) that contains all the constructed points so far.

Step 1: Construct the real line, $\mathscr{C}_0(1)$, and $\mathscr{C}_1(0)$

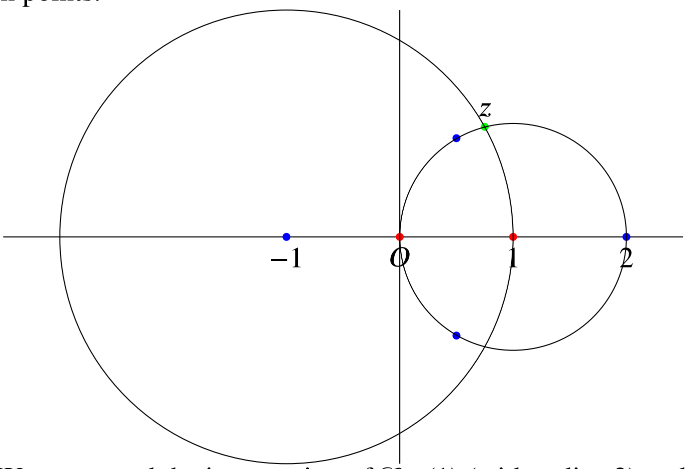


On the right, we have the happy-sad subfield $\mathbb{Q}[i][\sqrt{3}]$ which contains the intersections of all the lines and circles we've drawn so far. We *knew* from Proposition 3.54 that we could get the blue points in the intersection of the circles by adding a square root, in this case $\sqrt{3}$.

Step 2: Construct now $\mathscr{C}_1(-1)$ and $\mathscr{C}_{-1}(1)$



On the right, we have the happy-sad subfield $\mathbb{Q}[i][\sqrt{3}][\sqrt{5}]$ which we show below that is the one that contains the intersections of all the lines and circles we've drawn so far. The blue points $\pm\sqrt{3}i$ were already in $\mathbb{Q}[i][\sqrt{3}]$. We compute the green points:

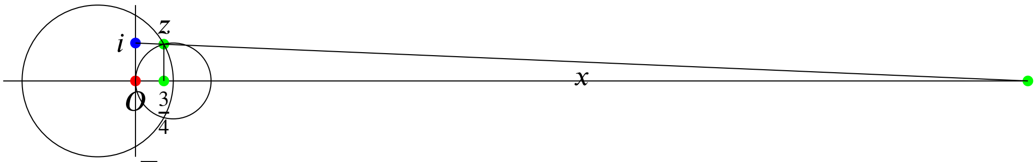


We computed the intersection of $\mathscr{C}_{-1}(1)$ (with radius 2) and $\mathscr{C}_1(0)$ (with radius 1) in Example 3.55: $z = \frac{3}{4} + \frac{\sqrt{15}}{4}i$. Note that $z \in \mathbb{Q}[i][\sqrt{3}][\sqrt{5}]$. (We could also use the field $\mathbb{Q}[i][\sqrt{3}][\sqrt{15}]$, it's actually the same field because $\sqrt{15} = \sqrt{3}\sqrt{5}$).

Again, we *knew* from Proposition 3.54 that we could get the green points in the intersection of the circles by adding a square root, in this case $\sqrt{5}$.

The line through i and $z = \frac{3}{4} + \frac{\sqrt{15}}{4}i$ meets the real axis at $3\sqrt{15} + 12$ (which by Proposition 3.50 we *knew* is a number in $\mathbb{Q}[i][\sqrt{3}][\sqrt{5}]$).

Why: We have two similar triangles:



So $\frac{x}{x+\frac{3}{4}} = \frac{\frac{\sqrt{15}}{4}}{1}$ Solving this we get $x = 3\sqrt{15} + \frac{45}{4}$ and the green point on the right is $x + \frac{3}{4} = 3\sqrt{15} + 12$.

> We can do this for any construction of any constructible number, step by step. And this means:...

Theorem 3.57. *Any constructible number is an element of a happy-sad subfield of $\mathbb{C}$ of the form $\mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]...[\sqrt{x_n}]$, where n is a natural number and $x_1 \in \mathbb{Q}[i]$, $x_2 \in \mathbb{Q}[i][\sqrt{x_1}]$, ..., $x_n \in \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]...[\sqrt{x_{n-1}}]$ are all real numbers.*

To prove the theorem, we do just as in the example above: we consider a step-by-step construction of the number, no matter how long, and we keep increasing the happy-sad subfields on the right:

$$
\begin{array}{ccc}
 & & \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}][\sqrt{x_n}] \\
 & & | \\
 & & \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}] \\
 & & | \\
 & & \cdots \\
 & & | \\
\text{Draw as many Lines and} & & \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}] \\
 & \rightsquigarrow & | \\
\text{Circles as you want} & & \color{green}{\mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]} \\
 & & | \\
 & & \color{blue}{\mathbb{Q}[i][\sqrt{x_1}]} \\
 & & | \\
 & & \color{red}{\mathbb{Q}[i]} \\
 & & | \\
 & & \mathbb{Q}
\end{array}
$$

You can imagine that you have a very big notepad, and you look at each step of the construction. You start your notepad with $\mathbb{Q}[i]$. As soon as a line and a circle or two circles intersect in a point that's not in $\mathbb{Q}[i]$, you compute x_1 (in the example above that's $x_1 = 3$) and you write it in your notepad. The existence of such x_1 is *guaranteed* by Propositions 3.52 and 3.54. And you patiently keep doing this: now for the first new point that appears that's not in $\mathbb{Q}[i][\sqrt{3}]$, you compute x_2 (in the example above that's $x_2 = 5$), and you continue until all the points appearing in the construction have been considered. The last x you compute is x_n.

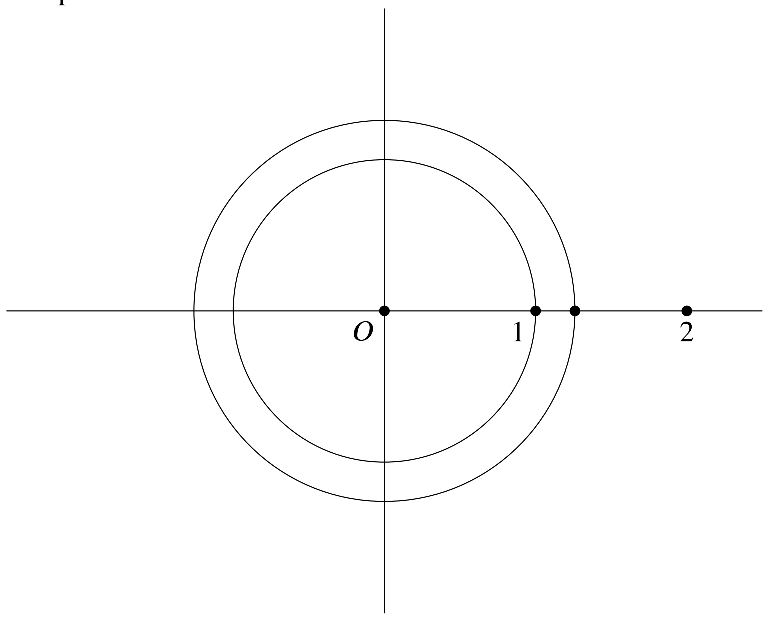
By construction, all the $x's$ are real numbers and $\color{blue}{x_1} \in \color{red}{\mathbb{Q}[i]}$, $\color{green}{x_2} \in \color{blue}{\mathbb{Q}[i][\sqrt{x_1}]}$, $x_3 \in \color{green}{\mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]}$, ... until the last one, $x_n \in \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}]$. This is exactly what the theorem says.

So, all the constructed points that are the intersections of the lines and circles that you drew “on the left", are complex numbers in the subfield $\mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}][\sqrt{x_n}]$ “on the right". This is how we “control" the constructible numbers.

Why does this imply that $\sqrt[3]{2}$ is not constructible? Let's see...[a]

[a] There are many ways to show this, but I'm writing here one that doesn't use any result from more advanced algebra like field theory.

Recall that there are three complex numbers z such that $z^3 = 2$. One is $\sqrt[3]{2}$, let's compute the other two:



Lemma 3.58. *Let $\mathbb{F}$ be a happy-sad subfield of $\mathbb{C}$. If z is any of the three complex numbers such that $z^3 = 2$, and $z \in \mathbb{F}$, then $\sqrt[3]{2}$ is in $\mathbb{F}$.*

Proof. This is because $Re(z) = \sqrt[3]{2}$ or $Re(z) = -\frac{1}{2}\sqrt[3]{2}$. But by Lemma 3.49 $Re(z) \in \mathbb{F}$, so in any case we get that $\sqrt[3]{2}$ is in $\mathbb{F}$.
($\sqrt[3]{2} = -\frac{1}{2}\sqrt[3]{2} \cdot (-2)$). □

Corollary 3.59. *If $\mathbb{F}$ is happy-sad, $\sqrt[3]{2} \notin \mathbb{F}$, and $x \in \mathbb{F}$, then $\sqrt[3]{2} \notin \mathbb{F}[\sqrt{x}]$.*

Proof. Let's assume that $\sqrt[3]{2} \in \mathbb{F}[\sqrt{x}]$ and show a contradiction. Let $a, b \in \mathbb{F}$ such that $\sqrt[3]{2} = a + b\sqrt{x}$, note that $b \neq 0$ because if $b = 0$ then $\sqrt[3]{2} = a \in \mathbb{F}$, so $\sqrt{x} = \frac{\sqrt[3]{2}-a}{b}$. Squaring the equality $\sqrt[3]{2} = a + b\sqrt{x}$ we get

$$(\sqrt[3]{2})^2 = a^2 + b^2x + 2ab\sqrt{x} = a^2 + b^2x + 2a(\sqrt[3]{2} - a)$$

Distributing, since $x \in \mathbb{F}$ and $\mathbb{F}$ is a subfield of $\mathbb{C}$ we get that $(\sqrt[3]{2})^2 = A + B\sqrt[3]{2}$, with $A, B \in \mathbb{F}$. Note that both A and B are different to 0:

- If $A = 0$ then $B = \sqrt[3]{2}$ would be in $\mathbb{F}$
- If $B = 0$ then $A = (\sqrt[3]{2})^2$ would be in $\mathbb{F}$, but then $2\sqrt[3]{2} = (\sqrt[3]{2})^4 = ((\sqrt[3]{2})^2)^2$ would be in $\mathbb{F}$ and then so would $\sqrt[3]{2}$.

We now square again:

$$2\sqrt[3]{2} = (\sqrt[3]{2})^4 = A^2 + B^2(\sqrt[3]{2})^2 + 2AB\sqrt[3]{2}$$

and then we get

$$-B^2(\sqrt[3]{2})^2 = A^2 + (2AB - 2)\sqrt[3]{2}$$

so

$$A + B\sqrt[3]{2} = (\sqrt[3]{2})^2 = \frac{A^2 + (2AB - 2)\sqrt[3]{2}}{-B^2}$$

that is

$$-B^2(A + B\sqrt[3]{2}) = A^2 + (2AB - 2)\sqrt[3]{2}$$

or equivalently

$$-B^2A - A^2 = (2AB - 2 + B^3)\sqrt[3]{2}$$

Now, if $2AB - 2 + B^3 \neq 0$, then we would get $\sqrt[3]{2} \in \mathbb{F}$. So $2AB - 2 + B^3 = 0$, and then $-B^2A - A^2 = 0$ too. Since $A \neq 0$, then we get $-B^2 = A$, and replacing $-2B^3 - 2 + B^3 = 0$, that is $(-B)^3 = 2$. By Lemma 3.58, since $-B \in \mathbb{F}$ this implies that $\sqrt[3]{2} \in \mathbb{F}$ which is a contradiction. □

Do you remember why $\sqrt[3]{2} \notin \mathbb{Q}$? Note that then $\sqrt[3]{2} \notin \mathbb{Q}[i]$ as well.

Theorem 3.60. $\sqrt[3]{2}$ *is not constructible.*

Proof. If $\sqrt[3]{2}$ was constructible, by Theorem 3.57 we have that

$$(**) \qquad \sqrt[3]{2} \in \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}][\sqrt{x_n}],$$

where n is a natural number and $x_1 \in \mathbb{Q}[i]$, $x_2 \in \mathbb{Q}[i][\sqrt{x_1}]$, $x_3 \in \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]$, ..., $x_n \in \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]...[\sqrt{x_{n-1}}]$ are all real numbers. But:

1. Since $\sqrt[3]{2} \notin \mathbb{Q}[i] \ni x_1$, by Corollary 3.59 then
   $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}]$
2. Since $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}] \ni x_2$, by Corollary 3.59 then
   $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}]$
3. Since $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}] \ni x_3$, by Corollary 3.59 then
   $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]$

   (...)

n. Since $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}] \ni x_n$, by Corollary 3.59 then
   $\sqrt[3]{2} \notin \mathbb{Q}[i][\sqrt{x_1}][\sqrt{x_2}][\sqrt{x_3}]...[\sqrt{x_{n-1}}][\sqrt{x_n}]$. This contradicts (**).

□

The question about the constructibility of $\sqrt[3]{2}$ was raised by the ancient Greeks, and answered more than two millenia later. Let's finish these notes with a quote from [1]:

> *The realization that there exists an algebra associated with constructions, and the development of field theory, are the two main accomplishments that allowed the mathematical community to answer two of the three major questions of antiquity concerning constructions.*

The *other two questions* are the trisection of an arbitrary angle and the squaring of the circle (i.e. constructing π). The method we used here to show that $\sqrt[3]{2}$ is not constructible is also relevant for answering the other two questions, though a bit more advanced theory is needed for those. You may encounter those in the future!

References

[1] Baragar, A., *A Survey of Classical and Modern Geometries: With Computer Activities*, Pearson Education (2001).

[2] Mumford D., *Pythagoras's rule*, `http://www.dam.brown.edu/people/mumford/blog/2015/Pythagoras.html`.

[3] Schwartz, R. *Life on the Infinite Farm*, AMS (2018). Available at `http://www.math.brown.edu/reschwar/99book.pdf`.

[4] Glaz Sarah, Sanders Mark, *Square Root of 2*, `https://www2.math.uconn.edu/~glaz/Square_Root_of_2/index.html`.

[5] Crabtree J., *Squaring the Circle, a Practical Approach*, Vinculum 53, 4, (2016), see also `https://www.jonathancrabtree.com/mathematics/squaring-the-circle-a-practical-approach/`