Analysis of the Simple Harmonic Oscillator

A foundational dynamical system

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1 1-dimensional simple harmonic oscillator: free and damped

The simple harmonic oscillator is one of the first problems solved using Newton's third law, $F = ma$. Usually this problem is framed as finding the 1-dimensional position of a mass on a spring, where the spring hangs from a fixed surface with a mass on the other end. The force is given by Hooke's law, which says that the restoring force of a spring is proportional to the distance from equilibrium, x, giving $F = -kx$, where k is the spring constant. Since a is the time derivative of position, Newton's third law can be written

$$
\begin{aligned}
-kx &= m\ddot{x} \\
\ddot{x} + \omega_0^2 x &= 0
\end{aligned}
$$

where $\omega_0 = \sqrt{k/m}$ is the *angular frequency* of the motion, or the number of cycles per second. x is often referred to as "displacement." In the case of a spring, ω is the rate at which the mass goes from its minimum displacement to its maximum displacement then returns to its minimum displacement. Because sine and cosine have the negative of themselves as their second derivatives, either can represent x and as a solution to 1a

$$x = A\sin(\omega_0 t - \delta) \tag{1a}$$
$$x = A\cos(\omega_0 t - \phi) \tag{1b}$$

Note these are are identical if $\delta - \phi = \pi/2$.

The damped harmonic oscillator adds a damping force to Newton's third law, $F_d = -b\dot{x}$. Damping effects, such as friction or air resistance, are always proportional to velocity. Now the equation of motion is

$$
\begin{aligned}
m\ddot{x} + b\dot{x} + kx &= 0 \\
\ddot{x} + 2\beta\dot{x} + \omega_0^2 x &= 0
\end{aligned}
$$

where $\beta = b/2m$. Now to solve the equations of motion, we consider the solution $x = e^{rt}$. Substituting we find

$$
\begin{aligned}
r^2 e^{rt} + 2\beta r e^{rt} + \omega_0^2 e^{rt} &= 0 \\
r^2 + 2\beta r + \omega_0^2 &= 0
\end{aligned}
$$

Thus,

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

There are three physically meaningful classifications of r: $r = -\beta$, r is real-valued ($\beta^2 > \omega_0^2$), or r is complex-valued ($\beta^2 < \omega_0^2$). These are referred to as critical damping, overdamping, and underdamping, respectively. In the critical or overdamped cases, all periodic behavior is removed from the system and the displacement goes directly to zero without switching sign. In the underdamped case, there is quasi-periodic behavior as the displacement oscillates around zero, eventually settling to $x = 0$. This is visualized in the phase diagrams below.

1.1 Phase diagrams

In the simple harmonic oscillator, the relevant state variables are the displacement, x, and the velocity $\dot{x}$. First we will consider the free harmonic oscillator, where the damping $b = 0$. Taking the solution $x(t) = A\sin(\omega_0 t - \delta)$, we find $\dot{x}(t) = A\omega_0 \cos(\omega_0 t - \delta)$. We can square both equations and divide the right-hand constants to get

$$\frac{x^2}{A^2} = \sin^2(\omega_0 t - \delta)$$
$$\frac{\dot{x}^2}{A^2\omega_0^2} = \cos^2(\omega_0 t - \delta)$$

Adding the two equations and using the relation $\sin^2(\theta) + \cos^2(\theta) = 1$ we recover the equation for an ellipse

$$\frac{x^2}{A^2} + \frac{\dot{x}^2}{A^2\omega_0^2} = 1$$

A few phase diagrams for the free harmonic oscillator are shown in Figure 1

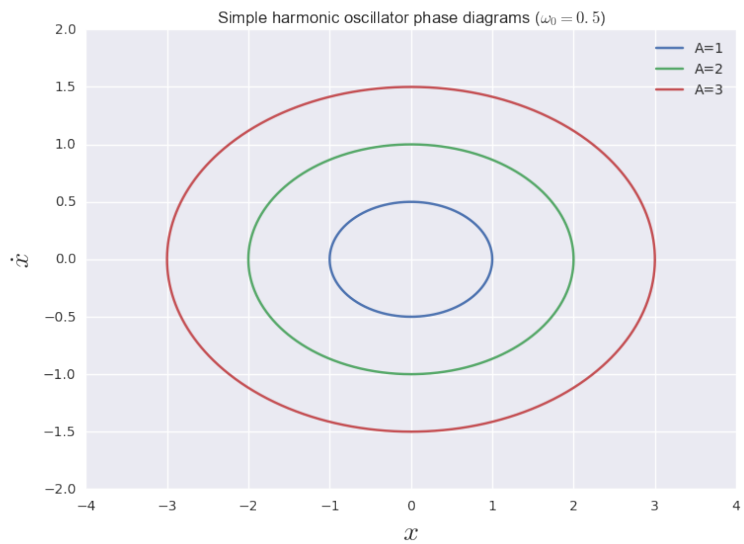

Figure 1: Phase portrait for some values of β for the underdamped harmonic oscillator

In the case of the underdamped harmonic oscillator, it can be shown that the solution is $x(t) = Ae^{-\beta t}\cos(\omega_1 t - \delta)$, where $\omega_1 = \sqrt{\omega_0^2 - \beta^2}$. Using the chain rule for derivatives, we find $\dot{x}(t) = -Ae^{-\beta t}(\beta\cos(\omega_1 t - \delta) + \omega_1 \sin(\omega_1 t - \delta))$. A few phase diagrams for the underdamped harmonic oscillator are shown in Figure 2.

Thornton, S. T., & Marion, J. B. (2004). *Classical Dynamics of Particles and Systems*. (5th ed.). Belmont, CA. Thompson Brooks/Cole.

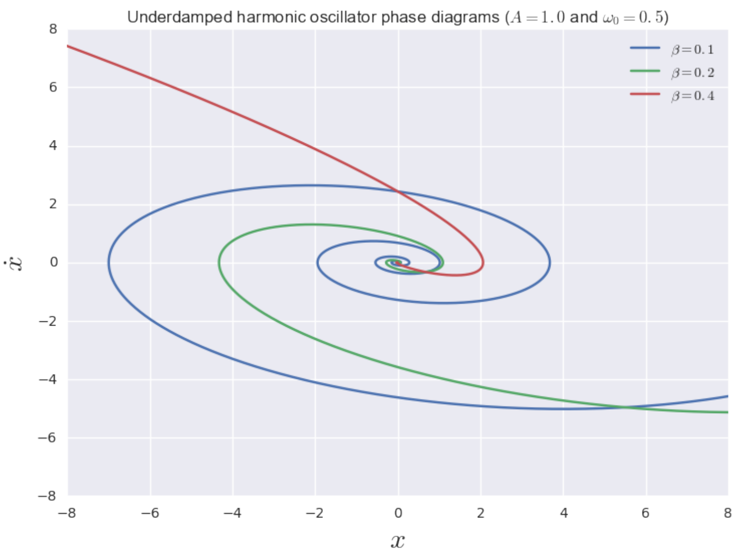

Figure 2: Phase portrait for some values of β for the underdamped harmonic oscillator