Let $G = (V, E)$ be an undirected graph on n vertices. A *clique* is a subset $S \subseteq V$ of vertices such that every pair of vertices in S are adjacent. The size of a clique is the number of vertices in the clique.

(a) Let $k \geq 1$ be an integer. How many distinct cliques of size k could there be in G?

(b) If G has a clique of size k, show that G has a clique of size ℓ for all $\ell \leq k$.

Solution.

(a) The number of distinct cliques of size k is at most equal to the number of subsets of V of size k, of which there are $\binom{n}{k}$, because all cliques are subsets of V. An example of when the number of distinct clique is equal to $\binom{n}{k}$ is in the complete graph K_n. All subsets of K_n of size k are cliques of size k, since all vertices in K_n are adjacent to all other vertices.

(b) Suppose that G has a clique of size k, and let $C \subseteq V$ be one of these cliques. Since C is a clique, for all $(u, v) \in C \times C$, u and v are adjacent. For any $l \leq k$, we can choose any subset $S \subseteq C$ of size l, and it will be a clique. This is because $S \times S \subseteq C \times C$, and thus for any $(u, v) \in S \times L$, the same pair (u, v) will also exist in C and therefore u and v will be adjacent.