Let $G = (V, E)$ be an undirected graph. A vertex v is a *cut vertex* if $G \setminus \{v\}$ has more connected components than G. In other words, v splits G into more connected components than G.

(a) Prove that every graph with at least two vertices has at least two vertices that are not cut vertices.

(b) If G is a simple and connected graph with *exactly* two vertices that are not cut vertices, then G must be a path.

Hint. *Firstly, show that every spanning tree in G has degree at most 2 (i.e. if T is a spanning tree in G, then every vertex in T has degree at most 2). Then, show that G cannot be a cycle.*

Solution.

(a) We can split G up into its connected components, and handle each one separately.

Lemma 1. *In a connected graph, every cut vertex is also a cut vertex of any of its spanning trees.*

Proof. Let v be a cut vertex of a connected graph C. By definition, removing v will split C into at least two connected components. This means that there exist vertices a, b in C such that after removing v, there will be no path between a and b using the edges in C.

If we consider the same vertices a, b on a spanning tree T of C, the removal of v will also lead to there no path existing between a and b, using the edges in S. This is because every edge in T is also in C. Therefore, removing v results in T being split into at least two connected components, and thus v is a cut vertex of any spanning tree. □

Lemma 2. *A tree with at least 2 vertices has at least 2 not cut vertices.*

Proof. All vertices with degree less than 2 are not cut. If a tree has n vertices, then it has $n - 1$ edges, and thus the sum of all the degrees is $2n - 2$ by the Handshake Lemma.

Assume for the sake of contradiction, that $n - 1$ vertices are cut. Since a vertex needs a degree of at least 2 to be cut, the sum of degrees is at least $2n - 2$. Thus the single not cut vertex must have a degree of 0, which is a contradiction since trees are connected. Therefore a tree cannot have $n - 1$ cut vertices.

Assume for the sake of contradiction, that n vertices are cut. The sum of degrees is at least $2n$ which is greater than the allowed sum of degrees $2n - 2$, and thus is a contradiction.

Therefore at most $n - 2$ vertices are cut, and equivalently, at least 2 vertices are not cut. □

Lemma 3. *A connected graph with at least 2 vertices has at least 2 not cut vertices.*

Proof. Trivially from Lemmas 1 and 2. □

If there exists a connected component C in G which contains more than 2 vertices, then G will have at least 2 not cut vertices (Lemma 3). Otherwise if all connected components contain only 1 vertex, then G has at least 2 isolated vertices. As isolated vertices are not cut, we can conclude that G has at least 2 not cut vertices.

(b) From Lemma 1, every cut vertex on G is also a cut vertex on a spanning tree T. Therefore T has at most 2 not cut vertices. Thus there can be at most 2 vertices in T with a degree less than 2. Therefore at least $n-2$ vertices in T have a degree of at least 2.

Let n be the number of vertices. The number of edges in T is therefore $n-1$, and hence the degree sum is $2n-2$ by the Handshake Lemma.

Assume for the sake of contradiction that there exists a vertex v in the spanning tree with a degree of at least 3. At least $n-3$ of the remaining $n-1$ edges must have a degree of at least 2. This brings the degree sum up to $3+2(n-3)$ which is equal to $2n-3$. The remaining two vertices have a degree of at least 1, bringing the degree sum up to $2n-1$ which is greater than that allowed by the Handshake Lemma, which is a contradiction.

Therefore any spanning tree in G has a degree of at most 2.

If G was a cycle, then there are always two paths between any two points, meaning that no removal of any vertex splits the graph into multiple connected components. Hence every vertex of G would be not cut. Since there are at least 3 vertices in a cycle, this contradicts the statement that G has exactly 2 not cut vertices.

Therefore G cannot be a cycle.

Assume for the sake of contradiction that at last one vertex in G had a degree of greater than 3. We could then construct a spanning tree from this vertex, including at least 3 of its adjacent edges, since it is always possible to expand a tree subgraph into a spanning tree. This contradicts the statement that any spanning tree in G has a degree of at most 2.

Therefore G has a degree of at most 2.

Since G is connected, it can either be a path, or a cycle. Since it is not a cycle, it must be a path.