Let f_1, f_2, g_1, g_2 be functions from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, and suppose that $f_1 \in O(g_1)$ and $f_2 \in O(g_2)$.

(a) Show that $f_1 + f_2 \in O(g_1 + g_2)$.

(b) Show that $f_1 \cdot f_2 \in O(g_1 \cdot g_2)$.

Solution.

Since $f_1 \in O(g_1)$, there exist positive constants c_1, n_1 such that

$$0 \leq f_1(n) \leq c_1 g_1(n) \quad \text{for all } n \geq n_1.$$

Similarly, since $f_2 \in O(g_2)$, there exist positive constants c_2, n_2 such that

$$0 \leq f_2(n) \leq c_2 g_2(n) \quad \text{for all } n \geq n_2.$$

(a) It suffices to show that there exist positive constants c, n_0 such that

$$0 \leq f_1(n) + f_2(n) \leq c g_1(n) + c g_2(n) \quad \text{for all } n \geq n_0.$$

Let $c = \max(c_1, c_2)$ and $n_0 = \max(n_1, n_2)$.
From the definition of $\max$, $c \geq c_1, c_2$ and $n_0 \geq n_1, n_2$.
If $n \geq n_0$, then $n \geq n_1$, thus $0 \leq f_1(n) \leq c_1 g_1(n)$ for all $n \geq n_0$.
Similarly $0 \leq f_2(n) \leq c_2 g_2(n)$ for all $n \geq n_0$.
Adding gives $0 \leq f_1(n) + f_2(n) \leq c_1 g_1(n) + c_2 g_2(n)$ for all $n \geq n_0$.
Since $c \geq c_1, c_2$, we deduce that $c_1 g_1(n) \leq c g_1(n)$ and $c_2 g_2(n) \leq c g_2(n)$.
Therefore $0 \leq f_1(n) + f_2(n) \leq c g_1(n) + c g_2(n)$ for all $n \geq n_0$, as required.

(b) It suffices to show that there exist positive constants c, n_0 such that

$$0 \leq f_1(n) \cdot f_2(n) \leq c g_1(n) \cdot g_2(n) \quad \text{for all } n \geq n_0.$$

Let $c = c_1 \cdot c_2$ and $n_0 = \max(n_1, n_2)$.
Like in part (a), $0 \leq f_1(n) \leq c_1 g_1(n)$ and $0 \leq f_2(n) \leq c_2 g_2(n)$ for all $n \geq n_0$.
As $f_1, f_2, g_1, g_2, c_1, c_2$ are all positive, multiplying gives a valid result.
Thus $0 \leq f_1(n) \cdot f_2(n) \leq c_1 g_1(n) \cdot c_2 g_2(n) \leq c g_1(n) \cdot g_2(n)$ for all $n \geq n_0$, as required.