

Problem 1.

y_t is a stationary process with the autocovariance function γ_k .

$$\bar{y} := \frac{1}{n} \sum_{t=1}^n y_t.$$

Show that

$$\text{Var}[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \gamma_k = \frac{1}{n} \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \gamma_k.$$

Solution.

$$\begin{aligned} \text{Var}[\bar{y}] &= \text{Cov}[\bar{y}, \bar{y}] \\ &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \text{Cov}[y_i, y_j] \\ &= \frac{1}{n^2} \sum_{i=1}^n \sum_{j=1}^n \gamma_{|i-j|} \end{aligned}$$

		i				
		1	2	3	\dots	n
j	1	γ_0	γ_1	γ_2	\dots	γ_{n-1}
	2	γ_1	γ_0	γ_1	\dots	γ_{n-2}
	3	γ_2	γ_1	γ_0	\dots	γ_{n-3}
	\vdots	\vdots	\vdots	\vdots	\ddots	\vdots
	n	γ_{n-1}	γ_{n-2}	γ_{n-3}	\dots	γ_0

$$\begin{aligned} \text{Var}[\bar{y}] &= \frac{1}{n^2} \left[n\gamma_0 + 2(n-1)\gamma_1 + 2(n-2)\gamma_2 + \dots + 2(n-(n-1))\gamma_{n-1} \right] \\ &= \frac{1}{n^2} \left[n\gamma_0 + 2 \sum_{k=1}^{n-1} (n-k)\gamma_k \right] \\ &= \frac{1}{n} \left[\gamma_0 + 2 \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \gamma_k \right] = \frac{\gamma_0}{n} + \frac{2}{n} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \gamma_k \\ &= \frac{1}{n} \left[\sum_{k=-(n-1)}^{-1} \left(1 - \frac{|k|}{n}\right) \gamma_k + \left(1 - \frac{|0|}{n}\right) \gamma_0 + \sum_{k=1}^{n-1} \left(1 - \frac{|k|}{n}\right) \gamma_k \right] \\ &= \frac{1}{n} \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \gamma_k \end{aligned}$$

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Problem 2.

Assume x_t is a stationary process and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$

- (a) Show that $\text{Cov}[y_t, y_{t-k}]$ is independent of t for all lags k .
 (b) Is y_t stationary?

Solution.

(a)

For stationary process x_t , let $\gamma_k = \text{Cov}[x_t, x_{t-k}]$

$$\text{Cov}[y_t, y_{t-k}] = \begin{cases} \text{Cov}[x_t, x_{t-k}] & t \text{ odd}, k \text{ even} \\ \text{Cov}[x_t, x_{t-k} + 3] & t \text{ odd}, k \text{ odd} \\ \text{Cov}[x_t + 3, x_{t-k} + 3] & t \text{ even}, k \text{ even} \\ \text{Cov}[x_t + 3, x_{t-k}] & t \text{ even}, k \text{ odd} \end{cases} = \gamma_k$$

Therefore, $\text{Cov}[y_t, y_{t-k}]$ is independent of t for all lags k .

Stationarity

- Strict Stationarity
 - Too strong for most applications
 - The whole probability structure must depend only on time differences
 - Second-order stationarity and an assumption of normality are sufficient to produce strict stationarity.
- Second-order Stationarity (Weak Stationarity)
 - Imposes conditions only on the first two moments of the series
 - The mean $E[x_t] = \mu$ is a fixed constant for all t
 - The autocovariances $\text{Cov}[x_t, x_{t-k}] = \gamma_k$ depend only on the time difference or time lag k for all t .

(b)

$$E[y_t] = \begin{cases} E[x_t] & t \text{ odd} \\ E[x_t] + 3 & t \text{ even} \end{cases}$$

The mean of the process $\{y_t\}$ is not a constant. Therefore, the process $\{y_t\}$ is nonstationary. ■

Problem 3.

Let $\{y_t\}$ be a stationary process with an autocovariance function γ_k .

(a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t .

(b) Show that $w_t = \nabla_2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.

Solution.

(a)

Since $\{y_t\}$ is a stationary process, it has constant mean μ over time and autocovariance function γ_k which does not depend on time t . We have $z_t = \nabla y_t = y_t - y_{t-1}$

$$\begin{aligned} \mathbf{E}[z_t] &= \mathbf{E}[y_t - y_{t-1}] \\ &= \mathbf{E}[y_t] - \mathbf{E}[y_{t-1}] \\ &= \mu - \mu = 0 \end{aligned}$$

The mean function of $\{z_t\}$ is constant over time t .

$$\begin{aligned} \text{Cov}[z_t, z_{t-k}] &= \text{Cov}[y_t - y_{t-1}, y_{t-k} - y_{t-k-1}] \\ &= \text{Cov}[y_t, y_{t-k}] + \text{Cov}[y_t, -y_{t-k-1}] + \text{Cov}[-y_{t-1}, y_{t-k}] + \text{Cov}[-y_{t-1}, -y_{t-k-1}] \\ &= \text{Cov}[y_t, y_{t-k}] - \text{Cov}[y_t, y_{t-k-1}] - \text{Cov}[y_{t-1}, y_{t-k}] + \text{Cov}[y_{t-1}, y_{t-k-1}] \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= -\gamma_{k+1} + 2\gamma_k - \gamma_{k-1} \quad := \gamma'_k \end{aligned}$$

The autocovariance function of $\{z_t\}$ only depends on time lag k . So we conclude that $\{z_t\}$ is stationary.

(b)

$$\begin{aligned} \mathbf{E}[w_t] &= \mathbf{E}[z_t - z_{t-1}] \\ &= \mathbf{E}[z_t] - \mathbf{E}[z_{t-1}] \\ &= 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{Cov}[w_t, w_{t-k}] &= \text{Cov}[z_t - z_{t-1}, z_{t-k} - z_{t-k-1}] \\ &= \text{Cov}[z_t, z_{t-k}] + \text{Cov}[z_t, -z_{t-k-1}] + \text{Cov}[-z_{t-1}, z_{t-k}] + \text{Cov}[-z_{t-1}, -z_{t-k-1}] \\ &= \text{Cov}[z_t, z_{t-k}] - \text{Cov}[z_t, z_{t-k-1}] - \text{Cov}[z_{t-1}, z_{t-k}] + \text{Cov}[z_{t-1}, z_{t-k-1}] \\ &= \gamma'_k - \gamma'_{k+1} - \gamma'_{k-1} + \gamma'_k \\ &= -\gamma'_{k+1} + 2\gamma'_k - \gamma'_{k-1} \end{aligned}$$

The mean function of $\{w_t\}$ is constant over time t . The autocovariance function of $\{w_t\}$ only depends on time lag k . So we conclude that $\{w_t\}$ is stationary. ■

Problem 4.

Let x_t be stationary with $E[x_t] = 0$, $\text{Var}[x_t] = 1$, autocorrelation function ρ_k . Define that μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are both deterministic and in function of t). Now we observe a time series formulated as

$$y_t = \mu_t + \sigma_t x_t.$$

- (a) Find the mean and autocovariance function of y_t .
- (b) Show that the autocorrelation of y_t depends only on the lag k . Is y_t stationary?
- (c) Let $\mu_t = \mu_0$ be a constant value. Justify that y_t is still nonstationary.

Solution.

(a)

$$\begin{aligned} E[y_t] &= E[\mu_t + \sigma_t x_t] \\ &= E[\mu_t] + E[\sigma_t x_t] \\ &= \mu_t + 0 = \mu_t \end{aligned}$$

$$\begin{aligned} \text{Cov}[y_t, y_{t-k}] &= \text{Cov}[\mu_t + \sigma_t x_t, \mu_{t-k} + \sigma_{t-k} x_{t-k}] \\ &= \text{Cov}[\sigma_t x_t, \sigma_{t-k} x_{t-k}] \\ &= \sigma_t \sigma_{t-k} \text{Cov}[x_t, x_{t-k}] \\ &= \sigma_t \sigma_{t-k} \text{Corr}[x_t, x_{t-k}] \sqrt{\text{Var}[x_t] \text{Var}[x_{t-k}]} \\ &= \sigma_t \sigma_{t-k} \text{Corr}[x_t, x_{t-k}] \text{Var}[x_t] \\ &= \sigma_t \sigma_{t-k} \rho_k \end{aligned}$$

(b)

$$\text{Var}[y_t] = \sigma_t^2 \underbrace{\rho_0}_1 = \sigma_t^2$$

$$\begin{aligned} \text{Corr}[y_t, y_{t-k}] &= \frac{\text{Cov}[y_t, y_{t-k}]}{\sqrt{\text{Var}[y_t] \text{Var}[y_{t-k}]}} \\ &= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{\sigma_t^2 \sigma_{t-k}^2}} \\ &= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k \end{aligned}$$

The autocorrelation of y_t depends only on the lag k .

y_t is not stationary because the mean function μ_t is not a constant over time t .

(c)

Let $\mu_t = \mu_0$ be a constant value. y_t is still nonstationary because the autocovariance function depend on time t .

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Problem 5.

Let x_t be the series of the "expected" measurements during the production process. Because the measuring tool itself won't be perfect, we observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the signal and e_t the measurement (white) noise. If x_t is stationary with the autocorrelation function ρ_k , show that y_t is also a stationary process with

$$\text{Corr}[y_t, y_{t-k}] = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \quad \text{for } k \geq 1.$$

$\frac{\sigma_x^2}{\sigma_e^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR** for short. The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t .

White Noise Process

The most fundamental example of a stationary process is a sequence of **independent and identically distributed** random variables, denoted as $\alpha_1, \dots, \alpha_t, \dots$, which we also assume to have **mean zero** and variance σ_α^2 . This process is strictly stationary and is referred to as a **white noise process**. Because independence implies that the α_t are uncorrelated, its autocovariance function is simply

$$\gamma_k = \text{E}[x_t, x_{t-k}] = \begin{cases} \sigma_\alpha^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Solution.

$$\begin{aligned} \text{E}[y_t] &= \text{E}[x_t + e_t] \\ &= \text{E}[x_t] + \text{E}[e_t] \\ &= \mu \quad \text{is a fixed constant} \end{aligned}$$

$$\begin{aligned} \text{Cov}[y_t, y_{t-k}] &= \text{Cov}[x_t + e_t, x_{t-k} + e_{t-k}] \\ &= \text{Cov}[x_t, x_{t-k}] + \text{Cov}[x_t, e_{t-k}] + \text{Cov}[e_t, x_{t-k}] + \text{Cov}[e_t, e_{t-k}] \\ &= \text{Corr}[x_t, x_{t-k}] \sigma_x^2 + \text{Cov}[e_t, e_{t-k}] \quad x_t \text{ and } e_t \text{ are independent} \\ &= \begin{cases} \rho_0 \sigma_x^2 + \sigma_e^2 & k = 0 \\ \rho_k \sigma_x^2 + 0 & k \geq 1 \end{cases} \\ &= \begin{cases} \sigma_x^2 + \sigma_e^2 = \text{Var}[y_t] & k = 0 \\ \rho_k \sigma_x^2 & k \geq 1 \end{cases} \end{aligned}$$

y_t is a stationary process with constant mean and autocovariances depending only on the time difference or time lag k for all t .

$$\begin{aligned}\text{Corr}[y_t, y_{t-k}] &= \frac{\text{Cov}[y_t, y_{t-k}]}{\sqrt{\text{Var}[y_t]\text{Var}[y_{t-k}]}} \\ &= \frac{\rho_k \sigma_x^2}{\text{Var}[y_t]} \\ &= \frac{\rho_k \sigma_x^2}{\sigma_x^2 + \sigma_e^2} \\ &= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}} \quad \text{for } k \geq 1.\end{aligned}$$

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Problem 6. Cyclical Behavior and Periodicity**Mixtures of periodic series with multiple frequencies and amplitudes:**

Suppose

$$y_t = \alpha_0 + \sum_{i=1}^q [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)],$$

where $\alpha_0, f_1, f_2, \dots, f_q$ are constants and $\alpha_1, \alpha_2, \dots, \alpha_q, \beta_1, \beta_2, \dots, \beta_q$ are independent random variables with zero means and variances

$$\text{Var}[\alpha_i] = \text{Var}[\beta_i] = \sigma_i^2.$$

Show that y_t is stationary and find its autocovariance function.

(Hint: show $\text{Cov}[y_t, y_s]$ depends only on $t - s$.)

Solution.

$$\begin{aligned} \mathbb{E}[y_t] &= \mathbb{E} \left[\alpha_0 + \sum_{i=1}^q [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)] \right] \\ &= \mathbb{E}[\alpha_0] + \mathbb{E} \left[\sum_{i=1}^q [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)] \right] \\ &= \mathbb{E}[\alpha_0] + \sum_{i=1}^q \left\{ \cos(2\pi f_i t) \underbrace{\mathbb{E}[\alpha_i]}_0 + \sin(2\pi f_i t) \underbrace{\mathbb{E}[\beta_i]}_0 \right\} \\ &= \mathbb{E}[\alpha_0] \quad \text{constant} \end{aligned}$$

Let

$$s = t - k$$

$$u_i(t) = \alpha_i \cos(2\pi f_i t) = \alpha_i c_{it}$$

$$v_i(t) = \beta_i \sin(2\pi f_i t) = \beta_i s_{it}$$

$$\begin{aligned} \text{Cov}[y_t, y_s] &= \text{Cov} \left[\left\{ \alpha_0 + \sum_{i=1}^q [u_i(t) + v_i(t)] \right\}, \left\{ \alpha_0 + \sum_{i=1}^q [u_i(s) + v_i(s)] \right\} \right] \\ &= \text{Cov} \left[\left\{ \sum_{i=1}^q [u_i(t) + v_i(t)] \right\}, \left\{ \sum_{i=1}^q [u_i(s) + v_i(s)] \right\} \right] \\ &= \sum_{i=1}^q \sum_{j=1}^q \{ \text{Cov}[u_i(t), u_j(s)] + \text{Cov}[u_i(t), v_j(s)] + \text{Cov}[v_i(t), u_j(s)] + \text{Cov}[v_i(t), v_j(s)] \} \\ &= \sum_{i=1}^q \sum_{j=1}^q \{ \text{Cov}[\alpha_i c_{it}, \alpha_j c_{js}] + \text{Cov}[\alpha_i c_{it}, \beta_j s_{js}] + \text{Cov}[\beta_i s_{it}, \alpha_j c_{js}] + \text{Cov}[\beta_i s_{it}, \beta_j s_{js}] \} \\ &= \sum_{i=1}^q \{ c_{it} c_{is} \text{Cov}[\alpha_i, \alpha_i] + s_{it} s_{is} \text{Cov}[\beta_i, \beta_i] \} \quad \alpha_1, \alpha_2, \dots, \alpha_q, \beta_1, \beta_2, \dots, \beta_q \text{ are independent r.v.} \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^q \{c_{it}c_{is} \text{Var}[\alpha_i] + s_{it}s_{is} \text{Var}[\beta_i]\} \\
&= \sum_{i=1}^q \{c_{it}c_{is} + s_{it}s_{is}\} \sigma_i^2 \\
&= \sum_{i=1}^q \{\cos(2\pi f_i t) \cos(2\pi f_i s) + \sin(2\pi f_i t) \sin(2\pi f_i s)\} \sigma_i^2 \\
&= \sum_{i=1}^q \{\cos(2\pi f_i t - 2\pi f_i s)\} \sigma_i^2 \\
&= \sum_{i=1}^q \{\cos(2\pi f_i k)\} \sigma_i^2
\end{aligned}$$

$\gamma_k = \text{Cov}[y_t, y_{t-k}] = \text{Cov}[y_t, y_s]$ depends only on $k = t - s$.

y_t is a stationary process with constant mean and autocovariances depending only on the time difference or time lag k for all t .

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Trigonometric Identities (Sum & Difference Identities)

$$\cos(\alpha \pm \beta) = \cos(\alpha) \cos(\beta) \mp \sin(\alpha) \sin(\beta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha) \cos(\beta) \pm \cos(\alpha) \sin(\beta)$$

$$\tan(\alpha \pm \beta) = \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha) \tan(\beta)}$$