Problem 1.

y_t is a is a stationary process with the autocovariance function γ_k.

$$\bar{y} := \frac{1}{n}\sum_{t=1}^{n} y_t.$$

Show that

$$\mathrm{Var}[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k = \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k.$$

Solution.

$$\begin{aligned}\mathrm{Var}[\bar{y}] &= \mathrm{Cov}[\bar{y},\bar{y}]\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{Cov}[y_i,y_j]\\ &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\gamma_{|i-j|}\end{aligned}$$

	$\gamma_{\lvert i-j \rvert}$	i 1	2	3	$\cdots$	n
j	1	γ_0	γ_1	γ_2	$\cdots$	γ_{n-1}
	2	γ_1	γ_0	γ_1	$\cdots$	γ_{n-2}
	3	γ_2	γ_1	γ_0	$\cdots$	γ_{n-3}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	n	γ_{n-1}	γ_{n-2}	γ_{n-3}	$\cdots$	γ_0

$$\begin{aligned}\mathrm{Var}[\bar{y}] &= \frac{1}{n^2}\left[n\gamma_0 + 2(n-1)\gamma_1 + 2(n-2)\gamma_2 + \cdots + 2(n-(n-1))\gamma_{n-1}\right]\\ &= \frac{1}{n^2}\left[n\gamma_0 + 2\sum_{k=1}^{n-1}(n-k)\gamma_k\right]\\ &= \frac{1}{n}\left[\gamma_0 + 2\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\right] \quad = \frac{\gamma_0}{n} + \frac{2}{n}\sum_{k=1}^{n-1}\left(1-\frac{k}{n}\right)\gamma_k\\ &= \frac{1}{n}\left[\sum_{k=-(n-1)}^{1}\left(1-\frac{|k|}{n}\right)\gamma_k + \left(1-\frac{|0|}{n}\right)\gamma_0 + \sum_{k=1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k\right]\\ &= \frac{1}{n}\sum_{k=-n+1}^{n-1}\left(1-\frac{|k|}{n}\right)\gamma_k\end{aligned}$$

■

Problem 2.

Assume x_t is a stationary process and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$

(a) Show that $\mathrm{Cov}[y_t, y_{t-k}]$ is independent of t for all lags k.

(b) Is y_t stationary?

Solution.

(a)

For stationary process x_t, let $\gamma_k = \mathrm{Cov}[x_t, x_{t-k}]$

$$\mathrm{Cov}[y_t, y_{t-k}] = \left\{ \begin{array}{ll} \mathrm{Cov}[x_t, x_{t-k}] & t \text{ odd, } k \text{ even} \\ \mathrm{Cov}[x_t, x_{t-k} + 3] & t \text{ odd, } k \text{ odd} \\ \mathrm{Cov}[x_t + 3, x_{t-k} + 3] & t \text{ even, } k \text{ even} \\ \mathrm{Cov}[x_t + 3, x_{t-k}] & t \text{ even, } k \text{ odd} \end{array} \right\} = \gamma_k$$

Therefore, $\mathrm{Cov}[y_t, y_{t-k}]$ is independent of t for all lags k.

Stationarity

- Strict Stationarity
  - Too strong for most applications
  - The whole probability structure must depend only on time differences
  - Second-order stationarity and an assumption of normality are sufficient to produce strict stationarity.
- Second-order Stationarity (Weak Stationarity)
  - Imposes conditions only on the first two moments of the series
  - The mean $\mathrm{E}[x_t] = \mu$ is a fixed constant for all t
  - The autocovariances $\mathrm{Cov}[x_t, x_{t-k}] = \gamma_k$ depend only on the time difference or time lag k for all t.

(b)

$$\mathrm{E}[y_t] = \begin{cases} x_t & t \text{ odd} \\ x_t + 3 & t \text{ even} \end{cases}$$

The mean of the process $\{y_t\}$ is not a constant. Therefore, the process $\{y_t\}$ is nonstationary.

■

Problem 3.

Let $\{y_t\}$ be a stationary process with an autocovariance function γ_k.

(a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t.

(b) Show that $w_t = \nabla_2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.

Solution.

(a)

Since $\{y_t\}$ is a stationary process, it has constant mean μ over time and autocovariance function γ_k which does not depend on time t. We have $z_t = \nabla y_t = y_t - y_{t-1}$

$$\begin{aligned} \mathrm{E}[z_t] &= \mathrm{E}[y_t - y_{t-1}] \\ &= \mathrm{E}[y_t] - \mathrm{E}[y_{t-1}] \\ &= \mu - \mu = 0 \end{aligned}$$

The mean function of $\{z_t\}$ is constant over time t.

$$\begin{aligned} \mathrm{Cov}[z_t, z_{t-k}] &= \mathrm{Cov}[y_t - y_{t-1}, y_{t-k} - y_{t-k-1}] \\ &= \mathrm{Cov}[y_t, y_{t-k}] + \mathrm{Cov}[y_t, -y_{t-k-1}] + \mathrm{Cov}[-y_{t-1}, y_{t-k}] + \mathrm{Cov}[-y_{t-1}, -y_{t-k-1}] \\ &= \mathrm{Cov}[y_t, y_{t-k}] - \mathrm{Cov}[y_t, y_{t-k-1}] - \mathrm{Cov}[y_{t-1}, y_{t-k}] + \mathrm{Cov}[y_{t-1}, y_{t-k-1}] \\ &= \gamma_k - \gamma_{k+1} - \gamma_{k-1} + \gamma_k \\ &= -\gamma_{k+1} + 2\gamma_k - \gamma_{k-1} \quad := \gamma'_k \end{aligned}$$

The autocovariance function of $\{z_t\}$ only depends on time lag k. So we conclude that $\{z_t\}$ is stationary.

(b)

$$\begin{aligned} \mathrm{E}[w_t] &= \mathrm{E}[z_t - z_{t-1}] \\ &= \mathrm{E}[z_t] - \mathrm{E}[z_{t-1}] \\ &= 0 - 0 = 0 \end{aligned}$$

$$\begin{aligned} \mathrm{Cov}[w_t, w_{t-k}] &= \mathrm{Cov}[z_t - z_{t-1}, z_{t-k} - z_{t-k-1}] \\ &= \mathrm{Cov}[z_t, z_{t-k}] + \mathrm{Cov}[z_t, -z_{t-k-1}] + \mathrm{Cov}[-z_{t-1}, z_{t-k}] + \mathrm{Cov}[-z_{t-1}, -z_{t-k-1}] \\ &= \mathrm{Cov}[z_t, z_{t-k}] - \mathrm{Cov}[z_t, z_{t-k-1}] - \mathrm{Cov}[z_{t-1}, z_{t-k}] + \mathrm{Cov}[z_{t-1}, z_{t-k-1}] \\ &= \gamma'_k - \gamma'_{k+1} - \gamma'_{k-1} + \gamma'_k \\ &= -\gamma'_{k+1} + 2\gamma'_k - \gamma'_{k-1} \end{aligned}$$

The mean function of $\{w_t\}$ is constant over time t. The autocovariance function of $\{w_t\}$ only depends on time lag k. So we conclude that $\{w_t\}$ is stationary.

■

Problem 4.

Let x_t be stationary with $\mathrm{E}[x_t] = 0$, $\mathrm{Var}[x_t] = 1$, autocorrelation function ρ_k . Define that μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are both deterministic and in function of t). Now we observe a time series formulated as

$$y_t = \mu_t + \sigma_t x_t.$$

(a) Find the mean and autocovariance function of y_t.
(b) Show that the autocorrelation of y_t depends only on the lag k. Is y_t stationary?
(c) Let $\mu_t = \mu_0$ be a constant value. Justify that y_t is still nonstationary.

Solution.
(a)

$$\begin{aligned}
\mathrm{E}[y_t] &= \mathrm{E}[\mu_t + \sigma_t x_t] \\
&= \mathrm{E}[\mu_t] + \mathrm{E}[\sigma_t x_t] \\
&= \mu_t + 0 = \mu_t
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}[y_t, y_{t-k}] =& \mathrm{Cov}[\mu_t + \sigma_t x_t, \mu_{t-k} + \sigma_{t-k} x_{t-k}] \\
&= \mathrm{Cov}[\sigma_t x_t, \sigma_{t-k} x_{t-k}] \\
&= \sigma_t \sigma_{t-k} \mathrm{Cov}[x_t, x_{t-k}] \\
&= \sigma_t \sigma_{t-k} \mathrm{Corr}[x_t, x_{t-k}] \sqrt{\mathrm{Var}[x_t]\mathrm{Var}[x_{t-k}]} \\
&= \sigma_t \sigma_{t-k} \mathrm{Corr}[x_t, x_{t-k}] \mathrm{Var}[x_t] \\
&= \sigma_t \sigma_{t-k} \rho_k
\end{aligned}$$

(b)

$$\mathrm{Var}[y_x] = \sigma_t^2 \underbrace{\rho_0}_{1} = \sigma_t^2$$

$$\begin{aligned}
\mathrm{Corr}[y_t, y_{t-k}] &= \frac{\mathrm{Cov}[y_t, y_{t-k}]}{\sqrt{\mathrm{Var}[y_t]\mathrm{Var}[y_{t-k}]}} \\
&= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sqrt{(\sigma_t^2 \rho_k)(\sigma_{t-k}^2 \rho_k)}} \\
&= \frac{\sigma_t \sigma_{t-k} \rho_k}{\sigma_t \sigma_{t-k}} = \rho_k
\end{aligned}$$

The autocorrelation of y_t depends only on the lag k.
y_t is not stationary because the mean function μ_t is not a constant over time t.

(c)

Let $\mu_t = \mu_0$ be a constant value. y_t is still nonstationary because the autocovariance function depend on time t.

■

Problem 5.

Let x_t be the series of the "expected" measurements during the production process. Because the measuring tool itself won't be perfect, we observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the signal and e_t the measurement (white) noise. If x_t is stationary with the autocorrelation function ρ_k, show that y_t is also a stationary process with

$$\text{Corr}[y_t, y_{t-k}] = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \quad \text{for } k \geq 1.$$

$\frac{\sigma_e^2}{\sigma_x^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR** for short. The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t.

White Noise Process

The most fundamental example of a stationary process is a sequence of **independent and identically distributed** random variables, denoted as $\alpha_1, \ldots, \alpha_t, \ldots$, which we also assume to have **mean zero** and variance σ_α^2. This process is strictly stationary and is referred to as a **white noise process**. Because independence implies that the α_t are uncorrelated, its autocovariance function is simply

$$\gamma_k = \text{E}[x_t, x_{t-k}] = \begin{cases} \sigma_\alpha^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Solution.

$$\begin{aligned} \text{E}[y_t] &= \text{E}[x_t + e_t] \\ &= \text{E}[x_t] + \text{E}[e_t] \\ &= \mu \quad \text{is a fixed constant} \end{aligned}$$

$$\begin{aligned} \text{Cov}[y_t, y_{t-k}] &= \text{Cov}[x_t + e_t, x_{t-k} + e_{t-k}] \\ &= \text{Cov}[x_t, x_{t-k}] + \text{Cov}[x_t, e_{t-k}] + \text{Cov}[e_t, x_{t-k}] + \text{Cov}[e_t, e_{t-k}] \\ &= \text{Corr}[x_t, x_{t-k}]\sigma_x^2 + \text{Cov}[e_t, e_{t-k}] \quad x_t \text{ and } e_t \text{ are independent} \\ &= \begin{cases} \rho_0\sigma_x^2 + \sigma_e^2 & k = 0 \\ \rho_k\sigma_x^2 + 0 & k \geq 1 \end{cases} \\ &= \begin{cases} \sigma_x^2 + \sigma_e^2 = \text{Var}[y_t] & k = 0 \\ \rho_k\sigma_x^2 & k \geq 1 \end{cases} \end{aligned}$$

y_t is a stationary process with constant mean and autocovariances depending only on the time difference or time lag k for all t.

$$\begin{aligned}
\mathrm{Corr}[y_t, y_{t-k}] &= \frac{\mathrm{Cov}[y_t, y_{t-k}]}{\sqrt{\mathrm{Var}[y_t]\mathrm{Var}[y_{t-k}]}} \\
&= \frac{\rho_k \sigma_x^2}{\mathrm{Var}[y_t]} \\
&= \frac{\rho_k \sigma_x^2}{\sigma_x^2 + \sigma_e^2} \\
&= \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}} \quad \text{for } k \geq 1.
\end{aligned}$$

■

Problem 6. Cyclical Behavior and Periodicity

Mixtures of periodic series with multiple frequencies and amplitudes:
Suppose

$$y_t = \alpha_0 + \sum_{i=1}^{q} \left[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)\right],$$

where $\alpha_0, f_1, f_2, \ldots, f_q$ are constants and $\alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_q$ are independent random variables with zero means and variances

$$\text{Var}[\alpha_i] = \text{Var}[\beta_i] = \sigma_i^2.$$

Show that y_t is stationary and find its autocovariance function.
(Hint: show $\text{Cov}[y_t, y_s]$ depends only on $t - s$.

Solution.

$$\begin{aligned}
\text{E}[y_t] &= \text{E}\left[\alpha_0 + \sum_{i=1}^{q} \left[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)\right]\right] \\
&= \text{E}[\alpha_0] + \text{E}\left[\sum_{i=1}^{q} \left[\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)\right]\right] \\
&= \text{E}[\alpha_0] + \sum_{i=1}^{q} \left\{ \cos(2\pi f_i t) \underbrace{\text{E}[\alpha_i]}_{0} + \sin(2\pi f_i t) \underbrace{\text{E}[\beta_i]}_{0} \right\} \\
&= \text{E}[\alpha_0] \quad \text{constant}
\end{aligned}$$

Let

$$\begin{aligned}
s &= t - k \\
u_i(t) &= \alpha_i \cos(2\pi f_i t) = \alpha_i c_{it} \\
v_i(t) &= \beta_i \sin(2\pi f_i t) = \beta_i s_{it}
\end{aligned}$$

$$\begin{aligned}
\text{Cov}[y_t, y_s] &= \text{Cov}\left[\left\{\alpha_0 + \sum_{i=1}^{q} [u_i(t) + v_i(t)]\right\}, \left\{\alpha_0 + \sum_{i=1}^{q} [u_i(s) + v_i(s)]\right\}\right] \\
&= \text{Cov}\left[\left\{\sum_{i=1}^{q} [u_i(t) + v_i(t)]\right\}, \left\{\sum_{i=1}^{q} [u_i(s) + v_i(s)]\right\}\right] \\
&= \sum_{i=1}^{q} \sum_{j=1}^{q} \left\{\text{Cov}[u_i(t), u_j(s)] + \text{Cov}[u_i(t), v_j(s)] + \text{Cov}[v_i(t), u_j(s)] + \text{Cov}[v_i(t), v_j(s)]\right\} \\
&= \sum_{i=1}^{q} \sum_{j=1}^{q} \left\{\text{Cov}[\alpha_i c_{it}, \alpha_j c_{js}] + \text{Cov}[\alpha_i c_{it}, \beta_j s_{js}] + \text{Cov}[\beta_i s_{it}, \alpha_j c_{js}] + \text{Cov}[\beta_i s_{it}, \beta_j s_{js}]\right\} \\
&= \sum_{i=1}^{q} \left\{c_{it} c_{is} \text{Cov}[\alpha_i, \alpha_i] + s_{it} s_{is} \text{Cov}[\beta_i, \beta_i]\right\} \quad \alpha_1, \alpha_2, \ldots, \alpha_q, \beta_1, \beta_2, \ldots, \beta_q \text{ are independent r.v.}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{q} \{c_{it}c_{is}\mathrm{Var}[\alpha_i] + s_{it}s_{is}\mathrm{Var}[\beta_i]\} \\
&= \sum_{i=1}^{q} \{c_{it}c_{is} + s_{it}s_{is}\}\, \sigma_i^2 \\
&= \sum_{i=1}^{q} \{\cos(2\pi f_i t)\cos(2\pi f_i s) + \sin(2\pi f_i t)\sin(2\pi f_i s)\}\, \sigma_i^2 \\
&= \sum_{i=1}^{q} \{\cos(2\pi f_i t - 2\pi f_i s)\}\, \sigma_i^2 \\
&= \sum_{i=1}^{q} \{\cos(2\pi f_i k)\}\, \sigma_i^2
\end{aligned}
$$

$\gamma_k = \mathrm{Cov}[y_t, y_{t-k}] = \mathrm{Cov}[y_t, y_s]$ depends only on $k = t - s$.
y_t is a stationary process with constant mean and autocovariances depending only on the time difference or time lag k for all t.

■

Trigonometric Identities (Sum & Difference Identities)

$$
\begin{aligned}
\cos(\alpha \pm \beta) &= \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) \\
\sin(\alpha \pm \beta) &= \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta) \\
\tan(\alpha \pm \beta) &= \frac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}
\end{aligned}
$$