Problem 1.

Suppose the annual sales (in millions) of company A follow an AR(2) model:

$$y_t = 5 + 1.1y_{t-1} - 0.5y_{t-2} + a_t \tag{1}$$

where $\sigma_a^2 = 2$.

(a) Show that the ψ_1 in the random shock form is also 1.1.

(b) If the sales for 2005, 2006, and 2007 were 9, 11, and 10, respectively, forecast the sales for 2008 and 2009.

(c) Calculate the 95% confidence interval of the 2008 forecast in (b).

(d) If we now know the real sales of 2008 is 12, update your forecast for 2009.

Solution.

(a)

$$\begin{aligned} y_t &= 5 + 1.1y_{t-1} - 0.5y_{t-2} + a_t \\ &= 5 + 1.1(5 + 1.1y_{t-2} - 0.5y_{t-3} + a_{t-1}) - 0.5y_{t-2} + a_t \\ &= 5 + 5.5 + 1.21y_{t-2} - 0.55y_{t-3} + 1.1a_{t-1} - 0.5y_{t-2} + a_t \\ &\vdots \end{aligned} \tag{2}$$

Put y_t in random shock form:

$$y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots \tag{3}$$

If we then equate coefficients of a_t, we get the recursive relationships

$$\left.\begin{aligned} \psi_0 &= 1 \\ \psi_1 - \phi_1\psi_0 &= 0 \\ \psi_j - \phi_1\psi_{j-1} - \phi_2\psi_{j-2} &= 0 \text{ for } j = 2, 3\ldots \end{aligned}\right\} \tag{4}$$

$$\begin{aligned} \psi_1 - (1.1)(1) &= 0 \\ \psi_1 &= 1.1 \end{aligned} \tag{5}$$

(b)

$$\begin{aligned} \hat{y}_{2007}(1) = \hat{y}_{2008} &= 5 + 1.1y_{2007} - 0.5y_{2006} \\ &= 5 + 1.1(10) - 0.5(11) \\ &= 10.5 \end{aligned} \tag{6}$$

$$\begin{aligned} \hat{y}_{2007}(2) = \hat{y}_{2009} &= 5 + 1.1\hat{y}_{2008} - 0.5y_{2007} \\ &= 5 + 1.1(10.5) - 0.5(10) \\ &= 11.55 \end{aligned} \tag{7}$$

(c)

The 95% confidence interval of $\hat{y}_{2008}$ is

$$\left(\hat{y}_{2007+1} - z_{0.05/2}\sqrt{\mathrm{Var}[e_{2007}(1)]}, \hat{y}_{2007+1} + z_{0.05/2}\sqrt{\mathrm{Var}[e_{2007}(1)]}\right) \tag{8}$$

Since we have $z_{0.025} = 1.96$, and

$$\mathrm{Var}[e_{2007}(1)] = \sigma_a^2\{1\} = 2 \tag{9}$$

Therefore,

$$\begin{aligned} &= \hat{y}_{2007+1} \pm z_{0.05/2}\sqrt{\mathrm{Var}[e_{2007}(1)]} \\ &= 10.5 \pm (1.96)(\sqrt{2}) \\ &= 10.5 \pm 2.77 \end{aligned} \tag{10}$$

Thus, the 95% confidence interval of $\hat{y}_{2008}$ is

$$(7.73, 13.27) \tag{11}$$

(d)

Given that the sales in 2008 turn out to be \$12 million, update the forecast for 2009. With the updating rule:

$$\hat{y}_{T+1}(l) = \hat{y}_T(l+1) + \psi_1 a_{T+1} \tag{12}$$

where a_{T+1} is the new residual as the time rolls to $T+1$, and can be estimated as

$$\hat{a}_{T+1} = y_{T+1} - \hat{y}_T(1)$$

Thus,

$$\begin{aligned} \hat{y}_{2007+1}(1) &= \hat{y}_{2007}(1+1) + \psi_1 a_{2007+1} \\ &= 11.55 + 1.1(12 - 10.5) \\ &= 13.2 \end{aligned} \tag{13}$$

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