Problem 1.

Show that for an MA(1) process

- $\max_{-\infty<\theta<\infty} \rho_1 = 0.5$
- $\min_{-\infty<\theta<\infty} \rho_1 = -0.5$

White Noise Process

The most fundamental example of a stationary process is a sequence of **independent and identically distributed** random variables, denoted as $\alpha_1, \ldots, \alpha_t, \ldots$, which we also assume to have **mean zero** and variance σ_α^2. This process is strictly stationary and is referred to as a **white noise process**. Because independence implies that the α_t are uncorrelated, its autocovariance function is simply

$$\gamma_k = \mathbf{E}[x_t, x_{t-k}] = \begin{cases} \sigma_\alpha^2 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

Thus, the autocorrelation function of white noise has a particularly simple form

$$\rho_k = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}.$$

Solution.

Consider the MA(1) model $y_t = a_t + \theta a_{t-1}$.

$$\mathbf{E}[y_t] = \mathbf{E}[a_t + \theta a_{t-1}] = 0 \tag{1}$$

$$\begin{aligned} \gamma_0 &= \mathrm{Var}[y_t] \\ &= \mathrm{Var}[a_t + \theta a_{t-1}] \\ &= \mathrm{Var}[a_t] + \theta^2 \mathrm{Var}[a_{t-1}] \\ &= \mathrm{Var}[a_t] + \theta^2 \mathrm{Var}[a_{t-1}] \\ &= (1+\theta^2)\sigma_a^2 \end{aligned} \tag{2}$$

$$\begin{aligned} \gamma_1 &= \mathrm{Cov}[y_t, y_{t+1}] \\ &= \mathrm{Cov}[a_t + \theta a_{t-1}, a_{t+1} + \theta a_t] \\ &= \mathrm{Cov}[a_t, a_{t+1}] + \mathrm{Cov}[a_t, \theta a_t] + \mathrm{Cov}[\theta a_{t-1}, a_{t+1}] + \mathrm{Cov}[\theta a_{t-1}, \theta a_t] \\ &= 0 + \theta\sigma_a^2 + 0 + 0 \\ &= \theta\sigma_a^2 \end{aligned} \tag{3}$$

$$\begin{aligned}\rho_1 &= \frac{\gamma_1}{\gamma_0} \\ &= \frac{\theta\sigma_a^2}{(1+\theta^2)\sigma_a^2} \\ &= \frac{\theta}{1+\theta^2}\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}\rho_1 &= \frac{(1+\theta^2)-\theta(2\theta)}{(1+\theta^2)^2} \\ &= \frac{1-\theta^2}{(1+\theta^2)^2}\end{aligned} \tag{5}$$

Set the first derivative to zero

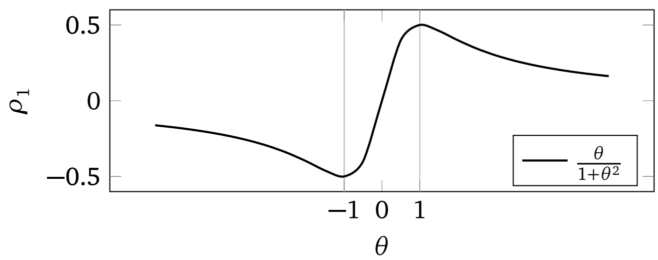
$$\frac{\mathrm{d}}{\mathrm{d}\theta}\rho_1 = \frac{1-\theta^2}{(1+\theta^2)^2} \stackrel{\text{set}}{=} 0 \tag{6}$$

Solving the equation we have

$$\theta = \pm 1 \tag{7}$$

So we have

$$\begin{aligned}\max_{\theta}\rho_1 &= \rho_1(\theta = -1) \\ &= -0.5 \\ \min_{\theta}\rho_1 &= \rho_1(\theta = 1) \\ &= 0.5\end{aligned} \tag{8}$$



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Problem 2.

For an AR(2) process $y_t - 1.0y_{t-1} + 0.5y_{t-2} = a_t$:

1. Calculate ρ_1.
2. Using ρ_0 and ρ_1 as starting values and the difference equation form for the autocorrelation function, calculate the values for ρ_k for $k = 2, \ldots, 15$.

Solution.

The ACF of an AR(2) Process

For an AR(2) process $y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t$:

$$\begin{aligned}
\gamma_k &= \mathrm{Cov}[y_t, y_{t+k}] \\
&= \mathrm{Cov}[y_t, (\phi_1 y_{t-1+k} + \phi_2 y_{t-2+k} + a_{t+k})] \\
&= \mathrm{Cov}[y_t, \phi_1 y_{t-1+k}] + \mathrm{Cov}[y_t, \phi_2 y_{t-2+k}] + \mathrm{Cov}[y_t, a_{t+k}] \\
&= \mathrm{Cov}[y_t, \phi_1 y_{t-1+k}] + \mathrm{Cov}[y_t, \phi_2 y_{t-2+k}] + \mathrm{Cov}[(\phi_1 y_{t-1} + \phi_2 y_{t-2} + a_t), a_{t+k}] \\
&= \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \mathrm{Cov}[a_t, a_{t+k}] \\
&= \begin{cases} \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} + \sigma_a^2 & k = 0 \\ \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} & k \neq 0 \end{cases}
\end{aligned} \tag{9}$$

For $k = 0$

$$\begin{aligned}
\mathrm{Var}[y_t] = \gamma_0 &= \phi_1 \gamma_{-1} + \phi_2 \gamma_{-2} + \sigma_a^2 \\
&= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_a^2
\end{aligned} \tag{10}$$

For $k = 1, 2, \ldots$

$$\gamma_k = \phi_1 \gamma_{k-1} + \phi_2 \gamma_{k-2} \tag{11}$$

Divide (11) through by γ_0 to obtain the difference equation for the ACF of the process:

$$\rho_k = \phi_1 \rho_{k-1} + \phi_2 \rho_{k-2} \tag{12}$$

For $k = 1$

$$\begin{aligned}
\rho_1 &= \phi_1 \rho_0 + \phi_2 \rho_{-1} \\
\rho_1 &= \phi_1 \rho_0 + \phi_2 \rho_1 \\
(1 - \phi_2)\rho_1 &= \phi_1 \\
\rho_1 &= \frac{\phi_1}{1 - \phi_2}
\end{aligned} \tag{13}$$

using the initial condition $\rho_0 = 1$.

For an AR(2) process $y_t - 1.0y_{t-1} + 0.5y_{t-2} = a_t$:

- $\phi_1 = 1$
- $\phi_2 = -0.5$

For $k = 1, 2, \ldots 15$, the difference equation for the ACF of the process:

$$\rho_k = \rho_{k-1} + (-0.5)\rho_{k-2} \tag{14}$$

For $k = 1$

$$\rho_1 = \frac{1}{1+0.5} = 2/3 \tag{15}$$

k	ρ_k
0	1
1	0.666666667
2	0.166666667
3	-0.166666667
4	-0.25
5	-0.166666667
6	-0.041666667
7	0.041666667
8	0.0625
9	0.041666667
10	0.010416667
11	-0.010416667
12	-0.015625
13	-0.010416667
14	-0.002604167
15	0.002604167

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Problem 3.

Put the following four models in B notation, and check whether it is stationary and invertible.

1. $y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}$.
2. $y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}$
3. $y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t$
4. $y_t - y_{t-1} = a_t - 0.5a_{t-1}$

Problem 4.

For each of the models of Problem 3, obtain:

(a) The first three ψ_j weights of the model form: $y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$

(b) The first three π_j weights of the model form: $y_t = \pi_1 y_{t-1} + \pi_2 y_{t-2} + \cdots + a_t$

(c) $\text{Var}[y_t]$, assuming that $\sigma_a^2 = 1.0$

Solution.

1. MA(2)

For MA(q) model

$$y_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q} \tag{16}$$

Using the backshift operator

$$(1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q)a_t = y_t \tag{17}$$

We define the **MA characteristic polynomial(moving average operator)**

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q \tag{18}$$

and the corresponding **MA characteristic equation**

$$\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - \cdots - \theta_q B^q = 0 \tag{19}$$

It can be shown that MA(q) model is **invertible**; that is, there are coefficients π_i such that

$$y + t = \pi_1 y_{t-1} + \pi_2 y_{t-2} + \pi_3 y_{t-3} + \cdots + a_t \tag{20}$$

if and only if the roots of the MA characteristic equation exceed 1 in modulus.

For the MA(2) process:

$$y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2} \tag{21}$$

B Operator Form

$$\begin{aligned} y_t &= \theta(B)a_t \\ &= (1 - 1.3B + 0.4B^2)a_t \end{aligned} \tag{22}$$

Stationarity

$$\begin{aligned} \mathbf{E}[y_t] &= \mathbf{E}[a_t - 1.3a_{t-1} + 0.4a_{t-2}] \\ &= 0 \quad \text{constant} \end{aligned} \tag{23}$$

$$\begin{aligned} \gamma_k &= \mathrm{Cov}[y_t, y_{t-k}] \\ &= \mathrm{Cov}[(a_t - 1.3a_{t-1} + 0.4a_{t-2}), (a_{t-k} - 1.3a_{t-1-k} + 0.4a_{t-2-k})] \\ &= \begin{cases} (1 + 1.3^2 + 0.4^2)\sigma_a^2 & k = 0 \\ (-1.3 + (-1.3)0.4)\sigma_a^2 & k = 1 \\ (-1.3)\sigma_a^2 & k = 2 \\ 0 & k > 2 \end{cases} \end{aligned} \tag{24}$$

The mean function of y_t is constant over time t. The autocovariance function of y_t only depends on time lag k. So we conclude that y_t is stationary.
Moving average processes are always stationary.

Invertibility

We can obtain the MA characteristic equation

$$\begin{aligned} \theta(B) = 1 - 1.3B + 0.4B^2 = 0 \\ (1 - 0.5B)(1 - 0.8B) = 0 \end{aligned} \tag{25}$$

The roots of the MA characteristic equation exceed 1 in modulus. Thus, The MA(2) process is invertible.

ψ weights

$$y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2} \tag{26}$$

$$\begin{aligned} \psi_0 &= 1 \\ \psi_1 &= -1.3 \\ \psi_2 &= 0.4 \\ \psi_3 &= 0 \end{aligned} \tag{27}$$

π weights

Infinite autoregressive representation

$$\pi(B)y_t = y_t - \sum_{j=1}^{\infty} \pi_j y_{t-j} = a_t \tag{28}$$

where

$$\pi(B) = \theta^{-1}(B) \tag{29}$$

The weights π are determined from the relation $\theta(B)\pi(B) = 1$ to satisfy

$$\pi_j = \theta_1 \pi_{j-1} + \theta_2 \pi_{j-2} + \cdots + \theta_q \pi_{j-q} \quad j > 0 \tag{30}$$

with $\pi_0 = -1, \pi_j = 0$ for $j < 0$, from which the weights π_j can easily be computed recursively in terms of θ_i

$$\begin{aligned}
\pi_0 &= -1 \\
\pi_1 &= \theta_1 \pi_0 + \theta_2 \pi_{-1} \\
&= (1.3)(-1) + (-0.4)(0) \\
&= -1.7 \quad -1.3 \\
\pi_2 &= \theta_1 \pi_1 + \theta_2 \pi_0 \\
&= (1.3)(-1.7) + (-0.4)(-1) \\
&= -1.81 \quad -1.29 \\
\pi_3 &= \theta_1 \pi_2 + \theta_2 \pi_1 \\
&= (1.3)(-1.81) + (-0.4)(-1.7) \\
&= -1.673 \quad -1.157
\end{aligned} \tag{31}$$

Variance

$$\begin{aligned}
\mathrm{Var}[y_t] = \gamma_0 &= (1 + 1.3^2 + 0.4^2)\sigma_a^2 \\
&= 1 + 1.3^2 + 0.4^2 \\
&= 2.85
\end{aligned} \tag{32}$$

2. ARMA(1,2) / MA(1)

$$y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2} \tag{33}$$

B Operator Form

In operator form

$$\begin{aligned}(1-0.5B)y_t &= (1-1.3B+0.4B^2)a_t \\ (1-0.5B)y_t &= (1-0.8B)(1-0.5B)a_t \\ \phi(B)y_t &= \theta(B)a_t \\ y_t &= \frac{\theta(B)}{\phi(B)}a_t\end{aligned} \tag{34}$$

The ARMA(1,2) process can be reduced to an MA(1) preocess.

$$\begin{aligned}y_t &= (1-0.8B)a_t \\ y_t &= \theta(B)a_t\end{aligned} \tag{35}$$

Stationarity

A stationary solution to (33) exists if and only if all the roots of the AR characteristic equation $\phi(B) = 0$ exceed unity in modulus.

$$\begin{aligned}\phi(B) = 1-0.5B &= 0 \\ B &= 2\end{aligned} \tag{36}$$

The root of the AR characteristic equation $\phi(B) = 0$ exceed unity in modulus. Thus, the process is stationary.
An MA process is always stationary.

Invertibility

The roots of $\theta(B) = 0$ must lie outside the unity circle if the process is to be invertible.

$$\begin{aligned}\theta(B) = 1-1.3B+0.4B^2 &= 0 \\ (1-0.8B)(1-0.5B) &= 0\end{aligned} \tag{37}$$

$$\begin{aligned}B_1 &= 2.0 \\ B_2 &= 1.25\end{aligned} \tag{38}$$

The root of the MA characteristic equation $\theta(B) = 0$ exceed unity in modulus. Thus, the process is invertible.

ψ weights

Moving average representation

$$y_t = \psi(B)a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{39}$$

where

$$\psi(B) = \frac{\theta(B)}{\phi(B)} \tag{40}$$

The weights ψ are determined from the relation $\psi(B)\phi(B) = \theta(B)$ to satisfy

$$\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \cdots + \phi_p\psi_{j-p} - \theta_j \quad j > 0 \tag{41}$$

with $\psi_0 = 1, \psi_j = 0$ for $j < 0$, and $\theta_j = 0$ for $j > q$

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi_1\psi_0 - \theta_1 \\
&= (0.5)(1) - (1.3) \\
&= -0.8 \\
\psi_2 &= \phi_1\psi_1 - \theta_2 \\
&= (0.5)(-0.8) - (-0.4) \\
&= 0 \\
\psi_3 &= \phi_1\psi_2 - \theta_3 \\
&= (0.5)(0) - (0) \\
&= 0
\end{aligned} \tag{42}$$

π weights

Infinite autoregressive representation

$$\pi(B)y_t = y_t - \sum_{j=1}^{\infty} \pi_j y_{t-j} = a_t \tag{43}$$

where

$$\pi(B) = \frac{\phi(B)}{\theta(B)} \tag{44}$$

The weights π are determined from the relation $\theta(B)\pi(B) = \phi(B)$ to satisfy

$$\pi_j = \theta_1\pi_{j-1} + \theta_2\pi_{j-2} + \cdots + \theta_q\pi_{j-q} + \phi_j \quad j > 0 \tag{45}$$

with $\pi_0 = -1, \pi_j = 0$ for $j < 0$, and $\phi_j = 0$ for $j > p$

$$\begin{aligned}
\pi_0 &= -1 \\
\pi_1 &= \theta_1 \pi_0 + \theta_2 \pi_{-1} + \phi_1 \\
&= (1.3)(-1) + (-0.4)(0) + (0.5) \\
&= -0.8 \\
\pi_2 &= \theta_1 \pi_1 + \theta_2 \pi_0 + \phi_2 \\
&= (1.3)(-0.8) + (-0.4)(-1) + (0) \\
&= -0.64 \\
\pi_3 &= \theta_1 \pi_2 + \theta_2 \pi_1 + \phi_3 \\
&= (1.3)(-0.64) + (-0.4)(-0.8) + (0) \\
&= -0.512
\end{aligned} \tag{46}$$

Variance

$$\begin{aligned}
\mathrm{Var}[y_t] &= \mathrm{Var}[\sum_{j=0}^{\infty} \psi_j a_{t-j}] \\
&= (\psi_0^2 + \psi_1^2 + \cdots)\,\mathrm{Var}[a_t] \\
&= 1^2 + (-0.8)^2 \\
&= 1.64
\end{aligned} \tag{47}$$

3. AR(2)

An autoregressive model of order p, abbreviated AR(p), is of the form

$$y_t = \phi_1 y_t t - 1 + \phi_2 y_t t - 2 + \cdots + \phi_p y_t t - p + a_t. \tag{48}$$

Using the backshift operator

$$(1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_q B^q) y_t = a_t \tag{49}$$

We define the **AR characteristic polynomial (autoregressive operator)**

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_q B^q \tag{50}$$

and the corresponding **AR characteristic equation**

$$\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_q B^q = 0 \tag{51}$$

The process $\phi(B) y_t = a_t$ can be written as

$$y_t = \phi^{-1}(B) a_t \equiv \psi(B) a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{52}$$

provided that the right-side expression is convergent. Using the factorization

$$\phi(B) = (1 - G_1 B)(1 - G_2 B) \cdots (1 - G_p B) \tag{53}$$

where $G_1^{-1}, G_2^{-1}, \ldots, G_p^{-1}$ are the roots of equation $\phi(B) = 0$, and expanding $\phi^{-1}(B)$ in partial fractions yields

$$y_t = \phi^{-1}(B) a_t = \sum_{i=1}^{p} \frac{K_i}{1 - G_i B} a_t \tag{54}$$

Hence, if $\psi(B) = \phi^{-1}(B)$ is to be convergent series for $|B| < 1$, that is if the weights $\psi_j = \sum_{i=1}^{\infty} K_i G_i^j$ are to be absolutely summable so that AR(p) process is stationary, we must have $\left|G_i\right| < 1$, for $i = 1, \ldots, p$. Equivalently, the roots of the **AR characteristic equation** $\phi(B) = 0$ must lie outside the unity circle.

For the AR(2) preocess:

$$y_t - 1.5 y_{t-1} + 0.6 y_{t-2} = a_t \tag{55}$$

B Operator Form

$$\phi(B) y_t = (1 - 1.5B + 0.6B^2) y_t = a_t \tag{56}$$

Stationarity

For stationarity, the roots of

$$\phi(B) = 1 - 1.5B + 0.6B^2 = 0 \tag{57}$$

must lie outside the unity circle, which implies that the parameters ϕ_1, ϕ_2 must lie in the triangular region

$$\begin{aligned} \phi_2 + \phi_1 &< 1 \\ \phi_2 - \phi_1 &< 1 \\ -1 < \phi_2 &< 1 \end{aligned} \tag{58}$$

Check

$$\begin{aligned} -0.6 + 1.5 = 0.9 &< 1 \\ -0.6 - 1.5 = -2.1 &< 1 \\ -1 < -0.6 &< 1 \end{aligned} \tag{59}$$

Therefore, the process is stationary

Invertibility

Pure AR models are always invertible (since they contain no MA terms).
Thus, the process is invertible.

ψ weights

Infinite moving average representation

$$y_t = \psi(B)a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{60}$$

where

$$\psi(B) = \frac{1}{\phi(B)} \tag{61}$$

The weights ψ are determined from the relation $\psi(B)\phi(B) = 1$ to satisfy

$$\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \cdots + \phi_p\psi_{j-p} \quad j > 0 \tag{62}$$

with $\psi_0 = 1, \psi_j = 0$ for $j < 0$, from which the weights ψ_j can be computed recursively in terms of the ϕ_i.

$\psi_1 = 1.5$

$\psi_2 = \frac{33}{20}$

$\psi_3 = \frac{63}{40}$

$$\begin{aligned}
\psi_0 &= 1 \\
\psi_1 &= \phi_1\psi_0 + \phi_2\psi_{-1} \\
&= (0.5)(1) + (-0.6)(0) \\
&= 0.5 \\
\psi_2 &= \phi_1\psi_1 + \phi_2\psi_0 \\
&= (0.5)(0.5) + (-0.6)(1) \\
&= -0.35 \\
\psi_3 &= \phi_1\psi_2 + \phi_2\psi_1 \\
&= (0.5)(-0.35) + (-0.6)(0.5) \\
&= -0.475
\end{aligned} \tag{63}$$

π weights

$$y_t = 1.5y_{t-1} - 0.6y_{t-2} + a_t \tag{64}$$

$$\begin{aligned}
\pi_0 &= -1 \\
\pi_1 &= 1.5 \\
\pi_2 &= -0.6 \\
\pi_3 &= 0
\end{aligned} \tag{65}$$

Variance

The variance of AR(2) process is

$$\begin{aligned}
\mathrm{Var}[y_t] &= \frac{\sigma_a^2}{1 - \rho_1\phi_1 - \rho_2\phi_2} \\
&= \frac{1-\phi_2}{1+\phi_2}\frac{\sigma_a^2}{(1-\phi_2)^2 - \phi_1^2}
\end{aligned} \tag{66}$$

So we have

$$\mathrm{Var}[y_t] = \frac{1+0.6}{1-0.6}\frac{1}{(1+0.6)^2 - (-1.5)^2} \tag{67}$$

4. ARMA(1,1)

$$y_t - y_{t-1} = a_t - 0.5a_{t-1} \tag{68}$$

B Operator Form

$$\begin{aligned}(1-B)y_t &= (1-0.5B)a_t \\ \phi(B)y_t &= \theta(B)a_t\end{aligned} \tag{69}$$

Stationarity

$$\phi(B) = 1 - B = 0 \tag{70}$$

The root of the AR characteristic equation does not lie outside the unity circle. Thus the process is not stationary.

Invertibility

$$\theta(B) = 1 - 0.5B = 0 \tag{71}$$

The root of the MA characteristic equation $B = 2$ lie outside the unity circle. Thus the process is invertible.

ψ weights

Infinite moving average representation

$$y_t = \psi(B)a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{72}$$

$$\begin{aligned}y_t &= y_{t-1} + a_t - 0.5a_{t-1} \\ &= y_{t-2} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\ &= y_{t-3} + a_{t-2} - 0.5a_{t-3} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\ &\vdots \\ &= a_t + \cdots + 0.5a_{t-1} + 0.5a_{t-2} + \cdots + 0.5a_1 + 0.5a_0 + \cdots - 0.5a_{t-\infty}\end{aligned} \tag{73}$$

$$\begin{aligned}\psi_0 &= 1 \\ \psi_1 &= 0.5 \\ \psi_2 &= 0.5 \\ \psi_3 &= 0.5\end{aligned} \tag{74}$$

π weights

Infinite autoregressive representation

$$\pi(B)y_t = y_t - \sum_{j=1}^{\infty} \pi_j y_{t-j} = a_t \tag{75}$$

where

$$\pi(B) = \frac{\phi(B)}{\theta(B)} \tag{76}$$

The weights π are determined from the relation $\theta(B)\pi(B) = \phi(B)$ to satisfy

$$\pi_j = \theta_1 \pi_{j-1} + \theta_2 \pi_{j-2} + \cdots + \theta_q \pi_{j-q} + \phi_j \quad j > 0 \tag{77}$$

with $\pi_0 = -1, \pi_j = 0$ for $j < 0$, and $\phi_j = 0$ for $j > p$

$$\begin{aligned}
\pi_0 &= -1 \\
\pi_1 &= \theta_1 \pi_0 + \phi_1 \\
&= (0.5)(-1) + (1) \\
&= 0.5 \\
\pi_2 &= \theta_1 \pi_1 + \phi_2 \\
&= (0.5)(0.5) + (0) \\
&= 0.25 \\
\pi_3 &= \theta_1 \pi_2 + \phi_3 \\
&= (0.5)(0.25) + (0) \\
&= 0.125
\end{aligned} \tag{78}$$

Variance

$$\begin{aligned}
\mathrm{Var}[y_t] = \gamma_0 &= \mathrm{Var}[y_{t-1} + a_t - 0.5a_{t-1}] \\
&= \mathrm{Var}[y_{t-1}] + 0.25\sigma_a^2
\end{aligned} \tag{79}$$

$$\begin{aligned}
y_t &= y_{t-1} + a_t - 0.5a_{t-1} \\
&= y_{t-2} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\
&= y_{t-3} + a_{t-2} - 0.5a_{t-3} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\
&\vdots \\
&= y_0 + a_t + 0.5a_{t-1} + 0.5a_{t-2} + \cdots + 0.5a_1 - 0.5a_0
\end{aligned} \tag{80}$$

$$\begin{aligned}
\mathrm{Var}[y_t] &= \mathrm{Var}[y_0 + a_t + 0.5a_{t-1} + 0.5a_{t-2} + \cdots + 0.5a_1 - 0.5a_0] \\
&= (1 + 0.25t)\sigma_a^2 + \mathrm{Var}(y_0) \\
&= 1 + 0.25t
\end{aligned} \tag{81}$$

$$\begin{aligned}
y_t &= y_{t-1} + a_t - 0.5a_{t-1} \\
&= y_{t-2} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\
&= y_{t-3} + a_{t-2} - 0.5a_{t-3} + a_{t-1} - 0.5a_{t-2} + a_t - 0.5a_{t-1} \\
&\vdots \\
&= a_t + \cdots + 0.5a_{t-1} + 0.5a_{t-2} + \cdots + 0.5a_1 + 0.5a_0 + \cdots - 0.5a_{-\infty}
\end{aligned} \tag{82}$$

$$
\begin{aligned}
\mathrm{Var}[y_t] &= \mathrm{Var}[a_t + \cdots + 0.5a_{t-1} + 0.5a_{t-2} + \cdots + 0.5a_1 + 0.5a_0 + \cdots - 0.5a_{-\infty}] \\
&= (1 + \sum_{i=1}^{\infty} 0.25)\sigma_a^2 \\
&= 1 + \sum_{i=1}^{\infty} 0.25 \\
&= 1 + (0.25)\infty
\end{aligned} \tag{83}
$$

■

Problem 5.

Consider y_t a stationary preocess. Show that if $\rho_1 < 0.5$, $(1-B)y_t$ has a larger variance than does y_t.

Solution.

$$\begin{aligned}
\mathrm{Var}[(1-B)y_t] &= \mathrm{Var}[y_t - y_{t-1}] \\
&= \mathrm{Var}[y_t] + \mathrm{Var}[y_{t-1}] - 2\mathrm{Cov}[y_t, y_{t-1}] \\
&= \gamma_0 + \gamma_0 - 2\gamma_1 \\
&= (2-2\rho_1)\gamma_0 \\
&= \underbrace{2(1-\rho_1)}_{>1}\gamma_0 \\
&> \gamma_0
\end{aligned} \tag{84}$$

We then conclude that $(1-B)y_t$ has a larger variance than does y_t. ■

Problem 6.

Consider an AR(1) process satisfying $y_t = \phi y_{t-1} + e_t$, where ϕ can be **any** number and e_t is a white noise process such that e_t is independent of the past $y_{t-1}, y_{t-2}, \dots$. Let y_0 be a random variable with mean μ_0 and variance σ_0.

(a) For $t > 0$, show that

$$y_t = e_t + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots + \phi^{t-1} e_1 + \phi^t y_0$$

(b) Show that $\mathbf{E}[y_t] = \phi^t \mu_0$, for $t > 0$.

(c) Show that for $t > 0$, we have

$$Var[y_t] = \begin{cases} \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 & \phi \neq 1 \\ t\sigma_e^2 + \sigma_0^2 & \phi = 1. \end{cases}$$

(d) Assuming $\mu_0 = 0$, show that, we must have $\phi \neq 1$ to make y_t stationary.

(e) following (d) and supposing that $\mu_0 = 0$ and y_t is stationary, show that $\mathrm{Var}[y_t] = \frac{\sigma_e^2}{1-\phi^2}$ and we must have $|\phi| < 1$.

Solution.

(a)

$$\begin{aligned} y_t &= \phi y_{t-1} + e_t \\ &= \phi(\phi y_{t-2} + e_{t-1}) + e_t \\ &= \phi^2 y_{t-2} + \phi e_{t-1} + e_t \\ &= \phi^2(\phi y_{t-3} + e_{t-2}) + \phi e_{t-1} + e_t \\ &= \phi^3 y_{t-3} + \phi^2 e_{t-2} + \phi e_{t-1} + e_t \\ &\vdots \\ &= \phi^t y_0 + \phi^{t-1} e_1 + \cdots + \phi e_{t-1} + e_t \end{aligned} \tag{85}$$

as required.

(b)

$$\begin{aligned} \mathbf{E}[y_t] &= \mathbf{E}[\phi^t y_0 + \phi^{t-1} e_1 + \cdots + \phi e_{t-1} + e_t] \\ &= \mathbf{E}[\phi^t y_0] \\ &= \phi^t \mathbf{E}[y_0] \\ &= \phi^t \mu_0 \end{aligned} \tag{86}$$

(c)

$$\begin{aligned} \mathrm{Var}[y_t] &= \mathrm{Var}[\phi^t y_0 + \phi^{t-1} e_1 + \cdots + \phi e_{t-1} + e_t] \\ &= \phi^{2t}\sigma_0^2 + (\phi^2 + \phi^4 + \phi^6 + \phi^{2(t-1)})\sigma_e^2 \\ &= \phi^{2t}\sigma_0^2 + \sigma_e^2 \sum_{k=1}^{t-1} \phi^{2k} \\ &= \begin{cases} \phi^{2t}\sigma_0^2 + \frac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 & \phi \neq 1 \\ \sigma_0^2 + t\sigma_e^2 & \phi = 1. \end{cases} \end{aligned} \tag{87}$$

Geometric Series

The sum of a n-term (finite) geometric series is given by:

$$S_n = \begin{cases} \frac{a_1(1-r^n)}{1-r} & r \neq 1 \\ a_1 n & r = 1 \end{cases}$$

with initial value $a = a_1$ and common ratio r.

(d) If $\phi = 1$, we have

$$\begin{aligned} \mathrm{Var}[y_t] &= \mathrm{Var}[y_{t-1} + e_t] \\ &= \mathrm{Var}[y_{t-1}] + \sigma_e^2 \end{aligned}$$

which is against the stationarity. Therefore, we must have $\phi \neq 1$ to make y_t stationary.

(e)

$$\begin{aligned} \mathrm{Var}[y_t] &= \mathrm{Var}[\phi y_{t-1} + e_t] \\ &= \phi^2 \mathrm{Var}[y_{t-1}] + \sigma_e^2 \end{aligned}$$

Due to the requirement of stationarity, we have

$$\begin{aligned} \mathrm{Var}[y_t] &= \phi^2 \mathrm{Var}[y_t] + \sigma_e^2 \\ (1-\phi^2)\mathrm{Var}[y_t] &= \sigma_e^2 \\ \mathrm{Var}[y_t] &= \frac{\sigma_e^2}{1-\phi^2} \end{aligned}$$

Since the variance $\mathrm{Var}[y_t]$ must be positive, we must have $|\phi| < 1$.

■