National Taiwan University
Institute of Industrial Engineering
Jakey BLUE

Time Series Analytics

111-1 Homework #04

Due at 23h59, October 16, 2022; files uploaded to NTU-COOL

1. (10%) Show that for an MA(1) process
   $\max_{-\infty<\theta<\infty} \rho_1 = 0.5$ and $\min_{-\infty<\theta<\infty} \rho_1 = -0.5$

2. (10%) For an AR(2) process $y_t - 1.0y_{t-1} + 0.5y_{t-2} = a_t$:
   (a) Calculate ρ_1.
   (b) Using ρ_0 and ρ_1 as starting values and the difference equation form for the autocorrelation function, calculate the values of ρ_k for $k = 2, \dots, 15$.

3. (20%) Put the following four models in B notation, and check whether it is stationary and invertible.
   (1) $y_t = a_t - 1.3a_{t-1} + 0.4a_{t-2}$
   (2) $y_t - 0.5y_{t-1} = a_t - 1.3a_{t-1} + 0.4a_{t-2}$
   (3) $y_t - 1.5y_{t-1} + 0.6y_{t-2} = a_t$
   (4) $y_t - y_{t-1} = a_t - 0.5a_{t-1}$

4. (20%) For each of the models of Exercise 3, obtain:
   (a) The first three ψ_j weights of the model form: $y_t = a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \cdots$
   (b) The first three π_j weights of the model form: $y_t = \pi_1 y_{t-1} + \pi_2 y_{t-2} + \cdots + a_t$
   (c) $\mathrm{V}[y_t]$, assuming that $\sigma_a^2 = 1.0$

5. (10%) Consider y_t a stationary process. Show that if $\rho_1 < \frac{1}{2}$, $(1-B)y_t$ has a larger variance than does y_t.

6. (20%) Consider an AR(1) process satisfying $y_t = \phi y_{t-1} + e_t$, where ϕ can be **any** number and e_t is a white noise process such that e_t is independent of the past $y_{t-1}, y_{t-2}, \dots$. Let y_0 be a random variable with mean μ_0 and variance σ_0^2.
   (a) For $t > 0$, show that
   $y_t = e_t + \phi e_{t-1} + \phi^2 e_{t-2} + \phi^3 e_{t-3} + \cdots + \phi^{t-1} e_1 + \phi^t y_0$.
   (b) Show that $\mathrm{E}[y_t] = \phi^t \mu_0$, for $t > 0$.
   (c) Show that for $t > 0$, we have

$$\mathrm{V}[y_t] = \begin{cases} \dfrac{1-\phi^{2t}}{1-\phi^2}\sigma_e^2 + \phi^{2t}\sigma_0^2 & \text{, for } \phi \neq 1, \\ t\sigma_e^2 + \sigma_0^2 & \text{, for } \phi = 1. \end{cases}$$

   (d) Assuming $\mu_0 = 0$, show that, we must have $\phi \neq 1$ to make y_t stationary.
   (e) Following (d) and supposing that $\mu_0 = 0$ and y_t is stationary, show that $\mathrm{V}[y_t] = \frac{\sigma_e^2}{1-\phi^2}$ and we must have $|\phi| < 1$.