



Time Series Analytics

111-1 Homework #03

Due at 23h59, October 2, 2022; files uploaded to NTU-COOL

1. (10%) y_t is a stationary process with the autocovariance function γ_k . Define $\bar{y} = \frac{1}{n} \sum_{t=1}^n y_t$. Show that

$$V[\bar{y}] = \frac{\gamma_0}{n} + \frac{2}{n} \sum_{k=1}^{n-1} \left(1 - \frac{k}{n}\right) \gamma_k = \frac{1}{n} \sum_{k=-n+1}^{n-1} \left(1 - \frac{|k|}{n}\right) \gamma_k.$$

2. (10%) Assume x_t is a stationary process and define $y_t = \begin{cases} x_t & \text{for odd } t \\ x_t + 3 & \text{for even } t \end{cases}$
 (a) Show that $\text{COV}[y_t, y_{t-k}]$ is independent of t for all lags k .
 (b) Is y_t stationary?
3. (10%) Let y_t be a stationary process with an autocovariance function γ_k .
 (a) Show that $z_t = \nabla y_t = y_t - y_{t-1}$ is stationary by finding the mean and autocovariance function for z_t .
 (b) Show that $w_t = \nabla^2 y_t = z_t - z_{t-1} = y_t - 2y_{t-1} + y_{t-2}$ is stationary.
4. (15%) Let x_t be stationary with $E[x_t] = 0, V[x_t] = 1$, autocorrelation function ρ_k . Define that μ_t is a nonconstant function and σ_t is a positively nonconstant **function** (that is to say: μ_t and σ_t are both deterministic and in function of t). Now we observe a time series formulated as

$$y_t = \mu_t + \sigma_t x_t.$$

 (a) Find the mean and autocovariance function of y_t .
 (b) Show that the autocorrelation of y_t depends only on the lag k . Is y_t stationary?
 (c) Let $\mu_t = \mu_0$ be a constant value. Justify that y_t is still nonstationary.
5. (5%) Let x_t be the series of the “expected” measurements during the production process. Because the measuring tool itself won’t be perfect, we observe $y_t = x_t + e_t$, assuming x_t and e_t are independent. In general, we call x_t the **signal** and e_t the **measurement (white) noise**.

If x_t is stationary with the autocorrelation function ρ_k , show that y_t is also a stationary process with

$$\text{corr}(y_t, y_{t-k}) = \frac{\rho_k}{1 + \frac{\sigma_e^2}{\sigma_x^2}}, \text{ for } k \geq 1.$$

$\frac{\sigma_x^2}{\sigma_e^2}$ is usually referred to as the **signal-to-noise ratio**, or **SNR** for short.

The larger the SNR, the closer the autocorrelation function of the observed series y_t is to the autocorrelation function of the desired signal x_t .

6. (10%) Suppose $y_t = \alpha_0 + \sum_{i=1}^q [\alpha_i \cos(2\pi f_i t) + \beta_i \sin(2\pi f_i t)]$, where $\alpha_0, f_1, f_2, \dots, f_q$ are constants and $\alpha_1, \alpha_2, \dots, \alpha_q, \beta_1, \beta_2, \dots, \beta_q$ are independent random variables with zero means and variances $V[\alpha_i] = V[\beta_i] = \sigma_i^2$. Show that y_t is stationary and find its autocovariance function.

(Hint: show $\text{COV}[y_t, y_s]$ depends only on $t - s$.)