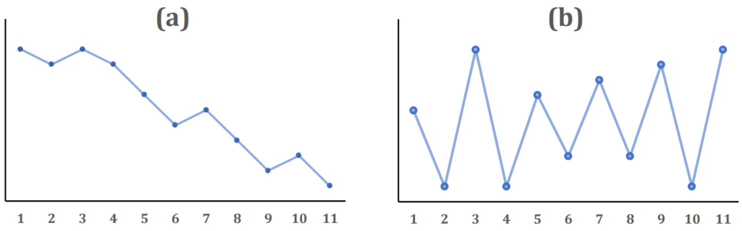
Problem 1.

The two time series plots are shown below. For each of the series, describe the sample autocorrelations $\hat{\rho}_1$ and $\hat{\rho}_2$ using the terms strongly positive, moderately positive, near zero, moderately negative, or strongly negative. Do you need to know the scale of measurement for the series to answer?



Solution.
The scale of measurement is not needed to describe the autocorrelations.

(a)

$\hat{\rho}_1$

moderately positive
The neighboring values are in general on the same side of the mean but in a decreasing trend with fluctuation.

$\hat{\rho}_2$

moderately positive
The alternationing values are in general on the same side of the mean but in a decreasing trend with fluctuation.

(b)

$\hat{\rho}_1$

moderately negative
The neighboring values are always on the different sides of the mean with similar (not always the same) distances.

$\hat{\rho}_2$

moderately positive
The alternationing values are always on the same side of the mean with similar (not always the same) distances.

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Problem 2.

Identify the (p, d, q) for following ARIMA models and calculate the $\mathbf{E}[\nabla y_t]$ and $\mathrm{Var}[\nabla y_t]$.

(a) $y_t = 3 + y_{t-1} + a_t - 0.75a_{t-1}$

(b) $y_t = 10 + 1.25y_{t-1} - 0.25y_{t-2} + a_t - 0.1a_{t-1}$

(c) $y_t = 5 + 2y_{t-1} - 1.7y_{t-2} + 0.7y_{t-3} + a_t - 0.5a_{t-1} + 0.25a_{t-2}$

Solution.

(a) $y_t = 3 + y_{t-1} + a_t - 0.75a_{t-1}$

$$\begin{aligned}\nabla y_t &= y_t - y_{t-1} \\ &= 3 + a_t - 0.75a_{t-1}\end{aligned}$$

So the model is a stationary, invertible, IMA(1,1) model with $\theta_1 = 0.75$, and $L = 3$

$\mathbf{E}[\nabla y_t]$

$$\begin{aligned}\mathbf{E}[\nabla y_t] &= \mathbf{E}[3 + a_t - 0.75a_{t-1}] \\ &= 3\end{aligned}$$

$\mathrm{Var}[\nabla y_t]$

$$\begin{aligned}\mathrm{Var}[\nabla y_t] &= \mathrm{Var}[3 + a_t - 0.75a_{t-1}] \\ &= [1 + 0.75^2]\sigma_a^2\end{aligned}$$

(b) $y_t = 10 + 1.25y_{t-1} - 0.25y_{t-2} + a_t - 0.1a_{t-1}$

$$\begin{aligned}\nabla y_t &= y_t - y_{t-1} \\ &= 10 + 0.25y_{t-1} - 0.25y_{t-2} + a_t - 0.1a_{t-1} \\ \nabla y_t - 0.25\nabla y_{t-1} &= 10 + a_t - 0.1a_{t-1}\end{aligned}$$

So the model is a stationary, invertible, ARIMA(1,1,1) model with

$$\begin{aligned}\phi &= 0.25 \\ \theta &= 0.1 \\ L &= 10\end{aligned}$$

$$
\begin{aligned}
\nabla y_t &= y_t - y_{t-1} \\
&= 10 + 0.25y_{t-1} - 0.25y_{t-2} + a_t - 0.1a_{t-1} \\
&= 10 + 0.25\nabla y_{t-1} + a_t - 0.1a_{t-1} \\
&= 10 + 0.25(10 + 0.25\nabla y_{t-2} + a_{t-1} - 0.1a_{t-2}) + a_t - 0.1a_{t-1} \\
&= \cdots
\end{aligned}
$$

E$[\nabla y_t]$

$$
\begin{aligned}
\mathbf{E}[\nabla y_t] &= \mathbf{E}[10 + 0.25(10 + 0.25\nabla y_{t-1})] \\
&= \mathbf{E}[10 + 0.25(10) + (0.25)^2(10 + 0.25\nabla y_{t-2})] \\
&= \mathbf{E}[10 + 0.25(10) + (0.25)^2(10) + (0.25)^3(10) + \cdots] \\
&= \frac{10}{1-0.25} \\
&= \frac{40}{3}
\end{aligned}
$$

Var$[\nabla y_t]$

$$
\begin{aligned}
\mathrm{Var}[\nabla y_t] &= \frac{1-2\phi\theta+\theta^2}{1-\phi^2}\sigma_a^2 \\
&= \frac{1-2(0.25)(0.1)+(0.1)^2}{1-(0.25)^2}\sigma_a^2 \\
&= 1.024\sigma_a^2
\end{aligned}
$$

(c) $y_t = 5 + 2y_{t-1} - 1.7y_{t-2} + 0.7y_{t-3} + a_t - 0.5a_{t-1} + 0.25a_{t-2}$

$$
\begin{aligned}
\nabla y_t &= y_t - y_{t-1} \\
&= 5 + y_{t-1} - 1.7y_{t-2} + 0.7y_{t-3} + a_t - 0.5a_{t-1} + 0.25a_{t-2} \\
&= 5 + y_{t-1} - y_{t-2} - 0.7y_{t-2} + 0.7y_{t-3} + a_t - 0.5a_{t-1} + 0.25a_{t-2} \\
&= 5 + \nabla y_{t-1} - 0.7\nabla y_{t-2} + a_t - 0.5a_{t-1} + 0.25a_{t-2}
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
\nabla y_t - \nabla y_{t-1} + 0.7\nabla y_{t-2} &= 5 + a_t - 0.5a_{t-1} + 0.25a_{t-2} \\
(1 - B + 0.7B^2)\nabla y_t &= 5 + (1 - 0.5B + 0.25B^2)a_t
\end{aligned}
$$

For the AR(2) part, we have

$$
\begin{aligned}
\phi_1 &= 1 \\
\phi_2 &= -0.7
\end{aligned}
$$

Stationarity conditions for the ARMA(2) model:

$$\begin{aligned}\phi_1 + \phi_2 &< 1 \\ \phi_2 - \phi_1 &< 1 \\ |\phi_2| &< 1\end{aligned} \tag{1}$$

Check the stationarity conditions

$$\begin{aligned}(1) + (-0.7) &< 1 \\ (-0.7) - (1) &< 1 \\ |-0.7| &< 1\end{aligned}$$

So the model is a stationary ARIMA(2,1,2) model

$\mathbf{E}[\nabla y_t]$

$$\begin{aligned}\mathbf{E}[\nabla y_t] &= \mathbf{E}[5 + \nabla y_{t-1} - 0.7\nabla y_{t-2} + a_t - 0.5a_{t-1} + 0.25a_{t-2}] \\ &= 5 + \mathbf{E}[\nabla y_{t-1}] - 0.7\mathbf{E}[\nabla y_{t-2}]\end{aligned}$$

Due to stationarity, $\mathbf{E}[\nabla y_t]$ is a constant for all t. Thus we have

$$\begin{aligned}\mathbf{E}[\nabla y_t] &= 5 + \mathbf{E}[\nabla y_t] - 0.7\mathbf{E}[\nabla y_t] \\ \mathbf{E}[\nabla y_t] &= \frac{5}{0.7} \\ &= \frac{50}{7} = \mu_w\end{aligned}$$

$\mathrm{Var}[\nabla y_t]$

Let

$$w_t = \nabla y_t$$

$$\tilde{w}_t = w_t - \mu_w \quad = \nabla y_t - \mu_w$$

Thus we have $\mathbf{E}[\tilde{w}_t] = 0$ and

$$\begin{aligned}\phi(B)\tilde{w}_t &= w_t - \mu_w \\ &= \nabla y_t - \mu_w = \theta(B)a_t\end{aligned}$$

For $\tilde{w}_t$, the moving average representation

$$\tilde{w}_t = \psi(B)a_t = \sum_{j=0}^{\infty} \psi_j a_{t-j} \tag{2}$$

where

$$\psi(B) = \frac{\theta(B)}{\phi(B)} \tag{3}$$

The weights ψ are determined from the relation $\psi(B)\phi(B) = \theta(B)$ to satisfy

$$\psi_j = \phi_1\psi_{j-1} + \phi_2\psi_{j-2} + \cdots + \phi_p\psi_{j-p} - \theta_j \quad j > 0 \tag{4}$$

with $\psi_0 = 1, \psi_j = 0$ for $j < 0$, and $\theta_j = 0$ for $j > q$

$$\begin{aligned}
\psi_0 &= 1\\
\psi_1 &= \phi_1\psi_0 + \phi_2\psi_{-1} - \theta_1\\
&= (1)(1) + (-0.7)(0) - (0.5)\\
&= 0.5\\
\psi_2 &= \phi_1\psi_1 + \phi_2\psi_0 - \theta_2\\
&= (1)(0.5) + (-0.7)(1) - (-0.25)\\
&= 0.05
\end{aligned}$$

The autocovariance function may be expressed as

$$\gamma_k = \phi_1\gamma_{k-1} + \cdots + \phi_p\gamma_p - \sigma_a^2(\theta_k\psi_0 + \theta_{k+1}\psi_1 + \cdots + \theta_p\psi_{q-k}) \tag{5}$$

with the convention that $\theta_0 = -1$.
For $k = 0, 1, 2$

$$\begin{aligned}
\gamma_0 &= \phi_1\gamma_1 + \phi_2\gamma_2 - \sigma_a^2(\theta_0\psi_0 + \theta_1\psi_1 + \theta_2\psi_2)\\
\gamma_1 &= \phi_1\gamma_0 + \phi_2\gamma_1 - \sigma_a^2(\theta_1\psi_0 + \theta_2\psi_1)\\
\gamma_2 &= \phi_1\gamma_1 + \phi_2\gamma_0 - \sigma_a^2(\theta_2\psi_0)
\end{aligned}$$

$$\begin{aligned}
\gamma_0 &= (1)\gamma_1 + (-0.7)\gamma_2 - \sigma_a^2(-1 + (0.5)(0.5) + (-0.25)(0.05))\\
\gamma_1 &= (1)\gamma_0 + (-0.7)\gamma_1 - \sigma_a^2((0.5)(1) + (-0.25)(0.5))\\
\gamma_2 &= (1)\gamma_1 + (-0.7)\gamma_0 - \sigma_a^2((-0.25)(1))
\end{aligned}$$

$$\begin{aligned}
\gamma_0 &= \gamma_1 + (-0.7)\gamma_2 - \sigma_a^2(-0.7625)\\
\gamma_1 &= \gamma_0 + (-0.7)\gamma_1 - \sigma_a^2(0.375)\\
\gamma_2 &= \gamma_1 + (-0.7)\gamma_0 - \sigma_a^2(-0.25)
\end{aligned}$$

Solving the equations, the varianceγ_0 of the process $\tilde{w}_t$ is obtained as

$$\gamma_0 \approx 1.563\sigma_a^2$$

For $w_t = \nabla y_t$

$$\gamma_0(w_t) = \mathrm{Var}[w_t] = \mathrm{Var}[\tilde{w}_t + \mu_w] = \mathrm{Var}[\tilde{w}_t] = \gamma_0(\tilde{w}_t) \quad \approx 1.563\sigma_a^2$$

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Problem 3.

Suppose that $y_t = A + Bt + x_t$, where x_t is a random walk. First suppose that A and B are constants.

(a) Is y_t stationary?

(b) Is ∇y_t stationary?

Now let A and B be random variables that are independent of the random walk x_t.

(c) Is y_t stationary?

(d) Is ∇y_t stationary?

Solution.

(a)

$$\begin{aligned} \mathbf{E}[y_t] &= \mathbf{E}[A + Bt + x_t] \\ &= A + Bt \end{aligned}$$

which depends on t. So y_t is not stationary.

(b)

$$\begin{aligned} \nabla y_t &= y_t - y_{t-1} \\ &= A + Bt + x_t - (A + B(t-1) + x_{t-1}) \\ &= B + x_t - x_{t-1} \end{aligned}$$

$$\begin{aligned} \mathbf{E}[\nabla y_t] &= \mathbf{E}[B + x_t - x_{t-1}] \\ &= B \end{aligned}$$

which is a constant.

$$\begin{aligned} \mathrm{Cov}[\nabla y_t, \nabla y_{t-k}] &= \mathrm{Cov}[B + x_t - x_{t-1}, B + x_{t-k} - x_{t-k-1}] \\ &= \mathrm{Cov}[x_t - x_{t-1}, x_{t-k} - x_{t-k-1}] \\ &= \mathrm{Cov}[x_t, x_{t-k}] + \mathrm{Cov}[x_t, -x_{t-k-1}] \\ &\quad + \mathrm{Cov}[-x_{t-1}, x_{t-k}] + \mathrm{Cov}[-x_{t-1}, -x_{t-k-1}] \\ &= \begin{cases} 2\sigma_a^2 & k = 0 \\ -\sigma_a^2 & k = 1 \\ 0 & k > 1 \end{cases} \end{aligned}$$

random walk

$\nabla x_t = a_t . + L$

which does not depend on t.
So ∇y_t is stationary.

(c)

$$\begin{aligned}\mathbf{E}[y_t] &= \mathbf{E}[A + Bt + x_t] \\ &= \mathbf{E}[A] + \mathbf{E}[B]t\end{aligned}$$

which depends on t. So y_t is not stationary.

(d)

$$\begin{aligned}\nabla y_t &= y_t - y_{t-1} \\ &= A + Bt + x_t - (A + B(t-1) + x_{t-1}) \\ &= B + x_t - x_{t-1}\end{aligned}$$

$$\begin{aligned}\mathbf{E}[\nabla y_t] &= \mathbf{E}[B + x_t - x_{t-1}] \\ &= \mathbf{E}[B]\end{aligned}$$

which is a constant.

$$\begin{aligned}\mathrm{Cov}[\nabla y_t, \nabla y_{t-k}] &= \mathrm{Cov}[B + x_t - x_{t-1}, B + x_{t-k} - x_{t-k-1}] \\ &= \mathrm{Cov}[B, B] \\ &\quad + \mathrm{Cov}[x_t, x_{t-k}] + \mathrm{Cov}[x_t, -x_{t-k-1}] \\ &\quad + \mathrm{Cov}[-x_{t-1}, x_{t-k}] + \mathrm{Cov}[-x_{t-1}, -x_{t-k-1}] \\ &= \begin{cases} \mathrm{Var}[B] + 2\sigma_a^2 & k = 0 \\ \mathrm{Var}[B] + (-\sigma_a^2) & k = 1 \\ \mathrm{Var}[B] & k > 1 \end{cases}\end{aligned}$$

which does not depend on t.
So ∇y_t is stationary.

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Problem 4.

Given a stationary process y_t, show that if $\rho_1 < 0.5$, ∇y_t has a larger variance than does y_t

Solution.

$$
\begin{aligned}
\operatorname{Var}[\nabla y_t] &= \operatorname{Var}[(1-B)y_t] \\
&= \operatorname{Var}[y_t - y_{t-1}] \\
&= \operatorname{Var}[y_t] + \operatorname{Var}[y_{t-1}] - 2\operatorname{Cov}[y_t, y_{t-1}] \\
&= \gamma_0 + \gamma_0 - 2\gamma_1 \\
&= (2 - 2\rho_1)\gamma_0 \\
&= \underbrace{2(1-\rho_1)}_{>1}\gamma_0 \\
&> \gamma_0
\end{aligned} \tag{6}
$$

We then conclude that ∇y_t has a larger variance than does y_t. ■