Problem 1.

Given the model $y_t = a + bt + c_t + x_t$, where a, b are constants, c_t is deterministic and periodic with period s and x_t is a SARIMA$(p,0,q)\times(P,1,Q)_s$. What is the model for $w_t = y_t - y_{t-s}$?

Solution.
For x_t:

$$\phi_p(B)\Phi_P(B^s)\nabla_s x_t = c + \theta_q(B)\Theta_Q(B^s)a_t \tag{1}$$

$$\begin{aligned} w_t &= y_t - y_{t-s} \\ &= (a + bt + c_t + x_t) - (a + b(t-s) + c_{t-s} + x_{t-s}) \\ &= bs + c_t - c_{t-s} + x_t - x_{t-s} \\ &= bs + x_t - x_{t-s} \\ &= bs + \nabla_s x_t \end{aligned} \tag{2}$$

Therefore w_t is ARMA$(p,q)\times(P,Q)_s$ with constant term of bs. ■

Problem 2.

Identify the following as certain multiplicative SARIMA models:

(a) $y_t = 0.5y_{t-1} + y_{t-4} - 0.5y_{t-5} + a_t - 0.3a_{t-1}$

(b) $y_t = y_{t-1} + y_{t-12} - y_{t-13} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13}$

Solution.
(a)

$$\begin{aligned} y_t - y_{t-4} &= 0.5(y_{t-1} - y_{t-5}) + a_t - 0.3a_{t-1} \\ (y_t - y_{t-4}) - 0.5(y_{t-1} - y_{t-5}) &= a_t - 0.3a_{t-1} \\ (\nabla_4 y_t) - 0.5(\nabla_4 y_{t-1}) &= a_t - 0.3a_{t-1} \\ (1-0.5B)\nabla_4 y_t &= (1-0.3)a_t \end{aligned} \tag{3}$$

We can see that y_t can be modeled as ARIMA$(1,0,1)\times(0,1,0)_4$, with $\phi_1 = 0.5$ and $\theta_1 = 0.3$.

(b)

$$\begin{aligned} y_t &= y_{t-1} + y_{t-12} - y_{t-13} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13} \\ \nabla y_t &= \nabla y_{t-12} + a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13} \\ \nabla y_t - \nabla y_{t-12} &= a_t - 0.5a_{t-1} - 0.5a_{t-12} + 0.25a_{t-13} \\ \nabla\nabla_{12} y_t &= (1 - 0.5B - 0.5B^{12} + 0.25B^{13})a_t \\ \nabla\nabla_{12} y_t &= (1-0.5B)(1-0.5B^{12})a_t \end{aligned} \tag{4}$$

We can see that y_t is an ARIMA$(0,1,1)\times(0,1,1)_{12}$, with $\Theta_1 = 0.5$, and $\theta_1 = 0.5$. ■

Problem 3.

If the characteristic polynomial of an AR time series model is

$$(1-1.6B+0.7B^2)(1-0.8B^{12})$$

(a) Is the model stationary?

(b) Identify the model as a certain SARIMA model.

Solution.

(a)

We have the characteristic functions

$$\begin{aligned}\Phi(B) &= (1-0.8B^{12}) \\ \phi(B) &= (1-1.6B+0.7B^2)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\Phi &= 0.8 \\ \phi_1 &= 1.6 \\ \phi_2 &= -0.7\end{aligned} \tag{6}$$

We can see that

$$\begin{aligned}\Phi_1 &= 0.8 & < 1 \\ \phi_1+\phi_2 &= 0.9 & < 1 \\ \phi_2-\phi_1 &= -2.3 & < 1 \\ \left|\phi_2\right| &= 0.7 & < 1\end{aligned} \tag{7}$$

The roots of the characteristic function are greater than 1 in absolute value. Therefore, the model is stationary.

(b)

This is a SARIMA$(2,0,0)\times(1,0,0)_{12}$ model with the form

$$\phi_2(B)\Phi_1(B^{12})y_t = a_t$$

■

Problem 4.

Suppose $y_t = y_{t-4} + a_t$ with

$$y_t = a_t \text{ for } t = 1,2,3,4.$$

(a) Find the variance function for y_t.

(b) Find the autocorrelation function for y_t.

(c) Identify the model for y_t as a certain SARIMA model.

Solution.

$$\begin{aligned} y_t &= y_{t-4} + a_t \\ &= y_{t-8} + a_{t-4} + a_t \\ &= y_{t-12} + a_{t-8} + a_{t-4} + a_t \\ &\vdots \end{aligned} \tag{8}$$

Let $t = 4k + i$ where $i = 1,2,3,4$ and $k = 0,1,2,3\ldots$

$$y_t = \underbrace{a_i + a_{4+i} + a_{8+i} + \cdots + a_{4k+i}}_{k+1 \text{ items}} \tag{9}$$

(a)

$$\begin{aligned} \mathrm{Var}[y_t] &= \mathrm{Var}[a_i + a_{4+i} + a_{8+i} + \cdots + a_{4k+i}] \\ &= (k+1)\sigma_a^2 \end{aligned} \tag{10}$$

(b)

Let $s = 4h + j$ where $j = 1,2,3,4$ and $h = 0,1,2,3\ldots$

$$\begin{aligned} \mathrm{Cov}[y_t, y_s] &= \mathrm{Cov}[\{a_i + a_{4+i} + a_{8+i} + \cdots + a_{4k+i}\}, \{a_j + a_{4+j} + a_{8+j} + \cdots + a_{4h+j}\}] \\ &= \begin{cases} 0 & i \neq j \\ (\min\{h,k\}+1)\sigma_a^2 & i = j \end{cases} \end{aligned} \tag{11}$$

$$\mathrm{Corr}[y_t, y_s] = \begin{cases} 0 & i \neq j \\ \frac{\min\{h,k\}+1}{\sqrt{(h+1)(k+1)}} & i = j \end{cases} \tag{12}$$

(c)

This is a SARIMA$(0,0,0) \times (0,1,0)_4$ model with the form

$$\nabla_4^1 y_t = a_t$$

■

Problem 5.

Consider the famous time series data "co2" (monthly carbon dioxide through 11 years in Alert, Canada).

(a) Fit a deterministic regression model in terms of months and time. Are the regression coefficients significant? What is the adjusted R-squared? (Note that the month variable should be treated as categorical and transformed into 11 dummy variables.)

(b) Identify, estimate the SARIMA model for the co2 level.

(c) Compare the two models above, what do you observe?