Convex Sets (III)

Lecture 3, Convex Optimization (Part a)

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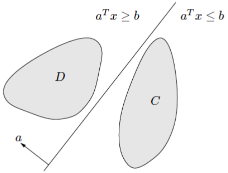
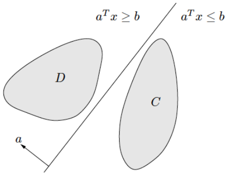
Separating Hyperplane Theorem (1/2)

Separating Hyperplane

The hyperplane $\{x \mid a^T x = b\}$ is called a **separating hyperplane** for the sets C and D, or is said to **separate** the sets C and D if $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.



Separating Hyperplane Theorem (2/2)



Separating Hyperplane Theorem

Suppose C and D are two convex sets that do not intersect, i.e., $C \cap D = \varnothing$. Then, there exist $a \neq 0$ and b such that the hyperplane $\{x \mid a^T x = b\}$ separates C and D.

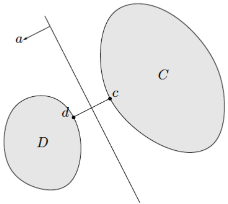
Separating Hyperplane Theorem – Proof of a Special Case (1/2)

- Consider that C and D are both convex, closed, and bounded.
- Assume that the **Euclidean distance** between C and D, defined as

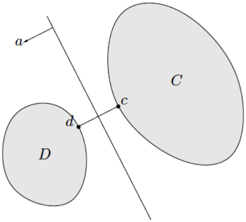
$$\textbf{dist}(C, D) = \inf \{||u - v||_2 \mid u \in C, v \in D\},$$

  is positive.
- Since C and D are both closed and bounded, there exist $c \in C$ and $d \in D$ such that

$$||c - d||_2 = \textbf{dist}(C, D).$$



Separating Hyperplane Theorem – Proof of a Special Case (2/2)



- Let

$$a = d - c, \quad b = \frac{\|d\|_2^2 - \|c\|_2^2}{2}.$$

- Then, it can be shown that the affine function

$$f(x) = a^T x - b = (d - c)^T \left(x - \frac{d + c}{2} \right)$$

is nonpositive on C and nonnegative on D.

Example – A Convex Set and An Affine Set

- Suppose $C \subseteq \mathbf{R}^n$ is convex and $D \subseteq \mathbf{R}^n$ is affine, i.e., $D = \{Fu + g | u \in \mathbf{R}^m\}$, where $F \in \mathbf{R}^{n \times m}$, $g \in \mathbf{R}^n$.
- Suppose C and D are disjoint, so by the separating hyperplane theorem there are $a \neq 0$ and b such that $a^T x \leq b$ for all $x \in C$ and $a^T x \geq b$ for all $x \in D$.
- $\because a^T x \geq b$ for all $x \in D$, $\therefore a^T F u \geq b - a^T g$ for all $u \in \mathbf{R}^m$.
- But a linear function is bounded below on $\mathbf{R}^m$ only when it is zero, so we conclude $a^T F = 0$ (and hence, $b \leq a^T g$).
- Thus we conclude that there exists $a \neq 0$ such that $F^T a = 0$ and $a^T x \leq a^T g$ for all $x \in C$.

Strict Separation of Convex Sets

Strict separation

For two sets $C, D \subseteq \mathbf{R}^n$, if there exists $a \in \mathbf{R}^n, b \in \mathbf{R}$ such that

$$a^T x < b \ \forall x \in C \text{ and } a^T x > b \ \forall x \in D,$$

then C and D are said to be **strictly separable**, and the hyperplane $\{x \mid a^T x = b\}$ is called **strict separation** of C and D.

- Remark: The separating hyperplane theorem only dictates that two disjoint convex sets are separated by a hyperplane. A strict separation is not guaranteed (even when the sets are closed).

Example – A Point and A Closed Convex Set

- Let C be a closed convex set and $x_0 \notin C$. Then there exists a hyperplane that strictly separates $\{x_0\}$ from C.
- Proof idea:
  - The two sets C and $B(x_0, \epsilon)$ do not intersect for some $\epsilon > 0$.
  - Apply the separating hyperplane theorem on C and $B(x_0, \epsilon)$ (getting a^T and b).
  - The affine function

$$f(x) = a^T x - b - \epsilon ||a||_2/2$$

    strictly separates C and $\{x_0\}$.
- Corollary: A closed convex set is the intersection of all halfspaces that contain it.

Convex Sets as Intersection of Halfspaces (Revisit)

- We have seen that the intersection of (possibly infinite) halfspaces is convex.
- It will be shown that a converse is true: every closed convex set S is the intersection of (usually infinite) halfspaces.
- A closed convex set S is the intersection of all halfspaces that contain it:

$$S = \bigcap_{\substack{S \subseteq \mathcal{H} \subseteq \mathbf{R}^n \\ \mathcal{H} \text{ is a halfspace}}} \mathcal{H}.^1$$

[1] In the text book, it was written as $S = \bigcap \{\mathcal{H} \mid \mathcal{H} \text{ halfspace}, S \subseteq \mathcal{H}\}$.

Converse of Separating Hyperplane Theorems

- Question: If there exists a hyperplane that separates convex sets C and D, does this imply C and D are disjoint?
  - (No. Consider $C = D = \{0\} \subseteq \mathbf{R}$.)
- Suppose C and D are convex sets, with C open, and there exists an affine function f that is nonpositive on C and nonnegative on D. Then C and D are disjoint.
  - *Hint: f is negative on C.*

Theorem

Any two convex sets, at least one of which is open, are disjoint if and only if there exists a separating hyperplane.

Supporting Hyperplanes (1/2)

Supporting hyperplanes

Suppose $C \subseteq \mathbf{R}^n$, and x_0 is a point in its boundary $\mathbf{bd}\ C$, i.e.,

$$x_0 \in \mathbf{bd}\ C = \mathbf{cl}\ C \backslash \mathbf{int}\ C.^2$$

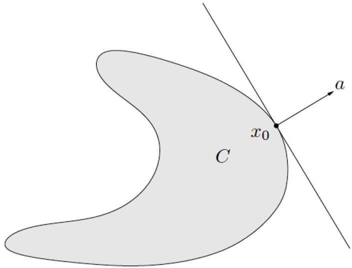
If $a \neq 0$ satisfies $a^T x \leq a^T x_0$ for all $x \in C$, then the hyperplane $\{x \mid a^T x = a^T x_0\}$ is called a **supporting hyperplane** to C at the point x_0.

- This is equivalent to the statement that $\{x_0\}$ and C are separated by the hyperplane $\{x \mid a^T x = a^T x_0\}$.

[2] The notation $\mathbf{cl}$ means the closure of a set, a concept that will be formally introduced soon.

Supporting Hyperplanes (2/2)

- The hyperplane is tangent to C at x_0, and the halfspace $\{x \mid a^T x \leq a^T x_0\}$ contains C.



Supporting Hyperplane Theorem

Supporting Hyperplane Theorem

For any nonempty convex set C, and any $x_0 \in \mathbf{bd}\ C$, there exists a supporting hyperplane to C at x_0.

Proof: Use the separating hyperplane theorem.

- If $\mathbf{int}\ C \neq \varnothing$: then by applying the separating hyperplane theorem on $\{x_0\}$ and $\mathbf{int}\ C$, the statement is proved.
- If $\mathbf{int}\ C = \varnothing$: then C lies in an affine set of dimension less than n. Then any hyperplane that contains this affine set contains both C and x_0 and therefore is a supporting hyperplane.

Converse of the Supporting Hyperplane Theorem

Converse of the Supporting Hyperplane Theorem

If a set C is closed, has nonempty interior, and has a supporting hyperplane at any $x_0 \in \mathbf{bd}\ C$, then C is convex.