Convex Sets (I)

Lecture 1, Convex Optimization (Part b)

National Taiwan University

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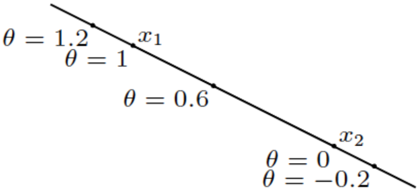
Line

Line

Let $x_1, x_2 \in \mathbf{R}^n$ and $x_1 \neq x_2$. The set of all points

$$\{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbf{R}\}$$

is called a **line** passing through x_1 and x_2.



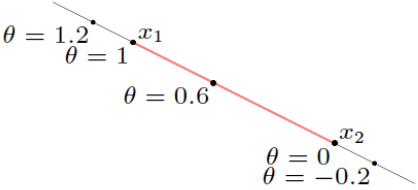
Line Segment

Line Segment

Let $x_1, x_2 \in \mathbf{R}^n$ and $x_1 \neq x_2$. The set of all points

$$\{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbf{R}, 0 \leq \theta \leq 1\}$$

is called a **(closed) line segment** between x_1 and x_2.



Line and Line Segment

Line and Line Segment

Let $x_1, x_2 \in \mathbf{R}^n$ and $x_1 \neq x_2$. The set of all points

$$\{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbf{R}\}$$

is called a **line** passing through x_1 and x_2. The set of all points

$$\{\theta x_1 + (1-\theta)x_2 \mid \theta \in \mathbf{R}, 0 \leq \theta \leq 1\}$$

is called a **(closed) line segment** between x_1 and x_2.

Another interpretation:

$$y = x_2 + \theta(x_1 - x_2)$$

is the sum of the **base point** x_2 and the **direction** $x_1 - x_2$ scaled by the parameter θ.

Affine Sets

Affine Sets

A set $C \subseteq \mathbf{R}^n$ is **affine** if the line through any two distinct points in C lies in C. That is,

$$x_1, x_2 \in C, \theta \in \mathbf{R} \quad \Rightarrow \quad \theta x_1 + (1-\theta)x_2 \in C.$$

Affine Combination

Let $x_1, x_2, \cdots, x_k \in \mathbf{R}^n$. Then, a point of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$ is referred to as an **affine combination** of the points $x_1, x_2, \cdots, x_k$.

Affine Combinations

Affine Combination

Let $x_1, x_2, \cdots, x_k \in \mathbf{R}^n$. Then, a point of the form

$$\theta_1 x_1 + \cdots + \theta_k x_k$$

with $\theta_1 + \cdots + \theta_k = 1$ is referred to as an affine combination of the points $x_1, x_2, \cdots, x_k$.

Property

A set is affine if and only if it contains every affine combination of its points.

Affine Sets

Affine Sets and Subspaces

If $C \subseteq \mathbf{R}^n$ is an affine set and $x_0 \in C$, then the set

$$V = C - x_0 = \{x - x_0 \mid x \in C\}$$

is a subspace[1] of $\mathbf{R}^n$.

Proof:

[1] Note that the subspace V associated with C does not depend on the choice of x_0.

Dimension of Affine Sets

Dimension of Affine Sets

The **dimension** of an affine set C is defined as the dimension of the subspace $V = C - x_0$ where x_0 is any element of C.

Example: Solution set of linear equations (1/2)

Solution set of linear equations

The solution set of a system of linear equations

$$C = \{x \mid Ax = b\}$$

where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ is an affine set.

Proof:

Example: Solution set of linear equations (2/2)

Solution set of linear equations

The solution set of a system of linear equations

$$C = \{x \mid Ax = b\}$$

where $A \in \mathbf{R}^{m \times n}$ and $b \in \mathbf{R}^m$ is an affine set.

- The subspace associated with the affine set C is the nullspace of A.
- Converse: every affine set can be expressed as the solution set of a system of linear equations.

Affine Hull

Affine Hull

The set of all affine combinations of points in some set $C \subseteq \mathbf{R}^n$ is called the **affine hull** of C, denoted **aff** C:

$$\textbf{aff } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}.$$

The affine hull is the smallest affine set that contains C:

- If S is any affine set with $C \subseteq S$, then $\textbf{aff}\, C \subseteq S$.

Affine Dimension

Affine Dimension

The **affine dimension** of C, a subset of $\mathbf{R}^n$, is defined by the dimension of its affine hull.

Example

Let $C = \left\{x \in \mathbf{R}^2 \mid x_1^2 + x_2^2 = 1\right\}$. What is the affine dimension of C?

Interior

Interior point

An element $x \in C \subseteq \mathbf{R}^n$ is called an **interior point** of C if there exists an $\epsilon > 0$ for which

$$\{y \mid ||y - x||_2 \leq \epsilon\}$$

is a subset of C.

Interior

The set of all interior points of C is called the **interior** of C, denoted **int** C:

$$\textbf{int} C = \{y \mid y \in C \text{ and } y \text{ is an interior point of } C\}$$

Interior Points and Interior – Example

Example 1

Let $C = \{x \mid 1 \leq x \leq 2\} \subseteq \mathbf{R}$. Then $x = 1.001 \in C$ is an interior point of C while $x = 1$ is not.
The interior of C is $\mathbf{int}\ C = \{x \mid 1 < x < 2\}$

Example 2

Let $C = \left\{x \in \mathbf{R}^3 \mid x_1^2 + x_2^2 + x_3^2 \leq 1\right\}$. Then $x = (0.9, 0, 0) \in C$ is an interior point of C while $x = (1, 0, 0)$ is not.

Relative Interior

Consider a set $C \subseteq \mathbf{R}^n$ whose affine dimension is less than n. That is, $\mathbf{aff} C \neq \mathbf{R}^n$. What is the interior of C?

Relative Interior

The **relative interior** of the set C, denoted **relint**C, is defined as its interior relative to **aff** C:

$$\textbf{relint } C = \{x \in C \mid B(x, r) \cap \textbf{aff } C \subseteq C \text{ for some } r > 0\}$$

where $B(x, r) = \{y \mid ||y - x||_2 \leq r\}$.

Relative Interior – An Example

- Consider a square in the (x_1, x_2)-plane in $\mathbf{R}^3$, defined as

$$C = \left\{x \in \mathbf{R}^3 \mid -1 \le x_1 \le 1, -1 \le x_2 \le 1, x_3 = 0\right\}.$$

- Its affine hull is the (x_1, x_2)-plane, i.e., $\mathbf{aff}\, C = \left\{x \in \mathbf{R}^3 \mid x_3 = 0\right\}$.
- The interior of C is $\mathbf{int}\ C = \varnothing$.
- The relative interior of C is

$$\textbf{relint}\ C = \left\{x \in \mathbf{R}^3 \mid -1 < x_1 < 1, -1 < x_2 < 1, x_3 = 0\right\}.$$

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- Affine dimension and relative interior

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- Convex sets
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- Cones
- Conic combination

3 Examples of convex and affine sets (I)
- Simple examples
- Hyperplanes and halfspaces
- Euclidean balls and ellipsoids

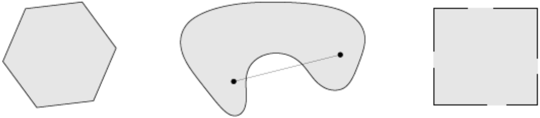
Convex Sets

Convex Set

A set C is **convex** if the line segment between any two points in C lies in C. That is, for any $x_1, x_2 \in C$ and any θ with $0 \leq \theta \leq 1$, we have

$$\theta x_1 + (1-\theta)x_2 \in C.$$

Example: which of following is convex?



Example: Every affine set is also convex. Any line segment is also convex.

Convex Combination

Convex combination

A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$, where $\theta_1 + \cdots + \theta_k = 1$ and $\theta_i \geq 0$, $i = 1, \cdots, k$, is called a **convex combination** of the points $x_1, \cdots, x_k$.

Property

A set is convex if and only if it contains every convex combination of its points.

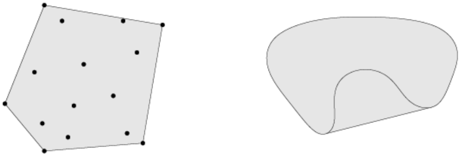
Convex Hull

Convex Hull

The **convex hull** of a set C, denoted **conv** C, is the set of all convex combinations of points in C:

$$\textbf{conv } C = \{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \cdots, k,\ \theta_1 + \cdots + \theta_k = 1\}.$$

Property: The convex hull **conv** C is always convex. It is the smallest convex set that contains C.



Generalized Definitions of Convex Combinations

- Infinite sum:
  - If C is convex and let $x_1, x_2, \cdots \in C$, then $\sum_{i=1}^{\infty} \theta_i x_i \in C$ where $\theta_i \geq 0, i = 1, 2, \cdots$ and $\sum_{i=1}^{\infty} \theta_i = 1$.
- Integral:
  - Let C be a convex set. Consider a function $p : \mathbf{R}^n \to \mathbf{R}$ that satisfies $p(x) \geq 0, \forall x \in C$ and $\int_C p(x)dx = 1$. Then $\int_C p(x)x\ dx \in C$.
- Probability distributions (most general form)
  - Suppose $C \subseteq \mathbf{R}^n$ is convex and x is a random vector with $x \in C$ with probability one. Then $\mathbf{E}[x] \in C$.

On Various Types of "Combinations"

Compare "linear combination," "affine combination," and "convex combination". All of these three types of combinations can be defined as the set $\{\theta_1 x_1 + \cdots + \theta_k x_k\}$ with certain constraints on the coefficients $\theta_1, \cdots, \theta_k$.

Type	Constraints on θ_i	Set of all combinations
linear combination	$\theta_1, \cdots, \theta_k \in \mathbf{R}$	span
affine combination	$\theta_1 + \cdots + \theta_k = 1$	affine hull
convex combination	$\theta_1 + \cdots + \theta_k = 1,\ \theta_i \geq 0$	convex hull

Cones

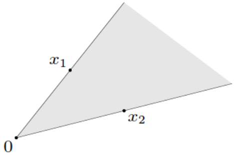
Cone

A set C is called a **cone** if for every $x \in C$ and $\theta \geq 0$ we have $\theta x \in C$. The set C is also said to be **nonnegative homogeneous**.

Convex Cone

A set C is called a **convex cone** if it is convex and is a cone. That is, for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$ we have

$$\theta_1 x_1 + \theta_2 x_2 \in C.$$



Conic Combination

Conic combination

A point of the form $\theta_1 x_1 + \cdots + \theta_k x_k$ with $\theta_1, \cdots, \theta_k \geq 0$ is called a **conic combination** (or a **nonnegative linear combination**) of $x_1, x_2, \cdots, x_k$.

- Property: If x_i are in a convex cone C, then every conic combination of x_i is in C.
- Property: A set C is a convex cone if and only if it contains all conic combinations of its elements.
- Generalized definitions: The idea of conic combination can be generalized to infinite sums and integrals.

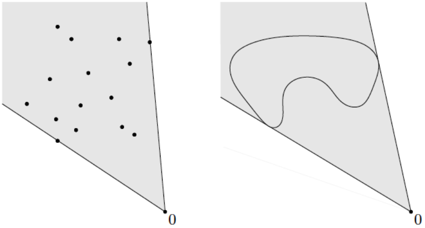
Conic Hull

Conic Hull

The **conic hull** of a set C is the set of all conic combinations of points in C:

$$\{\theta_1 x_1 + \cdots + \theta_k x_k \mid x_i \in C,\ \theta_i \geq 0,\ i = 1, \cdots, k\}.$$

Property: The conic hull of a set C is the smallest convex cone that contains C.



On Various Types of "Combinations"

Compare "linear combination," "affine combination," "convex combination," and "conic combination". All of these four types of combinations can be defined as the set $\{\theta_1 x_1 + \cdots + \theta_k x_k\}$ with certain constraints on the coefficients $\theta_1, \cdots, \theta_k$.

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affine combination	$\theta_1 + \cdots + \theta_k = 1$	affine hull
convex combination	$\theta_1 + \cdots + \theta_k = 1,\ \theta_i \geq 0$	convex hull
conic combination	$\theta_1, \ldots, \theta_k \geq 0$	conic hull

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Some Simple Examples of Affine / Convex Sets / Cones

- The empty set $\varnothing$ is affine (and hence convex).
- Any single point (i.e., **singleton**) $\{x_0\}$ is affine (and convex).
- The whole space $\mathbf{R}^n$ is affine (and convex).
- Any subspace is affine, and a convex cone.
- Any line is affine. If it passes through zero, it is a subspace, and also a convex cone.
- A line segment is convex, but is in general not affine.
- A **ray**, having the form $\{x_0 + \theta v \mid \theta \geq 0\}$, where $v \neq 0$, is convex but not affine. If $x_0 = 0$, then it is a convex cone.

Hyperplane

Hyperplane

A **hyperplane** is a set of the form

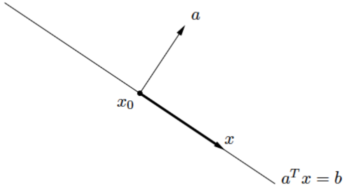
$$\left\{x \;\middle|\; a^T x = b\right\}$$

where $a \in \mathbf{R}^n$, $a \neq 0$, and $b \in \mathbf{R}$.

- A hyperplane is the solution set of a nontrivial linear equation among components of x. Thus, a hyperplane is affine.
- The vector a is called the **normal vector** of the hyperplane. Every point in the hyperplane has a constant inner product with the normal vector a.
- The constant $b \in \mathbf{R}$ determines the offset of the hyperplane from 0.

Hyperplane

- The hyperplane $\{x \mid a^T x = b\}$ can be rewritten as $\{x \mid a^T (x - x_0) = 0\}$, where x_0 is any point in the hyperplane.



- Further, we can write

$$\{x \mid a^T (x - x_0) = 0\} = x_0 + a^{\perp}$$

where $a^{\perp}$ denotes the orthogonal complement of a: $a^{\perp} = \{v \mid a^T v = 0\}$.

Halfspaces

A hyperplane divides $\mathbf{R}^n$ into two **halfspaces**.

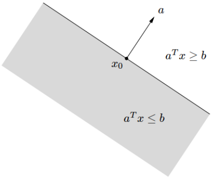
Halfspaces

A **(closed) halfspace** is a set of the form

$$\left\{x \;\middle|\; a^T x \leq b\right\},$$

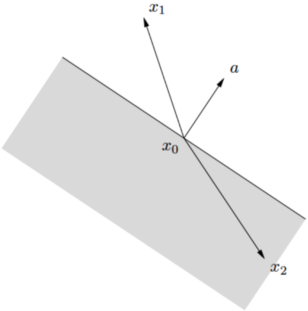
where $a \in \mathbf{R}^n, a \neq 0$, and $b \in \mathbf{R}$.

- A halfspace is the solution set of one (nontrivial) linear inequality.
- Halfspaces are convex, but not affine.



Halfspaces

- The halfspace $\{x \mid a^T x \le b\}$ can also be rewritten as
$$\{x \mid a^T(x - x_0) \le 0\},$$
where x_0 is any point on the **associated hyperplane** (i.e., $a^T x_0 = b$).



Halfspaces

- The boundary[2] of the halfspace $\{x \mid a^T x \leq b\}$ is the hyperplane $\{x \mid a^T x = b\}$.
- The set

$$\{x \mid a^T x < b\}$$

is the interior of the halfspace $\{x \mid a^T x \leq b\}$. It is called an **open halfspace**.

[2] A formal definition of boundary will be given somewhere else

Euclidean Balls

Euclidean Ball

A **Euclidean ball** (or just **ball**) in $\mathbf{R}^n$ has the form

$$B(x_c, r) = \{x \mid ||x - x_c||_2 \leq r\} = \left\{x \mid (x - x_c)^T (x - x_c) \leq r^2\right\}$$

where $r > 0$ and $||\cdot||_2$ denotes the Euclidean norm.
The vector x_c is the **center** of the ball. The scalar r is its **radius**.

- $B(x_c, r)$ consists of all points within a distance r of the center x_c.
- The Euclidean ball can be rewritten as

$$B(x_c, r) = \{x_c + ru \mid u \in \mathbf{R}^n, ||u||_2 \leq 1\}.$$

Euclidean Balls

Property

A Euclidean ball is a convex set.

Proof:

Ellipsoid

Ellipsoid

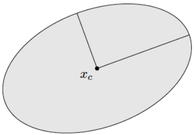
An ellipsoid has the form

$$\mathcal{E} = \left\{ x \;\middle|\; (x - x_c)^T P^{-1} (x - x_c) \leq 1 \right\},$$

where P is symmetric and positive definite: $P = P^T \succ 0$.
The vector $x_c \in \mathbf{R}^n$ is the center of the ellipsoid.

- The lengths of the semi-axes of $\mathcal{E}$ are given by $\sqrt{\lambda_i}$ where λ_i are the eigenvalues of P.
- A ball is an ellipsoid with $P = r^2 I$.
- An ellipsoid is convex.

Ellipsoid



- The ellipsoid $\mathcal{E} = \left\{x \mid (x - x_c)^T P^{-1}(x - x_c) \leq 1\right\}$ can be rewritten as

$$\mathcal{E} = \{x_c + Au \mid u \in \mathbf{R}^n, ||u||_2 \leq 1\}$$

  where A is square and nonsingular.
- W.l.o.g., we can assume A is symmetric and positive definite (by taking $A = P^{1/2}$).

Degenerate Ellipsoid

- If A is symmetric positive semidefinite but singular, then the set $\mathcal{E} = \{x_c + Au \mid ||u||_2 \leq 1\}$ is called a **degenerate ellipsoid**.
- Its affine dimension is rank A.
- Degenerate ellipsoids are also convex.