Problem 1. Testing for completely random dropouts

Let P_{ij} denote the probability that the $i-$th unit drops out at time t_j, $j = 1, \dots, m$.
Under the assumption of completely random dropouts, the probability P_{ij} may depend on time, treatment, or other explanatory variables, but cannot depend on the observed measurements $y_i = (y_{i1}, \dots, y_{i\,m_i})$.

Testing Method:

(a) Choose the score function $h_k(y_1, \dots, y_k)$ so that extreme values constitute evidence against completely random dropouts. A sensible choice is

$$h_k(y_1, \dots, y_k) = \sum_{j=1}^{k} \omega_j y_j.$$

(b) For each of $k = 1, \dots, (m-1)$, define

$$R_k = \{i : m_i \geq k\},$$
$$r_k = \{i : m_i = k\},$$

and compute the set of scores $h_{ik} = h_k(y_{i1\dots,y_{ik}})$ for $i \in R_k$.

(c) If $1 \leq |r_k| \leq |R_k|$, test the hypothesis that the r_k's scores so defined are a random sample from the "populations" of R_k's scores.

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Remark:

1. The implicit assumption that the separated $p-$values are mutually independent is valid precisely because once a unit drops out, it never returns.

2. A natural test statistics is $\bar{h}_k = \frac{1}{|r_k|} \sum_{\{j \in r_k\}} h_{jk}$. Under the assumption of completely random dropouts,

$$\bar{h}_k \sim N\left(\overline{H}_k, \frac{|R_k| - |r_k|}{(|R_k| - 1)|r_k|} \sum_{\{j \in R_k\}} (h_{jk} - \overline{H}_k)^2 / |R_k|\right),$$

where

$$\overline{H}_k = \frac{1}{|R_k|} \sum_{\{j \in r_k\}} h_{jk}.$$

- When $|R_k|$ or $|r_k|$ is small, evaluate the randomization distribution of $\bar{h}_k$ under the null hypothesis.
- Alternative method ...

3. The Final stage consists of analyzing the resulting set of $p-$values via
   (a) Empirical distribution of the $p-$values
   (b) Kolmogorov-Smirnov statistic $D_+ = \sup |\hat{F}_n(p) - p|$

Given a finite population of size N, with individual values $\{X_i\}_{i=1}^N$,

and a set of sample of size n, drawn from the population without replacement, with values $\{X_i\}_{i=1}^n$.

Let σ^2 be the population variance:

$$\sigma^2 = \mathbf{Var}[X_i] = \frac{1}{N}\sum_{i=1}^{N}(X_i - \mu),$$

where $\mu = \sum_{i=1}^N X_i$ is the population mean.

Let $\bar{X} = \frac{1}{n}S_n = \frac{1}{n}\sum_{i=1}^n X_i$ be the sample mean based on the sample set.

Since every pair (X_i, X_j) for $i \neq j$ has the same joint distribution, we have

$$\mathbf{Var}[S_n] = \sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{Cov}[X_i, X_j],$$

where

$$\mathbf{Cov}[X_i, X_j] = \begin{cases} \sigma^2 & i = j \\ c & i \neq j \end{cases}.$$

Thus,

$$\mathbf{Var}[S_n] = n\sigma^2 + n(n-1)c.$$

which applies to the case $n = N$ as well. Notice that S_N is a constant (equal to the sum of all N values in the population). It follows that

$$0 = \mathbf{Var}[S_N] = N\sigma^2 + N(N-1)c.$$

Solve the equation above for

$$c = -\frac{\sigma^2}{N-1}.$$

Hence,

$$\mathbf{Var}[S_n] = n\sigma^2\left(1 - \frac{n-1}{N-1}\right) = \frac{N-n}{N-1}\cdot n\sigma^2$$

and

$$\mathbf{Var}[\bar{X}] = \frac{N-n}{N-1}\cdot\frac{\sigma^2}{n}.$$

The factor $\dfrac{N-n}{N-1}$ is the Finite Population Correction Factor (FPC).

Problem 2. Generalized estimating equations under a random missing mechanism:

$$P(R_{ij} = 1 | R_{ij-1} = 1, H_{im}) = P(R_{ij} = 1 | R_{ij-1} = 1, H_{ij})$$

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Basic GEE method when dropouts are completely random:

$$S_\beta(\beta, \alpha) = \sum_i^n \left(\frac{\partial b}{\partial \theta_i}\right)^T \mathrm{Var}(Y_i)^{-1}(Y_i - \mu_i) = 0$$

Let $\mathbf{P} = \mathrm{diag}(\mathbf{P})$, $\mathrm{diag}(\mathbf{P}) = \sum_i P_{ij}$ with $P_{ij} = P(R_{ij} = 1 | R_{ij-1} = 1, H_{ij})$.

$$\mathbf{P} = \prod_{j=1}^{k} \hat{P}_{ij}$$

When $\mathbf{P}$'s are themselves estimated from the data using an assumed random dropout model, the estimators of $\mathbf{b}$ obtained from the following extended estimating equation are consistent:

$$\mathbf{S}^*(\mathbf{b}, a) = \mathbf{b} - n\hat{\boldsymbol{\beta}}^T \mathrm{Var}(Y)^{-1} \mathbf{P}^{-1}(Y - \boldsymbol{\mu}) = 0$$

Problem 3.