Generalized Estimating Equations (GEE), which is a multivariate analogue of quasi-likelihood.

$S_\beta(\beta,\alpha)=\sum_{i=1}^n\left(\frac{\partial\mu_i}{\partial\beta}\right)^T[\mathrm{Var}\left(Y_i\right)]^{-1}\left(Y_i-\mu_i\right)$, where $\mu_i=h\left(x_{ij}^T\beta\right), \mathrm{Var}\left(Y_i\right)=\mathrm{Var}\left(Y_i;\beta,\alpha\right)$

$S_\alpha(\beta,\alpha)=\sum_{i=1}^n\left(\frac{\partial\eta_i}{\partial\alpha}\right)^T H_i^{-1}\left(\omega_i-\eta_i\right)$, where $\omega_i=\left(R_{i1}R_{i2},\cdots,R_{i1}R_{im_i},\cdots,R_{i1}^2,\cdots,R_{im_i}^2\right)$

$\eta_i=\mathbb{E}[\omega_i\,|\,(\beta,\alpha)]$, and $H_i=\mathrm{Var}(\omega_i)$, with $R_{ij}=\frac{Y_{ij}-\mu_{ij}}{\sqrt{\mathrm{Var}(Y_{ij})}}$

The estimator, say $(\hat\beta,\hat\alpha)$ of (β,α) is defined to be the solution of the above equations, i.e. $S_\beta(\hat\beta,\hat\alpha)=0$ and $S_\alpha(\hat\beta,\hat\alpha)=0$.

Problem 5.

Theorem 3.1.

Under the regularity conditions, $n^{\frac{1}{2}}\left[\begin{pmatrix}\hat\beta\\ \hat\alpha\end{pmatrix}-\begin{pmatrix}\beta\\ \alpha\end{pmatrix}\right]\xrightarrow{d}N(0,\Sigma)$, where Σ can be estimated by

$$\left(\frac{1}{n}\sum_{i=1}^n C_i^TB_i^{-1}D_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n C_i^TB_i^{-1}V_{0i}B_i^{-1}C_i\right)\left(\frac{1}{n}\sum_{i=1}^n D_i^TB_i^{-1}C_i\right)^{-1},$$

where $C_i=\begin{pmatrix}\frac{\partial\mu_i}{\partial\beta} & 0\\ 0 & \frac{\partial\eta_i}{\partial\alpha}\end{pmatrix}, B_i=\begin{pmatrix}\mathrm{Var}\left(Y_i\right) & 0\\ 0 & H_i\end{pmatrix}, D_i=\begin{pmatrix}\frac{\partial\mu_i}{\partial\beta} & \frac{\partial\mu_i}{\partial\alpha}\\ \frac{\partial\eta_i}{\partial\beta} & \frac{\partial\eta_i}{\partial\alpha}\end{pmatrix}$,

and $V_{0i}=\begin{pmatrix}y_i-\mu_i\\ \omega_i-\eta_i\end{pmatrix}^{\otimes 2}$.

—

Hint

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix}&=\begin{pmatrix}S_\beta(\hat\beta,\hat\alpha)\\ S_\alpha(\hat\beta,\hat\alpha)\end{pmatrix}\\ &=\begin{pmatrix}S_\beta(\beta,\alpha)\\ S_\alpha(\beta,\alpha)\end{pmatrix}+\left.\begin{pmatrix}\frac{\partial S_\beta(\beta,\alpha)}{\partial\beta} & \frac{\partial S_\beta(\beta,\alpha)}{\partial\alpha}\\ \frac{\partial S_\alpha(\beta,\alpha)}{\partial\beta} & \frac{\partial S_\alpha(\beta,\alpha)}{\partial\alpha}\end{pmatrix}\right|_{\begin{pmatrix}\hat\beta^*\\ \hat\alpha^*\end{pmatrix}}\left[\begin{pmatrix}\hat\beta\\ \hat\alpha\end{pmatrix}-\begin{pmatrix}\beta\\ \alpha\end{pmatrix}\right],\end{aligned}$$

where $\begin{pmatrix}\hat\beta^*\\ \hat\alpha^*\end{pmatrix}$ lies on the line segment between $\begin{pmatrix}\beta\\ \alpha\end{pmatrix}$ and $\begin{pmatrix}\hat\beta\\ \hat\alpha\end{pmatrix}$.

By the first order tylor expansion,

$$\begin{aligned}n^{1/2}\left[\begin{pmatrix}\hat\beta\\ \hat\alpha\end{pmatrix}-\begin{pmatrix}\beta\\ \alpha\end{pmatrix}\right]&=-n\left.\begin{pmatrix}\frac{\partial S_\beta(\beta,\alpha)}{\partial\beta} & \frac{\partial S_\beta(\beta,\alpha)}{\partial\alpha}\\ \frac{\partial S_\alpha(\beta,\alpha)}{\partial\beta} & \frac{\partial S_\alpha(\beta,\alpha)}{\partial\alpha}\end{pmatrix}^{-1}\right|_{\begin{pmatrix}\hat\beta^*\\ \hat\alpha^*\end{pmatrix}}n^{-1/2}\begin{pmatrix}S_\beta(\beta,\alpha)\\ S_\alpha(\beta,\alpha)\end{pmatrix}\\ &=-nV^{*-1}n^{-1/2}\begin{pmatrix}S_\beta(\beta,\alpha)\\ S_\alpha(\beta,\alpha)\end{pmatrix}.\end{aligned}$$

We have

$$
\begin{aligned}
V^* \xrightarrow{p} V &= \mathbb{E}\left[\begin{pmatrix} \dfrac{\partial S_\beta(\beta,\alpha)}{\partial \beta} & \dfrac{\partial S_\beta(\beta,\alpha)}{\partial \alpha} \\ \dfrac{\partial S_\alpha(\beta,\alpha)}{\partial \beta} & \dfrac{\partial S_\alpha(\beta,\alpha)}{\partial \alpha} \end{pmatrix}\right] \\
&= \begin{pmatrix} \left(\dfrac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}(Y_i)]^{-1} \left(\dfrac{\partial \mu_i}{\partial \beta}\right) & \left(\dfrac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}(Y_i)]^{-1} \left(\dfrac{\partial \mu_i}{\partial \alpha}\right) \\ \left(\dfrac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} \left(\dfrac{\partial \eta_i}{\partial \beta}\right) & \left(\dfrac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} \left(\dfrac{\partial \eta_i}{\partial \alpha}\right) \end{pmatrix} \\
&= \mathbb{E}[C_i^T B_i^{-1} D_i],
\end{aligned}
$$

which is estimated by $\left(\frac{1}{n}\sum_{i=1}^n C_i^T B_i^{-1} D_i\right)$.

—

$$
\begin{aligned}
S_\beta(\beta,\alpha) &= \sum_{i=1}^n \left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}\,(Y_i)]^{-1} (Y_i - \mu_i) \\
&= \sum_{i=1}^n U_i.
\end{aligned}
$$

$$
\begin{aligned}
S_\alpha(\beta,\alpha) &= \sum_{i=1}^n \left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} (\omega_i - \eta_i) \\
&= \sum_{i=1}^n Z_i.
\end{aligned}
$$

By CLT,

$$
n^{-1/2} S_\beta(\beta,\alpha) = \sqrt{n}\frac{1}{n}\sum_{i=1}^n U_i = \sqrt{n}(\bar{U}_i - \mathbb{E}[U_i]) \xrightarrow{d} N(0, \sigma_U^2),
$$

where

$$
\begin{aligned}
\sigma_U^2 &= \mathrm{Var}[U_i] \\
&= \mathrm{Var}[\left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}\,(Y_i)]^{-1} (Y_i - \mu_i)] \\
&= \left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}\,(Y_i)]^{-2} \left(\frac{\partial \mu_i}{\partial \beta}\right) \mathrm{Var}(Y_i) \\
&= \left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}\,(Y_i)]^{-1} \left(\frac{\partial \mu_i}{\partial \beta}\right)
\end{aligned}
$$

Similarly,

$$
n^{-1/2} S_\alpha(\beta,\alpha) = \sqrt{n}\frac{1}{n}\sum_{i=1}^n Z_i = \sqrt{n}(\bar{Z}_i - \mathbb{E}[Z_i]) \xrightarrow{d} N(0, \sigma_Z^2),
$$

where

$$
\begin{aligned}
\sigma_Z^2 &= \mathrm{Var}[Z_i] \\
&= \mathrm{Var}[\left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} (\omega_i - \eta_i)] \\
&= \left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-2}\, \mathrm{Var}[\omega_i] \left(\frac{\partial \eta_i}{\partial \alpha}\right) \\
&= \left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} \left(\frac{\partial \eta_i}{\partial \alpha}\right).
\end{aligned}
$$

Thus,

$$n^{-1/2}\begin{pmatrix} S_\beta(\beta,\alpha) \\ S_\alpha(\beta,\alpha) \end{pmatrix} \xrightarrow{d} N(0,\Sigma_S),$$

where

$$\Sigma_S = \begin{pmatrix} \sigma_U^2 & \sigma_{UZ}^2 \\ \sigma_{UZ}^2 & \sigma_Z^2 \end{pmatrix},$$

with

$$\sigma_{UZ}^2 = \mathbf{Cov}(U_i, Z_i) = \mathbb{E}[U_i Z_i]$$

$$= \mathbb{E}\left[\left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}\,(Y_i)]^{-1}\,(Y_i - \mu_i)\cdot\left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1}\,(\omega_i - \eta_i)\right].$$

Since

$$\begin{aligned} &\mathbb{E}(C_i^T B_i^{-1} V_{0i} B_i^{-1} C_i) \\ &= \mathbb{E}\left[C_i^T B_i^{-1}\begin{pmatrix} (y_i-\mu_i)^2 & (y_i-\mu_i)(\omega_i-\eta_i) \\ (y_i-\mu_i)(\omega_i-\eta_i) & (\omega_i-\eta_i)^2 \end{pmatrix} B_i^{-1} C_i\right], \end{aligned}$$

where

$$C_i^T B_i^{-1} = \begin{pmatrix} \left(\frac{\partial \mu_i}{\partial \beta}\right)^T [\mathrm{Var}(Y_i)]^{-1} & 0 \\ 0 & \left(\frac{\partial \eta_i}{\partial \alpha}\right)^T H_i^{-1} \end{pmatrix},$$

we can estimate Σ_S with $\left(\frac{1}{n}\sum_{i=1}^n C_i^T B_i^{-1} V_{0i} B_i^{-1} C_i\right)$.

By Slutsky Theorem,

$$n^{1/2}\left[\begin{pmatrix} \hat\beta \\ \hat\alpha \end{pmatrix} - \begin{pmatrix} \beta \\ \alpha \end{pmatrix}\right] \xrightarrow{d} N(0,\Sigma),$$

where $\Sigma = V^{-T}\Sigma_S V^{-1}$.

Therefore, Σ can be estimated by

$$\left(\frac{1}{n}\sum_{i=1}^n C_i^T B_i^{-1} D_i\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n C_i^T B_i^{-1} V_{0i} B_i^{-1} C_i\right)\left(\frac{1}{n}\sum_{i=1}^n D_i^T B_i^{-1} C_i\right)^{-1},$$