Transition Models

$t'_{ij}s$ are assumed to be equally spaced.

Let $H_i = \{y_k, k = 1, \cdots, j-1\}$.

Consider

$$f(y_{ij} \mid H_{ij}, \alpha, \beta) = \exp\left\{\frac{y_{ij} - \psi(\theta_{ij})}{\phi} + c(y_{ij}, \phi)\right\},$$

where $\psi(\theta_{ij})$ and $c(y_{ij}, \phi)$ are known functions.

One has

$$\mu_{ij}^c = E\left[y_{ij} \mid H_{ij}\right] = \psi'\left(\theta_{ij}\right)$$

and

$$V_{ij}^c = V\left[y_{ij} \mid H_{ij}\right] = \psi''\left(\theta_{ij}\right)\phi$$

with

$$h(\mu_{ij}^c) = x_{ij}^T\beta + \sum_{r=1}^{s} f_r(H_{ij}; \alpha) \text{ for suitable functions } f_r(\cdot)'s,$$

and

$$v_{ij}^c = v(\mu_{ij}^c)\phi.$$

Problem 1. Fitting transition models: (A markov model of order q)

By

$$L_i(y_{i1}, \cdots, y_{im_i}) = f(y_{i1}, \cdots, y_{iq}) \prod_{j=q+1}^{m_i} f(y_{ij} \mid y_{i\,j-1}, \cdots, y_{i\,j-q}), i = 1, \cdots, n,$$

one can get the likelihood function

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(y_{i1}, \cdots, y_{iq}) \prod_{j=q+1}^{m_i} f(y_{ij} \mid H_{ij}, \alpha, \beta),$$

where

$$H_{ij} = \{y_{i\,j-1}, \cdots, y_{i\,j-q}\}.$$

Since the term $f\left(y_{i1}, \cdots, y_{iq}\right)$ is always unavailable, the estimators of (α, β) are obtained via maximizing the conditional likelihood

$$\prod_{i=1}^{n} \prod_{j=q+1}^{m_i} f\left(y_{ij} \mid H_{ij}, \alpha, \beta\right).$$

Let $\delta = (\alpha, \beta)$.
Show that the log-conditional likelihood or conditional score function has the form

$$S^c(\delta) = \sum_{i=1}^{n} \sum_{j=(q+1)}^{m_i} \frac{\partial \mu_{ij}^c}{\partial \delta} {v_{ij}^c}^{-1}(y_{ij} - \mu_{ij}^c).$$

Problem 2. Ordered Categorical data

Y: ordinal response with categories labeled $1, 2, \cdots, k$.
Let

$$F(a \mid x) = P(Y \leq a \mid x),$$

where $a = 1, \cdots, (k-1), x = \left(x_1, \cdots, x_p\right)^T$.
Proportional odds model:

$$\operatorname{logit} F(a \mid x) = \theta_a + x^T \beta, \quad a = 1, \cdots, (k-1).$$

Define $Y^* = \left(Y_1^*, \cdots, Y_{k-1}^*\right)$ with $Y_a^* = 1_{(Y \leq a)}$.
Then,

$$\operatorname{logit} F(a \mid x) = \operatorname{logit} P\left(Y_a^* = 1 \mid x\right).$$

Y	1	2	3	$\cdots$	$k-1$	k
Y_1^*	1	0	0	$\cdots$	0	0
Y_2^*	1	1	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$				$\vdots$
Y_{k-1}^*	1	1	1	$\cdots$	1	0

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Example:
Assume that

$$\operatorname{logit} P\left(Y_j \leq b \mid Y_{i\,j-1} = a\right) = \theta_\alpha + {x_i}^T \beta_\alpha, \quad a, b = 1, \cdots, (k-1).$$

It can be derived that

$$\operatorname{logit} P(Y_{ij} \leq b \mid Y_{i\,j-1}^* = y_{i\,j-1}^*) = \theta_b + \sum_{l=1}^{k-1} \alpha_{ab} y_{i(j-1)l}^* + x_{ij}^T (\beta + \sum_{l=1}^{k-1} r_l y_{i(j-1)l}^*),$$

where $\begin{cases} \theta_{kb} = \theta_b, \\ \alpha_{lb} = \theta_{lb} - \theta_{k+1\,b}, \\ \beta_k = \beta, \\ r_l = \beta_l - \beta_{l+1} \end{cases}.$

Problem 3. Log-linear transition models for count data

$Y_{ij} \mid \left(H_{ij}, x_{ij}\right) \sim$ Poissom $\left(\mu_{ij}^c\right)$.

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Model 1. Wong (1986) proposed that

$$\mu_{ij}^c = \exp\left(x_{ij}^T \beta\right)\left\{1 + \exp\left(-\alpha_0 - \alpha_1 y_{i\,j-1}\right)\right\},$$

$\alpha_0, \alpha_1 > 0$, where β is the influence of x_{ij} as $y_{i\,j-1} = 0$.

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Remark. When $y_{i\,j-1} > 0, \mu_{ij}^c$ decreases as $y_{i\,j-1}$ increases. A negative association is implied between the prior and current responses.

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Model 2. $\mu_{ij}^c = \exp\left(x_{ij}^T \beta + \alpha y_{i\,j-1}\right)$.

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Properties:

1. μ_{ij}^c increases as an exponential function of time as $\alpha > 0$.
2. When $\exp\left(x_{ij}^T \beta\right) = \mu$ and $\alpha < 0$, it leads to a stationary process.

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Moded 3.

$$\mu_{ij} = \exp\left(x_{ij}{}^T \beta + \alpha\left\{\ln\left(y_{i\,j-1}^*\right) - x_{i\,j-1}^T \beta\right\}\right),$$

where $y_{i\,j-1}^* = \max\left\{y_{i\,j-1}, d\right\}, 0 < d < 1$.

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$$\text{Property: } \begin{cases} \alpha = 0: \text{ it reduces to an oedinary log-tinear model.} \\ \alpha < 0: \text{ negative correlation between } y_{i\,j-1} \text{ and } y_{ij} \\ \alpha > 0: \text{ positive correlation between } y_{i\,j-1} \text{ and } y_{ij} \end{cases}$$