Transition Models

$t'_{ij}s$ are assumed to be equally spaced.

Let $H_i = \{y_k, k = 1, \cdots, j-1\}$.

Consider

$$f(y_{ij} \mid H_{ij}, \alpha, \beta) = \exp\left\{\frac{y_{ij} - \psi(\theta_{ij})}{\phi} + c(y_{ij}, \phi)\right\},$$

where $\psi(\theta_{ij})$ and $c(y_{ij}, \phi)$ are known functions.

One has

$$\mu_{ij}^c = E\left[y_{ij} \mid H_{ij}\right] = \psi'\left(\theta_{ij}\right)$$

and

$$V_{ij}^c = V\left[y_{ij} \mid H_{ij}\right] = \psi''\left(\theta_{ij}\right)\phi$$

with

$$h(\mu_{ij}^c) = x_{ij}^T \beta + \sum_{r=1}^{s} f_r(H_{ij}; \alpha) \text{ for suitable functions } f_r(\cdot)'s,$$

and

$$v_{ij}^c = v(\mu_{ij}^c)\phi.$$

Problem 1. Fitting transition models: (A markov model of order q)

By

$$L_i(y_{i1}, \cdots, y_{im_i}) = f(y_{i1}, \cdots, y_{iq}) \prod_{j=q+1}^{m_i} f(y_{ij} \mid y_{i\,j-1}, \cdots, y_{i\,j-q}), i = 1, \cdots, n,$$

one can get the likelihood function

$$L(\alpha, \beta) = \prod_{i=1}^{n} f(y_{i1}, \cdots, y_{iq}) \prod_{j=q+1}^{m_i} f(y_{ij} \mid H_{ij}, \alpha, \beta),$$

where

$$H_{ij} = \{y_{i\,j-1}, \cdots, y_{i\,j-q}\}.$$

Since the term $f\left(y_{i1}, \cdots, y_{iq}\right)$ is always unavailable, the estimators of (α, β) are obtained via maximizing the conditional likelihood

$$\prod_{i=1}^{n} \prod_{j=q+1}^{m_i} f\left(y_{ij} \mid H_{ij}, \alpha, \beta\right).$$

Let $\delta = (\alpha, \beta)$.

Show that the log-conditional likelihood or conditional score function has the form

$$S^c(\delta) = \sum_{i=1}^{n} \sum_{j=(q+1)}^{m_i} \frac{\partial \mu_{ij}^c}{\partial \delta} v_{ij}^{c\,-1}(y_{ij} - \mu_{ij}^c).$$

Problem 2.   Ordered Categorical data

Y: ordinal response with categories labeled $1, 2, \cdots, k$.

Let

$$F(a \mid x) = P(Y \leq a \mid x),$$

where $a = 1, \cdots, (k-1), x = (x_1, \cdots, x_p)^T$.

Proportional odds model:

$$\text{logit} \, F(a \mid x) = \theta_a + x^T \beta, \quad a = 1, \cdots, (k-1).$$

Define $Y^* = (Y_1^*, \cdots, Y_{k-1}^*)$ with $Y_a^* = 1_{(Y \leq a)}$.

Then,

$$\text{logit} \, F(a \mid x) = \text{logit} \, P(Y_a^* = 1 \mid x).$$

Y	1	2	3	$\cdots$	$k-1$	k
Y_1^*	1	0	0	$\cdots$	0	0
Y_2^*	1	1	0	$\cdots$	0	0
$\vdots$	$\vdots$	$\vdots$				$\vdots$
Y_{k-1}^*	1	1	1	$\cdots$	1	0

. . . .

Example:

Assume that

$$\text{logit} \, P\left(Y_j \leq b \mid Y_{i\,j-1} = a\right) = \theta_{ab} + x_i{}^T \beta_a, \quad a, b = 1, \cdots, (k-1).$$

It can be derived that

$$\text{logit} \, P(Y_{ij} \leq b \mid Y_{i\,j-1}^* = y_{i\,j-1}^*) = \theta_b + \sum_{l=1}^{k-1} \alpha_{lb} y_{i(j-1)l}^* + x_{ij}^T \left(\beta + \sum_{l=1}^{k-1} r_l y_{i(j-1)l}^*\right),$$

where $\begin{cases} \theta_{kb} = \theta_b, \\ \alpha_{lb} = \theta_{lb} - \theta_{l+1\,b}, \\ \beta_k = \beta, \\ r_l = \beta_l - \beta_{l+1} \end{cases}$.

$$
\begin{aligned}
\text{logit} \, P(Y_{ij} \leq b \mid Y_{i\,j-1}^* = y_{i\,j-1}^*) = \theta_b &+ (\theta_{1b} - \theta_{2\,b}) y_{i(j-1)1}^* \\
&+ (\theta_{2b} - \theta_{3\,b}) y_{i(j-1)2}^* \\
&+ \cdots \\
&+ (\theta_{ab} - \theta_{a+1\,b}) y_{i(j-1)a}^* \\
&+ (\theta_{a+1\,b} - \theta_{a+2\,b}) y_{i(j-1)(a+1)}^* \\
&+ \cdots \\
&+ (\theta_{(k-1)b} - \theta_{k\,b}) y_{i(j-1)(k-1)}^* \\
&+ x_{ij}^T \{ \beta \\
&+ (\beta_1 - \beta_2) y_{i(j-1)1}^*
\end{aligned}
$$

$$+(\beta_2 - \beta_3)y^*_{i(j-1)2}$$
$$+\cdots$$
$$+(\beta_a - \beta_{a+1})y^*_{i(j-1)a}$$
$$+\cdots$$
$$+(\beta_{k-1} - \beta_k)y^*_{i(j-1)(k-1)}\},$$

$$\begin{aligned}
\operatorname{logit} P(Y_{ij} \leq b \mid Y_{i\,j-1} = a) = \theta_b &+ (\theta_{1b} - \theta_{2\,b}) \\
&+ (\theta_{2b} - \theta_{3\,b}) \\
&+ \cdots \\
&+ (\theta_{ab} - \theta_{a+1\,b}) \\
+ x_{ij}^T \{\beta & \\
&+ (\beta_1 - \beta_2) \\
&+ (\beta_2 - \beta_3) \\
&+ \cdots \\
&+ (\beta_a - \beta_{a+1})\}, \\
= \theta_b &+ \theta_{1b} - \theta_{a+1\,b} + x_{ij}^T\{\beta + \beta_1 - \beta_{a+1}\}.
\end{aligned}$$

Problem 3. Log-linear transition models for count data

$Y_{ij} \mid (H_{ij}, x_{ij}) \sim \text{Poissom } (\mu_{ij}^c)$.

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Model 1. Wong (1986) proposed that

$$\mu_{ij}^c = \exp\left(x_{ij}^T \beta\right) \{1 + \exp\left(-\alpha_0 - \alpha_1 y_{i\,j-1}\right)\},$$

$\alpha_0, \alpha_1 > 0$, where β is the influence of x_{ij} as $y_{i\,j-1} = 0$.

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Remark. When $y_{i\,j-1} > 0, \mu_{ij}^c$ decreases as $y_{i\,j-1}$ increases. A negative association is implied between the prior and current responses.

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Model 2. $\mu_{ij}^c = \exp\left(x_{ij}^T \beta + \alpha y_{i\,j-1}\right)$.

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Properties:

1. μ_{ij}^c increases as an exponential function of time as $\alpha > 0$.

2. When $\exp\left(x_{ij}^T \beta\right) = \mu$ and $\alpha < 0$, it leads to a stationary process.

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Problem 4.

Moded 3.

$$\mu_{ij} = \exp\left(x_{ij}{}^T\beta + \alpha\left\{\ln\left(y_{i\,j-1}^*\right) - x_{i\,j-1}^T\beta\right\}\right),$$

where $y_{i\,j-1}^* = \max\left\{y_{i\,j-1}, d\right\}, 0 < d < 1$.

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Property: $\begin{cases} \alpha = 0 : \text{ it reduces to an oedinary log-tinear model.} \\ \alpha < 0 : \text{ negative correlation between } y_{i\,j-1} \text{ and } y_{ij} \\ \alpha > 0 : \text{ positive correlation between } y_{i\,j-1} \text{ and } y_{ij} \end{cases}$

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Application to a size-independent branching process:

$\exp(x_{ij}^T\beta) = \mu$

$$y_{ij} : \text{the number of individuals in the } i\text{-th population at generation } j$$
$$Z_k(y_{i\,j-1}) : \text{the number of offspring for person } k \text{ in generation } (j-1)$$

For $y_{i\,j-1} > 0$,

$$y_{ij} = \sum_{k=1}^{y_{i(j-1)}} Z_k(y_{i(j-1)}),$$

where

$$Z_k \overset{iid}{\sim} Poisson\left(\left(\frac{\mu}{y_{ij-1}^*}\right)^{1-\alpha}\right).$$

One can get

$$\mu_{ij}^c = \mu \cdot \left(\frac{y_{i(j-1)}}{\mu}\right)^\alpha.$$

Property:

- $\alpha < 0$: the sample paths oscillate back and forth about their long-term average level.

- $\alpha > 0$: the sample path have sharper peaks and broader valleys.