Problem 1.

If E is a measurable subset of $[0, 2\pi]$, then

$$\int_E \cos^2(nx + u_n)\, dx \to \frac{|E|}{2}, \qquad \text{as } n \to \infty$$

for any sequence $\{u_n\}$.

Since $E \subseteq \cup I_k$, where $I_k = [a_k, b_k]$ is an interval. It suffices to show that, for any interval I,

$$\int_I \cos^2(nx + u_n)\, dx \to \frac{|I|}{2}, \qquad \text{as } n \to \infty.$$

Let

$$\begin{aligned} f_n &= \cos^2(nx + u_n) \\ &= \frac{1}{2}\left[1 + \cos(2nx + 2u_n)\right] \\ &= \frac{1}{2} + \frac{1}{2}\left[\cos(2nx + v_n)\right] \\ &= \frac{1}{2} + \frac{1}{2}\left[\cos(2nx)\cos(v_n) - \sin(2nx)\sin(v_n)\right] \\ &= \frac{1}{2} + \frac{1}{2}\cos(2nx)\cos(v_n) - \frac{1}{2}\sin(2nx)\sin(v_n). \end{aligned}$$

By Riemann-Lebesgue Lemma, we have

$$\lim_{n\to\infty} \int_I \sin(2nx)\sin(v_n)\, dx = \lim_{n\to\infty} \int_I \cos(2nx)\cos(v_n)\, dx = 0.$$

$$\begin{aligned} \lim_{n\to\infty} \int_I f_n &= \lim_{n\to\infty} \int_I \frac{1}{2} + \frac{1}{2}\lim_{n\to\infty} \int_I \cos(2nx)\cos(v_n) - \frac{1}{2}\lim_{n\to\infty} \int_I \sin(2nx)\sin(v_n) \\ &= \lim_{n\to\infty} \int_I \frac{1}{2} + 0 + 0 \\ &= \frac{1}{2}|I|. \end{aligned}$$

Thus,

$$\cos^2(\alpha) = \frac{1}{2}[1 + \cos(2\alpha)]$$
$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

Riemann-Lebesgue Lemma

Let f be integrable over (a, b). It can be shown that

$$\lim_{n\to\infty} \int_a^b f(x)\sin nx\, dx = \lim_{n\to\infty} \int_a^b f(x)\cos nx\, dx = 0.$$

Problem 2. Cantor-Lebesgue's Theorem

Prove the Cantor-Lebesgue Theorom: If

$$\sum_{n=0}^{\infty} A_n(x) = \sum_{n=0}^{\infty}(a_n \cos nx + b_n \sin nx)$$

converges for x in a set of positive measure (or in particular for all x), then $a_n, b_n \to 0$.

Let

$$A_n(x) = a_n \cos(nx) + b_n \sin(nx).$$

Firstly, we see that there exist r_n and θ_n such that

$$\begin{aligned} a_n &= r_n \cos(\theta_n) \\ b_n &= r_n \sin(\theta_n) \end{aligned}$$

If $a_n = 0$ and $b_n = 0$, we can choose $r_n = 0$ and $\theta_n = 0$.

If $a_n \neq 0$ or $b_n \neq 0$, we can write:

$$a_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cdot a_n, \quad b_n = \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \cdot b_n$$

so

$$\begin{aligned} r_n &= \sqrt{a_n^2 + b_n^2}, \\ \cos(\theta_n) &= \frac{a_n}{\sqrt{a_n^2 + b_n^2}}, \\ \sin(\theta_n) &= \frac{b_n}{\sqrt{a_n^2 + b_n^2}}. \end{aligned}$$

Now, if we substitute in the definition of f_n and we apply the trigonometric formula for the subtraction of angles for the $\cos(x)$ function, we obtain:

$$\begin{aligned} A_n(x) &= r_n \cos(\theta_n)\cos(nx) + r_n \sin(\theta_n)\sin(nx) \\ &= r_n[\cos(\theta_n)\cos(nx) + \sin(\theta_n)\sin(nx)] \\ &= r_n \cos(nx - \theta_n). \end{aligned}$$

If $\sum A_n(x)$ converges for all x in E, with $|E| > 0$, then $A_n(x) \to 0$ on E.

By Egorov's Theorom, this implies that $A_n(x) \to 0$ uniformly on some $E' \subset E$, with $|E'| > 0$. Thus,

$$\begin{aligned} r_n \cos(nx - \theta_n) &\xrightarrow{u} 0 && \text{on } E' \\ r_n^2 \cos^2(nx - \theta_n) &\xrightarrow{u} 0 && \text{on } E' \\ \Rightarrow r_n^2 \int_{E'} \cos^2(nx - \theta_n) = \int_{E'} r_n^2 \cos^2(nx - \theta_n) &\xrightarrow{u} 0. \end{aligned}$$

By P.01, $\int_{E'} \cos^2(nx - \theta_n) \to \frac{1}{2}|E'|$. It is necessary that $r_n^2 \to 0$, implying that $a_n, b_n \to 0$, as required.

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Let

$$f(x) = \sum_{n=0}^{\infty} A_n(x) = \sum_{n=0}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

For $m \in \mathbb{N}$, it can be shown that

$$\begin{aligned}\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx &= \sum_{n=0}^{\infty} \int_{-\pi}^{\pi} [a_n \cos(nx) \cos(mx) + b_n \sin(nx) \cos(mx)]\ dx \\ &= \int_{-\pi}^{\pi} [a_n \cos(mx) \cos(mx)]\ dx \\ &= a_m \pi\end{aligned}$$

Since

$$\int_{-\pi}^{\pi} f(x) \cos(mx)\, dx \to 0,$$

we have $a_m \pi \to 0, \forall m \in \mathbb{N}$. Therefore, $a_n \to 0$, as required.