Let f be any measurable function defined on a set E. If f exists and is finite, we say that f is Lebesgue integrable, or simply integrable, on E and write $f \in L(E)$. Thus,

$$L(E) = \left\{ f : \int_E f \text{ is finite} \right\}.$$

Theorem 5.5

(i) If f and g are measurable and $0 \le g \le f$ on E, then $\int_E g \le \int_E f$. In particular, $\int_E (\inf f) \le \int_E f$.

(ii) If f is nonnegative and measurable on E and $\int_E f$ is finite, then $f < +\infty$ a.e. in E.

(iii) Let E_1 and E_2 be measurable and $E_1 \subset E_2$. If f is nonnegative and measurable on E_2, then $\int_{E_1} f \le \int_{E_2} f$.

Proof:

Parts (i) and (iii) follow from the relations $R(g, E) \subset R(f, E)$ and $R(f, E_1) \subset R(f, E_2)$, respectively.

To prove (ii), we may assume that $|E| > 0$. If $f = +\infty$ in a subset E_1 of E with positive measure, then by (iii) and (i), we have $\int_E f \ge \int_{E_1} f \ge \int_{E_1} a = a|E_1|$, no matter how large a is. This contradicts the finiteness of $\int_E f$.

Theorem 5.22

If $f \in L(E)$, then f is finite a.e. in E.

Proof: If $f \in L(E)$, then $|f| \in L(E)$, and the result follows from Theorem 5.5(ii).

Theorem 5.36 (Lebesgue's Dominated Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If there exists $\phi \in L(E)$ such that $|f_k| \le \phi$ a.e. in E for all k, then $\int_E f_k \to \int_E f$.

Problem 1. Zygmund p109 exercise 04

If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and that

$$\int_0^1 x^k f(x)\, dx \to 0.$$

Let $g_k(x) = x^k f(x)$ and $E = (0,1)$. We have $g_k(x)$ measurable on E, thus $\int_E g_k$ exists.

For $x \in (0,1)$, $x_k \leq 1$, so $g(x) = x^k f(x) \leq f(x), \forall k \in \mathbb{N}$. Hence,

$$\int_E g_k \leq \int_E f < \infty,$$

implying that $g_k(x) = x^k f(x) \in L(0,1)$.

....

Since $f \in L(E)$, f is finite a.e. in E.

Besides, for all $x \in E$, $x^k \to 0$, as $k \to \infty$.

Thus, $g_k(x) = x^k f(x) \to 0$ a.e in E. Additionally, $|g_k| \leq |f|$, while $f \in L(E)$.

Therefore, by Theorem 5.36 (Lebesgue's Dominated Convergence Theorem), we have

$$\int_E g_k(x)\, dx \to \int_E 0\, dx = 0.$$

Problem 2. Zygmund p109 exercise 05

Use Egorov's theorem to prove the bounded convergence theorem.

Given $f \in L(E)$, $\forall \epsilon > 0, \exists \delta > 0$, s.t. $\forall F \subseteq E$ with $|F| < \delta$, $\int_F |f| < \epsilon$.

By Egorov's Theorem,

given $\epsilon > 0$, find a closed subset F of E such that

- $|E \setminus F| < \delta_\epsilon$, thus $\int_{E \setminus F} f < \epsilon$;
- and $f_k \xrightarrow{u} f$ on F, thus $\int_F f_k \to \int_F f$ (By Uniform Convergence Theorem).

Additionally, since there is a finite constant M such that $|f_k| \leq M$ a.e. in E, then $|f_k| \leq M$ a.e. in $E \setminus F$, implying that

$$\int_{E \setminus F} f_k \leq \int_{E \setminus F} M = M|E \setminus F| \leq M\delta_\epsilon.$$

Hence,

$$\begin{aligned}
\int_E f - \int_E f_k &= \left(\int_F f + \int_{E \setminus F} f\right) - \left(\int_F f_k + \int_{E \setminus F} f_k\right) \\
&= \left(\int_F f - \int_F f_k\right) + \left(\int_{E \setminus F} f - \int_{E \setminus F} f_k\right) \\
&\leq \left(\int_F f - \int_F f_k\right) + \left(\int_{E \setminus F} f + \int_{E \setminus F} f_k\right)
\end{aligned}$$

It follows that:

$$\begin{aligned}
\lim_{k \to \infty} \left| \int_E f - \int_E f_k \right| &\leq \lim_{k \to \infty} \left| \int_F f - \int_F f_k \right| + (\epsilon + M\delta_\epsilon) \\
&= \epsilon + M\delta_\epsilon.
\end{aligned}$$

Choose $\delta_\epsilon \leq \frac{\epsilon}{M}$, then $\lim_{k \to \infty} \left| \int_E f - \int_E f_k \right| \leq 2\epsilon$.

Letting $\epsilon \to 0$, we can conclude that $\int_E f \to \int_E f_k$.

Corollary 5.37 (Bounded Convergence Theorem)

Let f_k be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If $|E| < +\infty$ and there is a finite constant M such that $|f_k| \le M$ a.e. in E, then

$$\int_E f_k \to \int_E f.$$

Theorem 4.17 (Egorov's Theorem)

Suppose that $\{f_k\}$ is a sequence of measurable functions that converges almost everywhere in a set E of finite measure to a finite limit f. Then, given $\varepsilon > 0$, there is a closed subset F of E such that $|E - F| < \varepsilon$ and $\{f_k\}$ converge uniformly to f on F.

Theorem 5.23

(i) If both $\int_E f$ and $\int_E g$ exist, and if $f \le g$ a.e. in E, then $\int_E f \le \int_E g$. Moreover, if f and g are functions with $f = g$ a.e. in E and $\int_E f$ exists, then $\int_E g$ exists, and $\int_E f = \int_E g$.

(ii) If $\int_{E_2} f$ exists and E_1 is a measurable subset of E_2, then $\int_{E_1} f$ exists.

Theorem 5.24

If $\int_E f$ exists and $E = \bigcup_k E_k$ is the countable union of disjoint measurable sets E_k, then

$$\int_E f = \sum_k \int_{E_k} f.$$

Theorem 5.33 (Uniform Convergence Theorem)

Let $f_k \in L(E)$ for $k = 1, 2, \dots$, and let $\{f_k\}$ converge uniformly to f on E where $|E| < +\infty$. Then $f \in L(E)$ and

$$\int_E f_k \to \int_E f.$$

Problem 3. Zygmund p109 exercise 06

Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x, y). Show that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

Problem 4. Zygmund p109 exercise 09-10

- If $p > 0$ and $|f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).
- If $p > 0$, $|f - f_k|^p \to 0$, and $|f_k|^p \le M$ for all k, show that $|f|^p \le M$.