Problem 1. Zygmund p58 exercise 01 不計分

(a) There is an analogue for bases different from 10 of usual decimal expansion of number. If b is an integer larger than 1 and $0 < x < 1$, show that there exist integral coefficient c_k, $0 \leq c_k < b$, such that $x = \sum_{k=1}^{\infty} c_k b^{-k}$. Furthermore, show that expansion is unique unless $x = cb^{-k}$, in which case there are two expansions.
(b) When $b = 3$, the expansion is called the triadic or ternary expansion of x. Show that Canter set consist of point in $[0, 1]$ which has triadic representation such that c_k is either 0 or 2, namely,

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{k=1}^{\infty} c_k 3^{-k}, c_k \in \{0, 2\}\}.$$

Solution.

(a)

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Input: x, b
Output: {c_k}_{k=1}^∞
initialization;
for n = 1, 2, ..., ∞ do
    Let c_n to be the largest integral coefficient, s.t. ∑_{k=1}^n c_k b^{-k} ≤ x;
return {c_k}_{k=1}^∞;
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Input: x, b
Output: $\{c_k\}_{k=1}^{\infty}$
initialization;
for $n = 1, 2, \ldots, \infty$ **do**
  Let c_n to be the largest integral coefficient, s.t. $\sum_{k=1}^{n} c_k b^{-k} \leq x$;
return $\{c_k\}_{k=1}^{\infty}$;

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⋮

Cantor Set

[a]Consider the closed interval $[0,1]$. The first stage of the construction is to subdivide $[0,1]$ into thirds and remove the interior of the middle third; that is, remove the open interval $\left(\frac{1}{3}, \frac{2}{3}\right)$. Each successive step of the construction is essentially the same. Thus, at the second stage, we subdivide each of the remaining two intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$ into thirds and remove the interiors, $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, of their middle thirds. We continue the construction for each of the remaining intervals. The subset of [0, 1] that remains after infinitely many such operations is called the Cantor set C: thus, if C_k denotes the union of the intervals left at the k-th stage, then

$$C = \bigcap_{k=1}^{\infty} C_k.$$

[a]Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis.* CRC, 2015, pp. 42–43.

Limit Point

A point x is a limit point of the set E if every neighborhood of x contains a point $x \neq y$ such that $y \in E$.[a]

In other words, x is a limit point of E if $\exists$ a sequence $\{x_n\} \in E$, s.t. $x_n \to x$ and $x_n \neq x$.[b]

[a]W. Rudin. *Principles of Mathematical Analysis.* McGraw-Hill, 1976, p. 32.
[b]Wheeden and Zygmund, see n. *a*, pp. 3–4.

Perfect Set

A closed set E is said to be a perfect set if every point of E is a limit point of E.[a]

In other words, A closed set E is said to be a perfect set if $\forall x \in E, \forall \epsilon > 0, (B(x, \epsilon) \setminus \{x\}) \cap E \neq \emptyset$.

[a]Ibid., p. 7.

Theorem 1.7

(i) The intersection of any number of closed sets is closed.

(ii) The union of any number of open sets is open.

Cantor Set is perfect

To prove that Cantor Set C is a perfect set, we need to show that it is closed and every point in the set is a limit point of the set.

Since each C_k is closed, it follows from Theorem 1.7 that C is closed.

Then show that every point in C is a limit point of the set:

Case 1. Let $x \in C$ be an endpoint of the interval $I_k \subseteq C_k$. Consider the intervals $I_k^i \subseteq C_{k+i}$ with endpoint x, let x_1 be the other endpoint of $I_k^1 \subseteq C_{k+1}$, x_2 be the other endpoint of $I_k^2 \subseteq C_{k+2}$,..., x_n be the other endpoint of $I_k^n \subseteq C_{k+n}$. Thus, $|x_n - x| = (\frac{1}{3})^{k+n}$.

We have

$$x_n \to x,$$
$$x_n \neq x,$$
$$x_n \in C.$$

Therefore, x is a limit point of C.

Case 2. Suppose $x \in C$ is not an endpoint of any interval consisting C. $\forall n \in \mathbb{N}$, we have $x \in (a_n, b_n)$, where $a_n \in C$ and $|a_n - x| < (\frac{1}{3})^n$. Let $x_n = a_n$, $\{x_n\}$ is the squence s.t.

$$x_n \to x,$$
$$x_n \neq x,$$
$$x_n \in C.$$

Thus, x is a limit point of C.

We can conclude that every point in C is a limit point of the set. Therefore, C is perfect.

Problem 2. Zygmund p58 exercise 03

Construct a two-dimensional Cantor set in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which forms a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Solution.

Let D_0 be the unit square $\{(x, y) : 0 \leq (x, y) \leq 1\}$. Let D_k be the set remaining after k steps. Let $D = \bigcap_{k=1}^{\infty} D_k$ be the resulting set.

(a) Show that D has plane measure zero:

Since D is covered by the intervals in any D_k, we have

$$|D|_e \leq |D_k|_e = \left(\frac{4}{9}\right)^k.$$

Let $k \to \infty$, we have $|D|_e = 0$.

(b)

$$\begin{aligned}
D &:= \bigcap_{k=1}^{\infty} D_k && \text{(By definition} \\
&= \bigcap_{k=1}^{\infty} C_k \times C_k && \text{(By definition} \\
&= \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right) && \text{(To be proved} \\
&= C \times C. && \text{(By definition}
\end{aligned}$$

To prove that $\bigcap_{k=1}^{\infty} C_k \times C_k = (\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k)$, first show that $\bigcap_{k=1}^{\infty} C_k \times C_k \subseteq (\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k)$:

For all

$$(x,y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$$

we have

$$(x,y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, we have

$$x \in C_k, \forall k \in \mathbb{N},$$
$$y \in C_k, \forall k \in \mathbb{N}.$$

Thus,

$$x \in \bigcap_{k=1}^{\infty} C_k,$$
$$y \in \bigcap_{k=1}^{\infty} C_k.$$

Therefore,

$$(x,y) \in \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right), \forall (x,y) \in D,$$

which means

$$\bigcap_{k=1}^{\infty} C_k \times C_k \subseteq \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right).$$

Next, prove that $(\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k) \subseteq \bigcap_{k=1}^{\infty} C_k \times C_k$. For all

$$(x,y) \in \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right)$$

we have

$$x \in C_k, \forall k \in \mathbb{N},$$
$$y \in C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, this implies that

$$(x,y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

Since it satisfies the definition of intersection,we have $(x,y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$, implying that

$$\left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right) \subseteq \bigcap_{k=1}^{\infty} C_k \times C_k.$$

Therefore, we can conclude that the two sets are equal:

$$\bigcap_{k=1}^{\infty} C_k \times C_k = \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right).$$

(c) Prove that it is a perfect set:

Since C is perfect, $D = C \times C$ is perfect, by the property of perfect set.

Cartesian product of two perfect sets

To prove that the Cartesian product of two perfect sets in $\mathbb{R}$ is a perfect set in $\mathbb{R}^2$:

Let E_1 and E_2 be perfect sets in $\mathbb{R}$, consider $E = E_1 \times E_2$, which is the product of E_1 and E_2.

Since both E_1 and E_2 are perfect sets, they are closed in $\mathbb{R}$. The product of two closed sets is also closed. Thus, $E = E_1 \times E_2$ is closed in $\mathbb{R}^2$.

Suppose $(x, y) \in E = E_1 \times E_2$, by definition, $x \in E_1$, and $y \in E_2$. Since both E_1 and E_2 are perfect sets, every point in the set is the limit point of the set respectively. In other words, $\forall x \in E_1$, we have a sequence $\{x_n\}$ s.t.

$$x_n \to x, x_n \neq x, x_n \in E_1.$$

Similarly, $\forall y \in E_2$, we have a sequence $\{y_n\}$ s.t.

$$y_n \to y, y_n \neq y, y_n \in E_2.$$

Thus, $\forall (x, y) \in E = E_1 \times E_2$, we have a sequence $\{(x_n, y_n)\}$ s.t.

$$\begin{aligned} &(x_n, y_n) \to (x, y), \\ &(x_n, y_n) \neq (x, y), \\ &(x_n, y_n) \in E. \end{aligned}$$

This means that every point in E is a limit point of E. Hence, we can conclude that E is perfect.

It is proved that the product of two perfect sets in $\mathbb{R}$ is a perfect set.

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Problem 3. Zygmund p59 exercise 04

Construct a subset of $[0, 1]$ in the same manner as the Cantor set by removing from each remaining interval a subinterval of relative length $\theta, 0 < \theta < 1$. Show that the resulting set is perfect and has measure zero.

Solution.

If C'_k denotes the union of the intervals left at the k-th stage, then the resulting set is

$$C' = \bigcap_{k=1}^{\infty} C'_k.$$

(a) To prove that the set C' is a perfect set, we need to show that it is closed and every point in the set is a limit point of the set.

Since each C'_k is closed, it follows from Theorem 1.7 that C' is closed.

Then show that every point in C' is a limit point of the set:

Case 1. Let $x \in C'$ be an endpoint of $I_k \subseteq C'_k$. Consider the intervals $I_k^i \subseteq C'_{k+i}$ with endpoint x, let x_1 be the other endpoint of $I_k^1 \subseteq C'_{k+1}$, x_2 be the other endpoint of $I_k^2 \subseteq C'_{k+2}$,..., x_n be the other endpoint of $I_k^n \subseteq C'_{k+n}$. Thus, $|x_n - x| = (\frac{1}{3})^{k+n}$.

We have the squence $\{x_n\}$ s.t.

$$\begin{aligned} &x_n \to x, \\ &x_n \neq x, \\ &x_n \in C'. \end{aligned}$$

Therefore, x is a limit point of C'.

Case 2. Suppose $x \in C'$ is not an endpoint of any interval consisting C'. $\forall n \in \mathbb{N}$, we have $x \in (a_n, b_n)$, where $a_n \in C'$ and $|a_n - x| < (\frac{1-\theta}{2})^n$. Let $x_n = a_n$, then $\{x_n\}$ is the squence s.t.

$$\begin{aligned} &x_n \to x, \\ &x_n \neq x, \\ &x_n \in C'. \end{aligned}$$

Thus, x is a limit point of C'.

We can conclude that every point in C' is a limit point of the set. Therefore, C' is perfect.

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(b) Show that C' has plane measure zero:

Since C' is covered by the intervals in any C'_k, we have

$$|C'|_e \leq |C'_k|_e = (1-\theta)^k.$$

Let $k \to \infty$, we have $|C'|_e = 0$.

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