Problem 1. Zygmund p76 exercise 05

Give an example to show that $\varphi(f(x))$ may not be measurable if φ and f are measurable and finite. (Let F be the Cantor-Lebesgue function and let f be its inverse, suitably defined. Let φ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$, where F is the Cantor-Lebesgue function, and consider $f = g^{-1}$.) Cf. Exercise 22.

Problem 2.

Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

Solution.

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$
$$\Updownarrow$$
$$|\{x|f(x) \neq \chi_{[0,1]}(x)\}| = 0.$$

. . . .

Suppose, for the sake of contradiction, that $\exists f$ on $\mathbb{R}$ s.t.

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

Without loss of generality, $f(x) = 1, x \in [0, 1]$.

By the definition of continuous everywhere, $\forall \epsilon > 0, \exists \delta > 0$, s.t. $|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$, which means $|f(x) - 1| < \epsilon$.

$$\Rightarrow f(x) \neq 0 \text{ on } [-\delta, 0]$$
$$\Rightarrow f(x) \neq \chi_{[0,1]} \text{ on } [-\delta, 0], \delta > 0.$$

Thus, $|\{x|f(x) \neq \chi_{[0,1]}(x)\}| \neq 0$, which contradicts the assumption that $f(x) = \chi_{[0,1]}(x)$ a.e.

Therefore, we conclude that there is no everywhere continuous function f on $\mathbb{R}$ such that $f(x) = \chi_{[0,1]}(x)$ a.e.

$\blacksquare$

Problem 3.

Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}$, $\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $|\Gamma| = 0$.

Solution.

It suffices to prove that $|\Gamma|_e = 0$. Since R^d is a countable union of almost disjoint cubes of side length 1, it is enough to show that $|\Gamma'|_e = 0$, where

$$\Gamma' = \{(x, y) \in [0, 1]^d \times \mathbb{R} : y = f|_{[0,1]^d}(x)\}.$$

Since we know $R = \bigsqcup_{k\in\mathbb{Z}}[k, k + 1)$, it follows that

$$\Gamma' = \bigsqcup_{k\in\mathbb{Z}} \{(x, y) \in [0, 1]^d \times [k, k + 1) : y = f|_{[0,1]^d}(x)\}.$$

Again, it is sufficient to prove that $|\Gamma''|_e = 0$, where

$$\Gamma'' = \{(x, y) \in [0, 1]^d \times [0, 1) : y = f|_{[0,1]^d}(x)\}.$$

For every $n \in \mathbb{N}$, we have $[0, 1) = \bigsqcup_{j=1}^n I_j$, where $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right)$ for all $j \in \{1, 2, \ldots, n\}$.
Since we know

$$\Gamma'' = \bigsqcup_{j=1}^{n} \left\{(x, y) \in [0, 1]^d \times I_j : y = f|_{[0,1]^d}(x)\right\},$$

and $f|_{[0,1]^d}$ is measurable on $[0, 1]^d$, it follows that

$$\begin{aligned}
|\Gamma''|_e &\le \sum_{j=1}^{n} \left| \left\{(x, y) \in [0, 1]^d \times I_j : y = f|_{[0,1]^d}(x)\right\} \right|_e \\
&= \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \times I_j \right|_e \\
&= \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \right| \cdot |I_j| \\
&= \frac{1}{n} \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \right| \\
&= \frac{1}{n} \left| \bigsqcup_{j=1}^{n} f|^{\text{pre}}_{[0,1]^d}(I_j) \right| \\
&= \frac{1}{n} \left| f|^{\text{pre}}_{[0,1]^d}\left([0, 1)\right) \right| \le \frac{1}{n} \left| [0, 1]^d \right| = \frac{1}{n}
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have

$$|\Gamma''|_e \le \lim_{n\to\infty} \frac{1}{n} = 0.$$

Therefore, we obtain $|\Gamma''|_e = 0$.

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