MATH 6337: Homework 3 Solutions

3.10. If E_1 and E_2 are measurable, show that $|E_1 \cup E_2| + |E_1 \cap E_2| = |E_1| + |E_2|$.

Solution. We may assume that both $|E_1|, |E_2| < +\infty$, or else the result is trivially true. Otherwise, since

$$(E_1 \cup E_2) \setminus E_2 = E_1 \setminus (E_1 \cap E_2),$$

$E_2 \subseteq (E_1 \cup E_2)$, and $(E_1 \cap E_2) \subseteq E_1$, we have

$$|E_1 \cup E_2| - |E_2| = |E_1| - |E_1 \cap E_2|,$$

hence the result.[1] □

3.12. If E_1 and E_2 are measurable subsets of $\mathbb{R}^1$, show that $E_1 \times E_2$ is a measurable subset of $\mathbb{R}^2$ and $|E_1 \times E_2| = |E_1|\,|E_2|$. (Interpret $0 \cdot \infty$ as 0.) [Hint: Use a characterization of measurability.]

Solution. If E_1, E_2 are closed intervals in $\mathbb{R}$, then $E_1 \times E_2$ is a closed interval in $\mathbb{R}^2$ and is therefore measurable with measure $|E_1|\,|E_2|$. If E_1, E_2 are open sets in $\mathbb{R}$, then they are countable unions of nonoverlapping intervals, so their product is a countable union of nonoverlapping intervals in $\mathbb{R}^2$ and is therefore measurable. Moreover, since the intervals are nonoverlapping, we have

$$|E_1 \times E_2| = \left| \bigcup_i I_i \times \bigcup_j J_j \right| = \left| \bigcup_{i,j} I_i \times J_j \right| = \sum_{i,j} |I_i|\,|J_j| = |E_1|\,|E_2|.$$

If E_1, E_2 are G_δ sets in $\mathbb{R}$, then they can each be written as the intersection of a decreasing sequence of open sets; so $E_1 \times E_2$ can be written as the intersection of a decreasing sequence of open sets in $\mathbb{R}^2$, so the product set is also G_δ and therefore measurable. Since measure is continuous from above, we have

$$|E_1 \times E_2| = \left| \lim_{k\to\infty} G_k \times H_k \right| = \lim_{k\to\infty} |G_k|\,|H_k| = |E_1|\,|E_2|.$$

We will presently show that the product of a null set and any other set in $\mathbb{R}$ is a null set in $\mathbb{R}^2$. From this it follows that the product of any two measurable sets in $\mathbb{R}$ is measurable in $\mathbb{R}^2$, since a measurable set can be written as $H \setminus Z$ for a G_δ set H and a null set Z, so that

$$E_1 \times E_2 = (H_1 \setminus Z_1) \times (H_2 \setminus Z_2) = (H_1 \times H_2) \setminus \big((Z_1 \times H_2) \cup (H_1 \times Z_2)\big).$$

$H_1 \times H_2$ is a G_δ set, and, by the argument following, $Z_1 \times H_2$ and $H_1 \times Z_2$ are null sets, so $E_1 \times E_2$ is the difference of a G_δ and a null set, so it is measurable. Moreover, since

[1] We need that $|E_1|, |E_2| < +\infty$ in order to subtract measures. It is possible to do this problem without using any subtraction, though.

$|E_1| = |H_1|$ and $|E_2| = |H_2|$, we have

$$|E_1 \times E_2| = \left|(H_1 \times H_2) \setminus \big((Z_1 \times H_2) \cup (H_1 \times Z_2)\big)\right| = |H_1 \times H_2| = |H_1|\,|H_2|\,.$$

Now suppose $|E_1| = 0$ and $|E_2| < +\infty$. Then, for every $\varepsilon > 0$, there exist countable covers $S_1 = \{I_i\}_{i=1}^{\infty}$ of E_1 and $S_2 = \{J_j\}_{j=1}^{\infty}$ of E_2 by closed intervals in $\mathbb{R}$ such that

$$\sum_{i=1}^{\infty} v(I_i) < \varepsilon \quad \text{and} \quad \sum_{j=1}^{\infty} v(J_j) < |E_2| + \varepsilon.$$

Then $S_1 \times S_2 = \{I_i \times J_j : I_i \in S_1, J_j \in S_2\}$ is a countable cover of $E_1 \times E_2$ by closed intervals in $\mathbb{R}^2$ such that

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v(I_i \times J_j) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} v(I_i)v(J_j) = \sum_{i=1}^{\infty} v(I_i) \sum_{j=1}^{\infty} v(J_j) < (|E_2|+\varepsilon) \sum_{i=1}^{\infty} v(I_i) < \varepsilon(|E_2|+\varepsilon).$$

Thus, $|E_1 \times E_2|_e = 0$, so $E_1 \times E_2$ is measurable and $|E_1 \times E_2| = 0$. On the other hand, if $|E_2| = +\infty$, then partition E_2 into the disjoint sets[2]

$$E_2^k = \{x \in E_2 : k - 1 \le x < k\}\,, \quad k \in \mathbb{Z}.$$

Then by the above argument, $E_1 \times E_2^k$ is a set of measure zero in $\mathbb{R}^2$. Since the countably many sets $E_1 \times E_2^k$ are disjoint, we have

$$|E_1 \times E_2| = \left|\bigcup_{k=-\infty}^{\infty} E_1 \times E_2^k\right| = \sum_{k=-\infty}^{\infty} \left|E_1 \times E_2^k\right| = 0.$$

$\square$

3.13. Define the *inner measure* of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that

(i) $|E|_i \le |E|_e$ and

(ii) if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

Solution.

(i) By monotonicity of outer measure, $|F| = |F|_e \le |E|_e$ for all closed subsets $F \subseteq E$. Thus, $\sup |F| = |E|_i \le |E|_e$.

(ii) Suppose E is measurable: then for every $\varepsilon > 0$ there exists a closed subset $F \subseteq E$ such that $|E \setminus F|_e < \varepsilon$. Since E is measurable and $|F| \le |E| < +\infty$, $|E \setminus F|_e = |E| - |F| < \varepsilon$. Thus, $|E|_e = |E| \le \sup |F| + \varepsilon = |E|_i + \varepsilon$ for all $\varepsilon > 0$, so $|E|_e \le |E|_i$, which yields equality by part (i).

[2]This is a common technique used in arguments involving subsets of $\mathbb{R}^n$ with infinite measure. It works because $\mathbb{R}^n$ is what we call a σ-*finite* measure space; in other words, we can partition it into countably many disjoint subsets which have finite measure.

Now suppose that $|E|_i = |E|_e$: then given $\varepsilon > 0$ there exists an open set G and a closed set F such that $F \subseteq E \subseteq G$ and such that $|G| - |E|_e < \varepsilon$ and $|E|_e - |F| < \varepsilon$; hence $|E \setminus F|_e \leq |G \setminus F| = |G| - |F| < 2\varepsilon$, so E is measurable.

$\square$

3.18. Prove that outer measure is *translation invariant*; that is, if $E_h = \{x + h : x \in E\}$ is the translate of E by $h \in \mathbb{R}^n$, show that $|E_h|_e = |E|_e$. If E is measurable, show that E_h is also measurable.

Solution. Given a countable cover $S = \{I_i\}_{i=1}^{\infty}$ of E by closed intervals, observe that $S_h = \{(I_i)_h\}_{i=1}^{\infty}$ is a countable cover of E_h by closed intervals. Also observe that $\sigma(S) = \sigma(S_h)$ because $v(I_i) = v((I_i)_h)$. Thus

$$\mathcal{S} = \{\sigma(S) : S \text{ is a countable cover of } E \text{ by closed intervals}\} \subseteq \\ \{\sigma(S) : S \text{ is a countable cover of } E_h \text{ by closed intervals}\} = \mathcal{S}_h,$$

so $\inf \mathcal{S} \geq \inf \mathcal{S}_h$. Repeating the argument for E_h and $(E_h)_{-h} = E$ gives us the opposite inequality, so $|E|_e = \inf \mathcal{S} = \inf \mathcal{S}_h = |E_h|_e$.

If E is measurable, then for every $\varepsilon > 0$ there exists an open set $G \supseteq E$ such that $|G \setminus E|_e < \varepsilon$. Then G_h is an open set such that $|G_h \setminus E_h|_e = |(G \setminus E)_h|_e = |G \setminus E| < \varepsilon$, so E_h is also measurable.[3] $\square$

[3]Often there are different ways of expressing the same argument using different characterizations of measurability. We could also argue: if E is measurable, then $E = H \setminus Z$ for some G_δ set H and some null set Z. Then $E_h = (H \setminus Z)_h = H_h \setminus Z_h$. H_h is a G_δ set and Z_h is a null set, so E_h is measurable.