Problem 1. Zygmund p76 exercise 05

Give an example to show that $\varphi(f(x))$ may not be measurable if φ and f are measurable and finite. (Let F be the Cantor–Lebesgue function and let f be its inverse, suitably defined. Let φ be the characteristic function of a set of measure zero whose image under F is not measurable.) Show that the same may be true even if f is continuous. (Let $g(x) = x + F(x)$, where F is the Cantor–Lebesgue function, and consider $f = g^{-1}$.) Cf. Exercise 22.

Problem 2.

Let $\chi_{[0,1]}$ be the characteristic function of $[0, 1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

Solution.

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$
$$\Updownarrow$$
$$|\{x|f(x) \neq \chi_{[0,1]}(x)\}| = 0.$$

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Suppose, for the sake of contradiction, that $\exists f$ on $\mathbb{R}$ s.t.

$$f(x) = \chi_{[0,1]}(x) \text{ almost everywhere.}$$

Without loss of generality, $f(x) = 1, x \in [0, 1]$.

By the definition of continuous everywhere, $\forall \epsilon > 0, \exists \delta > 0$, s.t. $|x - 0| < \delta \Rightarrow |f(x) - f(0)| < \epsilon$, which means $|f(x) - 1| < \epsilon$.

$$\Rightarrow f(x) \neq 0 \text{ on } [-\delta, 0]$$
$$\Rightarrow f(x) \neq \chi_{[0,1]} \text{ on } [-\delta, 0], \delta > 0.$$

Thus, $|\{x|f(x) \neq \chi_{[0,1]}(x)\}| \neq 0$, which contradicts the assumption that $f(x) = \chi_{[0,1]}(x)$ a.e.

Therefore, we conclude that there is no everywhere continuous function f on $\mathbb{R}$ such that $f(x) = \chi_{[0,1]}(x)$ a.e.

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Problem 3.

Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}$, $\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $|\Gamma| = 0$.