Real Analysis Homework 4

Due Date: October 11

Choose one of the two problems as your homework.

Problem 1

(a) Suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^d$ and that
$$\sum_{k=1}^{\infty} |E_k| < \infty$$
Let $E = \limsup_{k\to\infty} E_k$. Prove that $|E| = 0$.

(b) Given an irrational x, one can show (using the pigeonhole principle, for example) that there exists infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}$$
However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that for every $\epsilon > 0$, one has
$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$
is a set of measure zero.

Problem 2

(a) Let E be a subset of $\mathbb{R}$ with $|E|_e > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that
$$|E \cap I|_e \geq \alpha |I|_e$$
Loosely speaking, this estimate shows that E contains almost a whole interval.

(b) Suppose E is a measurable subset of $\mathbb{R}$ with $|E| > 0$. Prove that the **difference set** of E, which is defined by
$$E - E = \{x - y \in \mathbb{R} : x, y \in E\}$$
contains an open interval centered at the origin.