Real Analysis I Homework Solution 6

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Problem 1

Give an example to show that $\phi \circ f$ may not be measurable, where ϕ is measurable and finite and f is continuous and finite.

Proof. Let F be the Cantor-Lebesgue function and define the function g on $[0,1]$ by

$$g(x) = F(x) + x$$

Clearly, the function g is continuous since it's the sum of two continuous functions and is strictly increasing since it's the sum of a monotone increasing and a strictly increasing function. Moreover, we know

$$g\Big([0,1]\Big) = [0,2]$$

since we have

$$g(0) = 0 \text{ and } g(1) = 2$$

We know

$$[0,1] \setminus \mathcal{C} = \bigsqcup_{n\in\mathbb{N}} (a_n, b_n)$$

where (a_n, b_n) is the interval deleted in the construction of $\mathcal{C}$. Furthermore, we have

$$\Big|g\Big((a_n,b_n)\Big)\Big| = g(b_n) - g(a_n) = b_n - a_n = \Big|(a_n,b_n)\Big|$$

Hence, we know

$$\begin{aligned}
\Big|[0,2] \setminus g(\mathcal{C})\Big| &= \left|g\left(\bigsqcup_{n\in\mathbb{N}} (a_n,b_n)\right)\right| \\
&= \left|\bigsqcup_{n\in\mathbb{N}} g\Big((a_n,b_n)\Big)\right| = \sum_{n=1}^{\infty} \Big|g\Big((a_n,b_n)\Big)\Big| = \sum_{n=1}^{\infty} \Big|(a_n,b_n)\Big| = 1
\end{aligned}$$

Since we have

$$\Big|[0,2] \setminus g(\mathcal{C})\Big| = \Big|[0,2]\Big| - |g(\mathcal{C})| = 2 - |g(\mathcal{C})|$$

it follows that

$$|g(\mathcal{C})| = 1$$

Since any set of real numbers with positive outer measure contains a subset that fails to be measurable, there exists a non-measurable set $\mathcal{N}$ such that $\mathcal{N} \subseteq g(\mathcal{C})$. Let

$$M = g^{\text{pre}}(\mathcal{N})$$

Then we know

$$g(M) = \mathcal{N}$$

Obviously, we have $M \subseteq \mathcal{C}$. Therefore, M is measurable since we know $|M| = 0$. Define

$$f = g^{-1} \qquad \phi = \chi_M$$

Clearly, f is continuous and finite, and ϕ is measurable and finite. Moreover, $\phi \circ f$ is non-measurable since there exists a measurable set $\{1\}$ such that

$$(\phi \circ f)^{\text{pre}}\Big(\{1\}\Big) = f^{\text{pre}}\left(M\right) = \mathcal{N}$$

$\square$

Problem 2

Let $\chi_{[0,1]}$ be the characteristic function of $[0,1]$. Show that there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere}$$

Proof. Suppose for the sake of contradiction that there is an everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere}$$

Let $\epsilon = \dfrac{1}{2}$. Then we know there exists $0 < \delta_\epsilon < 1$ such that

$$|x| < \delta_\epsilon \Longrightarrow |f(x) - f(0)| < \epsilon$$

Since $f(x) = \chi_{[0,1]}(x)$ almost everywhere, it follows that there exist $x_0 \in (-\delta_\epsilon, 0)$ and $x_1 \in [0, \delta_\epsilon)$ such that

$$f(x_0) = \chi_{[0,1]}(x_0) \qquad f(x_1) = \chi_{[0,1]}(x_1)$$

otherwise we have

$$0 < 2\delta_\epsilon = \Big|(-\delta_\epsilon, \delta_\epsilon)\Big| \leq \left|\Big\{x \in \mathbb{R} : f(x) \neq \chi_{[0,1]}(x)\Big\}\right| = 0$$

which is a contradiction. Hence, we obtain

$$1 = |f(x_0) - f(x_1)| \leq |f(x_0) - f(0)| + |f(0) - f(x_1)| < 2\epsilon = 1$$

which is clearly a contradiction. Therefore, there is no everywhere continuous function f on $\mathbb{R}$ such that

$$f(x) = \chi_{[0,1]}(x) \quad \text{almost everywhere}$$

$\square$

Problem 3

Let $\Gamma \subset \mathbb{R}^d \times \mathbb{R}$, $\Gamma = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : y = f(x)\}$, and assume f is measurable on $\mathbb{R}^d$. Show that Γ is a measurable subset of $\mathbb{R}^{d+1}$, and $|\Gamma| = 0$.

Proof. It suffices to prove that $|\Gamma|_e = 0$. Since $\mathbb{R}^d$ is a countable union of almost disjoint cubes of side length 1, it is enough to show that $|\Gamma'|_e = 0$, where

$$\Gamma' = \left\{(x, y) \in [0,1]^d \times \mathbb{R} : y = f|_{[0,1]^d}(x)\right\}$$

Since we know $\mathbb{R} = \bigsqcup_{k\in\mathbb{Z}} [k, k+1)$, it follows that

$$\Gamma' = \bigsqcup_{k\in\mathbb{Z}} \left\{(x, y) \in [0,1]^d \times [k, k+1) : y = f|_{[0,1]^d}(x)\right\}$$

Again, it is sufficient to prove that $|\Gamma''|_e = 0$, where

$$\Gamma'' = \left\{(x, y) \in [0,1]^d \times [0, 1) : y = f|_{[0,1]^d}(x)\right\}$$

For every $n \in \mathbb{N}$, we have $[0, 1) = \bigsqcup_{j=1}^{n} I_j$, where $I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right)$ for all $j \in [n]$. Since we know

$$\Gamma'' = \bigsqcup_{j=1}^{n} \left\{(x, y) \in [0,1]^d \times I_j : y = f|_{[0,1]^d}(x)\right\}$$

and $f|_{[0,1]^d}$ is measurable on $[0, 1]^d$, it follows that

$$\begin{aligned}
|\Gamma''|_e &\leq \sum_{j=1}^{n} \left|\left\{(x, y) \in [0,1]^d \times I_j : y = f|_{[0,1]^d}(x)\right\}\right|_e \\
&\leq \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \times I_j \right|_e \\
&\leq \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \right| \cdot |I_j| \\
&= \frac{1}{n} \cdot \sum_{j=1}^{n} \left| f|^{\text{pre}}_{[0,1]^d}(I_j) \right| \\
&= \frac{1}{n} \cdot \left| \bigsqcup_{j=1}^{n} f|^{\text{pre}}_{[0,1]^d}(I_j) \right| = \frac{1}{n} \cdot \left| f|^{\text{pre}}_{[0,1]^d}\big([0, 1)\big) \right| \leq \frac{1}{n} \cdot \left|[0, 1]^d\right| = \frac{1}{n}
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, we have

$$|\Gamma''|_e \leq \lim_{n \to \infty} \frac{1}{n} = 0$$

Therefore, we obtain

$$|\Gamma''|_e = 0$$

$\square$