Real Analysis, Stein and Shakarchi

Chapter 2 Integration Theory*

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1 Exercises

1. *Proof.* □

2. *Proof.* □

3. *Proof.* □

4. *Proof.* □

5. *Proof.* □

6. *Proof.* □

7. *Proof.* □

8. *Proof.* □

9. *Proof.* □

10. *Proof.* □

11. *Proof.* □

12. *Proof.* □

13. **Give an example of two measurable sets A and B such that $A+B$ is not measurable.**

    Proof. As hint, consider $A = \{0\} \times [0,1]$ and $B = \mathcal{N} \times \{0\}$, they are of measure zero. If $A+B$ is measurable, then by Fubini's theorem, $(A+B)^y \equiv \mathcal{N}$ is measurable for a.e. $y \in [0,1]$. □

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***Remark* 1.** The hypothesis $A+B$ is measurable in Brunn-Minkowski inequality is needed.

14. *Proof.* $\square$

15. *Proof.* $\square$

16. *Proof.* $\square$

17. *Proof.* $\square$

18. *Proof.* $\square$

19. *Proof.* $\square$

20. *Proof.* $\square$

21. **Suppose that f and g are measurable functions on R^d. (a) Prove that $f(x-y)g(y)$ is measurable on R^{2d}.**

    (b) Show that if f and g are integrable on R^d, then $f(x-y)g(y)$ is integrable on R^{2d}.

    (c) Recall the definition of the convolution of f and g given by

    $$(f*g)(x) = \int^{\mathbb{R}^d} f(x-y)g(y)\,dy.$$

    Show that $f*g$ is well defined for a.e. x.

    (d) Show that $f*g$ is integrable whenever f and g are integrable, and that

    $$\|f*g\|_{L^1(R^d)} \leq \|f\|_{L^1(R^d)}\|g\|_{L^1(R^d)}$$

    with equality if f and g are non-negative.

    (e) The Fourier transform of an integrable function f is dened by

    $$\widehat{f}(\xi) = \int_{R^d} f(x)e^{2\pi i x\cdot\xi}\,dx.$$

    Check that $\widehat{f}$ is bounded and is a continuous function of ξ. Prove that for each one has

    $$\widehat{f*g}(\xi) = \widehat{f}(\xi)\widehat{g}(\xi).$$

    Proof. $\square$

***Remark* 2.** See Exercise 1.17 in Book IV for a L^p space setting.

22. **Let $f \in L^1(\mathbb{R}^d)$ and**

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi}\, dx.$$

Prove that $\widehat{f}$ is uniformly continuous and vanishes at infinity. (This is Riemann-Lebesgue lemma).

Remark 3. Consult [4, Section I.4.1] or [3, Exercise 1.6] for showing the Fourier transform map from $L^1(\mathbb{R}^d)$ to $C_0(\mathbb{R}^d)$ is not surjective. For $L^1(\mathbb{T})$ case, consult Section 4.3.1 of Book IV.

Proof. The continuity is directly from LDCT. Note that

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\xi}\, dx = \int_{\mathbb{R}} -f(y - \frac{1}{2\xi}) e^{-2\pi i y\xi}\, dy.$$

So $\widehat{f}(\xi) = \frac{1}{2}\int_{\mathbb{R}} \left[f(x) - f(x - \frac{1}{2\xi})\right] e^{-2\pi i x\xi}\, dx$ and hence the LDCT implies the decay at ∞. The uniform continuity is an easy consequence of decay at ∞ and continuity. $\square$

23. **As an application of the Fourier transform, show that there does not exist a function $I \in L^1(\mathbb{R}^d)$ such that**

$$f * I = f \text{ for all } f \in L^1(\mathbb{R}^d).$$

Proof. (1st proof, need Fubini's theorem and Riemann-Lebesgue lemma) If this is true, then $\widehat{f} = \widehat{f * I} = \widehat{f}\widehat{I}$. Since there exists $f \in L^1$ such that $\widehat{f} \not\equiv 0, I \equiv 1$, which contradicts to the Riemann-Lebesgue lemma.

(2nd proof, use "translation is continuous in L^1 norm", also see Exercise 1.17 in Book IV) If this is true, then every L^∞_c function (essential bounded and has compact support) f is going to be uniformly continuous since $f * I$ is. But this is not true (e.g. $f = \chi_{[0,1]^d}$).

(3rd proof, use the notion of absolutely continuity of integral, that is, Propostion 1.12) If such I, then $\chi_{B(0,r)} * I = \chi_{B(0,r)}$ for each $r > 0$. So there exists $x_r \in \chi_{B(0,r)}$ such that $1 = \chi_{B(0,r)} * I(x_r) = \int_{B(x_r,r)} I \geq 1$. Then $\int_{B(0,2r)} |I| \geq 1$ for each r, but this contradicts to Proposition 1.12. $\square$

Remark 4. The 2nd and 3rd proof are taken from [2, p.284-285].

24. **(a) Show that $f * g$ is uniformly continuous when $f \in L^1$ and $g \in L^\infty$**

(b) If $f \in L^1$ and $g \in L^1 \cap L^\infty$, prove that $(f * g)(x) \to 0$ as $|x| \to \infty$.

Proof. (a) May assume $g \not\equiv 0$. By translation is continuous in L^1 norm, there is $\delta > 0$ such that, $\|f(\cdot + h - f(\cdot)\|_{L^1} < \epsilon / \|g\|_{L^\infty}$ whenever $h < \delta$. Then for all $x \in \mathbb{R}^d$, Hölder's inequality implies that

$$|(f * g)(x+h) - (f * g)(x)| \leq \|f(\cdot + h - f(\cdot)\|_{L^p} \|g\|_{L^q} < \epsilon,$$

whenever $h < \delta$.

(b) By Young's inequality, $f * g \in L^1$. Then by (a) and Exercise 6, $(f * g)(x) \to 0$ as $|x| \to \infty$. $\square$

Remark 5. On 2017.09.23, I saw this problem which ask whether $f * g$ is continuous for $f, g \in L^1(\mathbb{T})$. The answer is no by constructing explicit example $f(x) = g(x) = x^{-3/4} \chi_{(0,\pi]}(x)$. There is another answer states a non-trivial fact for me:

"Salem and Zygmund proved that convolution map $L^1(\mathbb{T}) \times L^1(\mathbb{T}) \to L^1(\mathbb{T})$ is onto.

This was shown to hold for all locally compact groups by Paul Cohen in 1959. This result was the starting point of an entire industry establishing "factorization theorems".

A nice survey on this topic is Jan Kisynski, On Cohen's proof of the Factorization Theorem, Annales Polonici Mathematici 75, 2 (2000), 177-192."

25. **Show that for each $\epsilon > 0$, the function $F(\xi) = (1 + |\xi|^2)^{-\epsilon}$ is the Fourier transform of an L^1 function. (Note that $F \notin L^1$ if $\epsilon \leq \frac{d}{2}$)**

[Hint: $K_\delta(x) = e^{-\pi|x|^2/\delta} \delta^{-d/2}$, consider $f(x) = \int_0^\infty K_\delta(x) e^{-\pi\delta} \delta^{\epsilon - 1}\, d\delta$.]

Proof. Since $f \geq 0$, Fubini-Tonelli's theorem implies that

$$\int_{\mathbb{R}^d} f(x)\, dx = \int_0^\infty \int_{\mathbb{R}^d} K_\delta(x)\, dx\, e^{-\pi\delta} \delta^{\epsilon-1}\, d\delta = \int_0^\infty e^{-\pi\delta}\delta^{\epsilon-1}\, d\delta = \Gamma(\epsilon)\pi^{-\epsilon} < \infty$$

Again, the Fubini-Tonelli theorem implies that

$$\begin{aligned} \widehat{f}(\xi) &= \int_{\mathbb{R}^d} f(x) e^{-2\pi x \cdot \xi}\, dx = \int_0^\infty \int_{\mathbb{R}^d} e^{-\pi|x|^2/\delta} e^{-2\pi x\cdot\xi} \delta^{-d/2}\, dx\, e^{-\pi\delta}\delta^{\epsilon-1}\, d\delta \\ &= \int_0^\infty e^{-\pi\delta(|\xi|^2+1)} \delta^{\epsilon-1}\, d\delta = \Gamma(\epsilon)\Big(\pi(1+|\xi|^2)\Big)^{-\epsilon}. \end{aligned}$$

$\square$

Remark 6. One can also construct functions on $\mathbb{T}$ with arbitrary slow decay for their Fourier transform, see [1, Section 3.3].

Problems

1. *Proof.* $\square$

2. *Proof.* $\square$

3. *Proof.* $\square$

4. *Proof.* $\square$

5. *Proof.* $\square$

References

[1] Grafakos, Loukas. Classical Fourier Analysis. 3rd ed., Springer, 2014.

[2] Jones, Frank. Lebesgue Integration on Euclidean Space. Revised ed., Jones and Bartlett Learning, 2001.

[3] Linares, Felipe, and Gustavo Ponce. Introduction to Nonlinear Dispersive Equations. 2nd ed., Springer, 2015.

[4] Stein, Elias M., and Guido L. Weiss. Introduction to Fourier Analysis on Euclidean Spaces. Vol. 1. Princeton University Press, 1971.