4.2 Semicontinuous Functions

[a] Let f be defined on E, and let x_0 be a limit point of E that lies in E. Then f is said to be upper semicontinuous at x_0 if

$$\limsup_{\substack{x \to x_0 \\ x \in E}} f(x) \leq f(x_0).$$

We will usually abbreviate this by saying that f is usc at x_0. Note that if $f(x_0) = +\infty$, then f is automatically usc at x_0; otherwise, the statement that f is usc at x_0 means that given $M > f(x_0)$, there exists $\delta > 0$ such that $f(x) < M$ for all $x \in E$ that lie in the ball $|x - x_0| < \delta$. Intuitively, this means that near x_0, the values of f do not exceed $f(x_0)$ by a fixed amount.

Similarly, f is said to be lower semicontinuous at x_0, or lsc at x_0, if

$$\liminf_{\substack{x \to x_0 \\ x \in E}} f(x) \geq f(x_0).$$

Thus, if $f(x_0) = -\infty$, f is lsc at x_0, while if $f(x_0) > -\infty$, the definition amounts to saying that given $m < f(x_0)$, there exists $\delta > 0$ such that $f(x) > m$ if $x \in E$ and $|x - x_0| < \delta$. Equivalently, f is lsc at x_0 if and only if $-f$ is usc at x_0.

It follows that f is continuous at x_0 if and only if $|f(x_0)| < +\infty$ and f is both usc and lsc at x_0.

[a] Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis.* CRC, 2015, p. 69.

Problem 1. Zygmund p77 exercise 11

Let f be defined on $\mathbb{R}^n$, and let $B(x)$ denote the open ball $\{y : |x-y| < r\}$ with center x and fixed radius r. Show that the function $g(x) = \sup\{f(y) : y \in B(x)\}$ is lsc (lower semi-continuous), and that the function $h(x) = \inf\{f(y) : y \in B(x)\}$ is usc (upper semi-continuous) on $\mathbb{R}^n$. Is the same true for the closed ball $\{y : |x - y| \leq r\}$?

(a) lsc

To prove by contradiction, we assume that $g(x) = \sup\{f(y) : y \in B(x)\}$ is not lsc, to be specific, $\exists x_0 \in \mathbb{R}^n$

$$\liminf_{\substack{x \to x_0 \\ x \in \mathbb{R}^n}} g(x) < g(x_0),$$

that is,

$$\liminf_{\substack{x \to x_0 \\ x \in \mathbb{R}^n}} \sup\{f(y) : y \in B(x)\} < \sup\{f(y) : y \in B(x_0)\}.$$

Thus, by "lim inf", $\exists x_n \to x_0$, $\exists \epsilon > 0$ s.t.

$$\sup\{f(y) : y \in B(x_n)\} < \sup\{f(y) : y \in B(x_0)\} - \epsilon.$$

By "sup", $\exists y \in B(x_0)$, s.t.

$$\begin{aligned} f(y) &> \sup\{f(y) : y \in B(x_0)\} - \epsilon/2 \\ &> \sup\{f(y) : y \in B(x_n)\} + \epsilon/2. \end{aligned}$$

Note that $\exists n0$ s.t. $|x_{n0} - x_0| < r - |y - x_0|$, we have

$$|y - x_{n0}| < |y - x_0| + |x_{n0} - x_0| < r,$$

implying that $y \in B(x_{n_0})$.

Thus,

$$\begin{aligned} f(y) &> \sup\{f(z) : z \in B(x_{n_0})\} + \epsilon/2 \\ &> f(y) + \epsilon/2, \end{aligned}$$

which is a contradiction.

Therefore, we can conclude that $g(x)$ is lower semicontinuous on $\mathbb{R}^n$.

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(b) usc:

Since $f(x)$ is lsc if and only if $-f(x)$ is usc, to show that $h(x)$ is usc is to show that $-h(x)$ is usc.

$$\begin{aligned} -h(x) = -\inf\{f(y) : y \in B(x)\} &= \sup\{-f(y) : y \in B(x)\} \\ &= \sup\{\phi(y) : y \in B(x)\} \end{aligned}$$

which is lsc by (1).

Thus, we can see that $h(x)$ is usc on $\mathbb{R}^n$.

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(c)

For the closed ball $\bar{B} = \{y : |x - y| \leq r\}$, the same properties hold.

Problem 2. Zygmund p77 exercise 12

If $f(x)$, $x \in \mathbb{R}^1$, is continuous at almost every point of an interval $[a, b]$, show that f is measurable on $[a, b]$.

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Generalize this to functions defined in $\mathbb{R}^n$.
For a constructive proof, use the subintervals of a sequence of partitions to define a sequence of simple measurable functions converging to f a.e. in $[a, b]$. Use Theorem 4.12. See also the proof of Theorem 5.54.

Since $f(x)$ is continuous at almost every point of $[a, b]$,

there exists $Z \subset [a, b]$ and s.t. $|Z| = 0$ and f is continuous on $E = [a, b] \setminus Z$.

Note that Z is measurable, and $E = [a, b] \setminus Z$ is also measurable.

For any finite α, we have

$$\{x \in [a, b] : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \sqcup \{x \in Z : f(x) > \alpha\}.$$

Note that $\{x \in E : f(x) > \alpha\}$ is measurable since f is continuous, thus measurable on E.

Since $\{x \in Z : f(x) > \alpha\} \subseteq Z$, thus $|\{x \in Z : f(x) > \alpha\}| = 0$, implying that $\{x \in Z : f(x) > \alpha\}$ is also measurable. Thus $\{x \in [a, b] : f(x) > \alpha\}$ is measurable. This means f is measurable on $[a, b]$.

Problem 3. Zygmund p78 exercise 15

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < +\infty$. If $|f_k(x)| \leq M_x < +\infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed set $F \subset E$ and a finite M such that $|E - F| < \varepsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

Let

$$E_m := \{|f_k(x)| \leq m, \forall k \in \mathbb{N}\} = \bigcap_{k=1}^{\infty} \{|f_k(x)| \leq m\}.$$

Notice that E_M being measurable implies that its complement E_M^c is measurable (By Thm 3.17).

Since $E_m \nearrow$, and $\forall x \in E, \exists M_x$ s.t

$$x \in \{|f_k(x)| \leq M_x \, \forall k \in \mathbb{N}\} = E_{M_x}$$

We have $E_m \to E$. Hence, by Thm 3.26, we have $\lim_{m\to\infty} |E_m| = |E| < +\infty$.

Thus, $\forall \epsilon > 0$, there exists $M < \infty$, s.t. $\{|f_k(x)| \leq M\}$ for all k and all $x \in E_M$, and

$$|E| - |E_M| = |E \setminus E_M| < \epsilon/2.$$

Since f_k is measurable, by Thm 4.1, we have $\{|f_k(x)| \leq m\}$ measurable for all k, thus, by Thm 3.18, $E_m = \bigcap_{k=1}^{\infty} \{|f_k(x)| \leq m\}$.

For the measurable E_M^c, given $\epsilon > 0$, there exists an open set G s.t.

$$E_M^c \subset G \text{ and } |G \setminus E_M^c| < \epsilon/2.$$

Let $F = G^c$ be a close set, we have

$$F \subset E_M \text{ and } |E_M \setminus F| = |E_M \setminus G^c| < \epsilon/2.$$

Then,

$$|E \setminus F| = |E \setminus E_M| + |E_M \setminus F| < \epsilon$$

and $|f_k(x)| \leq M$ for all k and all $x \in F$ as required.

Theorem 3.26. *Let $\{E_k\}_{k=1}^{\infty}$ be a sequence of measurable sets.*

(i) If $E_k \subset E$, then $\lim_{k\to\infty} |E_k| = |E|$.

(ii) If $E_k \supset E$ and $|E_k| < +\infty$ for some k, then $\lim_{k\to\infty} |E_k| = |E|$.