Measure and Integration: Solutions of CW2

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Problem 1 of Sheet 5

a) Left (f_n) and (g_n) be sequences of integrable functions with $f_n(x) \to f(x)$ and $g_n(x) \to g(x)$ for almost every x and the limits f and g being also integrable. Prove that if $|f_n| \leq g_n$ and $\int g_n \to \int g$, then also $\int f_n \to \int f$. Hint: Mimick the proof of the dominant convergence theorem from lectures.

b) Let (f_n) be a sequence of integrable functions with $f_n(x) \to f(x)$ for almost every x and the limit f being integrable. Prove that

$$\int |f_n - f| \to 0 \quad \text{if and only if} \quad \int |f_n| \to \int |f|\,.$$

Solution

a) We start from

$$|f_n(x) - f(x)| \leq g_n(x) + g(x)$$

which holds for a.e. x by the triangle inequality (note the assumptions imply $|f(x)| \leq g(x)$ for a.e. x). After changing g on a set of measure zero, the sequence of functions given by $h_n(x) = g(x) + g_n(x) - |f_n(x) - f(x)|$ is non-negative and we can apply Fatou's Lemma to produce the inequality

$$\liminf_{n\to\infty} \int \left(g(x) + g_n(x) - |f_n(x) - f(x)|\right) dx \geq \int 2g(x)dx\,.$$

Now since we are assuming $\int g_n \to \int g$ and that $\int g < \infty$ we obtain, just as in the proof in class

$$\limsup_{n\to\infty} \int \left(|f_n(x) - f(x)|\right) dx \leq 0\,,$$

from which we deduce $\lim_{n\to 0} \int |f_n(x) - f(x)|\, dx = 0$ and by the triangle inequality that $\int f_n \to \int f$. For the application in part b) we note that we actually proved the stronger statement that $\int |f_n - f| \to 0$.

b) Suppose $\int |f_n - f| \to 0$. By the triangle inequality for the integral and the pointwise reverse triangle inequality we have

$$\left| \int \Big(|f_n(x)| - |f(x)|\Big) dx \right| \leq \int \Big| |f_n(x)| - |f(x)| \Big|\, dx \leq \int \Big| f_n(x) - f(x) \Big|\, dx$$

and using the assumption the left hand side goes to zero. It follows that $\int |f_n| \to \int |f|$.

Suppose now $\int |f_n| \to \int |f|$. The idea is to use part a). We have $f_n(x) \to f(x)$ for almost every x and hence $|f_n(x)| \to |f(x)|$ for almost every x. If we set $g_n(x) = |f_n(x)|$ and $g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} |f_n(x)| = |f(x)|$ it is easily checked that all assumptions of part a) are satisfied, in particular $f_n(x) \leq |f_n(x)| = |g_n(x)|$ and $\int |f_n| \to \int |f|$. The stronger statement proven in part a) then implies $\int |f_n - f| \to 0$ as desired.

Problem 4 of Sheet 5

This question looks at the invariance properties of the Lebesgue integral (Stein-Shakarchi, page73-74). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. Define the translation

$$f_h(x) := f(x-h) \quad \text{for some } h \in \mathbb{R}^d .$$

a) Prove that if f is integrable, then so is f_h and

$$\int_{\mathbb{R}^d} f(x-h)\,dx = \int_{\mathbb{R}^d} f(x)\,dx .$$

Hint: First show this for the characteristic function $f = \chi_E$, then for simple functions. Finally use the approximation theorem from lectures and the monotone convergence theorem.

b) Prove that if f is integrable, then so are $f(\delta x)$ for $\delta > 0$ and $f(-x)$ with the relations

$$\delta^d \int_{\mathbb{R}^d} f(\delta x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx \quad \text{and} \quad \int_{\mathbb{R}^d} f(-x)\,dx = \int_{\mathbb{R}^d} f(x)\,dx$$

Hint: Proceed exactly as in a).

c) Prove that if f is integrable, then

$$\|f_h - f\|_{L^1} \to 0 \quad \text{as } h \to 0.$$

Hint: Approximate f by a continuous function of compact support g and write $f_h - f = (f_h - g_h) + (g_h - g) + (g - f)$.

d) Prove that if f is integrable, then

$$\|f(\delta x) - f(x)\|_{L^1} \to 0 \quad \text{as } \delta \to 1.$$

Solution

a) First, from the translation invariance of the Lebesgue measure (discussed in class) we recall that the set E and the translated set $E_h = \{x + h \mid x \in E\}$ have the same measure $m(E_h) = m(E)$. Since $f_h = \chi_{E_h}$ the identity for the integral follows. Second, by the linearity of the integral the identity then also holds for finite linear combinations of characteristic function, hence for simple functions. Third, an arbitrary non-negative function can be approximated by a strictly increasing sequence of simple functions (φ_n). But then both (φ_n) and $((\varphi_h)_n)$ are increasing sequences of simple functions converging to f and f_h respectively. The desired identity clearly holds for each φ_n and $(\varphi_h)_n$ and the MCT implies that it also holds for f and f_h. Fourth and finally, a general integrable f can be decomposed into its positive and negative part which are both integrable and for which the identity has already been established. Using linearity once more the result follows.

b) Indeed exactly the same as in a).

c) We follow the hint and for $\epsilon > 0$ prescribed we find a continuous function of compact support g with $\|f - g\|_{L^1} < \epsilon$. The assumptions on g imply that $g_h - g$ with $|h| < 1$ is (uniformly) compactly supported and continuous, hence $g_h - g$ is dominated by an integrable function and the DCT implies $\lim_{h\to 0} \int |g(x-h) - g(x)|dx = 0$. Therefore we can find a δ (depending on ϵ) with $\|g_h - g\| < \epsilon$ for $|h| < \delta$. Finally, by part a) we have $\|f_h - g_h\|_{L^1} = \|f - g\|_{L^1} < \epsilon$ and hence

$$\|f_h - f\|_{L^1} \leq 3\epsilon \quad \text{for all } |h| < \delta$$

which establishes that the desired limit exists and is equal to zero.

d) We adapt the proof in c). Since we are interested in the limit $\delta \to 1$ we a priori restrict to $\frac{1}{2} < \delta < 2$. Set $f_\delta(x) = f(\delta x)$, pick g continuous and compactly supported with $\|f - g\|_{L^1} < \epsilon$ and decompose $\|f_\delta - f\|_{L^1} = \|f_\delta - g_\delta\|_{L^1} + \|g_\delta - g\|_{L^1} + \|g - f\|_{L^1}$. We have that $g(\delta x) - g(x)$ is (uniformly) compactly supported and hence $\lim_{\delta\to 1} \int |g(\delta x) - g(x)|dx = 0$ by DCT. We can hence choose η such that $|\delta - 1| < \eta$ implies $\|g_\delta - g\|_{L^1} < \epsilon$. Finally, we have $\|f_\delta - g_\delta\|_{L^1} = \delta^{-d}\|f - g\|_{L^1}$ by part b) so that in total

$$\|f_\delta - f\|_{L^1} \leq \delta^{-d}\epsilon + 2\epsilon \leq 2^d\epsilon + 2\epsilon .$$

Problem 1 of Sheet 6

There are several ways in which a sequence of real valued measurable functions (f_n) can converge to a limiting function f: For instance, pointwise, pointwise a.e. or uniformly. In lectures, we also introduced the L^1-norm and hence the notion of $f_n \to f$ in L^1.

a) Discuss the convergence of the following sequences of functions with regard to the aforementioned notions:
   i. $f_n = \frac{1}{n}\chi_{(0,n)}$
   ii. $f_n = \chi_{(n,n+1)}$
   iii. $f_n = n \cdot \chi_{[0,1/n]}$

b) Give an example of a sequence (f_n) which converges to f in L^1 but not pointwise a.e.
HINT: Construct a sequence of indicator functions travelling over and over through the interval $[0,1]$ with decreasing length.

c) There is a further notion of convergence, which plays an important role in probability: We say that (f_n) is **Cauchy in measure** if for every $\epsilon > 0$ we have

$$m(\{x \mid |f_m(x) - f_n(x)| \geq \epsilon\}) \to 0 \quad \text{as } m, n \to \infty.$$

We say that (f_n) converges to f in measure if for every $\epsilon > 0$ we have

$$m(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \text{as } n \to \infty.$$

Prove that if $f_n \to f$ in L^1, then $f_n \to f$ in measure. Revisit a) to see that the converse does not hold.

Solution

a) The function in i) converges to the zero function uniformly. It has L^1-norm equal to 1 for all n and hence does not converge to the zero function in L^1.

The function in *ii*) converges to zero pointwise but not uniformly. The L^1-norm is equal to 1 for all n and hence the function does not converge to the zero function in L^1.

The function in *iii*) converges to zero pointwise almost everywhere (everywhere except at $x = 0$). The L^1-norm is again equal to 1 for all n so f_n does not converge in L^1 to the zero function.

b) Define the following sequence of functions. We let

- $f_1 = \chi_{[0,1]}$.
- $f_2 = \chi_{[0,1/2]}$ and $f_3 = \chi_{[1/2,1]}$
- $f_4 = \chi_{[0,1/4]}$ and $f_5 = \chi_{[1/4,1/2]}$ and $f_6 = \chi_{[1/2,3/4]}$ and $f_7 = \chi_{[3/4,1]}$
- ...

I leave it to you to find an explicit expression for the general f_n.

It is clear from the definition that f_n converges to zero in L^1. However, f_n does not converge pointwise to 0 for any x. This is because if $f_n(x) \to 0$ for some $x \in [0,1]$ then one could find N such that $|f_n(x)| < \frac{1}{2}$ for any $n \geq N$. However, one can clearly find an n larger than N such that $|f_n(x)| = 1$ holds.

c) This is a consequence of the Chebychev inequality: For fixed $\epsilon > 0$ we have

$$m(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \leq \frac{1}{\epsilon} \int |f_n(x) - f(x)|$$

and the right hand side goes to zero as $n \to \infty$ proving that $f_n \to f$ in measure.

The example i) in a) converges to zero in measure but not in L^1.

Problem 3 of Sheet 6

Suppose $I \subset \mathbb{R}$ is open and that $f : I \times E \to \mathbb{R}$, $(t, x) \mapsto f(t, x)$ for E a measurable subset of $\mathbb{R}^d$. Suppose f is in $L^1(E)$ for all t, differentiable in t for all x with the derivative satisfying $|\partial_t f(t, x)| \leq g(x)$ for some $g \in L^1(E)$. Then

$$\frac{d}{dt} \int_E f(t, x)\, dx = \int_E \frac{\partial f}{\partial t}(t, x) < \infty\,.$$

Solution

We establish the identity at fixed $t \in I$. We first define the difference quotient

$$h_n(t, x) = \frac{f(t + t_n, x) - f(t, x)}{t_n}$$

where (t_n) is any sequence converging to zero with $t_n \neq 0$ for all n and such that $t + t_n \in I$. Clearly $h_n(t, x) \to \partial_t f(t, x)$ for all $x \in E$. Hence $\partial_t f(t, x)$ is a measurable function on E and in view of the assumption $|\partial_t f(t, x)| \leq g(x)$ also integrable. By the mean value theorem

$$h_n(t, x) = f'\left(\tilde{t}, x\right)$$

for a $\tilde{t} \in (t, t + t_n) \subset I$ and hence by assumption $|h_n(t, x)| \leq g(x)$ for sufficiently large n. It follows that the dominant convergence theorem applies to $h_n(t, x)$ producing

$$\lim_{n\to\infty} \int_E h_n(t, x) = \int_E \frac{\partial f}{\partial t}(t, x) < \infty\,.$$

Finally, observe that

$$\lim_{n\to\infty} \int_E h_n(t, x) = \lim_{n\to\infty} \frac{\int_E f(t + t_n, x) - \int_E f(t, x)}{t_n} = \frac{d}{dt} \int_E f(t, x)\, dx\,,$$

the last step following since we proved the result for any sequence (t_n) converging to zero.

Problem 5 of Sheet 7

Let $a, b > 0$ be positive constant and consider

$$f(x) = \begin{cases} x^a \sin(x^{-b}) & \text{for } 0 < x \le 1, \\ 0 & \text{if } x = 0 \end{cases}$$

a) Prove that f is of bounded variation if $[0,1]$ if and only if $a > b$.

b) Set $a = b$. Construct for each $\alpha \in (0,1)$ a function that satisfies the Lipschitz condition of exponent α

$$|f(x) - f(y)| \le M_\alpha |x-y|^\alpha$$

but which is not of bounded variation. Here M_α is a constant.
Remark: Recall that in lectures we saw that if the above holds with $\alpha = 1$, then f is of bounded variation.
HINT: Estimate $|f(x+h) - f(x)|$ for $h > 0$ in two different ways (one being the mean value theorem, the other being by $C(x+h)^a$. Consider then the cases $x^{a+1} \ge h$ and $x^{a+1} < h$.).

Solution

a) We first establish that if $a > b$, then f is of BV. We have

$$f'(x) = ax^{a-1} \sin\left(x^{-b}\right) - bx^{a-b-1} \cos\left(x^{-b}\right)$$

for $x \in (0,1)$ and f' is also integrable in $[0,1]$ in view of

$$|f'(x)| \le a \cdot x^{a-1} + b \cdot x^{a-b-1} \quad \text{and hence} \quad \int_0^1 dx |f'(x)| dx \le 1 + \frac{b}{a-b}.$$

It follows that for any partition $0 = x_0 < x_1 < ... < x_N = 1$ we have by the FT for the Lebesgue integral

$$\sum_{i=1}^N |f(x_i) - f(x_{i-1})| = \sum_{i=1}^N \Big| \int_{x_{i-1}}^{x_i} dt f'(t)\, dt \Big| \le \sum_{i=1}^N \int_{x_{i-1}}^{x_i} |f'(x)| dx \le \int_0^1 |f'(x)| dx \le 1 + \frac{b}{a-b}.$$

We now establish that if $a \le b$ then f is not of bounded variation. To do this, we construct a sequence of partitions whose variation is unbounded.

For $n \ge 1$ we define $x_n = \left(\frac{2}{\pi n}\right)^{1/b}$. Note that $|f(x_n)| = \left(\frac{2}{\pi n}\right)^{a/b}$ if n is odd, while $f(x_n) = 0$ if n is even. For any $N \ge 1$ we now consider a partition of $[0,1]$ by $N+1$ intervals of the form

$$0 < x_N < x_{N-1} < ... < x_2 < x_1 < 1.$$

The variation of this partition is (dropping the left and right outermost interval)

$$Var_f(\mathcal{P}) \ge \sum_{n=1}^{N-1} |f(x_{n+1}) - f(x_n)| \ge \sum_{\substack{n=1 \\ n \text{ odd}}}^{N-1} \left(\frac{2}{\pi n}\right)^{a/b} + \sum_{\substack{n=1 \\ n \text{ even}}}^{N-1} \left(\frac{2}{\pi(n+1)}\right)^{a/b}$$

Both sums on the right hand side diverge as $N \to \infty$ if $a \le b$, the case $a = b$ being the borderline case (harmonic series). Hence the total variation of f on $[0,1]$ is not bounded.

b) Let us fix $\alpha \in (0,1)$ and set $b = a$, so $f(x) = x^a \sin x^{-a}$ on $(0,1]$ and $f(0) = 0$. We will choose $a(\alpha)$ such that the α-Lipschitz condition holds. Wlog we fix $y > x$ and set $y = x + h$ with $0 < h \le 1$ and $x \in [0,1)$.

We estimate $|f(x+h) - f(x)| \le (x+h)^a + x^a \le 2(x+h)^a$ for $x \in [0,1)$ and also $|f(x+h) - f(x)| \le |f'(\tilde{x})| h$ for some $\tilde{x} \in (x, x+h)$ by the mean value theorem. By part a) we estimate f' to obtain $|f(x+h) - f(x)| \le \frac{2a}{x} h$ for $x \in (0,1]$.

Now if $h \le x^{a+1}$ then $\frac{1}{x} \le h^{-\frac{1}{a+1}}$ then $x \ne 0$ and the second estimate yields $|f(x+h) - f(x)| \le \frac{2a}{x} h \le 2ah^{1-\frac{1}{a+1}}$. Hence if we set $\alpha := 1 - \frac{1}{a+1} = \frac{a}{a+1}$ we satisfy the desired Lipschitz condition in this range of h. If $h > x^{a+1}$ the first estimate yields the desired Lipschitz condition via (using $h \le 1$ and $a > 0$)

$$|f(x+h) - f(x)| \le 2(h^{\frac{1}{a+1}} + h)^a \le 2(h^{\frac{1}{a+1}} + h^{\frac{1}{a+1}})^a \le 2 \cdot 2^a h^\alpha.$$