Math 6338 : Real Analysis II

Mid-term Exam 1
13 February 2012

Instructions: Answer all of the problems.

1. Let X be a real normed space. Suppose that $C \subset X$ is a convex subset which contains 0 and has the property that

$$\bigcup_{t>0} tC = X$$

For every $x \in X$ define

$$Q_C(x) = \inf\{t > 0 : x \in tC\}$$

Show that the map $Q_C : X \to \mathbb{R}$ is a sublinear functional. Namely it satisfies the following properties:

(i) $Q_C(x+y) \leq Q_C(x) + Q_C(y)$;

(ii) $Q_C(tx) = tQ_C(x)$ for all $x \in X$ and all $t \in \mathbb{R}_+$.

Solution: For $x \in X$, let

$$S(x) = \{t > 0 : x \in tC\}.$$

In particular, note that since $0 \in C$ we have that $S(0) = (0, \infty)$, and thus $Q_C(0) = 0$. To prove the result in question, it suffices to show that

(i) $S(x+y) \supset S(x) + S(y)$;

(ii) $S(\lambda x) = \lambda S(x)$ for all $x \in X$ and all $\lambda > 0$.

Using these two conditions and the definition of Q_C, the result follows.

We now turn to proving (i). Let $t \in S(x)$ and $s \in S(y)$. Consider the elements $u = t^{-1}x$ and $v = s^{-1}y$. As $u, v \in C$ and C is convex it follows that

$$\frac{1}{t+s}(x+y) = \frac{t}{t+s}u + \frac{s}{t+s}v \in C.$$

This means that $x + y \in (t+s)C$, and so $t + s \in S(x+y)$. This proves that $S(x+y) \supset S(x) + S(y)$.

Consider the second claim (ii). Suppose that $t \in S(\lambda x)$, then we have that $\lambda x \in tC$, or equivalently, $\lambda^{-1}tx \in C$, which means that $\lambda^{-1}t \in S(x)$, and so

$$t = \lambda\lambda^{-1}t \in \lambda S(x).$$

This proves the inclusion $S(\lambda x) \subset \lambda S(x)$. To see the other inclusion, suppose that $s \in \lambda S(x)$, then there must exist a $t \in S(x)$ such that $s = \lambda t$. Since $t = \lambda^{-1}s \in S(x)$, this means that $x \in \lambda^{-1}sC$, or $\lambda x \in sC$, so $s \in S(\lambda x)$, i.e. $\lambda S(x) \subset S(\lambda x)$.

2. Let X be a real normed space, and let $C \subset X$ be a convex open set which contains 0. If $x_0 \notin C$, then show that there exits a bounded linear functional $\tilde{\varphi} : X \to \mathbb{R}$ such that

(i) $\tilde{\varphi}(x_0) = 1$;

(ii) $\tilde{\varphi}(v) < 1$ for all $v \in C$.

Hint: Apply Problem 1.

Solution: The idea is to apply the Hahn-Banach Theorem with the sublinear functional from Problem 1. First note that if C is a convex open set which contains 0, then we have $X = \bigcup_{t>0} tC$. Note that we have $\bigcup_{t>0} tC \subset X$. Now let $x \in X$, we need to show that $x \in \rho C$ from some value of $\rho > 0$. Note that the map $t \to tx$ is continuous. Since C is a neighborhood of 0, there exists some $\rho > 0$ such that for all $t \in [-\rho, \rho]$ we have that $tx \in C$, in particular, $\rho x \in C$ or $x \in \rho^{-1}C$. We now can apply Problem 1.

Let $Y = \mathbb{R}x_0 = \{rx_0 : r \in \mathbb{R}\}$. This is clearly a linear subspace of X. Define $\varphi : Y \to \mathbb{R}$ by $\varphi(rx_0) = r$. Then, we have that $\varphi(x_0) = 1$, and the idea will be to extend the functional φ to all of X. We next note that

$$\varphi(y) \leq Q_C(y) \quad \forall y \in Y.$$

If $y = rx_0$ with $r < 0$, then this is immediate since $\varphi(y) = r < 0 \leq Q_C(y)$ since $Q_C(y) \geq 0$ always. If $y = rx_0$ with $r \geq 0$, then we have that

$$\varphi(y) = r \leq rQ_C(x_0) = Q_C(y)$$

here we used that since $x_0 \notin C$, we must have that $Q_C(x_0) \geq 1$.

So we have that φ is a linear functional with $\varphi(y) \leq Q_C(y)$ for all $y \in Y$. By the Hahn-Banach Theorem we can find an extension $\tilde{\varphi} : X \to \mathbb{R}$ such that $\tilde{\varphi}|_Y = \varphi$ and $\tilde{\varphi}(x) \leq Q_C(x)$ for all $x \in X$. Now we obviously have that $\tilde{\varphi}(x_0) = \varphi(x_0) = 1$. And we also have that $\tilde{\varphi}(v) < 1$ for all $v \in C$.

It remains to prove that $\tilde{\varphi}$ is bounded, and since $\tilde{\varphi}$ is linear to do so it will be enough to prove that it is continuous at the origin. Namely, we need to prove that for $\epsilon > 0$ there is some open set U_ϵ containing the origin such that

$$|\tilde{\varphi}(u)| < \epsilon \quad u \in U_\epsilon.$$

Take $U_\epsilon = \epsilon C \cap -\epsilon C$, which is clearly open and contains the origin. Note that for all $u \in U_\epsilon$ we have that $\pm u \in \epsilon C$, or equivalently, $\epsilon^{-1}(\pm u) \in C$. This implies that

$$Q_C(\pm u) < \epsilon$$

and since $\tilde{\varphi}(u) \leq Q_C(u)$ for all u, we have that $|\tilde{\varphi}(u)| < \epsilon$ for all $u \in U_\epsilon$.

3. Let X be a normed vector space over the real numbers. Suppose that $A, B \subset X$ are non-empty convex sets with A open and $A \cap B = \emptyset$. Show that there exists a bounded linear

functional $\phi : X \to \mathbb{R}$ and a real number α such that

$$\phi(a) < \alpha \leq \phi(b) \quad \forall a \in A, b \in B.$$

Hint: Apply Problem 2 with $C = A - B + b_0 - a_0$.

Solution: Fix points $a_0 \in A$ and $b_0 \in B$ and define the set

$$C = A - B + b_0 - a_0 = \{a - b + b_0 - a_0 : a \in A, b \in B\}$$

It is easy to see that C is convex and contains the origin. We also have that

$$C = \bigcup_{b \in B} (A + b_0 - a_0)$$

so C is open. Consider the vector $x_0 = b_0 - a_0$, and since $A \cap B = \emptyset$, we have that $x_0 \notin C$. Now apply Problem 2, to find a bounded linear map $\tilde{\varphi}$ such that $\tilde{\varphi}(x_0) = 1$ and $\tilde{\varphi}(v) < 1$ for all $v \in C$. Now compute that

$$1 > \tilde{\varphi}(a - b + b_0 - a_0) = \tilde{\varphi}(a) - \tilde{\varphi}(b) + \tilde{\varphi}(b_0) - \tilde{\varphi}(a_0)$$

Using that $\tilde{\varphi}(b_0) - \tilde{\varphi}(a_0) = 1$ we have that

$$\tilde{\varphi}(a) < \tilde{\varphi}(b) \quad a \in A, b \in B.$$

Set $\alpha = \inf_{b \in B} \tilde{\varphi}(b)$, then we have that

$$\tilde{\varphi}(a) \leq \alpha \leq \tilde{\varphi}(b) \quad a \in A, b \in B.$$

So, to prove the desired result, we need to simply show that

$$\tilde{\varphi}(a) < \alpha \quad a \in A.$$

Suppose to the contrary that there exists a $a_1 \in A$ such that $\tilde{\varphi}(a_1) = \alpha$. Since the map $t \to a_1 + tx_0$ is continuous, there is some $\epsilon > 0$ such that

$$a_1 + tx_0 \in A \quad \forall t \in [-\epsilon, \epsilon].$$

In particular, we have that

$$\tilde{\varphi}(a_1 + \epsilon x_0) \leq \alpha,$$

which is equivalent to $\alpha + \epsilon \leq \alpha$, which is impossible.

4. Suppose that X is a normed linear space over $\mathbb{C}$. Show that the set of all bounded linear functionals on X, denoted by X^*, is a Banach space with norm given by

$$\|f\|_{X^*} = \sup_{x \neq 0} \frac{|f(x)|}{\|x\|_X}.$$

Solution: It is easy to see that $\| \cdot \|_{X^*}$ is actually a norm. The real issue here is completeness of the space X^*. So suppose that $\{f_n\}$ is a Cauchy sequence in X^*. Namely, $\|f_n - f_m\|_{X^*} \to 0$ as n and m get very large. Fix $x \in X$, and observe that

$$|f_n(x) - f_m(x)| \leq \|x\|_X \|f_n - f_m\|_{X^*}$$

which implies that $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{C}$. We can then define

$$f(x) = \lim_{n\to\infty} f_n(x)$$

since $\mathbb{C}$ is complete. Since limits are linear and each f_n is linear we have that f is a linear functional on X. It only remains to show that f is a bounded linear functional. For any $x \in X$ with $x \neq 0$ we have that for $\epsilon > 0$ that

$$|f(x) - f_N(x)| < \epsilon \|x\|_X$$

when $n \geq N$. Also, since f_n is Cauchy in X^* we have that $\|f_n\|$ is a bounded sequence, i.e. $\sup_n \|f_n\|_{X^*} = C < \infty$. Thus we have that

$$|f(x)| \leq |f(x) - f_N(x) + f_N(x)| \leq (\epsilon + C)\|x\|_X$$

And since ϵ was arbitrary, we have that $|f(x)| \leq C\|x\|_X$, and so f is a bounded linear functional. This then proves that X^* is complete and hence a Banach space.

5. Let H be a Hilbert space over $\mathbb{R}$. Show that H is isometrically isomorphic to H^*.

Solution: This is essentially a Corollary of the Riesz Representation Theorem. We need to find the map that carries H^* to H. So, let $f \in H^*$, then f is a bounded linear functional on H, and so by the Riesz Representation Theorem, we have that there exists a unique $z \in H$ such that

$$f(x) = \langle x, z \rangle_H$$

and moreover $\|f\|_{H^*} = \|z\|_H$. Now define a map from $\imath : H^* \to H$ given by $\imath(f) = z$ where z is the unique element obtained by the Riesz Representation Theorem. We first observe that $\imath$ is linear. If $f, g \in H^*$ then we have that

$$\imath(f + rg) = z_{f+rg} \quad \imath(f) = z_f \quad \imath(rg) = z_{rg}$$

and for any $x \in H$, $\langle x, z_{f+rg} \rangle_H = (f + rg)(x) = f(x) + rg(x) = \langle x, z_f \rangle_H + r \langle x, z_g \rangle_H$, so we have that $\imath(f + rg) = \imath(f) + r\imath(g)$, and $\imath$ is linear. We also have that $\imath$ is injective, since if $\imath(f) = \imath(g) = w$, then we have that $f(x) = \langle x, w \rangle_H = g(x)$ for all $x \in H$, and so $f = g$. It is also true that $\imath$ is surjective, since we have that the linear functional $f_u(x) = \langle x, u \rangle_H$ has the property that $\imath(f_u) = u$. So $\imath : H^* \to H$ is an isomorphism between the linear spaces H^* and H. But, the Riesz Representation Theorem then gives that

$$\|\imath f\|_H = \|z\|_H = \|f\|_{H^*}.$$

6. Let X and Y be Banach spaces and let $T : X \to Y$ be bounded and linear. Show that there is a constant $c > 0$ such that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$ if and only if $\ker T = \{0\}$ and $\operatorname{Ran} T$ is closed.

Solution: Suppose that $\|Tx\|_Y \geq c\|x\|_X$ for all $x \in X$. Then if we have $x \in \ker T$ we have that

$$0 = \|Tx\|_Y \geq c\|x\|_X$$

which gives that $x = 0$, and so $\ker T = \{0\}$. Suppose that $y \in \overline{\operatorname{Ran} T}$, and let $y_n \in \operatorname{Ran} T$ with $y_n \to y$. Since $y_n \in \operatorname{Ran} T$, we have that $y_n = Tx_n$. Note that x_n is a Cauchy sequence since

$$\|y_n - y_m\|_Y = \|Tx_n - Tx_m\|_Y \geq c\|x_n - x_m\|_X.$$

Since X is complete, then we have that $x_n \to x$. But, since T is bounded, hence continuous, we have that $Tx_n \to Tx$ so $Tx = y$, and so $\overline{\operatorname{Ran} T} \subset \operatorname{Ran} T$, and so $\operatorname{Ran} T$ is closed.

Now suppose that $\ker T = \{0\}$ and $\operatorname{Ran} T$ is closed. Note that since T is bounded and linear and $\ker T = \{0\}$ we have that T is injective. Similarly, since $\operatorname{Ran} T$ is closed, we have that T is surjective. So T is bounded linear and bijective, and so by the Open Mapping Theorem (Bounded Inverse Theorem) we have that T^{-1} is bounded,

$$\|T^{-1}y\|_X \leq c\|y\|_Y.$$

Apply this inequality to $y = Tx$ to get the desired result.

7. Let $L^2[0,1]$ be the Lebesgue space of square integrable functions on the unit interval. Similarly, let $L^1[0,1]$ be the space of integrable functions on the unit interval. Prove that $L^2[0,1]$ is of the first category in $L^1[0,1]$.

Solution: Let $E_n = \{f \in L^2[0,1] : \int |f|^2\, dx \leq n\}$, and we have that $L^2[0,1] = \bigcup_{n=1}^{\infty} E_n$. We claim that E_n is closed in $L^1[0,1]$ but is nowwhere dense. This then proves the desired result.

It is easy to see that E_n is closed. If we have a sequence $\{f_k\} \in E_n$ with $f_k \to f$ in L^1, then we can find a subsequence f_{n_k} that converges pointwise almost everywhere. Then as a Corollary to Fatou's Lemma we have that

$$\|f\|^2_{L^2[0,1]} \leq n.$$

Now observe that the interior of E_n is empty. Suppose that this isn't true, then there would exist an $f_0 \in E_n$ and some $r > 0$ such that $D = \{g : \|f_0 - g\|_{L^1[0,1]} < r\} \subset E_n$. Let $f \in L^1$ and consider the function $g = f_0 + \frac{r}{2}\frac{f}{\|f\|_{L^1[0,1]}}$. We have that $g \in D$ and so $G \in E_n$, and so $g \in L^2[0,1]$, rearrangement then gives that

$$f = \frac{2\|f_0\|_1}{r}(g - f_0) \in L^2[0,1].$$

But $f \in L^1[0,1]$ was arbitrary, this implies that $L^2[0,1] = L^1[0,1]$, which is false. So we have that E_n has empty interior. This then gives that E_n is nowhere dense, and so $L^2[0,1]$ is meager in $L^1[0,1]$.

8. Let S be a closed linear subspace of $L^1[0,1]$. Suppose that for each $f \in S$ there exists a $p > 1$ such that $f \in L^p[0,1]$. Show that there exists a $q > 1$ such that $S \subset L^q[0,1]$

Solution: Since $S \subset L^1[0,1]$ and S is closed, it is complete since $L^1[0,1]$ is complete. Let q_n be a sequence decreasing to 1 and

$$E_{n,m} := \{f \in S : f \in L^{q_n}[0,1], \|f\|_{L^{q_n}} \leq m\}$$

where $m \in \mathbb{N}$. Then we have that

$$S = \bigcup_{n,m} E_{n,m}$$

since by hypothesis for each $f \in S$ there is a $p > 1$ such that $f \in L^p[0,1]$ and if $r > s$ then $L^r[0,1] \subset L^s[0,1]$ and choosing $q_n < p$ will place $f \in E_{n,m}$ since the norm of f will have to be smaller than some m.

Now each term in the union is closed, i.e. $E_{n,m} = \overline{E_{n,m}}$. Since if we take a sequence $\{f_k\} \subset E_{n,m}$ with $f_k \to f$, then $f \in E_{n,m}$. To see this, note that we are working in the topology defined by the L^1 norm, and so we can find a subsequence $\{f_{k_j}\}$ that converges to f almost everywhere on $[0,1]$. Then as a Corollary of Fatou's Lemma we have

$$\int_0^1 |f|^{q_n}\, dx \leq m.$$

By Baire's Category Theorem, there exists n_0, m_0 with $q_{n_0} > 1$ such that E_{n_0,m_0} has non-empty interior. So, there exists a ball of radius $\delta > 0$ centered at some point f such that

$$B_\delta(f) \subset E_{n_0,m_0}.$$

For, the moment, lets assume that $f = 0$, the general case can be handled similarly and by translation. Then we have that $B_\delta(0) \subset L^{q_{n_0}}[0,1]$.

Let $0 \neq g \in S$ be any function, then the function $\tilde{g} = \frac{\delta}{2}\frac{g}{\|g\|_{L^1}} \in B_\delta(0)$, and so $\tilde{g} \in L^{q_{n_0}}[0,1]$. But then since $L^{q_{n_0}}[0,1]$ is a linear space, $c\tilde{g} \in L^{q_{n_0}}[0,1]$ and so $g \in L^{q_{n_0}}[0,1]$, which gives that $S \subset L^{q_{n_0}}[0,1]$.