

Problem 1. Zygmund p58 exercise 01

(a) There is an analogue for bases different from 10 of usual decimal expansion of number. If b is an integer larger than 1 and $0 < x < 1$, show that there exist integral coefficient c_k , $0 \leq c_k < b$, such that $x = \sum_{k=1}^{\infty} c_k b^{-k}$. Furthermore, show that expansion is unique unless $x = cb^{-k}$, in which case there are two expansions.

(b) When $b = 3$, the expansion is called the triadic or ternary expansion of x . Show that Cantor set consist of point in $[0, 1]$ which has triadic representation such that c_k is either 0 or 2, namely,

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{k=1}^{\infty} c_k 3^{-k}, c_k \in \{0, 2\}\}.$$

Solution.

(a)

Input: x, b

Output: $\{c_k\}_{k=1}^{\infty}$

1 initialization;

2 **for** $n = 1, 2, \dots, \infty$ **do**

3 \lfloor Let c_n to be the largest integral coefficient, s.t. $\sum_{k=1}^n c_k b^{-k} \leq x$;

4 **return** $\{c_k\}_{k=1}^{\infty}$;

■

⋮

Cantor Set

^aConsider the closed interval $[0, 1]$. The first stage of the construction is to subdivide $[0, 1]$ into thirds and remove the interior of the middle third; that is, remove the open interval $(\frac{1}{3}, \frac{2}{3})$. Each successive step of the construction is essentially the same. Thus, at the second stage, we subdivide each of the remaining two intervals $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$ into thirds and remove the interiors, $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, of their middle thirds. We continue the construction for each of the remaining intervals. The subset of $[0, 1]$ that remains after infinitely many such operations is called the Cantor set C : thus, if C_k denotes the union of the intervals left at the k -th stage, then

$$C = \bigcap_{k=1}^{\infty} C_k.$$

^aRichard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis*. CRC, 2015, pp. 42–43.

Limit Point

A point x is a limit point of the set E if every neighborhood of x contains a point $x \neq y$ such that $y \in E$.^a

In other words, x is a limit point of E if \exists a sequence $\{x_n\} \in E$, s.t. $x_n \rightarrow x$ and $x_n \neq x$.^b

^aW. Rudin. *Principles of Mathematical Analysis*. McGraw-Hill, 1976, p. 32.

^bWheeden and Zygmund, see n. a, pp. 3–4.

Perfect Set

A closed set E is said to be a perfect set if every point of E is a limit point of E .^a

In other words, A closed set E is said to be a perfect set if $\forall x \in E, \forall \epsilon > 0, (B(x, \epsilon) \setminus \{x\}) \cap E \neq \emptyset$.

^aIbid., p. 7.

Theorem 1.7

- (i) The intersection of any number of closed sets is closed.
- (ii) The union of any number of open sets is open.

Cantor Set is perfect

To prove that Cantor Set C is a perfect set, we need to show that it is closed and every point in the set is a limit point of the set.

Since each C_k is closed, it follows from Theorem 1.7 that C is closed.

Then show that every point in C is a limit point of the set:

Case 1. Let $x \in C$ be an endpoint of the interval $I_k \subseteq C_k$. Consider the intervals $I_k^i \subseteq C_{k+i}$ with

endpoint x , let x_1 be the other endpoint of $I_k^1 \subseteq C_{k+1}$, x_2 be the other endpoint of $I_k^2 \subseteq C_{k+2}, \dots$, x_n be the other endpoint of $I_k^n \subseteq C_{k+n}$. Thus, $|x_n - x| = (\frac{1}{3})^{k+n}$.

We have

$$x_n \rightarrow x,$$

$$x_n \neq x,$$

$$x_n \in C.$$

Therefore, x is a limit point of C .

Case 2. Suppose $x \in C$ is not an endpoint of any interval consisting C . $\forall n \in \mathbb{N}$, we have $x \in (a_n, b_n)$, where $a_n \in C$ and $|a_n - x| < (\frac{1}{3})^n$. Let $x_n = a_n$, $\{x_n\}$ is the squence s.t.

$$x_n \rightarrow x,$$

$$x_n \neq x,$$

$$x_n \in C.$$

Thus, x is a limit point of C .

We can conclude that every point in C is a limit point of the set. Therefore, C is perfect.

Problem 2. Zygmund p58 exercise 03

Construct a two-dimensional Cantor set in the unit square $\{(x, y) : 0 \leq x, y \leq 1\}$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which forms a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $C \times C$.

Solution.

Let D_0 be the unit square $\{(x, y) : 0 \leq (x, y) \leq 1\}$. Let D_k be the set remaining after i steps. Let $D = \bigcap_{k=1}^{\infty} D_k$ be the resulting set.

(a) Show that D has plane measure zero:

Since D is covered by the intervals in any D_k , we have

$$|D|_e \leq |D_k|_e = \left(\frac{4}{9}\right)^k.$$

Let $k \rightarrow \infty$, we have $|D|_e = 0$.

(b)

$$\begin{aligned} D &:= \bigcap_{k=1}^{\infty} D_k && \text{(By definition)} \\ &= \bigcap_{k=1}^{\infty} C_k \times C_k && \text{(By definition)} \\ &= \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right) && \text{(To be proved)} \\ &= C \times C. && \text{(By definition)} \end{aligned}$$

To prove that $\bigcap_{k=1}^{\infty} C_k \times C_k = (\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k)$, first show that $\bigcap_{k=1}^{\infty} C_k \times C_k \subseteq (\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k)$:

For all

$$(x, y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$$

we have

$$(x, y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, we have

$$x \in C_k, \forall k \in \mathbb{N},$$

$$y \in C_k, \forall k \in \mathbb{N}.$$

Thus,

$$x \in \bigcap_{k=1}^{\infty} C_k,$$

$$y \in \bigcap_{k=1}^{\infty} C_k.$$

Therefore,

$$(x, y) \in \left(\bigcap_{k=1}^{\infty} C_k \right) \times \left(\bigcap_{k=1}^{\infty} C_k \right), \forall (x, y) \in D,$$

which means

$$\bigcap_{k=1}^{\infty} C_k \times C_k \subseteq \left(\bigcap_{k=1}^{\infty} C_k \right) \times \left(\bigcap_{k=1}^{\infty} C_k \right).$$

Next, prove that $(\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k) \subseteq \bigcap_{k=1}^{\infty} C_k \times C_k$. For all

$$(x, y) \in \left(\bigcap_{k=1}^{\infty} C_k \right) \times \left(\bigcap_{k=1}^{\infty} C_k \right)$$

we have

$$x \in C_k, \forall k \in \mathbb{N},$$

$$y \in C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, this implies that

$$(x, y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

Since it satisfies the definition of intersection, we have $(x, y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$, implying that

$$\left(\bigcap_{k=1}^{\infty} C_k \right) \times \left(\bigcap_{k=1}^{\infty} C_k \right) \subseteq \bigcap_{k=1}^{\infty} C_k \times C_k.$$

Therefore, we can conclude that the two sets are equal:

$$\bigcap_{k=1}^{\infty} C_k \times C_k = \left(\bigcap_{k=1}^{\infty} C_k \right) \times \left(\bigcap_{k=1}^{\infty} C_k \right).$$

(c) Prove that it is a perfect set:

Since C is perfect, $D = C \times C$ is perfect, by the property of perfect set.

Cartesian product of two perfect sets

To prove that the Cartesian product of two perfect sets in \mathbb{R} is a perfect set in \mathbb{R}^2 :

Let E_1 and E_2 be perfect sets in \mathbb{R} , consider $E = E_1 \times E_2$, which is the product of E_1 and E_2 .

Since both E_1 and E_2 are perfect sets, they are closed in \mathbb{R} . The product of two closed sets is also closed. Thus, $E = E_1 \times E_2$ is closed in \mathbb{R}^2 .

Suppose $(x, y) \in E = E_1 \times E_2$, by definition, $x \in E_1$, and $y \in E_2$. Since both E_1 and E_2 are perfect sets, every point in the set is the limit point of the set respectively. In other words, $\forall x \in E_1$, we have a sequence $\{x_n\}$ s.t.

$$x_n \rightarrow x, x_n \neq x, x_n \in E_1.$$

Similarly, $\forall y \in E_2$, we have a sequence $\{y_n\}$ s.t.

$$y_n \rightarrow y, y_n \neq y, y_n \in E_2.$$

Thus, $\forall (x, y) \in E = E_1 \times E_2$, we have a sequence $\{(x_n, y_n)\}$ s.t.

$$(x_n, y_n) \rightarrow (x, y),$$

$$(x_n, y_n) \neq (x, y),$$

$$(x_n, y_n) \in E.$$

This means that every point in E is a limit point of E . Hence, we can conclude that E is perfect.

It is proved that the product of two perfect sets in \mathbb{R} is a perfect set.

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Problem 3. Zygmund p59 exercise 04

Construct a subset of $[0, 1]$ in the same manner as the Cantor set by removing from each remaining interval a subinterval of relative length $\theta, 0 < \theta < 1$. Show that the resulting set is perfect and has measure zero.

Solution.

If C'_k denotes the union of the intervals left at the k -th stage, then the resulting set is

$$C' = \bigcap_{k=1}^{\infty} C'_k.$$

(a) To prove that the set C' is a perfect set, we need to show that it is closed and every point in the set is a limit point of the set.

Since each C'_k is closed, it follows from Theorem 1.7 that C' is closed.

Then show that every point in C' is a limit point of the set:

Case 1. Let $x \in C'$ be an endpoint of $I_k \subseteq C'_k$. Consider the intervals $I_k^i \subseteq C'_{k+i}$ with endpoint x , let x_1 be the other endpoint of $I_k^1 \subseteq C'_{k+1}$, x_2 be the other endpoint of $I_k^2 \subseteq C'_{k+2}$, ..., x_n be the other endpoint of $I_k^n \subseteq C'_{k+n}$. Thus, $|x_n - x| = (\frac{1}{3})^{k+n}$.

We have the sequence $\{x_n\}$ s.t.

$$x_n \rightarrow x,$$

$$x_n \neq x,$$

$$x_n \in C'.$$

Therefore, x is a limit point of C' .

Case 2. Suppose $x \in C'$ is not an endpoint of any interval consisting C' . $\forall n \in \mathbb{N}$, we have $x \in (a_n, b_n)$, where $a_n \in C'$ and $|a_n - x| < (\frac{1-\theta}{2})^n$. Let $x_n = a_n$, then $\{x_n\}$ is the sequence s.t.

$$x_n \rightarrow x,$$

$$x_n \neq x,$$

$$x_n \in C'.$$

Thus, x is a limit point of C' .

We can conclude that every point in C' is a limit point of the set. Therefore, C' is perfect.

(b) Show that C' has plane measure zero:

Since C' is covered by the intervals in any C'_k , we have

$$|C'|_e \leq |C'_k|_e = (1 - \theta)^k.$$

Let $k \rightarrow \infty$, we have $|C'|_e = 0$.

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