MATH 6337: Homework 8 Solutions

6.1.

(a) Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}$, $\{y : (x, y) \in E\}$ has $\mathbb{R}$-measure zero. Show that E has measure zero and that for almost every $y \in \mathbb{R}$, $\{x : (x, y) \in E\}$ has measure zero.

(b) Let $f(x, y)$ be nonnegative and measurable in $\mathbb{R}^2$. Suppose that for almost every $x \in \mathbb{R}$, $f(x, y)$ is finite for almost every y. Show that for almost every $y \in \mathbb{R}$, $f(x, y)$ is finite for almost every x.

Solution.

(a) Consider the indicator function $\mathbb{1}_E$ on $\mathbb{R}^2$. This is a nonnegative measurable function, so by Fubini/Tonelli we have

$$\iint_{\mathbb{R}^2} \mathbb{1}_E = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{1}_E(x, y)\, dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{1}_E(x, y)\, dx \right) dy.$$

For almost every $x \in \mathbb{R}$, we have

$$\int_{\mathbb{R}} \mathbb{1}_E(x, y)\, dy = |\{y : (x, y) \in E\}| = 0,$$

so all the integrals in the first line above are 0. Since

$$\iint_{\mathbb{R}^2} \mathbb{1}_E = |E|,$$

we have $|E| = 0$. Moreover, since

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \mathbb{1}_E(x, y)\, dx \right) dy = \int_{\mathbb{R}} |\{x : (x, y) \in E\}|\, dy$$

and since $|\{x : (x, y) \in E\}|$ is a nonnegative function of y whose integral over y is zero, it follows that $|\{x : (x, y) \in E\}| = 0$ for almost every y.

(b) Let E be the set of points on which f is not finite. This is a measurable subset of $\mathbb{R}^2$, and $E_x = \{y : (x, y) \in E\}$ has measure zero for almost every $x \in \mathbb{R}$, so by part (a) the result immediately follows.

$\square$

6.2. If f and g are measurable in $\mathbb{R}^n$, show that the function $h(x,y) = f(x)g(y)$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$. Deduce that if E_1 and E_2 are measurable subsets of $\mathbb{R}^n$, then their Cartesian product $E_1 \times E_2 = \{(x,y) : x \in E_1, y \in E_2\}$ is measurable in $\mathbb{R}^n \times \mathbb{R}^n$, and $|E_1 \times E_2| = |E_1|\,|E_2|$.

Solution. Fix a measurable function f, and let

$$\mathcal{G}_f = \{g : \mathbb{R}^n \to \mathbb{R}, \text{ is measurable} : f(x)g(y) \text{ is measurable}\}.$$

Let $E \subseteq \mathbb{R}^n$ be a measurable set, and let $\mathbb{1}_E$ be its indicator function. If $h(x,y) = f(x)\mathbb{1}_E(y)$, then

$$h^{-1}(a,\infty) = \begin{cases} E^c \cup f^{-1}(a,\infty), & a < 0 \\ E \cap f^{-1}(a,\infty), & a \geq 0 \end{cases}$$

is measurable for all a, so h is measurable; hence, $\mathcal{G}_f$ contains the indicator functions of measurable subsets of $\mathbb{R}^n$. Moreover, if $a,b \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{G}_f$, then $f(x) \cdot (ag_1(y) + bg_2(y)) = af(x)g_1(y) + bf(x)g_2(y)$, a sum of measurable functions; hence, $\mathcal{G}_f$ is closed under linear combinations. Finally, if $g_1, g_2, \ldots$ is an increasing sequence of functions in $\mathcal{G}_f$ which converges to the function g, then $f(x)g_1(y), f(x)g_2(y), \ldots$ is an increasing sequence of measurable functions converging to $f(x)g(y)$. Since a limit of measurable functions is measurable, $f(x)g(y) \in \mathcal{G}_f$; hence, $\mathcal{G}_f$ is closed under increasing limits. Any collection of functions on $\mathbb{R}^n$ satisfying these three properties contains all measurable functions on $\mathbb{R}^n$, so $\mathcal{G}_f$ contains all measurable functions. Since this is true for all measurable f, it follows that $f(x)g(y)$ is measurable for all measurable f and g.*

Alternatively, we could let $F(x,y) = f(x)$ and $G(x,y) = g(y)$. Then $\{(x,y) : F(x,y) > a\} = \{x : f(x) > a\} \times \mathbb{R}^n$ and $\{(x,y) : G(x,y) > a\} = \mathbb{R}^n \times \{y : g(y) > a\}$. These sets are measurable via repeated application of a lemma proved in Chapter 5 that $E \times \mathbb{R}$ is measurable for all measurable $E \subseteq \mathbb{R}^d$.†

Given this result, take $f = \mathbb{1}_{E_1}$ and $g = \mathbb{1}_{E_2}$. Then $h(x,y) = \mathbb{1}_{E_1 \times E_2}(x,y) = f(x)g(y)$ is measurable, so $E_1 \times E_2$ is measurable. Moreover, via Fubini/Tonelli we have

$$|E_1 \times E_2| = \iint_{\mathbb{R}^2} \mathbb{1}_{E_1 \times E_2} = \iint_{\mathbb{R}^2} \mathbb{1}_{E_1}(x)\mathbb{1}_{E_2}(y)\,dx\,dy = \int_{\mathbb{R}} \mathbb{1}_{E_1}(x) \int_{\mathbb{R}} \mathbb{1}_{E_2}(y)\,dy\,dx = |E_1|\,|E_2|.$$

□

*This is a common proof technique in analysis called a *monotone class argument*. It will come up again later in the section on abstract measure theory.

†Since the purpose of the problem is to show that the product of two general measurable sets is measurable, you cannot just cite that result to say that $F^{-1}(a,\infty)$ and $G^{-1}(a,\infty)$ are measurable.

6.4. Let f be measurable and periodic with period 1: $f(t+1) = f(t)$. Suppose that there is a finite c such that

$$\int_0^1 |f(a+t) - f(b+t)|\, dt \le c$$

for all a and b. Show that $f \in L(0,1)$. [Hint: Set $a = x$, $b = -x$, integrate with respect to x, and make the change of variables $\xi = x + t$, $\eta = -x + t$.]

Solution.

$$\begin{aligned} c &\ge \int_0^1 \int_0^1 |f(x+t) - f(-x+t)|\, dt\, dx \\ &= \frac{1}{2}\int_0^1 \int_{-\xi}^{\xi} |f(\xi) - f(\eta)|\, d\eta\, d\xi + \frac{1}{2}\int_1^2 \int_{\xi-1}^{1-\xi} |f(\xi) - f(\eta)|\, d\eta\, d\xi. \end{aligned}$$

By the periodicity of f, each integral will double in size by integrating η over the entire interval $[-1, 1]$. Thus,

$$\int_0^2 \int_{-1}^1 |f(\xi) - f(\eta)|\, d\eta\, d\xi \le 2c,$$

so (again by periodicity)

$$\int_0^1 \int_0^1 |f(\xi) - f(\eta)|\, d\eta\, d\xi \le \frac{c}{2}.$$

Thus, $|f(\xi) - f(\eta)|$ (hence $f(\xi) - f(\eta)$) is integrable over the square $[0,1] \times [0,1]$. By the result of Exercise 6.3, f is integrable over $(0,1)$.[‡] $\square$

[‡]Proof of 6.3: ...

6.5.

(a) If f is nonnegative and measurable on E and $\omega(y) = |\{x \in E : f(x) > y\}|$, $y > 0$, use Tonelli's theorem to prove that $\int_E f = \int_0^\infty \omega(y)\,dy$. [Hint: By definition of the integral, we have $\int_E f = |R(f,E)| = \iint_{R(f,E)} dx\,dy$. Use the observation in the proof of (6.11) that $\{x \in E : f(x) \ge y\} = \{x : (x,y) \in R(f,E)\}$, and recall that $\omega(y) = |\{x \in E : f(x) \ge y\}|$ unless y is a point of discontinuity of ω.]

(b) Deduce from this special case the general formula

$$\int_E f^p = p\int_0^\infty y^{p-1}\omega(y)\,dy \qquad (f \ge 0, 0 < p < \infty).$$

Solution.

(a) Since ω is a monotone decreasing function on $\mathbb{R}$, it has only countably many points of discontinuity, so $\omega(y) = |\{x \in E : f(x) \ge y\}|$ for almost all y. Thus,

$$\begin{aligned}
\int_0^\infty \omega(y)\,dy &= \int_0^\infty |\{x \in E : f(x) \ge y\}|\;dy \\
&= \int_0^\infty |\{x : (x,y) \in R(f,E)\}|\;dy \\
&= \int_0^\infty \int_{R(f,E)_y} dx\,dy \\
&= \iint_{R(f,E)} 1 = |R(f,E)| = \int_E f.
\end{aligned}$$

(See Theorem (6.8) on page 90 of the text for why $\int_0^\infty \int_{R(f,E)_y} dx\,dy = \iint_{R(f,E)} 1$.)

(b) Let $\omega_f(y)$ be the distribution function for f, and let $\omega_{f^p}(y)$ be the same for f^p. Then we have

$$\int_E f^p = \int_0^\infty \omega_{f^p}(y)\,dy = \int_0^\infty \omega_f(y^{1/p})\,dy.$$

Make the change of variables $u = y^{1/p}$. Then $du = \frac{1}{p}y^{(p-1)/p} = \frac{1}{p}u^{p-1}$, so

$$\int_E f^p = p\int_0^\infty u^{p-1}\omega_f(u)\,du.$$

$\square$

6.6. For $f \in L(\mathbb{R})$, define the *Fourier transform* $\widehat{f}$ of f by

$$\widehat{f}(x) = \int_{-\infty}^{+\infty} f(t)e^{-ixt}\,dt \qquad (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i\int F_1$.) Show that if f and g belong to $L(\mathbb{R})$, then

$$\widehat{f * g}(x) = \widehat{f}(x)\widehat{g}(x).$$

Solution.

$$\widehat{f * g}(x) = \int_{\mathbb{R}} \left[\int_{\mathbb{R}} f(u-t)g(t)\,dt\right] e^{-ixu}\,du = \int_{\mathbb{R}} g(t)\left[\int_{\mathbb{R}} f(u-t)e^{-ixu}\,du\right] dt,$$

where the second equality follows from Fubini. Then we have

$$\int_{\mathbb{R}} g(t)\left[\int_{\mathbb{R}} f(u-t)e^{-ixu}\,du\right] dt = \int_{\mathbb{R}} g(t)e^{-ixt}\left[\int_{\mathbb{R}} f(u-t)e^{-ix(u-t)}\,du\right] dt.$$

Changing variables $v = u - t$, we have

$$\int_{\mathbb{R}} g(t)e^{-ixt}\left[\int_{\mathbb{R}} f(u-t)e^{-ix(u-t)}\,du\right] dt = \int_{\mathbb{R}} g(t)e^{-ixt}\left[\int_{\mathbb{R}} f(v)e^{-ixv}\,dv\right] dt =$$
$$\left(\int_{\mathbb{R}} g(t)e^{-ixt}\,dt\right)\left(\int_{\mathbb{R}} f(v)e^{-ixv}\,dv\right) = \widehat{f}(x)\widehat{g}(x).$$

$\square$

6.10. Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$v_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\,dt.$$

[We also observe that the integral can be expressed in terms of the Γ-function: $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}\,dt, s > 0$.]

Solution. We proceed by induction, noting *a priori* that $v_1 = 2$. For $n = 2$, we have $v_2 = \pi$, which jibes with

$$2\cdot v_1 \cdot \int_0^1 \sqrt{1-t^2}\,dt = 4\cdot\frac{\pi}{4} = \pi.^{\S}$$

Now suppose the formula holds for the case $n-1$. Recall that the n-ball B^n is defined by $x_1^2 + \cdots + x_n^2 \le 1$. We have

$$v_n = \int\!\!\cdots\!\!\int_{B^n} 1 = \int\!\!\cdots\!\!\int_{x_1^2+\cdots+x_n^2\le 1} dx_1\cdots dx_n = \int_{-1}^1 \int\!\!\cdots\!\!\int_{x_2^2+\cdots+x_n^2\le 1-x_1^2} dx_1\,dx_2\cdots dx_n$$

Defining $y_j = x_j/\sqrt{1-x_1^2}$ for $j = 2,...,n$, and noting that $dy_j = dx_j/\sqrt{1-x_1^2}$, we make a change of variables:

$$\int_{-1}^1 \int\!\!\cdots\!\!\int_{y_2^2+\cdots+y_n^2\le 1} (1-x_1^2)^{(n-1)/2} dx_1\,dy_2\cdots dy_n = 2v_{n-1}\int_0^1 (1-t^2)^{(n-1)/2}\,dt.$$

Since all functions involved are bounded on the compact domain of integration, they are all integrable, so Fubini/Tonelli justifies the swapping of integrals. The last inequality is justified by the fact that the integrand is even and that $y_2^2 + \cdots + y_n^2 \le 1$ defines an $(n-1)$-ball of radius 1. □

[§] In case you're not familiar with this integral, you can compute it with the trig substitution $t = \cos\theta$. Geometrically, just observe that it is the area under the graph of the portion of the unit circle in the first quadrant of the plane.

6.11. Use Fubini's theorem to prove that

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \pi^{n/2}.$$

[Hint: For $n = 1$, write $\left(\int_{-\infty}^{+\infty} e^{-x^2}\, dx\right)^2 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2}\, dx\, dy$ and use polar coordinates. For $n > 1$, use the formula $e^{-|x|^2} = e^{-x_1^2} \cdots e^{-x_n^2}$ and Fubini's theorem to reduce to the case $n = 1$.]

Solution. Following the hint, in the case $n = 1$ we have

$$\left(\int_{\mathbb{R}} e^{-x^2}\, dx\right)^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-x^2-y^2}\, dx\, dy = \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\, dr\, d\theta = \pi \int_0^{\infty} e^{-u}\, du = \pi,$$

so

$$\int_{\mathbb{R}} e^{-x^2}\, dx = \pi^{1/2}.$$

Assume by induction that the proposition holds true for $n-1$; we'll show the case n. Write $x = (x_1, ..., x_n)$. Then we have, via induction and Fubini/Tonelli,

$$\int_{\mathbb{R}^n} e^{-|x|^2}\, dx = \int_{\mathbb{R}^n} \prod_{j=1}^{n} e^{-x_j^2} = \left(\int_{\mathbb{R}} e^{-x_1^2}\, dx_1\right)\left(\int_{\mathbb{R}^{n-1}} \prod_{j=2}^{n} e^{-x_j^2}\, dx_2 \cdots dx_n\right) =$$

$$\pi^{1/2}\pi^{(n-1)/2} = \pi^{n/2}.$$

$\square$