MATH 6337: Homework 5 Solutions

4.14. Let $f(x,y)$ be as in Exercise 13[*]. Show that, given $\varepsilon > 0$, there exists a closed set $F \subseteq E$ with $|E \setminus F| < \varepsilon$ such that $f(x,y)$ converges uniformly for $x \in F$ to $f(x)$ as $y \to 0$. [Hint: Follow the proof of Egorov's theorem, using the sets $E_{\varepsilon,1/m}$ defined in Exercise 13[†] for the sets E_m of (4.18).]

Solution. Fix $\varepsilon, \eta > 0$. For each $m \geq 1$, let

$$E_m := E_{\varepsilon,1/m} = \{|f(x,y) - f(x)| \leq \varepsilon \text{ for all } y < 1/m\}.$$

As showed in Problem 4.13, these sets are all measurable. Also, $E_m \subseteq E_{m+1}$. Since $f(x,y) \to f(x)$ for all $x \in E$ as $y \to 0$, $E_m \nearrow E$,[‡] so $|E_m| \nearrow |E|$. Moreover, since $|E| \leq 1 < +\infty$, $|E \setminus E_m| \searrow 0$. Choose m_0 such that $|E \setminus E_{m_0}| < \eta/2$, and choose a closed subset $F \subseteq E$ such that $|E_{m_0} \setminus F| < \eta/2$. Then F is a closed subset so that $|E \setminus F| < \eta$ and $|f(x,y) - f(x)| \leq \varepsilon$ for $y < 1/m_0$.

Now, given $\varepsilon > 0$, select closed sets $F_m \subset E$ and integers $M_{\varepsilon,m}$ for each $m \geq 1$ using the above procedure such that $|E \setminus F_m| < \varepsilon 2^{-m}$ and such that $|f(x,y) - f(x)| \leq 1/m$ in F_m if $y < 1/M_{\varepsilon,m}$. Then $F := \bigcap_{m=1}^{\infty} F_m$ is a closed set on which $f(x,y) \to f(x)$ uniformly. Moreover, $E \setminus F = \bigcup E \setminus F_m$, so $|E \setminus F| \leq \sum |E \setminus F_m| < \varepsilon$. $\square$

[*] That is, $f(x,y)$ is defined and continuous on $0 \leq x \leq 1$ and $0 < y \leq 1$, and that $f(x) = \lim_{y \to 0} f(x)$ exists and is finite for x in a measurable subset $E \subseteq [0,1]$.

[†] $E_{\varepsilon\delta} := \{x \in E : |f(x,y) - f(x)| \leq \varepsilon \text{ for all } y < \delta\}$

[‡] By definition, E is the set on which $f(x,y) \to f(x)$ as $y \to 0$, so there's no need to say that $E_m \nearrow E \setminus Z$ for some set Z of zero measure.

4.15. Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < +\infty$. If $|f_k(x)| \leq M_x < +\infty$ for all k for each x, show that given $\varepsilon > 0$ there is a closed $F \subseteq E$ and a finite M such that $|E \setminus F| < \varepsilon$ and $|f_k(x)| \leq M$ for all k and all $x \in F$.

Solution. If $f(x) = \sup_k |f_k(x)| \leq M_x$ for all $x \in E$, then f is a measurable function on E. Given $\varepsilon > 0$, since f is measurable and finite on E, use Lusin's theorem to choose a closed set F so that $|E \setminus F| < \varepsilon/2$ and $f|_F$ is continuous. Since E is finite in measure, choose $R > 0$ so that $\left|E \setminus \overline{B_R(0)}\right| < \varepsilon/2$.[§] Then $F \cap \overline{B_R(0)}$ is a compact set on which f is continuous, so f obtains a maximum value M on that set. It follows that

$$\left|E \setminus (F \cap \overline{B_R(0)})\right| < \varepsilon,$$

where $F \cap \overline{B_R(0)}$ is a closed set on which $|f_k| \leq f \leq M$.

An alternate proof (which doesn't require a compactness argument): Define $E_m = \{f \leq m\}$, for $m \in \mathbb{N}$, where $f(x) = \sup_k f_k(x)$. Then $E_m \nearrow E$ since f is finite-valued. Thus, $|E_m| \nearrow |E|$, and since $|E| < +\infty$, $|E \setminus E_m| \searrow 0$, so given $\varepsilon > 0$ there exists M such that $|E \setminus E_M| < \varepsilon/2$. Moreover, there is a closed set $F \subseteq E_M$ such that $|E_M \setminus F| < \varepsilon/2$. Therefore, $|E \setminus F| < \varepsilon$, and $|f_k| \leq M$ for all k. □

4.16. Prove that $f_k \xrightarrow{m} f$ on E if and only if given $\varepsilon > 0$ there exists K such that $|\{|f - f_k| > \varepsilon\}| < \varepsilon$ if $k > K$. Give an analogous Cauchy criterion.

Solution. If $f_k \xrightarrow{m} f$ on E, then for all $\alpha, \varepsilon > 0$ there exists K such that $|\{|f - f_k| > \alpha\}| < \varepsilon$ if $k > K$. Taking $\alpha = \varepsilon$ proves the forward direction. For the reverse direction, given $\alpha, \varepsilon > 0$, let K_α be such that $|\{|f - f_k| > \alpha\}| < \alpha$ if $k > K_\alpha$, and define K_ε similarly. Let $\delta = \min(\alpha, \varepsilon)$, and let $k > \max(K_\alpha, K_\varepsilon)$. Then

$$\{|f - f_k| > \alpha\} \subset \{|f - f_k| > \delta\},$$

so we have

$$|\{|f - f_k| > \alpha\}| \leq |\{|f - f_k| > \delta\}| < \delta \leq \varepsilon,$$

so $f_k \xrightarrow{m} f$.

The analogous Cauchy criterion states that f_k converges in measure to some function f if and only if for every $\varepsilon > 0$ there exists K such that whenever $k, \ell > K$, we have

$$|\{|f_\ell - f_k| > \varepsilon\}| < \varepsilon.$$

□

[§]We need this step because it's possible to have unbounded sets of finite measure: consider $F = \bigcup_{j=1}^\infty [j, j + 1/j^2]$, which has measure $\pi^2/6$. However, we can restrict ourselves to some ball about the origin at the loss of ε measure because $|E| = \sum_{k=1}^\infty |(E \cap B_k(0)) \setminus B_{k-1}(0)|$, a convergent infinite sum, so the tails of the sum must converge to 0.

4.17. Suppose that $f_k \xrightarrow{m} f$ and $g_k \xrightarrow{m} g$ on E. Show that $f_k + g_k \xrightarrow{m} f + g$ on E and, if $|E| < +\infty$, that $f_k g_k \xrightarrow{m} fg$ on E. If, in addition, $g_k \to g$ on E with $g \neq 0$ a.e. and $|E| < +\infty$, show that $f_k/g_k \xrightarrow{m} f/g$ on E. [For the product $f_k g_k$, write

$$f_k g_k - fg = (f_k - f)(g_k - g) + f(g_k - g) + g(f_k - f).$$

Consider each term separately, using the fact that a function which is finite on E, $|E| < +\infty$, is bounded outside a subset of E with small measure.]

Solution. Let $\varepsilon > 0$, and let K be such that $|\{|f_k - f| > \varepsilon/2\}|, |\{|g_k - g| > \varepsilon/2\}| < \varepsilon/2$ whenever $k > K$. Then

$$|\{|(f_k + g_k) - (f + g)| > \varepsilon\}| \leq |\{|f_k - f| > \varepsilon/2\}| + |\{|g_k - g| > \varepsilon/2\}| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now suppose $|E| < +\infty$. Fix $M > 0$ such that we have

$$|\{|f| > M\}|, |\{|g| > M\}| < \varepsilon/6.$$

Now choose K such that

$$\left|\left\{|f_k - f| > \sqrt{\varepsilon/3}\right\}\right|, \left|\left\{|g_k - g| > \sqrt{\varepsilon/3}\right\}\right| < \varepsilon/6$$

and

$$|\{|f_k - f| > \varepsilon/3M\}|, |\{|g_k - g| > \varepsilon/3M\}| < \varepsilon/6.$$

By the pigeonhole principle, we have

$$\{|f_k g_k - fg| > \varepsilon\} \subset \{|f_k - f|\,|g_k - g| > \varepsilon/3\} \cup \{|f|\,|g_k - g| > \varepsilon/3\} \cup \{|g|\,|(f_k - f)| > \varepsilon/3\}.$$

Moreover

$$\{|f_k - f|\,|g_k - g| > \varepsilon/3\} \subset \left\{|f_k - f| > \sqrt{\varepsilon/3}\right\} \cup \left\{|g_k - g| > \sqrt{\varepsilon/3}\right\},$$
$$\{|f|\,|g_k - g| > \varepsilon/3\} \subset \{|f| > M\} \cup \{|g_k - g| > \varepsilon/3M\},$$

and

$$\{|g|\,|f_k - f| > \varepsilon/3\} \subset \{|g| > M\} \cup \{|f_k - f| > \varepsilon/3M\}.$$

It follows that $|\{|f_k g_k - fg| > \varepsilon\}| < 6\varepsilon/6 = \varepsilon$, as desired.

Let $h_k = 1/g_k$, so that $h_k \to h = 1/g$ a.e. in E. Since $|E| < +\infty$, it follows that $h_k \xrightarrow{m} h$.[¶] Then, applying the previous result, we have $f_k/g_k = f_k h_k \xrightarrow{m} fh = f/g$. □

[¶]See Theorem 4.21 on page 59 of the textbook.