Problem 1. Zygmund p77 exercise 11

Let f be defined on $\mathbb{R}^n$, and let $B(x)$ denote the open ball $\{y : |x-y| < r\}$ with center x and fixed radius r. Show that the function $g(x) = \sup\{f(y) : y \in B(x)\}$ is lsc (lower semi-continuous), and that the function $h(x) = \inf\{f(y) : y \in B(x)\}$ is usc (upper semi-continuous) on $\mathbb{R}^n$. Is the same true for the closed ball $\{y : |x-y| \leq r\}$?

(a)

lsc

(b)

usc

Problem 2. Zygmund p77 exercise 12

If $f(x)$, $x \in \mathbb{R}^1$, is continuous at almost every point of an interval $[a, b]$, show that f is measurable on $[a, b]$.

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Generalize this to functions defined in $\mathbb{R}^n$.
For a constructive proof, use the subintervals of a sequence of partitions to define a sequence of simple measurable functions converging to f a.e. in $[a, b]$. Use Theorem 4.12. See also the proof of Theorem 5.54.

Assume $f(x)$ is continuous at almost every point of $[a, b]$. Let $E = \{x \in [a, b] : f$ is continuous at $x\}$. Let $Z = [a, b] \setminus E$. Then $|Z| = 0$. Note that Z is measurable, and $E = [a, b] \setminus Z$ is measurable too. For any finite α, we have $\{x \in [a, b] : f(x) > \alpha\} = \{x \in E : f(x) > \alpha\} \cup \{x \in Z : f(x) > \alpha\}$. Note that $\{x \in E : f(x) > \alpha\}$ is measurable since f is continuous, thus measurable on E. Since $\{x \in Z : f(x) > \alpha\} \subset Z$, thus $|\{x \in Z : f(x) > \alpha\}| = 0$. Hence $\{x \in Z : f(x) > \alpha\}$ is also measurable. Thus $\{x \in [a, b] : f(x) > \alpha\}$ is measurable. This means f is measurable on $[a, b]$.

Problem 3. Zygmund p78 exercise 15

Let $\{f_k\}$ be a sequence of measurable functions defined on a measurable set E with $|E| < +\infty$. If $|f_k(x)| \le M_x < +\infty$ for all k for each $x \in E$, show that given $\varepsilon > 0$, there is a closed set $F \subset E$ and a finite M such that $|E - F| < \varepsilon$ and $|f_k(x)| \le M$ for all k and all $x \in F$.

Let $\epsilon > 0$. For each $n \in \mathbb{N}$, define

$$E_n := \{x \in E : |f_k(x)| \le n \text{ for all } k\} = \bigcap_{k=1}^{\infty} \{x \in E : |f_k(x)| \le n\}.$$

Note that each E_n is measurable since each f_k is measurable. Since each $M_x < \infty$, we have that $E_n \subseteq E$. By the Monotone Convergence Theorem for measure, $\lim_{n\to\infty} |E_n| = |E| < \infty$. Thus, there exists N such that

$$|E| - |E_M| = |E \setminus E_M| < \epsilon/2.$$

Let F be a closed set contained in E_M such that $|E_M \setminus F| < \epsilon/2$. Then $|E \setminus F| = |E \setminus E_M| + |E_M \setminus F| < \epsilon$, and $|f_k(x)| \le N$ for all k and all $x \in F$.