MATH 6337: HOMEWORK 7 SOLUTIONS

5.12. Give an example of a bounded, continuous function f on $(0,\infty)$ such that $\lim_{x\to\infty} f(x) = 0$ but $f \notin L^p(0,\infty)$ for any $p > 0$.

Solution. An example is

$$f(x) = \begin{cases} 1, & x \le e \\ \frac{1}{\ln(x)}, & x \ge e. \end{cases}$$

This function clearly satisfies the conditions of boundedness, continuity, and decay to 0 as $x \to \infty$. Observe that, for every $p > 0$, we have $\ln(x) \le x^{1/p}$ for x larger than some number K_p. Thus,

$$\int_{K_p}^{\infty} \frac{dx}{\ln(x)^p} \ge \int_{K_p}^{\infty} \frac{dx}{x} = +\infty,$$

so $f(x)$ cannot be in $L^p(0,\infty)$ for any $p > 0$. $\square$

5.15. Suppose that f is nonnegative and measurable on E and that ω is finite on $(0,\infty)$. If $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ is finite, show that

$$\lim_{a\to 0^+} a^p\omega(a) = \lim_{b\to+\infty} b^p\omega(b) = 0.$$

[Hint: Consider the integrals over $(a/2,a)$ and $(b/2,b)$.]

Solution. Since the integral $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ converges, we know that for all $\varepsilon > 0$ there exists a such that

$$\int_{a/2}^a \alpha^{p-1}\omega(\alpha)\,d\alpha \le \int_0^a \alpha^{p-1}\omega(\alpha)\,d\alpha < \varepsilon/2^p,$$

as tails of convergent integrals must converge to 0. Since α^{p-1} is a monotone increasing function and since $\omega(\alpha)$ is monotone decreasing, we have

$$(a/2)^p\omega(a) \le \int_{a/2}^a \alpha^{p-1}\omega(\alpha)\,d\alpha < \varepsilon/2^p,^*$$

so

$$a^p\omega(a) < \varepsilon.$$

Thus, $a^p\omega(a) \to 0$ as $a \to 0^+$.

The argument for $b^p\omega(b)$ is similar: since $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ converges, we know that for all $\varepsilon > 0$ there exists b such that

$$\int_{b/2}^b \alpha^{p-1}\omega(\alpha)\,d\alpha \le \int_{b/2}^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha < \varepsilon/2^p.$$

Since α^{p-1} is a monotone increasing function and since $\omega(\alpha)$ is monotone decreasing, we have

$$(b/2)^p\omega(b) \le \int_{b/2}^b \alpha^{p-1}\omega(\alpha)\,d\alpha < \varepsilon/2^p,$$

so

$$b^p\omega(b) < \varepsilon.$$

Thus, $b^p\omega(b) \to 0$ as $b \to +\infty$.† □

*The extra factor of $a/2$ comes from the length of the interval we are integrating over.

†It is not permissible to say "$\lim_{b\to+\infty} b^p\omega(b) = 0$ because $0\cdot\infty = 0$." This is a convention used in a previous problem to make the statement of the problem more concise: it is not a hard-and-fast rule of limits. The goal of this problem is to show that $\omega(b)$ approaches zero more rapidly than b^p grows to ∞.

5.16. Suppose that f is nonnegative and measurable on E and that ω is finite on $(0,\infty)$. Show that (5.51)—if $f \geq 0$ and $f \in L^p(E)$ ($|E|$ assumed finite), then $\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$—holds without any further restrictions—that is, f need not be in $L^p(E)$ and $|E|$ need not be finite—if we interpret

$$\int_0^\infty \alpha^p \, d\omega(\alpha) = \lim_{\substack{a\to 0^+\\ b\to+\infty}} \int_a^b \alpha^p \, d\omega(\alpha).$$

[Hint: Use $E_{ab} := \{x \in E : a < f(x) \leq b\}$, where a and b are finite, to obtain the relation $\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha)$. If either $\int_0^\infty \alpha^p \, d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ is finite, use (5.50)—if $f \in L^p(E)$, then $\lim_{\alpha\to+\infty} \alpha^p\omega(\alpha) = 0$—and the results of Exercises 14—if $f \in L^p(E)$, then $\lim_{a\to 0^+} a^p\omega(a) = 0$, where $|E| = +\infty$—and 15 to integrate by parts.]

Solution. Given that $f \geq 0$ and $|E|$ is not necessarily finite,[‡] we mimic the proof of (5.46) to show that if $\varphi(x) = |x|^p$, then $\int_{E_{ab}} \varphi(f) = -\int_a^b \varphi(\alpha)\,d\omega(\alpha)$. Write $f|_{E_{ab}}$ as the increasing limit of simple measurable functions via partitions of $[a,b] \subset [0,k]$ (starting with $k \geq b$) as provided in the proof of (4.13): observe that the norms of the partitions approach 0. Since φ is monotone increasing on nonnegative real numbers, it follows that $\varphi(f_k) \nearrow \varphi(f)$, so by the monotone convergence theorem we have $\int_{E_{ab}} \varphi(f_k) \to \int_{E_{ab}} \varphi(f)$. Moreover, since $\varphi(f_k)$ is a simple function on $[a,b]$, we have

$$\sum_j \varphi(\alpha_{j-1}^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})] \leq \int_{E_{ab}} \varphi(f_k) \leq \sum_j \varphi(\alpha_j^{(k)})[\omega(\alpha_j^{(k)}) - \omega(\alpha_{j-1}^{(k)})],$$

where $\alpha_j^{(k)}$ is the j^{th} partition element of $[a,b]$ in the partition used for the simple function f_k, and where j ranges so that $a < \alpha_j^{(k)} \leq b$. Since the norms of the partitions approach 0 as $k \to \infty$, $\int_{E_{ab}} \varphi(f_k) \to -\int_a^b \varphi(\alpha)\,d\omega(\alpha)$. Letting $a \to 0^+$, $b \to +\infty$ shows that

$$\int_E \varphi(f) = -\int_0^\infty \varphi(\alpha)\,d\omega(\alpha)$$

via the monotone convergence theorem without regard to the finiteness of either side.

Suppose that $\int_0^\infty \alpha^p \, d\omega(\alpha)$ is finite. Then $f \in L^p(E)$, so Exercise 14 and (5.50) state that $\lim\limits_{a\to 0^+} a^p\omega(a)$ and $\lim\limits_{b\to+\infty} b^p\omega(b) = 0$, so integrating by parts gives us

$$\int_a^b \alpha^p \, d\omega(\alpha) = \alpha^p\omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha)\,d\alpha \to -p\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha.$$

[‡]This assumption is key, as the proof of (5.46) uses $|E| < +\infty$ in a nontrivial way: the bounded convergence theorem. Since $|E|$ could be $+\infty$, we need a different convergence tool. We'll use the monotone convergence theorem here.

Conversely, suppose that $\int_0^\infty \alpha^{p-1}\omega(\alpha)\,d\alpha$ is finite. Then Exercise 15 states that $\lim\limits_{a\to 0^+} a^p\omega(a)$ and $\lim\limits_{b\to+\infty} b^p\omega(b) = 0$. Thus, integrating by parts again, we have

$$\alpha^p\omega(\alpha)\Big|_a^b - p\int_a^b \alpha^{p-1}\omega(\alpha)\,d\alpha = \int_a^b \alpha^p\,d\omega(\alpha).$$

Letting $a \to 0^+$ and $b \to +\infty$ in both cases gives us the desired equality. It therefore follows that one integral is finite if and only if the other is finite, and if they are finite, then they are equal (so if they are not finite, they are also both equal to $+\infty$, as f is nonnegative. $\square$

5.17. If $f \geq 0$ and $\omega(\alpha) \leq c(1+\alpha)^{-p}$ for all $\alpha > 0$, show that $f \in L^r$, $0 < r < p$.

Solution. From the above exercise, it suffices to show that

$$\int_0^\infty \alpha^{r-1}\omega(\alpha)\,d\alpha \leq c\int_0^\infty \frac{\alpha^{r-1}}{(1+\alpha)^p}\,d\alpha < +\infty$$

for all $r \in (0,p)$. The integral is improper only near ∞, and convergence there follows from the fact that

$$\frac{\alpha^{r-1}}{(1+\alpha)^p} < \frac{\alpha^{r-1}}{\alpha^p} = \frac{1}{\alpha^{p-r+1}}$$

for sufficiently large α. Since $r < p$, we have $p - r + 1 > 1$, so $\int_{K_p}^\infty \frac{1}{\alpha^{p-r+1}}\,d\alpha$ converges. $\square$

5.18. If $f \geq 0$, show that $f \in L^p$ if and only if $\displaystyle\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < +\infty$. [Use Exercise 16.]

Solution. By Exercise 16, we have that $f \in L^p$ if and only if $\displaystyle\int_0^{\infty} \alpha^{p-1}\omega(\alpha)\,d\alpha$ converges. Now observe that

$$\int_0^{\infty} \alpha^{p-1}\omega(\alpha)\,d\alpha = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha).$$

Since α^{p-1} is monotone increasing and $\omega(\alpha)$ is monotone decreasing, we have

$$\int_{2^k}^{2^{k+1}} 2^{k(p-1)}\omega(2^{k+1}) \leq \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha) \leq \int_{2^k}^{2^{k+1}} 2^{(k+1)(p-1)}\omega(2^k);$$

or

$$2^{-p}2^{(k+1)p}\omega(2^{k+1}) = 2^{kp}\omega(2^{k+1}) \leq \int_{2^k}^{2^{k+1}} \alpha^{p-1}\omega(\alpha) \leq 2^{(k+1)(p-1)+k}\omega(2^k) = 2^{kp+p-1}\omega(2^k).$$

Thus,

$$2^{-p}\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) = 2^{-p}\sum_{k=-\infty}^{\infty} 2^{(k+1)p}\omega(2^{k+1}) \leq \int_0^{\infty} \alpha^{p-1}\omega(\alpha)\,d\alpha \leq$$
$$\sum_{k=-\infty}^{\infty} 2^{(k+1)p+1}\omega(2^k) = 2^{p-1}\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k).$$

So the integral converges if and only if the sum converges. □

5.21. If $\int_A f = 0$ for every measurable subset A of a measurable set E, show that $f = 0$ a.e. in E.

Solution. $|f|$ (hence f) equals 0 a.e. in E because

$$\int_E |f| = \int_{\{f>0\}} f - \int_{\{f<0\}} f = 0$$

and $\{f > 0\}$ and $\{f < 0\}$ are measurable subsets of E. □