Math 6337 : Real Analysis I

Mid-term Exam 2
04 November 2011

Instructions: Answer all of the problems.

1. Use Fubini's Theorem to prove that

$$\lim_{A\to\infty}\int_0^A \frac{\sin x}{x}dx = \frac{\pi}{2}$$

Solution: Write

$$\frac{1}{x} = \int_0^\infty e^{-xt}dt,$$

and then consider the following integral,

$$\int_0^A \frac{\sin x}{x}dx = \int_0^A\int_0^\infty \sin x e^{-xt}dtdx = \int_0^\infty\int_0^A \sin x e^{-xt}dxdt.$$

Now evaluate the inner integral,

$$\int_0^A \sin x e^{-xt}dx = \frac{1}{t^2+1} - \frac{t}{t^2+1}e^{-At}\sin A - \frac{1}{t^2+1}e^{-At}\cos A.$$

Finally, then we integrate these expressions in t, to find

$$\int_0^A \frac{\sin x}{x}dx = \int_0^\infty \frac{1}{t^2+1}dt - \sin A\int_0^\infty \frac{t}{t^2+1}e^{-At}dt - \cos A\int_0^\infty \frac{e^{-At}}{t^2+1}dt.$$

From this identity the result easily follows via Fubini and some simple convergence theorems.

2. Let $f\in L^1(0,1)$. Show that

$$\lim_{k\to\infty}\int_{(0,1)} x^k f(x)\, dx = 0\,.$$

Solution: There are several ways to approach this problem. We present one of the more clever solutions of the class.

Since $f\in L^1(0,1)$, and by the absolute continuity of the integral, there exists a $\delta>0$ such that

$$\int_{(\delta,1)} |f(x)|\, dx < \epsilon.$$

For any $x\in(0,\delta]$, we have $\left|x^k f(x)\right| \le \delta^k\, |f(x)|$ and $\lim_k \left|x^k f(x)\right| \to 0$. Then we have

$$\begin{aligned}\limsup_k \left|\int_{(0,1)} x^k f(x)dx\right| &\le \limsup_k \left|\int_{(0,\delta)} x^k f(x)dx\right| + \limsup_k \left|\int_{(\delta,1)} x^k f(x)dx\right| \\ &\le \int_{(0,\delta)} \limsup_k \left|x^k f(x)\right| dx + \epsilon = \epsilon.\end{aligned}$$

3. For any function h on $[0,1]$, define the distribution function $\omega_h(\alpha) = |\{x : h(x) > \alpha\}|$. Let $1 < p < 2$, and assume that $f \in L^p([0,1])$, and that g is measurable with

$$\omega_{|g|}(\alpha) \leq -\frac{1}{\alpha^2}\int_0^\alpha t^2 d\omega_{|f|}(t) - \frac{1}{\alpha}\int_\alpha^\infty t d\omega_{|f|}(t).$$

Show that $g \in L^p([0,1])$.

Solution: To show that $g \in L^p([0,1])$, we will compute the following

$$\int_0^1 |g(x)|^p \, dx = p\int_0^\infty \alpha^{p-1}\omega_{|g|}(\alpha)d\alpha = -\int_0^\infty \alpha^p d\omega_{|g|}(\alpha).$$

One should first show that the distribution function of g is finite. Now, use the estimate on the distribution function of g to find

$$\begin{aligned} p\int_0^\infty \alpha^{p-1}\omega_{|g|}(\alpha)d\alpha &\leq -p\int_0^\infty \alpha^{p-3}\int_0^\alpha t^2 d\omega_{|f|}(t)d\alpha - p\int_0^\infty \alpha^{p-2}\int_\alpha^\infty t d\omega_{|f|}(t)d\alpha \\ &= -p\int_0^\infty t^2\left(\int_t^\infty \alpha^{p-3}d\alpha\right)d\omega_{|f|}(t) - p\int_0^\infty t\left(\int_0^t \alpha^{p-2}d\alpha\right)d\omega_{|f|}(t) \end{aligned}$$

Changing the order of integration can be justified by Fubini. Now, note that $1 < p < 2$, and by simple calculus we have

$$\int_t^\infty \alpha^{p-3}d\alpha = \frac{1}{2-p}t^{p-2} \quad \text{and} \quad \int_0^t \alpha^{p-2}d\alpha = \frac{1}{p-1}t^{p-1}.$$

Using this we see that

$$\int_0^1 |g(x)|^p \, dx = p\int_0^\infty \alpha^{p-1}\omega_{|g|}(\alpha)d\alpha \leq -\left(\frac{p}{2-p} + \frac{p}{p-1}\right)\int_0^\infty t^p d\omega_{|f|}(t).$$

But, this gives the following

$$\int_0^1 |g(x)|^p \, dx \leq \frac{p}{(2-p)(p-1)}\int_0^1 |f(x)|^p \, dx,$$

which gives the desired conclusion.

4. Suppose that $f \geq 0$ on $(0,\infty)$, $f \in L^p(0,\infty)$ and

$$F(x) = \frac{1}{x}\int_0^x f(t)dt.$$

Prove that for $1 < p < \infty$

$$\int_0^\infty F(x)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(t)^p dt$$

Hint: Write $xF(x) = \int_0^x f(t)t^a t^{-a}dt$ and use Hölder's Inequality.

Solution: We start with

$$\int_0^\infty F(x)^p dx = \int_0^\infty \frac{1}{x^p}\left(\int_0^x f(t)dt\right)^p dx.$$

Now use the hint, and write

$$\int_0^x f(t)dt = \int_0^x f(t)t^a t^{-a} dt$$

where $0 < a < \frac{1}{q} = 1 - \frac{1}{p}$. We apply Hölder's Inequality to this expression to get that

$$\int_0^x f(t)dt \leq \left(\int_0^x f(t)^p t^{ap} dt\right)^{\frac{1}{p}} \left(\int_0^x t^{-aq} dt\right)^{\frac{1}{q}} = \frac{x^{\frac{1}{q}-a}}{(1-aq)^{\frac{1}{q}}}\left(\int_0^x f(t)^p t^{ap} dt\right)^{\frac{1}{p}}.$$

Using this estimate we find

$$\begin{aligned}
\int_0^\infty F(x)^p dx &= \int_0^\infty \frac{1}{x^p}\left(\int_0^x f(t)dt\right)^p dx \\
&\leq \frac{1}{(1-aq)^{\frac{p}{q}}}\int_0^\infty x^{\frac{p}{q}-pa-p}\left(\int_0^x f(t)^p t^{ap} dt\right) dx \\
&= \frac{1}{(1-aq)^{\frac{p}{q}}}\int_0^\infty f(t)^p t^{ap}\left(\int_t^\infty x^{\frac{p}{q}-pa-p} dx\right) dt.
\end{aligned}$$

Now note that

$$\int_t^\infty x^{\frac{p}{q}-pa-p} dx = \int_t^\infty x^{-1-pa} dx = t^{-pa}\frac{1}{ap},$$

and so we have

$$\int_0^\infty F(x)^p dx \leq \frac{1}{ap(1-aq)^{\frac{p}{q}}}\int_0^\infty f(t)^p dt.$$

Choose now $a = \frac{1}{p+q}$, which is found by solving a minimization problem, and then simple algebra gives

$$\frac{1}{ap(1-aq)^{\frac{p}{q}}} = \left(\frac{p}{p-1}\right)^p.$$

5. Fix a positive integer N, and put $\xi = e^{\frac{2\pi i}{N}}$.

(a) Prove the orthogonality relation

$$\frac{1}{N}\sum_{n=1}^N \xi^{nk} = \begin{cases} 1 & : \quad k = 0 \\ 0 & : \quad 1 \leq k \leq N-1. \end{cases}$$

(b) Use this identity to prove that in a Hilbert space H, when $N \geq 3$ we have

$$\langle x, y\rangle_H = \frac{1}{N}\sum_{n=1}^N \|x + \xi^n y\|_H^2 \xi^n$$

(c) Show that more generally we have

$$\langle x, y\rangle_H = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta}y\|_H^2 e^{i\theta}\, d\theta$$

Solution: To prove (a), one can compute directly. Here is an easy way to do this. Note that the identity is obvious when $n = 0$ since

$$\frac{1}{N}\sum_{k=1}^{N} \xi^0 = \frac{1}{N}\sum_{k=1}^{N} 1 = 1.$$

To prove the other cases, note that the ξ^k, when $k = 1, \ldots, N$ are the zeros of the polynomial $p(z) = z^N - 1 = \prod_{k=1}^{N}(z - \xi^k)$. Comparing these two representations, we see that

$$\frac{1}{N}\sum_{k=1}^{N} \xi^k = 0,$$

which is the case of $n = 1$. The general case can be deduced from the identity

$$\sum_{k=1}^{N} x^k = \frac{1 - x^{N+1}}{1 - x}.$$

We now use (a) to prove (b). First, compute

$$\|x + \xi^n y\|_H^2 \quad = \quad \|x\|_H^2 + \|y\|_H^2 + \xi^{-n}\langle x, y\rangle_H + \xi^n\langle y, x\rangle_H$$

We then have

$$\frac{1}{N}\sum_{n=1}^{N} \|x + \xi^n y\|_H^2 \xi^n \quad = \quad \left(\|x\|_H^2 + \|y\|_H^2\right)\frac{1}{N}\sum_{n=1}^{N}\xi^n + \langle x, y\rangle_H \frac{1}{N}\sum_{n=1}^{N} 1 + \langle y, x\rangle_H \frac{1}{N}\sum_{n=1}^{N}\xi^{2n}.$$

By part (a), the first and third terms are zero, and the third term reduces to $\langle x, y\rangle_H$. Again for (c), compute

$$\|x + e^{i\theta}y\|_H^2 \quad = \quad \|x\|_H^2 + e^{-i\theta}\langle x, y\rangle_H + e^{i\theta}\langle y, x\rangle_H + \|y\|_H^2.$$

Norm integrate this equality with respect to $e^{i\theta}$ and find

$$\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta}y\|_H^2 e^{i\theta} d\theta \quad &= \quad \left(\|x\|_H^2 + \|y\|_H^2\right)\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta} d\theta + \langle y, x\rangle_H \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{2i\theta} d\theta \\
&+ \quad \langle x, y\rangle_H \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta \\
&= \quad \langle x, y\rangle_H.
\end{aligned}$$

The last inequality follows since $\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta} d\theta = 0$ for $k \neq 0$.

6. A sequence $\{f_k\}$ in a Hilbert space H converges weakly to f if and only if for any $h \in H$ we have

$$\lim_{k\to\infty} \langle f_k, h\rangle_H = \langle f, h\rangle_H.$$

Suppose that $\{f_k\}$ converges weakly to f and

$$\lim_{k\to\infty} \|f_k\|_H = \|f\|_H.$$

Show that

$$\lim_{k\to\infty} \|f_k - f\|_H = 0.$$

Solution: Compute the following

$$\begin{aligned}
\|f - f_k\|_H^2 &= \langle f - f_k, f - f_k\rangle_H \\
&= \langle f, f\rangle_H - \langle f, f_k\rangle_H - \langle f_k, f\rangle_H + \langle f_k, f_k\rangle_H \\
&= \langle f, f\rangle_H - \langle f, f_k\rangle_H - \langle f_k, f\rangle_H + \langle f, f\rangle_H + \langle f_k, f_k\rangle_H - \langle f, f\rangle_H
\end{aligned}$$

Note that

$$\langle f_k, f_k\rangle_H - \langle f, f\rangle_H \to 0$$

by the second hypothesis. And by the first hypothesis we have that

$$\langle f, f\rangle_H - \langle f, f_k\rangle_H \to 0$$

since f_k converges to f weakly (let $h = f$ in the definition of weak convergence).

7. Suppose that F and G are non-negative functions that satisfy the "relative distributional inequality"

$$|\{x : F(x) > \lambda; G(x) \leq c\lambda\}| \leq a\,|\{x : F(x) > b\lambda\}| \quad \forall\lambda > 0.$$

Assume that for some p, $0 < p < \infty$ that $F \in L^p(E)$ and that $a < b^p$. Then show that there exists a constant $C = C(a, b, c, p)$ such that

$$\int_E F(x)^p dx \leq C \int_E G(x)^p dx.$$

Solution: This is a simple application of computing the L^p norm by the distribution inequality. We have that

$$\begin{aligned}
|\{x : F(x) > \lambda\}| &= \left|\{x : F(x) > \lambda; G(x) \leq c\lambda\} \bigcup \{x : F(x) > \lambda; G(x) > c\lambda\}\right| \\
&\leq |\{x : F(x) \geq \lambda; G(x) \leq c\lambda\}| + |\{x : F(x) \geq \lambda; G(x) > c\lambda\}| \\
&\leq a\,|\{x : F(x) > b\lambda\}| + |\{x : G(x) > c\lambda\}|.
\end{aligned}$$

With the last inequality following by the hypothesis of the problem.

Now we use the following

$$\int_E H(x)^p dx = p\int_0^\infty \lambda^{p-1}\left|\{x : H(x) > \lambda\}\right| d\lambda.$$

$$\begin{aligned}
\int_E F(x)^p dx &= p\int_0^\infty \lambda^{p-1}\left|\{x : F(x) > \lambda\}\right| d\lambda \\
&\leq ap\int_0^\infty \lambda^{p-1}\left|\{x : F(x) > b\lambda\}\right| d\lambda + p\int_0^\infty \lambda^{p-1}\left|\{x : G(x) > c\lambda\}\right| d\lambda \\
&= \frac{a}{b^p}p\int_0^\infty \lambda^{p-1}\left|\{x : F(x) > \lambda\}\right| d\lambda + c^{-p}p\int_0^\infty \lambda^{p-1}\left|\{x : G(x) > \lambda\}\right| d\lambda \\
&= ab^{-p}\int_E F(x)^p dx + c^{-p}\int_E G(x)^p dx.
\end{aligned}$$

Since $ab^{-p} < 1$ and since $F \in L^p(E)$ we have

$$(1 - ab^{-p})\int_E F(x)^p dx \leq c^{-p}\int_E G(x)^p dx.$$

Rearrangement gives the desired inequality we $C(a, b, c, p) = \frac{c^{-p}}{1-ab^{-p}}$.