

Definition of σ -algebra

A nonempty collection Σ of subsets E is called a σ -algebra if it satisfies the following conditions^a:

- (i) $\setminus E \in \Sigma$ if $E \in \Sigma$
- (ii) $\cup_k E_k \in \Sigma$ if $E_k \in \Sigma, k = 1, 2, \dots$

^aRichard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis*. CRC, 2015, p. 49.

Borel σ -algebra

The smallest σ -algebra of subsets of \mathbb{R}^n containing all the open subsets of \mathbb{R}^n is called the Borel σ -algebra \mathfrak{B} of \mathbb{R}^n , and the sets in \mathfrak{B} are called Borel subsets of \mathbb{R}^n . Sets of type $F_\sigma, G_\delta, F_{\sigma\delta}, G_{\delta\sigma}$ (see p.6 in Section 1.3), etc., are Borel sets.^a

^aIbid., p. 49.

Problem 1. Zygmund p59 exercise 08

Show that the Borel σ -algebra \mathfrak{B} in \mathbb{R}^n is the smallest σ -algebra containing the closed sets in \mathbb{R}^n .

Solution.

To show that the Borel σ -algebra \mathfrak{B} in \mathbb{R}^n is the smallest σ -algebra containing the closed sets in \mathbb{R}^n , we need to show that

- (i) \mathfrak{B} contains the closed sets in \mathbb{R}^n .
- (ii) $\mathfrak{B} \subseteq A$ if A is a σ -algebra containing the closed sets in \mathbb{R}^n .

(i)

For every closed subset $E \in \mathbb{R}^n$, there is a open subset $\setminus E \in \mathbb{R}^n$. By definition of the Borel σ -algebra \mathfrak{B} , we have $\setminus E \in \mathfrak{B}$, implying that $E \in \mathfrak{B}$. Thus, \mathfrak{B} contains the closed sets in \mathbb{R}^n .

(ii)

Let A be a σ -algebra containing the closed sets in \mathbb{R}^n , the definition of σ -algebra implies that A is a σ -algebra containing all the open sets in \mathbb{R}^n

By definition of the Borel σ -algebra \mathfrak{B} , it is the smallest σ -algebra containing the open sets in \mathbb{R}^n . Thus, we have $\mathfrak{B} \subseteq A$.

Therefore, we can conclude that the Borel σ -algebra \mathfrak{B} in \mathbb{R}^n is the smallest σ -algebra containing the closed sets in \mathbb{R}^n .

■

Problem 2. Zygmund p59 exercise 09

If $\{E_k\}_{k=1}^\infty$ is a sequence of sets with $\sum_{k=1}^\infty |E_k|_e < +\infty$, show that $\limsup_{k \rightarrow \infty} E_k$ (and so also $\liminf_{k \rightarrow \infty} E_k$) has measure zero.

^aSuppose $\{E_k\}_{k=1}^\infty$ is a sequence of subsets:

$$\begin{aligned}\limsup_{k \rightarrow \infty} E_k &:= \lim_{n \rightarrow \infty} V_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k, \\ \liminf_{k \rightarrow \infty} E_k &:= \lim_{n \rightarrow \infty} B_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k.\end{aligned}$$

In other words,

$$\begin{aligned}V_n &= \bigcup_{k=n}^\infty E_k \searrow V = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k =: \limsup_{k \rightarrow \infty} E_k, \\ B_n &= \bigcap_{k=n}^\infty E_k \nearrow B = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k =: \liminf_{k \rightarrow \infty} E_k.\end{aligned}$$

^aWheeden and Zygmund, see n. a, p. 49.

Solution. By the definition of the limit of a sequence, let $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\sum_{k=n}^\infty |E_k|_e = \sum_{k=1}^\infty |E_k|_e - \sum_{k=1}^{n-1} |E_k|_e < \epsilon.$$

Since

$$\begin{aligned}\limsup_{k \rightarrow \infty} E_k &:= \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k \\ &\subseteq \bigcup_{k=N}^\infty E_k,\end{aligned}$$

we have

$$\begin{aligned}|\limsup_{k \rightarrow \infty} E_k|_e &\leq \left| \bigcup_{k=N}^\infty E_k \right|_e \\ &= \sum_{k=N}^\infty |E_k|_e < \epsilon.\end{aligned}$$

Let $\epsilon \rightarrow 0$, we have

$$|\limsup_{k \rightarrow \infty} E_k|_e = 0.$$

.....

By the definition, we have

$$\liminf_{k \rightarrow \infty} E_k \subseteq \limsup_{k \rightarrow \infty} E_k.$$

Thus,

$$|\liminf_{k \rightarrow \infty} E_k|_e \leq |\limsup_{k \rightarrow \infty} E_k|_e = 0.$$

Therefore, $|\liminf_{k \rightarrow \infty} E_k|_e = 0$.

■

Problem 3. Zygmund p59 exercise 09

Show that there exist sets $\{E_k\}_{k=1}^\infty$ such that $E_k \searrow E$, $|E_k|_e < +\infty$, and $\lim_{k \rightarrow \infty} |E_k|_e > |E|_e$ with strict inequality.

Problem 4. Zygmund p76 exercise 02

Let f be a simple function, taking distinct values on disjoint sets E_1, \dots, E_N . Show that f is measurable if and only if E_i is measurable for $1 \leq i \leq N$.

Problem 5. Zygmund p76 exercise 05

Give an example to show that $\phi(f(x))$ may not be measurable if ϕ and f are measurable and finite.

(Let F be the Cantor-Lebesgue function and let f be its inverse, suitably defined. Let ϕ be the characteristic function of a set of measure zero whose image under F is not measurable.)

—

Show that the same may be true even if f is continuous.

(Let $g(x) = x + F(x)$, where F is the Cantor-Lebesgue function, and consider $f = g^{-1}$.) Cf. Exercise 22.