Problem 1.

(a) Suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ and that

$$\sum_{k=1}^{\infty} |E_k| < +\infty.$$

Let $E = \limsup_{k\to\infty} E_k$. Prove that $|E| = 0$.

(b) Given an irrational x, one can show (using the pigeonhole principle, for example) that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that $\forall \epsilon > 0$,

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero.

Solution.

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Problem 2.

(a) Let E be a subset of $\mathbb{R}$ with $|E|_e > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$|E \cap I|_e \geq \alpha |I|_e.$$

Loosely speaking, this estimate shows that E contains almost a whole interval.

(b) Suppose E is a measurable subset of $\mathbb{R}$ with $|E| > 0$. Prove that the difference set of E, which is defined by

$$|E - E| = \{x - y \in \mathbb{R} \,|\, x, y \in E\}$$

contains an open interval centered at the origin.

Solution.

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