

Theorems 5.16

If $f_k, k = 1, 2, \dots$, are nonnegative and measurable, then

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \sum_{k=1}^{\infty} \int_E f_k.$$

Proof. The functions F_N defined by $F_N = \sum_{k=1}^N f_k$ are nonnegative and measurable, and increase to $\sum_{k=1}^{\infty} f_k$. Hence,

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \lim_{N \rightarrow \infty} \int_E F_N = \lim_{N \rightarrow \infty} \sum_{k=1}^N \int_E f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

Theorem 5.22

If $f \in L(E)$, then f is finite almost everywhere in E .

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent, we call the series conditionally convergent.

The Monotone Convergence Criterion for Real Sequences (Real-Analysis-4th-Ed-Royden)

Theorem 15: A monotone sequence of real numbers converges if and only if it is bounded.

Problem 1. Zygmund p109 exercise 13

- (a) Let $\{f_k\}$ be a sequence of measurable functions on E . Show that $\sum_k |f_k|$ converges absolutely a.e. in E if $\sum_k \int_E |f_k| < +\infty$. (Use Theorems 5.16 and 5.22.)

Since $|f_k|$ is nonnegative, and

$$\sum_k \int_E |f_k| < +\infty,$$

by Thm 5.16,

$$\int_E \left(\sum_{k=1}^{\infty} |f_k| \right) = \sum_{k=1}^{\infty} \int_E |f_k| < +\infty,$$

implying that $\sum_{k=1}^{\infty} |f_k| \in L(E)$.

By Thm 5.22, $\sum_{k=1}^{\infty} |f_k|$ is finite a.e in E . Notice that $\sum_{k=1}^{\infty} |f_k|$ is monotone increasing. We can conclude that $\sum_{k=1}^{\infty} |f_k|$ converges a.e. in E , by the Monotone Convergence Theorem for real sequences.

Given $\phi \geq 0$, let $L_\phi(E)$ denote the class of measurable functions f such that $\phi(f) \in L(E)$. If $\phi(\alpha) = |\alpha|^p$, $0 < p < \infty$, the standard notation is

$$L^p(E) = \left\{ f : \int_E |f|^p < +\infty \right\}, \quad 0 < p < \infty.$$

Note that $L^1(E) = L(E)$. We will systematically study the L^p classes in Chapter 8.

Theorem 5.51

If $0 < p < \infty$, $f \geq 0$, and $f \in L^p(E)$, then

$$\int_E f^p = - \int_0^\infty \alpha^p d\omega(\alpha) = p \int_0^\infty \alpha^{p-1} \omega(\alpha) d\alpha,$$

where the last integral may be interpreted as either a Lebesgue or an improper Riemann integral.

Problem 2. Zygmund p109 exercise 16

Suppose that f is nonnegative and measurable on E , and ω is finite on $(0, \infty)$. Show that Theorem 5.51 holds without any further restrictions (i.e., f need not be in $L^p(E)$ and $|E|$ need not be finite) if we interpret

$$\int_0^\infty \alpha^p d\omega(\alpha) = \lim_{\substack{a \rightarrow 0^+ \\ b \rightarrow \infty}} \int_a^b.$$

For the first part, use the sets E_{ab} to obtain the relation

$$\int_E f^p = - \int_0^\infty \alpha^p d\omega(\alpha).$$

If either $\int_0^\infty \alpha^p d\omega(\alpha)$ or $\int_0^\infty \alpha^{p-1} \omega(\alpha) d\alpha$ is finite, use Lemma 5.50 and the results of Exercises 14 or 15 to integrate by parts.

Problem 3. Zygmund p110 exercise 18

If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^\infty 2^{kp} \omega(2^k) < +\infty$. (Use Exercise 16.)