Theorems 5.16

If f_k, $k = 1, 2, \ldots$, are nonnegative and measurable, then

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \sum_{k=1}^{\infty} \int_E f_k.$$

Proof. The functions F_N defined by $F_N = \sum_{k=1}^{N} f_k$ are nonnegative and measurable, and increase to $\sum_{k=1}^{\infty} f_k$. Hence,

$$\int_E \left(\sum_{k=1}^{\infty} f_k \right) = \lim_{N \to \infty} \int_E F_N = \lim_{N \to \infty} \sum_{k=1}^{N} f_k = \sum_{k=1}^{\infty} \int_E f_k.$$

Theorem 5.22

If $f \in L(E)$, then f is finite almost everywhere in E.

A series $\sum a_n$ is called absolutely convergent if $\sum |a_n|$ is convergent.

If $\sum a_n$ is convergent and $\sum |a_n|$ is divergent, we call the series conditionally convergent.

The Monotone Convergence Criterion for Real Sequences (Real-Analysis-4th-Ed-Royden)

Theorem 15: A monotone sequence of real numbers converges if and only if it is bounded.

Problem 1. Zygmund p109 exercise 13

(a) Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum_k |f_k|$ converges absolutely a.e. in E if $\sum_k \int_E |f_k| < +\infty$. (Use Theorems 5.16 and 5.22.)

Since $|f_k|$ is nonnegative, and

$$\sum_k \int_E |f_k| < +\infty,$$

by Thm 5.16,

$$\int_E \left(\sum_{k=1}^{\infty} |f_k| \right) = \sum_{k=1}^{\infty} \int_E |f_k| < +\infty,$$

we have $\sum_{k=1}^{\infty} |f_k| \in L(E)$.

By Thm 5.22, $\sum_{k=1}^{\infty} |f_k|$ is finite a.e in E.

Notice that $\sum_{k=1}^{\infty} |f_k|$ is monotone increasing.

We can conclude that $\sum_{k=1}^{\infty} |f_k|$ converges a.e. in E, by the Monotone Convergence Theorem for real sequences.

Given $\phi \geq 0$, let $L_\phi(E)$ denote the class of measurable functions f such that $\phi(f) \in L(E)$. If $\phi(\alpha) = |\alpha|^p$, $0 < p < \infty$, the standard notation is

$$L^p(E) = \left\{ f : \int_E |f|^p < +\infty \right\}, \quad 0 < p < \infty.$$

Note that $L^1(E) = L(E)$. We will systematically study the L^p classes in Chapter 8.

Theorem 5.51

If $0 < p < \infty$, $f \geq 0$, and $f \in L^p(E)$, then

$$\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha,$$

where the last integral may be interpreted as either a Lebesgue or an improper Riemann integral.

Problem 2. Zygmund p110 exercise 18

If $f \geq 0$, show that $f \in L^p$ if and only if $\sum_{k=-\infty}^{\infty} 2^{kp}\omega(2^k) < +\infty$. (Use Exercise 16.)

By Exercise 16, we have that if f is nonnegative and measurable on E and that ω is finite on $(0, \infty)$, then

$$\int_E f^p = -\int_0^\infty \alpha^p \, d\omega(\alpha) = p \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha.$$

That is, it suffices to show that $f \in L^p$ if and only if

$$\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha < \infty.$$

Now observe that

$$\int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \alpha^{p-1} \omega(\alpha) \, d\alpha.$$

Since α^{p-1} is monotone increasing and $\omega(\alpha)$ is monotone decreasing, we have

$$\int_{2^k}^{2^{k+1}} 2^{k(p-1)} \omega(2^{k+1}) \, d\alpha \leq \int_{2^k}^{2^{k+1}} \alpha^{p-1} \omega(\alpha) \leq \int_{2^k}^{2^{k+1}} 2^{(k+1)(p-1)} \omega(2^k).$$

Simplifying, we get

$$\begin{aligned} 2^{k(p-1)+k} \omega(2^{k+1}) \, d\alpha &\leq \int_{2^k}^{2^{k+1}} \alpha^{p-1} \omega(\alpha) \leq 2^{(k+1)(p-1)+k} \omega(2^k) \\ 2^{(k+1)p-p} \omega(2^{k+1}) \, d\alpha &\leq \int_{2^k}^{2^{k+1}} \alpha^{p-1} \omega(\alpha) \leq 2^{kp+(p-1)} \omega(2^k). \end{aligned}$$

Therefore,

$$\begin{aligned} &2^{-p} \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) \quad \text{(Since } k \text{ ranges from } -\infty \text{ to } \infty \\ =&2^{-p} \sum_{k=-\infty}^{\infty} 2^{(k+1)p} \omega(2^{k+1}) \leq \int_0^\infty \alpha^{p-1} \omega(\alpha) \, d\alpha = \sum_{k=-\infty}^{\infty} \int_{2^k}^{2^{k+1}} \alpha^{p-1} \omega(\alpha) \, d\alpha \quad \leq 2^{p-1} \sum_{k=-\infty}^{\infty} 2^{kp} \omega(2k). \end{aligned}$$

So, $f \in L^p$ if and only if

$$\sum_{k=-\infty}^{\infty} 2^{kp} \omega(2^k) < +\infty.$$