Let f be any measurable function defined on a set E. If f exists and is finite, we say that f is Lebesgue integrable, or simply integrable, on E and write $f \in L(E)$. Thus,

$$L(E) = \left\{ f : \int_E f \text{ is finite} \right\}.$$

Theorem 5.5

(i) If f and g are measurable and $0 \le g \le f$ on E, then $\int_E g \le \int_E f$. In particular, $\int_E (\inf f) \le \int_E f$.

(ii) If f is nonnegative and measurable on E and $\int_E f$ is finite, then $f < +\infty$ a.e. in E.

(iii) Let E_1 and E_2 be measurable and $E_1 \subset E_2$. If f is nonnegative and measurable on E_2, then $\int_{E_1} f \le \int_{E_2} f$.

Proof:

Parts (i) and (iii) follow from the relations $R(g, E) \subset R(f, E)$ and $R(f, E_1) \subset R(f, E_2)$, respectively.

To prove (ii), we may assume that $|E| > 0$. If $f = +\infty$ in a subset E_1 of E with positive measure, then by (iii) and (i), we have $\int_E f \ge \int_{E_1} f \ge \int_{E_1} a = a|E_1|$, no matter how large a is. This contradicts the finiteness of $\int_E f$.

Theorem 5.22

If $f \in L(E)$, then f is finite a.e. in E.

Proof: If $f \in L(E)$, then $|f| \in L(E)$, and the result follows from Theorem 5.5(ii).

Theorem 5.36 (Lebesgue's Dominated Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If there exists $\phi \in L(E)$ such that $|f_k| \le \phi$ a.e. in E for all k, then $\int_E f_k \to \int_E f$.

Problem 1. Zygmund p109 exercise 04

If $f \in L(0,1)$, show that $x^k f(x) \in L(0,1)$ for $k = 1, 2, \ldots$, and that

$$\int_0^1 x^k f(x)\, dx \to 0.$$

Let $g_k(x) = x^k f(x)$ and $E = (0,1)$. We have $g_k(x)$ measurable on E, thus $\int_E g_k$ exists.

For $x \in (0,1)$, $x_k \leq 1$, so $g(x) = x^k f(x) \leq f(x), \forall k \in \mathbb{N}$. Hence,

$$\int_E g_k \leq \int_E f < \infty,$$

implying that $g_k(x) = x^k f(x) \in L(0,1)$.

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Since $f \in L(E)$, f is finite a.e. in E.

Besides, for all $x \in E$, $x^k \to 0$, as $k \to \infty$.

Thus, $g_k(x) = x^k f(x) \to 0$ a.e in E. Additionally, $|g_k| \leq |f|$, while $f \in L(E)$. Therefore, by Theorem 5.36 (Lebesgue's Dominated Convergence Theorem), we have

$$\int_E g_k(x)\, dx \to \int_E 0\, dx = 0.$$

Problem 2. Zygmund p109 exercise 05

Use Egorov's theorem to prove the bounded convergence theorem.

Problem 3. Zygmund p109 exercise 06

Let $f(x,y)$, $0 \leq x, y \leq 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and $\dfrac{\partial f(x,y)}{\partial x}$ is a bounded function of (x,y). Show that $\dfrac{\partial f(x,y)}{\partial x}$ is a measurable function of y for each x and

$$\frac{d}{dx}\int_0^1 f(x,y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\, dy.$$

Problem 4. Zygmund p109 exercise 09

If $p > 0$ and $|f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).

Problem 5. Zygmund p109 exercise 10

If $p > 0$, $|f - f_k|^p \to 0$, and $|f_k|^p \leq M$ for all k, show that $|f|^p \leq M$.