Problem 1.

(a) Suppose that $\{E_k\}_{k=1}^{\infty}$ is a countable family of measurable subsets of $\mathbb{R}^n$ and that

$$\sum_{k=1}^{\infty} |E_k| < +\infty.$$

Let $E = \limsup_{k\to\infty} E_k$. Prove that $|E| = 0$.

(b) Given an irrational x, one can show (using the pigeonhole principle, for example) that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\frac{p}{q}$, with relatively prime integers p and q such that $\forall \epsilon > 0$,

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero.

Solution.

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Problem 2.

(a) Let E be a subset of $\mathbb{R}$ with $|E|_e > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$|E \cap I|_e \geq \alpha |I|_e.$$

Loosely speaking, this estimate shows that E contains almost a whole interval.

(b) Suppose E is a measurable subset of $\mathbb{R}$ with $|E| > 0$. Prove that the difference set of E, which is defined by

$$E - E = \{x - y \in \mathbb{R} \,|\, x, y \in E\}$$

contains an open interval centered at the origin.

Solution.

(a)

For any $\alpha \in (0,1)$, let $I \subseteq \mathbb{R}$ be an open set s.t. $E \subseteq I$ and $|E|_e \geq \alpha |I|_e$, implying that

$$\alpha |I|_e \leq |E|_e \leq |I|_e.$$

Write the open set I as a countable union of disjoint open intervals:

$$I = \bigsqcup_{k=1}^{\infty} I_k.$$

Thus,

$$E = E \bigcap I = E \bigcap \left(\bigsqcup_{k=1}^{\infty} I_k \right) = \bigsqcup_{k=1}^{\infty} \left(E \bigcap I_k \right).$$

By the countable subadditivity of Lebesgue Outer Measure (Theorem 3.4), we have

$$|E|_e \leq \sum_{k=1}^{\infty} \left| E \bigcap I_k \right|_e.$$

Suppose, by way of contradiction, that $\forall I_k$,

$$\left| E \bigcap I_k \right|_e < \alpha |I_k|_e,$$

then we have

$$|E|_e \leq \sum_{k=1}^{\infty} \left| E \bigcap I_k \right|_e < \sum_{k=1}^{\infty} \alpha |I_k|_e = \alpha \sum_{k=1}^{\infty} |I_k|_e = \alpha |I|_e \leq |E|_e.$$

The second equality holds since I_k are disjoint and open.

Thus, it is implied that

$$|E|_e < |E|_e,$$

which is a contradiction. Therefore, our assumption that $\forall I_k, |E \bigcap I_k|_e < \alpha |I_k|_e$ must be false, which means that there exists an open interval I so that

$$|E \cap I|_e \geq \alpha |I|_e.$$

. . . .

(b)

$\exists G$ open s.t. $E \subseteq G$ and $|G| < |E|(1+\epsilon)$.

Since G is open, G can be written as a countable union of disjoint open intervals

$$G = \bigsqcup_{k=1}^{\infty} \mathring{I}_k.$$

Let $E_k = \mathring{I}_k \cap E$, $\{E_k\}_{k=1}^{\infty}$ is a sequence of disjoint measurable sets.

$$|G| = \sum_{k=1}^{\infty} |\mathring{I}_k|,$$
$$|E| = \sum_{k=1}^{\infty} |E_k|.$$

$$\begin{aligned} |G| = \sum_{k=1}^{\infty} |\mathring{I}_k| \quad &< |E|(1+\epsilon) \\ &= \left(\sum_{k=1}^{\infty} |E_k| \right)(1+\epsilon). \end{aligned} \tag{1}$$

$\exists k_0$ s.t. $|\mathring{I}_{k0}| < |E_{k0}|(1+\epsilon)$.

Suppose not; in other words, $|\mathring{I}_k| \geq |E_k|(1+\epsilon), \forall k$,

$$\sum_{k=1}^{\infty} |\mathring{I}_k| \geq \left(\sum_{k=1}^{\infty} |E_k| \right)(1+\epsilon),$$

contradicting (1).

Let $\epsilon = \frac{1}{3}$, we have

$$|\mathring{I}_{k0}| < |E_{k0}|(1+\frac{1}{3}) = \frac{4}{3}|E_{k0}|,$$
$$\frac{3}{4}|\mathring{I}_{k0}| < |E_{k0}|.$$

Let $E_{k0} + d = \{x + d \,|\, x \in E_{k0}\}$.

Claim: (to be proved by contradiction)

If $|d| \leq \frac{1}{2}|\mathring{I}_{k0}|$,

$$(E_{k0} + d) \cap E_{k0} \neq \varnothing.$$

$\Rightarrow$

$$\begin{aligned} \left(-\frac{1}{2}|\mathring{I}_{k0}|, \frac{1}{2}|\mathring{I}_{k0}| \right) &\subseteq \{x - y \,|\, x, y \in E_{k0}\} \\ &\subseteq \{x - y \,|\, x, y \in E\} \end{aligned}$$

$\blacksquare$

Proof of Claim

Claim:

If $|d| \leq \frac{1}{2}|\mathring{I}_{k0}|$,

$$(E_{k0} + d) \cap E_{k0} \neq \varnothing.$$

Proof. Suppose not; in other words, $E_{k0} + d$ and E_{k0} are disjoint measurable sets.

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| &= |(E_{k0} + d)| + |E_{k0}| \\ &= 2|E_{k0}|. \end{aligned}$$

Since $E_k = \mathring{I}_k \cap E$, we have

$$(E_{k0} + d) \cup E_{k0} \subseteq (\mathring{I}_{k0} + d) \cup \mathring{I}_{k0},$$

and

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| &\leq |(\mathring{I}_{k0} + d) \cup \mathring{I}_{k0}| \\ &= |\mathring{I}_{k0}| + |d| \\ &< \frac{3}{2}|\mathring{I}_{k0}|. \end{aligned}$$

Thus, we have

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| = 2|E_{k0}| < \frac{3}{2}|\mathring{I}_{k0}| \quad &< \frac{3}{2}|E_{k0}|(1 + \frac{1}{3}) \\ &= 2|E_{k0}|. \end{aligned}$$

This leads to a contradiction: $2|E_{k0}| < 2|E_{k0}|$.

Therefore, if $|d| \leq \frac{1}{2}|\mathring{I}_{k0}|$,

$$(E_{k0} + d) \cap E_{k0} \neq \varnothing.$$

□

Countable Subadditivity of Lebesgue Outer Measure

Theorem 3.4. *If $E = \bigcup_k E_k$ is a countable union of sets, then $|E|_e \leq \sum_k |E_k|_e$.*

Proof. We may assume that $|E_k|_e < +\infty$ for each $k = 1, 2, \ldots$, since otherwise, the conclusion is obvious. Fix $\varepsilon > 0$. Given k, choose intervals $I_j^{(k)}$ such that $E_k \subset \bigcup_j I_j^{(k)}$ and $\sum_j v(I_j^{(k)}) < |E_k|_e + \varepsilon 2^{-k}$.

Since $E \subset \bigcup_{j,k} I_j^{(k)}$, we have $|E|_e \leq \sum_{j,k} v(I_j^{(k)}) = \sum_k \sum_j v(I_j^{(k)})$. Therefore,

$$|E|_e \leq \sum_k (|E_k|_e + \varepsilon 2^{-k}) = \sum_k |E_k|_e + \varepsilon,$$

and the result follows by letting $\varepsilon \to 0$.[a] □

[a] Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis.* CRC, 2015, p. 42.