# MATH 6337: Homework 9 Solutions

**8.1.** For complex-valued, measurable $f$, $f = f_1 + i f_2$ with $f_1$ and $f_2$ real-valued and measurable, we have $\int_E f = \int_E f_1 + i \int_E f_2$. Prove that $\int_E f$ is finite if and only if $\int_E |f|$ is finite, and $\left|\int_E f\right| \leq \int_E |f|$. [Hint: Note that $\left|\int_E f\right| = \left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2}$, and use the fact that $(a^2+b^2)^{1/2} = a\cos(\alpha) + b\sin(\alpha)$ for an appropriate $\alpha$, while $(a^2+b^2)^{1/2} \geq |a\cos(\alpha) + b\sin(\alpha)|$ for all $\alpha$.]

*Solution.* If $\int_E f$ is finite, then both $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E |f_1|$ and $\int_E |f_2|$ are finite. Thus,

$$\int_E |f| = \int_E |f_1 + i f_2| \leq \int_E |f_1| + \int_E |f_2| < +\infty.$$

Conversely, if $\int_E |f|$ is finite, then so are $\int_E |f_1|$ and $\int_E |f_2|$ since $|f_1|, |f_2| \leq |f|$. Thus, $\int_E f_1$ and $\int_E f_2$ are finite, so $\int_E f = \int_E f_1 + i f_2$ is also finite.

Following the hint, choose $\alpha$ such that

$$\left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} = \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2.$$

Then

$$\left|\int_E f\right| = \left[\left(\int_E f_1\right)^2 + \left(\int_E f_2\right)^2\right]^{1/2} = \cos(\alpha)\int_E f_1 + \sin(\alpha)\int_E f_2 =$$
$$\int_E (f_1\cos(\alpha) + f_2\sin(\alpha)) \leq \int_E |f_1\cos(\alpha) + f_2\sin(\alpha)| \leq \int_E \sqrt{f_1^2 + f_2^2} = \int_E |f|.$$

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**8.6.** Prove the following generalization of Hölder's inequality: if $\sum_{i=1}^{k} 1/p_i = 1/r$ and $p_i, r \geq 1$, then

$$||f_1 \cdots f_k||_r \leq ||f_1||_{p_1} \cdots ||f_k||_{p_k}.$$

[See also Exercise 12 of Chapter 7.]

*Solution.* We prove this by induction. The $k = 2$ case is a consequence of Hölder's inequality: if $1/p_1 + 1/p_2 = 1/r$, then $r/p_1 + r/p_2 = 1$, so

$$||fg||_r^r = ||f^r g^r||_1 \leq ||f^r||_{p_1/r} \, ||g^r||_{p_2/r} = ||f||_{p_1}^r \, ||g||_{p_2}^r.$$

Now if $1/p_1 + \cdots + 1/p_k = 1/r$ for $k > 2$, we have

$$||f_1 \cdots f_k||_r \leq ||f_1 \cdots f_{k-1}||_s \, ||f_k||_{p_k} \leq ||f_1||_{p_1} \cdots ||f_k||_{p_k},$$

where $1/s = 1/r - 1/p_k = 1/p_1 + \cdots + 1/p_{k-1}$. $\square$

**8.7.** Show that when $0 < p < 1$, the neighborhoods $\left\{f : ||f||_p < \varepsilon\right\}$ of zero in $L^p(0,1)$ are not convex. [Hint: Let $f = \chi_{(0,\varepsilon^p)}$ and $g = \chi_{(\varepsilon^p, 2\varepsilon^p)}$. Show that $||f||_p = ||g||_p = \varepsilon$ but that $||f/2 + g/2||_p > \varepsilon$.]

*Solution.*

$$||f||_p = \left(\int_{(0,\varepsilon^p)} |1|^p\right)^{1/p} = \varepsilon$$

and similarly $||g||_p = \varepsilon$. However,

$$||f/2 + g/2||_p = \left(\int_{(0,2\varepsilon^p)} |1/2|^p\right)^{1/p} = \left(2\varepsilon^p 2^{-p}\right)^{1/p} = 2^{1/p-1}\varepsilon > \varepsilon$$

since $1/p > 1$. So the neighborhood $B_{\varepsilon+\eta}(0)$ is not convex for sufficiently small $\eta = \eta(p)$. $\square$

**8.8.** Prove the following *integral version* of Minkowski's inequality for $1 \le p < \infty$:

$$\left(\int \left|\int f(x,y)\,dx\right|^p dy\right)^{1/p} \le \int \left(\int |f(x,y)|^p\,dy\right)^{1/p} dx.$$

In other words,

$$\left|\left|\int f(x,y)\,dx\right|\right|_p \le \int ||f(x,y)||_p\,dx.$$

*Solution.* For $p = 1$, this is just Tonelli's theorem. For $p > 1$, we have

$$\begin{aligned}
\int \left|\int f(x,y)\,dx\right|^p dy &= \int \left|\int f(z,y)\,dz\right|^{p-1} \left|\int f(x,y)\,dx\right| dy \\
&= \int \left|\int f(x,y) \left(\int f(z,y)\,dz\right)^{p-1} dx\right| dy \\
&\le \int \left[\int |f(x,y)| \left|\int f(z,y)\,dz\right|^{p-1} dx\right] dy \\
&\overset{\text{Tonelli}}{=} \int \left[\int |f(x,y)| \left|\int f(z,y)\,dz\right|^{p-1} dy\right] dx \\
&\overset{\text{Hölder}}{\le} \int \left(\int |f(x,y)|^p\,dy\right)^{1/p} \left(\left|\int f(z,y)\,dz\right|^{(p-1)p'} dy\right)^{1/p'} dx \\
&= \left(\int \left(\int |f(x,y)|^p\,dy\right)^{1/p} dx\right) \left(\int \left|\int f(z,y)\,dz\right|^p dy\right)^{1/p'}.
\end{aligned}$$

Thus,

$$\int \left|\int f(x,y)\,dx\right|^p dy \le \left(\int \left(\int |f(x,y)|^p\,dy\right)^{1/p} dx\right) \left(\int \left|\int f(z,y)\,dz\right|^p dy\right)^{1/p'},$$

or

$$\left(\int \left|\int f(x,y)\,dx\right|^p dy\right)^{1-1/p'} = \left(\int \left|\int f(x,y)\,dx\right|^p dy\right)^{1/p} \le \int \left(\int |f(x,y)|^p\,dy\right)^{1/p} dx,$$

assuming $\int f(x,y)\,dy \in L^p(x)$. If $\int f(x,y)\,dy \notin L^p(x)$, then use the above result to show that $\left|\left|\int f_k(x,y)\,dy\right|\right|_p \le \int ||f_k(x,y)||_p\,dy$, $f_k = f\mathbb{1}_{||(x,y)|| \le k}$. Since $f_k \nearrow f$, by the monotone convergence theorem it follows that

$$\lim \left|\left|\int f_k(x,y)\,dy\right|\right|_p = \left|\left|\int f(x,y)\,dy\right|\right|_p \le \int ||f(x,y)||_p\,dy = \lim \int ||f_k(x,y)||_p\,dy,$$

so $\int ||f(x,y)||_p\,dy = +\infty$ too. $\square$

**8.9.** If $f$ is real-valued and measurable on $E$, define its *essential infimum* on $E$ by

$$\operatorname*{ess\,inf}_E f = \sup\left\{\alpha : |\{x \in E : f(x) < \alpha\}| = 0\right\}.$$

If $f \geq 0$, show that $\operatorname{ess\,inf}_E f = (\operatorname{ess\,sup}_E 1/f)^{-1}$.

*Solution.* Suppose $\operatorname{ess\,inf}_E f = 0$. Then for every $\alpha > 0$ we have $|\{x \in E : f(x) > \alpha\}| > 0$; thus, for every $0 < \beta < +\infty$ we have $|\{x \in E : 1/f(x) < \beta\}| > 0$, so $\operatorname{ess\,sup}_E 1/f = +\infty$. Interpreting $+\infty^{-1} = 0$, the proposition holds.

Now suppose $\operatorname{ess\,inf}_E f > 0$, so there exists $\alpha > 0$ such that $|\{x \in E : f(x) < \alpha\}| = 0$. Then

$$\begin{aligned}
\operatorname*{ess\,inf}_E f &= \sup\{\alpha > 0 : |\{x \in E : f(x) < \alpha\}| = 0\} \\
&= \sup\{1/\beta : |\{x \in E : f(x) < 1/\beta\}| = 0\} \\
&= \sup\{1/\beta : |\{x \in E : 1/f(x) > \beta\}| = 0\} \\
&= \big(\inf\{\beta : |\{x \in E : 1/f(x) > \beta\}| = 0\}\big)^{-1} \\
&= (\operatorname*{ess\,sup}_E 1/f)^{-1}.
\end{aligned}$$

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