Problem 1. Zygmund p111 exercise 28

Let E be a measurable set in $\mathbb{R}^n$ with $|E| < \infty$. Suppose that $f > 0$ a.e. in E and $f, \log f \in L^1(E)$. Prove that

$$\lim_{p \to 0^+} \left(\frac{1}{|E|} \int_E f^p \, dx \right)^{1/p} = \exp\left(\frac{1}{|E|} \int_E \log f \, dx \right).$$

(Start by using Theorem 5.36 to show that $\int_E f^p \, dx \to |E|$ as $p \to 0^+$. Note that $\int_E (f^p - 1)^{1/p} \, dx \to \log f$.)

Problem 2. Zygmund p111 exercise 29

Let f be measurable, nonnegative, and finite a.e. in a set E. Prove that for any nonnegative constant c,

$$\int_E e^{cf(x)} \, dx = |E| + c \int_0^\infty e^{c\alpha} \omega f(\alpha) \, d\alpha.$$

Deduce that $e^{cf} \in L^1(E)$ if $|E| < \infty$ and there exist constants C_1 and c_1 such that $c_1 > c$ and $\omega f(\alpha) \leq C_1 e^{-c_1 \alpha}$ for all $\alpha > 0$. We will study such an exponential integrability property in Section 14.5.