MATH 6337: Homework 10 Solutions

8.11. If $f_k \to f$ in L^p, $1 \le p < \infty$, $g_k \to g$ pointwise, and $||g_k||_\infty \le M$ for all k, prove that $f_k g_k \to fg$ in L^p.

Solution. By Minkowski's inequality,

$$||f_k g_k - fg||_p \le ||f_k g_k - f g_k||_p + ||f g_k - fg||_p \le M\, ||f_k - f||_p + ||f g_k - fg||_p \,.$$

Since $g_k \to g$ pointwise and since $|g_k| \le M$ a.e., it follows that $|g| \le M$ a.e., so $|g_k - g| \le 2M$ a.e. Thus,

$$\int |f g_k - fg|^p \le \int (2M)^p \, |f|^p < +\infty,$$

so $|f g_k - fg|^p \in L^1$. By the dominated convergence theorem, we then have

$$\lim_{k\to\infty} \int |f g_k - fg|^p = \int \lim_{k\to\infty} |f|^p \, |g_k - g|^p = 0,$$

so $||f g_k - fg||_p \to 0$. Since $||f_k - f||_p \to 0$ as well, we have

$$\lim_{k\to\infty} ||f_k g_k - fg||_p = 0,$$

as desired. □

8.12. Let $f, \{f_k\} \in L^p$. Show that if $||f - f_k||_p \to 0$, then $||f_k||_p \to ||f||_p$. Conversely, if $f_k \to f$ a.e. and $||f_k||_p \to ||f||_p$, $1 \le p < \infty$, show that $||f - f_k||_p \to 0$.

Solution. By Minkowski's inequality, $\left| ||f||_p - ||f_k||_p \right| \le ||f - f_k||_p$, so $||f - f_k||_p \to 0$ implies $||f||_p \to ||f_k||_p$. (If $p < 1$, we have $\left| ||f||_p - ||f_k||_p \right| \le c\, ||f - f_k||_p$ for some $c > 1$, so the argument still holds.) For the converse, since $p \ge 1$, we have

$$|f - f_k|^p \le 2^{p-1} \left(|f|^p + |f_k|^p \right),$$

so by Fatou's lemma and the fact that $f_k \to f$ a.e., we have

$$\begin{aligned}
\int 2^p \, |f|^p &= \int \liminf_k 2^{p-1} \left(|f|^p + |f_k|^p \right) - |f - f_k|^p \\
&\le \liminf_k \int 2^{p-1} \left(|f|^p + |f_k|^p \right) + \liminf_k - \int |f - f_k|^p \\
&= \lim_k \int 2^{p-1} \left(|f|^p + |f_k|^p \right) - \limsup_k \int |f - f_k|^p \\
&= \int 2^p \, |f|^p - \limsup_k \int |f - f_k|^p \,,
\end{aligned}$$

where the last equality follows by the dominated convergence theorem since $|f|^p$ and $|f_k|^p$ are in L^1. Thus, $\limsup_k \int |f - f_k|^p \le 0$, so $||f - f_k||_p \to 0$. □

8.13. Suppose that $f_k \to f$ a.e. and that $f_k, f \in L^p$, $1 < p < \infty$. If $\|f_k\|_p \le M < +\infty$, show that $\int f_k g \to \int fg$ for all $g \in L^{p'}$, $1/p + 1/p' = 1$.

Solution. First suppose $|E| < +\infty$: we may assume $|E| > 0$, $M > 0$, and $\|g\|_{p'} > 0$ or else the result is trivial. Let $\varepsilon > 0$. Then $g \in L^{p'}$, so $g^{p'} \in L^1$, so there exists $\delta > 0$ such that

$$\int_A \left|g^{p'}\right| < \varepsilon^{p'}$$

for all $|A| < \delta$. Since $f \in L^p$, f is finite a.e. in E; since $f_k \to f$ a.e., by Egorov there exists a closed subset $F \subseteq E$ such that $|E \setminus F| < \delta$ and $f_k \to f$ uniformly on F, so $|f_k(x) - f(x)| < \varepsilon$ for large enough k and all $x \in F$. Then

$$\begin{aligned}
\left|\int_E f_k g - \int_E fg\right| &\le \int_E |(f - f_k)g| \\
&= \int_F |(f - f_k)g| + \int_{E\setminus F} |(f - f_k)g| \\
&\le \varepsilon \int_F |g| + \|f - f_k\|_p \left(\int_{E\setminus F} |g|^{p'}\right)^{1/p'} \\
&\le \varepsilon \|\mathbb{1}_F\|_p \|g\|_{p'} + 2M\varepsilon \\
&\le \varepsilon(|E|^{1/p} \|g\|_{p'} + 2M),
\end{aligned}$$

where we use Hölder's inequality twice.

Now suppose $|E| = +\infty$: apply the result for $|E| < +\infty$ to obtain

$$\lim_k \int (f_k - f)g_j = 0$$

for all $j \ge 1$, where $g_j = g\mathbb{1}_{E\cap\{|x|<j\}}$. Since $g \in L^p(E)$ and $|g_j| \nearrow |g|$, by the monotone convergence theorem

$$\lim_j \int |g_j|^{p'} = \int |g|^{p'},$$

so we can find j such that

$$\int_{E\setminus\{|x|<j\}} |g|^{p'} = \int_E |g|^{p'} - \int_E |g_j|^{p'} < \varepsilon^{p'}.$$

Then by Hölder's inequality we have

$$\int |(f_k - f)g| \le \int |(f_k - f)g_j| + \int |(f_k - f)(g - g_j)| =$$

$$\int |(f_k - f)g_j| + \|f - f_k\|_p \|g - g_j\|_{p'} < \varepsilon + 2M\varepsilon.$$

$\square$

8.16. A sequence $\{f_k\}$ in L^p is said to *converge weakly* to a function f in L^p if $\int f_k g \to \int fg$ for all $g \in L^{p'}$. Prove that if $f_k \to f$ in L^p norm, $1 \le p \le \infty$, then $\{f_k\}$ converges weakly to f in L^p. Note by Exercise 15 that the converse is not true.

Solution. Given $g \in L^{p'}$, by Hölder's inequality we have

$$\left| \int_E f_k g - \int_E fg \right| \le \int_E |(f - f_k)g| \le ||f - f_k||_p \, ||g||_{p'} .$$

If $f_k \to f$ in L^p norm, then $\int_E f_k g \to \int_E fg$, so f_k converges weakly to f. □

8.17. Suppose that $f_k, f \in L^2$ and that $\int f_k g \to \int fg$ for all $g \in L^2$ (that is, $\{f_k\}$ converges weakly to f in L^2). If $||f_k||_2 \to ||f||_2$, show that $f_k \to f$ in L^2 norm.

Solution. Let $\langle f, g \rangle = \int fg$ be the inner product in L^2. Since $f_k \to f$ weakly in L^2, we have

$$\langle f_k, f \rangle \to \langle f, f \rangle = ||f||_2^2 \quad \text{and} \quad \langle f, f_k \rangle \to \langle f, f \rangle = ||f||_2^2 .$$

Then

$$||f_k - f||_2^2 = ||f||_2^2 - \langle f_k, f \rangle - \langle f, f_k \rangle + ||f_k||_2^2 .$$

Since $f_k \to f$ weakly, the middle terms limit to $-2\,||f||_2^2$. Since $||f_k||_2 \to ||f||_2$, the last term limits to $||f||_2^2$. Thus, $||f_k - f||_2^2 \to 0$, so $f_k \to f$ in L^2 norm. □

8.18. Prove the *parallelogram law* for L^2:

$$||f + g||^2 + ||f - g||^2 = 2\,||f||^2 + 2\,||g||^2 .$$

Is this true for L^p when $p \ne 2$? The geometric interpretation is that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of the sides.

Solution. Since L^2 is a Hilbert space, we have the identity $||h||^2 = \langle h, h \rangle$:

$$\begin{aligned} ||f + g||^2 + ||f - g||^2 &= \langle f + g, f + g \rangle + \langle f - g, f - g \rangle \\ &= (||f||^2 + \langle f, g \rangle + \langle g, f \rangle + ||g||^2) + (||f||^2 - \langle f, g \rangle - \langle g, f \rangle + ||g||^2) \\ &= 2(||f||^2 + ||g||^2). \end{aligned}$$

The identity need not hold if $p \ne 2$; there are many counterexamples. □

8.22. Let $\{\varphi_k\}$ be a complete orthonormal system in L^2, and let $m = \{m_k\}$ be a given sequence of numbers. If $f \in L^2$, $f \sim \sum c_k\varphi_k$, define Tf by $Tf \sim \sum m_k c_k \varphi_k$. Show that T is bounded on L^2, i.e., that there is a constant c independent of f such that $||Tf||_2 \leq c\,||f||_2$ for all $f \in L^2$, if and only if $m \in \ell^\infty$. Show that the smallest possible choice for c is $||m||_\infty$.

Solution. By Parseval's identity, we have

$$||Tf||^2 = \sum_{k=1}^{\infty} |m_k c_k|^2 .$$

If $||m||_\infty < +\infty$, then

$$||Tf||^2 \leq ||m||_\infty^2 \sum_{k=1}^{\infty} |c_k|^2 = ||m||_\infty^2 \, ||f||^2 .$$

If $||m||_\infty = +\infty$, for every $M > 0$ there exists $m_k > M$ for some k. Let $f = \varphi_k$: then

$$||Tf|| = |m_k| > M \, ||f|| .$$

Since M can be arbitrarily large, there is no constant c such that $||Tf|| \leq c\,||f||$ for all f.

If $||m||_\infty < +\infty$, for every $\varepsilon > 0$ there is some k such that $||m||_\infty - m_k < \varepsilon$. Let $f = \varphi_k$: then

$$||Tf|| = |m_k| .$$

So there is sequence of functions such that $||Tf|| \nearrow ||m||_\infty$, so $||m||_\infty$ is the smallest possible choice for c. $\square$