MATH 6337: Homework 11 Solutions

7.1. Let f be measurable in $\mathbb{R}^n$ and different from zero in some set of positive measure. Show that there is a positive constant c such that $f^*(x) \geq c\,|x|^{-n}$ for $|x| \geq 1$.

Solution. Let $E \subset \mathbb{R}^n$ be a bounded subset of positive measure on which $|f| > 0$, let $x \in \mathbb{R}^n$ have magnitude at least 1, and let Q' be the smallest cube centered at x containing E. If the side length of Q' is $k\,|x|$, take $c = \frac{\int_E |f|}{k^n}$. Then

$$f^*(x) = \sup_Q \frac{1}{|Q|} \int_Q |f| \geq \sup_Q \frac{1}{|Q|} \int_{Q \cap E} |f| \geq \frac{1}{|Q'|} \int_{Q' \cap E} |f| = \frac{\int_E |f|}{k^n\,|x|^n} = c\,|x|^{-n}.$$

□

7.2. Let $\varphi(x)$, $x \in \mathbb{R}^n$, be a bounded measurable function such that $\varphi(x) = 0$ for $|x| \geq 1$ and $\int \varphi = 1$. For $\varepsilon > 0$, let $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$. ($\varphi_\varepsilon$ is called an *approximation to the identity.*) If $f \in L(\mathbb{R}^n)$, show that

$$\lim_{\varepsilon \to 0} (f * \varphi_\varepsilon)(x) = f(x)$$

in the Lebesgue set of f.* [Note that $\int \varphi_\varepsilon = 1$ when $\varepsilon > 0$, so that

$$(f * \varphi_\varepsilon)(x) - f(x) = \int [f(x-y) - f(x)]\varphi_\varepsilon(y)\,dy.$$

Use (7.16).†]

Solution. Observe that if $x \geq \varepsilon$, then $\varphi_\varepsilon(x) = 0$. Then, by a change of variables $u = x/\varepsilon$, we have

$$\int_{|x|<\varepsilon} \varphi_\varepsilon(x)\,dx = \varepsilon^{-n} \int_{|x|<\varepsilon} \varphi(x/\varepsilon)\,dx = \int_{|u|<1} \varphi(u)\,du = 1.$$

Thus

$$|(f * \varphi_\varepsilon)(x) - f(x)| \leq \int_{|y|<\varepsilon} |f(x-y) - f(x)|\,|\varphi_\varepsilon(y)|\,dy.$$

Since φ is bounded (by, say, M), we have $\varphi_\varepsilon(y) = \varepsilon^{-n}\varphi(y/\varepsilon) \leq \varepsilon^{-n}M$. Then

$$\int_{|y|<\varepsilon} |f(x-y) - f(x)|\,|\varphi_\varepsilon(y)|\,dy \leq \frac{M}{\varepsilon^n} \int_{|y|<\varepsilon} |f(x-y) - f(x)|\,dy.$$

Let Q_ε be a cube centered at 0 contained in $B_\varepsilon(0)$. Then Q_ε is a cube with volume $c^n\varepsilon^n$, where $c \leq 2$, so

$$\frac{M}{\varepsilon^n} \int_{|y|<\varepsilon} |f(x-y) - f(x)|\,dy \leq \frac{Mc^n}{|Q_\varepsilon|} \int_{Q_\varepsilon} |f(x-y) - f(x)|\,dy.$$

After the change of variables $v = x - y$, we have

$$|(f * \varphi_\varepsilon)(x) - f(x)| \leq \frac{Mc^n}{|Q_\varepsilon|} \int_{Q_\varepsilon + x} |f(v) - f(x)|\,dv.$$

Then by (7.16), it follows that for every point x in the Lebesgue set of f, we have

$$|(f * \varphi_\varepsilon)(x) - f(x)| \to 0.$$

□

*That is, the set of points x at which $\lim_{Q \searrow x} \frac{1}{|Q|}\int_Q |f(y) - f(x)|\,dy = 0$.

†Let f be locally integrable in $\mathbb{R}^n$. Then at every point x of the Lebesgue set of f (in particular, almost everywhere), $\frac{1}{|S|}\int_S |f(y) - f(x)|\,dy \to 0$.

7.6. Show that if $\alpha > 0$, x^α is absolutely continuous on every bounded subinterval of $[0, \infty)$.

Solution. On the interval $[a, b]$ (or (a, b), etc.), $F(x) = x^\alpha$ is the sum of an indefinite integral and a constant:

$$F(x) = \alpha \int_a^x x^{\alpha-1}\, dx + a^\alpha.$$

Constants and indefinite integrals of integrable functions are absolutely continuous, as is the sum of absolutely continuous functions, so F is absolutely continuous provided $x^{\alpha-1}$ is integrable on $[a, b]$. This is clearly true unless $a = 0$ and $\alpha < 1$. However, in this case we have a convergent improper Riemann integral since $\alpha > 0$, and since the integrand is nonnegative, the Lebesgue integral exists and equals the Riemann integral. Thus, $x^{\alpha-1}$ is integrable on $[0, b]$, so we're done. $\square$

7.12. Use Jensen's inequality to prove that for $a, b \geq 0$; $p, q > 1$; and $(1/p) + (1/q) = 1$, we have

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

More generally, show that

$$a_1 \cdots a_N \leq \sum_{j=1}^{N} \frac{a_j^{p_j}}{p_j},$$

where $a_j \geq 0$, $p_j > 1$, and $\sum_{j=1}^{N}(1/p_j) = 1$. [Hint: Write $a_j = e^{x_j/p_j}$ and use the convexity of e^x.]

Solution. We prove the general case. We may assume none of the $a_j = 0$, or else the claim is immediate. Pick $x_j = p_j \log(a_j)$; then

$$a_1 \cdots a_N = e^{\sum x_j/p_j} = \exp\left(\frac{\sum x_j/p_j}{\sum 1/p_j}\right)$$

since $\sum 1/p_j = 1$. Since e^x is a convex function, we use Jensen's inequality:

$$\exp\left(\frac{\sum x_j/p_j}{\sum 1/p_j}\right) \leq \frac{\sum e^{x_j}/p_j}{\sum 1/p_j} = \sum \frac{e^{x_j}}{p_j} = \sum \frac{a_j^{p_j}}{p_j}.$$

$\square$

7.13. Prove Theorem (7.36):

(i) If φ_1 and φ_2 are convex in (a,b), then $\varphi_1+\varphi_2$ is convex in (a,b).

(ii) If φ is convex in (a,b) and c is a positive constant, then $c\varphi$ is convex in (a,b).

(iii) If φ_k, $k=1,2,...$, are convex in (a,b) and $\varphi_k \to \varphi$ in (a,b), then φ is convex in (a,b).

Solution. Let $x, y \in (a,b)$.

(i) Let $\varphi = \varphi_1 + \varphi_2$. Then for all $t \in [0,1]$, we have

$$\varphi(tx+(1-t)y) = \varphi_1(tx+(1-t)y) + \varphi_2(tx+(1-t)y) \leq$$
$$t\varphi_1(x) + (1-t)\varphi_1(y) + t\varphi_2(x) + (1-t)\varphi_2(y) = t\varphi(x) + (1-t)\varphi(y).$$

(ii) Let $\psi = c\varphi$. Then for all $t \in [0,1]$, we have

$$\psi(tx+(1-t)y) = c\varphi(tx+(1-t)y) \leq c(t\varphi(x)+(1-t)\varphi(y)) = t\psi(x) + (1-t)\psi(y).$$

(iii) For all $t \in [0,1]$ and all $k \geq 1$, we have

$$\varphi_k(tx+(1-t)y) \leq t\varphi_k(x) + (1-t)\varphi_k(y).$$

Taking limits, since $\varphi_k \to \varphi$ pointwise in (a,b), we have

$$\varphi(tx+(1-t)y) = \lim_{k\to\infty} \varphi_k(tx+(1-t)y) \leq \lim_{k\to\infty} t\varphi_k(x) + (1-t)\varphi_k(y) = t\varphi(x) + (1-t)\varphi(y).$$

$\square$