MA 108B PROBLEM SET 3 SOLUTIONS

Problem 1 (Wheeden–Zygmund Chapter 2 Problem 14)

We consider the pair of functions on $[-1,1]$ given by

$$f(x) = \begin{cases} 0\,, & -1 \le x < 0 \\ 1\,, & 0 \le x \le 1 \end{cases} \qquad \phi(x) = \begin{cases} 0\,, & -1 \le x \le 0 \\ 1\,, & 0 < x \le 1 \end{cases}$$

We have already seen in the example on page 34 of the textbook that the Riemann–Stieltjes integral $\int_{-1}^{1} f \,\mathrm{d}\phi$ does not exist. For any partition $\Gamma = \{-1 = x_0 < \cdots < x_m = 0\}$ of $[-1,0]$ we compute

$$S_\Gamma = \sum_{i=1}^{m} f(\xi_i)[\phi(x_i) - \phi(x_{i-1})] = 0$$

since ϕ is constant on $[-1,0]$. Similarly, for partitions Γ of $[0,1]$ we get

$$S_\Gamma = \sum_{i=1}^{m} f(\xi_i)[\phi(x_i) - \phi(x_{i-1})] = f(\xi_1) = 1\,.$$

Since the above sums do not depend on the partition, the Riemann–Stieltjes integral exists.

Problem 2 (Wheeden–Zygmund Chapter 2 Problem 16)

We first consider the two simplest cases of a jump singularity. Let $c \in [a,b]$ and consider the functions

$$f_c^+(x) = \begin{cases} 0\,, & a \le x < c \\ 1\,, & c \le x \le b \end{cases} \qquad f_c^-(x) = \begin{cases} 0\,, & a \le x \le c \\ 1\,, & c < x \le b\,. \end{cases}$$

Now suppose ϕ is a function of bounded variation which is continuous at c. Let $\Gamma = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be any partition. Then there is an i such that $c \in [x_{i-1}, x_i]$. Then for any selection of intermediate points ξ_i there is a $y \in \{x_{i-1}, x_i, x_{i+1}\}$ such that

$$S_\Gamma(f_c^\pm, \phi) = \sum_{i=1}^{m} f_c^\pm(\xi_i)[\phi(x_i) - \phi(x_{i-1})] = \phi(b) - \phi(y)\,.$$

Since ϕ is continuous at c and $|c-y| < 2|\Gamma|$ we see that $\int_a^b f \,\mathrm{d}\phi$ exists and equals $\phi(b) - \phi(c)$.

We now consider the general case where f has jump discontinuities at finitely many points $c_1, \ldots, c_k$ (and ϕ continuous at each c_i). Then we can write f as

$$f = g + \sum_{j=1}^{k} d_j f_{c_j}^+ + \sum_{j=1}^{k} e_j f_{c_j}^-$$

with some coefficients d_j, e_j and a continuous function g. To see this, start with c_1. If $f(c_1) = f(c_1-) \neq f(c_1+)$ then set $d_1 = 0, e_1 = f(c_1+) - f(c_1)$. Similarly if $f(c_1) = f(c_1+) \neq f(c_1-)$ then set $d_1 = f(c_1) - f(c_1-), e_1 = 0$. Finally if all three numbers are different, then set

$d_1 = f(c_1) - f(c_1-), e_1 = f(c_1+) - f(c_1)$. The function $f - d_1 f^+_{c_1} - e_1 f^-_{c_1}$ only has jump discontinuities at $c_2, \dots, c_k$. Proceed iteratively.

From the simple case above and Theorem 2.16 we conclude that

$$\lim_{|\Gamma|\to 0} S_\Gamma\big(\sum_j d_j f^+_{c_j} + \sum_j e_j f^-_{c_j}, \phi\big)$$

exists, and by Theorem 2.24 so does $\lim_{|\Gamma|\to 0} S_\Gamma(g,\phi)$. Evoking again Theorem 2.16 we can conclude that $\int_a^b f \,\mathrm{d}\phi$ exists.

Problem 3 (Wheeden–Zygmund Chapter 3 Problem 3)

With $D_0 = [0,1]\times[0,1]$ let D_k be the set of points remaining after k steps of the process and set $D = \bigcap_k D_k$. The set D_k is a union of 4^k closed squares, each of area 9^{-k}, i.e. $D_k = \bigcup_{j=1}^{4^k} Q^k_j$. Thus D_k is closed and as an intersection of closed sets, D is closed. By Theorem 3.18 (or Theorem 3.14) it is measurable. Since $|D| \le |D_k| \le 4^k 9^{-k}$ for all $k \ge 1$, it holds that $|D| = 0$.

Let $a^k_j, b^k_j, c^k_j, d^k_j$ be the corners of Q^k_j. Then each of these points will be the corner of some cube of D_{k+1} and thus $a^k_j, b^k_j, c^k_j, d^k_j \in D$. In particular D is not empty.

Since D is closed it remains to show that every point of D is a limit point of D in order for D to be perfect (and thus uncountable). Let $x \in D$, then for any $k \ge 1$ by the definition of D, there exists a cube $Q^k_{j_k}$ with $x \in Q^k_{j_k}$. To construct a sequence of points x_k in D that converges to x, choose for example

$$x_k = \begin{cases} a^k_{j_k}, & x \ne a^k_{j_k} \\ b^k_{j_k}, & x = a^k_{j_k} . \end{cases}$$

As seen above $x_k \in D$ and $|x - x_k| \le \operatorname{diam}(Q^k_j) = 3^{-k}\sqrt{2}$. Thus $x_k \to x$.

To show that $D = C\times C$ we note that $D_k = C_k \times C_k$. This is easy to see and can be proved formally by induction: For $k=1$, the statement clearly is true. Now let $x = (x_1, x_2) \in D_{k+1}$. Then x is in one of the four corner subsquares of some cube $x \in Q^k_{j_0}$ of D_k. The projection of $Q^k_{j_0}$ onto the x-axis lies by the induction hypotheses in C_k. Since x is in one of the corner subsquares, x_1 is in either the first third or the last third of the projection onto the x-axis and thus by definition in C_{k+1}. Similarly $x_2 \in C_{k+1}$ and arguing backwards we also see that $x = (x_1, x_2) \in D_{k+1}$ if $x_1 \in C_{k+1}$ and $x_2 \in C_{k+1}$. Finally,

$$D = \bigcap_k D_k = \bigcap_k (C_k \times C_k) = \Big(\bigcap_k C_k\Big) \times \Big(\bigcap_k C_k\Big) = C \times C .$$

Problem 4 (Wheeden–Zygmund Chapter 3 Problem 4)

Fix θ. With $D_0 = [0,1]$ let D_k be the set of points remaining after k steps of the process and set $D = \bigcap_k D_k$. The set D_k is a union of 2^k closed intervals, each of length $(\frac{1-\theta}{2})^k$, i.e. $D_k = \bigcup_{j=1}^{2^k} I^k_j$. Thus D_k is closed and as an intersection of closed sets, D is closed. By Theorem 3.18 (or Theorem 3.14) it is measurable. Since $|D| \le |D_k| \le 2^k(1-\theta)^k 2^{-k} = (1-\theta)^k$ for all $k \ge 1$, it holds that $|D| = 0$ (Note that $\lim_{k\to\infty}(1-\theta)^k = 0$).

Let a^k_j, b^k_j be the endpoints of I^k_j. Then each of these points will be the endpoint of some interval of D_{k+1} and thus $a^k_j, b^k_j \in D$. In particular D is not empty.

Since D is closed it remains to show that every point of D is a limit point of D in order for D to be perfect (and thus uncountable). Let $x \in D$, then for any $k \geq 1$ by the definition of D, there exists an interval $I^k_{j_k}$ with $x \in I^k_{j_k}$. To construct a sequence of points x_k in D that converges to x, choose for example

$$x_k = \begin{cases} a^k_{j_k}\,, & x \neq a^k_{j_k} \\ b^k_{j_k}\,, & x = a^k_{j_k}\,. \end{cases}$$

As seen above $x_k \in D$ and $|x - x_k| \leq \operatorname{diam}(I^k_j) = (1-\theta)^k 2^{-k}$. Thus $x_k \to x$.

Problem 5 (Wheeden–Zygmund Chapter 3 Problem 18)

First note that for any n-dimensional interval $I = \{x : a_i \leq x_i \leq b_i \, i = 1, \dots, n\}$, clearly

$$|I|_e = v(I) = \prod_{i=1}^{n}(b_i - a_i) = \prod_{i=1}^{n}(b_i + h - a_i - h) = v(I_h) = |I_h|_e\,.$$

Let $S = \{I_k\}$ be a countable cover of E consisting of intervals. Then $S' = \{(I_k)_h\}$ is a countable cover of E_h consisting of intervals and $\sigma(S) = \sum_k v(I_k) = \sum_k v((I_k)_h) = \sigma(S')$. Thus for any cover S of E there is a cover S' of E_h with $\sigma(S) = \sigma(S')$ and by the definition of the outer Lebesgue measure we conclude $|E|_e \geq |E_h|_e$. Noting that $E = (E_h)_{-h}$ and repeating the argument yields $|E_h|_e \geq |E|_e$ and thus $|E|_e = |E_h|_e$.

If E is measurable then for $\varepsilon > 0$ there exists an open G with $E \subset G$ and $|G - E|_e < \varepsilon$. The set G_h is open (for a formal argument note that $G_h = f^{-1}(G)$ for the continuous function $f(x) = x - h$) and $E_h \subset G_h$. Furthermore $G_h - E_h = (G - E)_h$ and thus $|G_h - E_h|_e = |G - E|_e < \varepsilon$, so E_h is measurable.