Real Analysis I Homework Solution 4

Jian-An Wang

https://jiananwang1007.wordpress.com · jiananwang1007@gmail.com

Problem 1(a)

Suppose $\{E_k\}_{k=1}^{\infty}$ is a countable family of subsets of $\mathbb{R}^d$ and that

$$\sum_{k=1}^{\infty} |E_k|_e < \infty$$

Let $E = \limsup_{k\to\infty}(E_k)$. Prove $|E|_e = 0$.

Proof. We have

$$|E|_e = \left| \bigcap_{n\in\mathbb{N}} \bigcup_{k\geq n} E_k \right|_e \leq \left| \bigcup_{k\geq n} E_k \right|_e \leq \sum_{k=n}^{\infty} |E_k|_e$$

for all $n \in \mathbb{N}$. Since we know

$$\sum_{k=1}^{\infty} |E_k|_e < \infty$$

it follows that

$$\lim_{n\to\infty} \sum_{k=n}^{\infty} |E_k|_e = 0$$

Therefore, we have

$$0 \leq |E|_e \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} |E_k|_e = 0$$

That is, we obtain

$$|E|_e = 0$$

□

Problem 1(b)

Prove that the set of those $x \in \mathbb{R}$ such that there exist infinitely many fractions $\dfrac{p}{q}$, with relatively prime integers p and q such that for every $\epsilon > 0$, we have

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero.

Proof. For every $m \in \mathbb{Z}$, we define

$$E_{m,q} = \left\{ x \in [m, m+1) : \left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}} \text{ for some } p \in \mathbb{Z} \right\}$$

for all $q \in \mathbb{N}$. Then we know

$$E_{m,q} \subseteq \overline{B_r\left(m + \frac{0}{q}\right)} \cup \overline{B_r\left(m + \frac{1}{q}\right)} \cup \cdots \cup \overline{B_r\left(m + \frac{q}{q}\right)}$$

where $r = \dfrac{1}{q^{2+\epsilon}}$. Hence, we have

$$|E_{m,q}|_e \leq \sum_{i=0}^{q} \left| \overline{B_r\left(m + \frac{i}{q}\right)} \right|_e = \sum_{i=0}^{q} \frac{2}{q^{2+\epsilon}} = \frac{2(q+1)}{q^{2+\epsilon}}$$

In other words, we obtain

$$\sum_{q=1}^{\infty} |E_{m,q}|_e \leq \sum_{q=1}^{\infty} \frac{2(q+1)}{q^{2+\epsilon}} < \infty$$

By (a), for every $m \in \mathbb{Z}$, we get

$$\left| \limsup_{q \to \infty} E_{m,q} \right| = 0$$

The set in the problem is clearly $\bigsqcup_{m \in \mathbb{Z}} \limsup_{q \to \infty} E_{m,q}$. Moreover, we know

$$\left| \bigsqcup_{m \in \mathbb{Z}} \limsup_{q \to \infty} E_{m,q} \right| = \sum_{m \in \mathbb{Z}} \left| \limsup_{q \to \infty} E_{m,q} \right| = 0$$

$\square$

Problem 2(a)

Let E be a subset of $\mathbb{R}$ with $|E|_e > 0$. Prove that for each $0 < \alpha < 1$, there exists an open interval I so that

$$|E \cap I|_e \geq \alpha |I|_e$$

Loosely speaking, this estimate shows that E contains almost a whole interval.

Proof. We split the problem into two cases.

Case 1: If $|E|_e < \infty$, then for every $0 < \epsilon < \dfrac{(1-\alpha)|E|_e}{\alpha}$, there exists a countable covering $E \subseteq \bigcup_{j\in\mathbb{N}} I_j^\epsilon$ by closed intervals such that

$$\sum_{j=1}^{\infty} |I_j^\epsilon| < |E|_e + \epsilon$$

since we know

$$|E|_e = \inf \sum_{j=1}^{\infty} |I_j|$$

where the infimum is taken over all countable coverings $E \subseteq \bigcup_{j\in\mathbb{N}} I_j$ by closed intervals. Hence, we have

$$\alpha \sum_{j=1}^{\infty} |I_j^\epsilon| < \alpha\left(|E|_e + \epsilon\right) \leq |E|_e \leq \sum_{j=1}^{\infty} |E \cap I_j^\epsilon|_e$$

since we know

$$E = \bigcup_{j\in\mathbb{N}} (E \cap I_j^\epsilon)$$

By the pigeonhole principle, there exists $j^* \in \mathbb{N}$ such that

$$\alpha|I_{j^*}^\epsilon| \leq |E \cap I_{j^*}^\epsilon|_e$$

Let

$$I = \operatorname{int} I_{i^*}^\epsilon$$

Then we have I is an open interval and

$$|I|_e = |I_{j^*}^\epsilon| \qquad |E \cap I_{j^*}^\epsilon|_e = |E \cap I|_e$$

That is, we obtain

$$|E \cap I|_e \geq \alpha|I|_e$$

Case 2: If $|E|_e = \infty$, then there exists $N \in \mathbb{N}$ such that

$$0 < \Big|E \cap [-N, N]\Big|_e < \infty$$

Let $E' = E \cap [-N, N]$. By **Case 1**, we know there exists an open interval I such that

$$\alpha|I|_e \leq |E' \cap I|_e \leq |E \cap I|_e$$

$\square$

Problem 2(b)

Suppose E is a measurable subset of $\mathbb{R}$ with $|E| > 0$. Prove that the **difference**

set of E, which is defined by

$$E - E = \{x - y \in \mathbb{R} : x, y \in E\}$$

contains an open interval centered at the origin.

Proof. Let $E' \subseteq E$ with $0 < |E'| < \infty$. It suffices to show that $E' - E'$ contains an open interval centered at the origin. Let $\alpha > \frac{1}{2}$. By (a), there exists an open interval I_α such that

$$|E_\alpha| \geq \alpha |I_\alpha|$$

where

$$E_\alpha = E' \cap I_\alpha$$

Suppose for the sake of contradiction that $E_\alpha - E_\alpha$ does not contain an open interval centered at the origin. That is, for every $r > 0$, there exists $a_r \in (-r, r)$ such that

$$a_r \notin E_\alpha - E_\alpha$$

In other words, we know

$$a_r \neq x - y$$

for all $x, y \in E_\alpha$. Hence, we have $y + a_r \notin E_\alpha$ for all $y \in E_\alpha$. That is, E_α and $E_\alpha + a_r$ are disjoint. Therefore, we obtain

$$2\alpha |I_\alpha| \leq 2|E_\alpha| = |E_\alpha| + |E_\alpha + a_r| = |E_\alpha \sqcup (E_\alpha + a_r)| \leq |I_\alpha \cup (I_\alpha + a_r)|$$

By the continuity of measure, we know

$$2\alpha |I_\alpha| \leq \lim_{r \to 0^+} |I_\alpha \cup (I_\alpha + a_r)| = |I_\alpha|$$

In other words, we have $\alpha \leq \frac{1}{2}$, which is a contradiction. Thus, $E_\alpha - E_\alpha$ contains an open interval centered at the origin. Since we know

$$E_\alpha - E_\alpha \subseteq E' - E'$$

it follows that $E' - E'$ also contains an open interval centered at the origin. $\square$