Problem 1. Zygmund p59 exercise 05

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the k-stage, each interval removed has length $\delta 3^{-k}$, where $0 < \delta < 1$. Show that the set has measure $1-\delta$.

Solution.

Construct a subset of $[0,1]$ in the same manner as the Cantor set, except that at the k-th stage, each interval removed has length $\delta 3^{-k}$, $0 < \delta < 1$. Let F_k denote the union of the intervals left at the k-th stage. Now show that the resulting set (Fat Cantor Set) $F = \bigcap_{k=1}^{\infty} F_k$ has positive measure $1-\delta$, and contains no intervals.

By construction,

$$|F_k| = 1 - \sum_{i=1}^{k} 2^{i-1}\delta(\frac{1}{3})^i.$$

Since

$$0 \leq |F|_e \leq |F_k|_e,$$

let $k \to \infty$, we have

$$|F|_e = \lim_{k\to\infty} |F_k|_e = 1-\delta.$$

Since F cannot contain an interval of length greater than $1/2^k$ for all k, so F contains no intervals.

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Problem 2. Zygmund p60 exercise 25

Construct a measurable subset in $[0,1]$ such that for every interval in $[0,1]$, both $E\cap I$ and $E^c \cap I$ have some property.

Solution.

Therefore, we have

$$|I \cap F^c| > 0, \forall I \subseteq [0,1].$$

However, by construction,

$$\exists I \subseteq [0,1] \text{ s.t. } |I \cap F^c| > 0.$$

Construct another such set on each subinterval of the complement of F, and get the resulting set E. Thus, E^c contains no intervals by construction. Therefore, $\forall I \subseteq [0,1]$, we have

$$|I \cap E| > 0,$$
$$|I \cap E^c| > 0.$$

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Problem 3.

Motivated by (3.7), define the inner measure of E by $|E|_i = \sup |F|$, where the supremum is taken over all closed subsets F of E. Show that

1. $|E|_i \leq |E|_e$,

2. if $|E|_e < +\infty$, then E is measurable if and only if $|E|_i = |E|_e$.

(Use Lemma 3.22.)

Theorem 3.6. *Let $E \subset \mathbb{R}^n$. Then, given $\varepsilon > 0$, there exists an open set G such that $E \subset G$ and $|G|_e \leq |E|_e + \varepsilon$. Hence,*

$$|E|_e = \inf |G|_e, \tag{3.7}$$

where the infimum is taken over all open sets G containing E.

Proof. We may assume that $|E_k|_e < +\infty$ for each $k = 1, 2, \ldots$, since otherwise, the conclusion is obvious. Fix $\varepsilon > 0$. Given k, choose intervals $I_j^{(k)}$ such that $E_k \subset \bigcup_j I_j^{(k)}$ and $\sum_j v(I_j^{(k)}) < |E_k|_e + \varepsilon 2^{-k}$.

Since $E \subset \bigcup_{j,k} I_j^{(k)}$, we have $|E|_e \leq \sum_{j,k} v(I_j^{(k)}) = \sum_k \sum_j v(I_j^{(k)})$. Therefore,

$$|E|_e \leq \sum_k (|E_k|_e + \varepsilon 2^{-k}) = \sum_k |E_k|_e + \varepsilon,$$

and the result follows by letting $\varepsilon \to 0$.[a] □

[a] Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis.* CRC, 2015, p. 42.

Problem 4.

Construct a continuous function f such that f on $[0, 1]$ is not of bounded variation on any interval. (Hint: Modify the Cantor-Lebesgue function) [Zygmund p49 exercise 26]

Problem 5.

Show that there are disjoint sets $E_i \subset \mathbb{R}$, where $i = 1, 2, \ldots$, such that $|\bigcup_{i=1}^{\infty} E_i|_e < \sum_{i=1}^{\infty} |E_i|_e$. [Zygmund p48 exercise 20]

Solution.

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