

**Definition of  $\sigma$ -algebra**

A nonempty collection  $\Sigma$  of subsets  $E$  is called a  $\sigma$ -algebra if it satisfies the following conditions<sup>a</sup>:

- (i)  $\setminus E \in \Sigma$  if  $E \in \Sigma$
- (ii)  $\cup_k E_k \in \Sigma$  if  $E_k \in \Sigma, k = 1, 2, \dots$

<sup>a</sup>Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis*. CRC, 2015, p. 49.

**Borel  $\sigma$ -algebra**

The smallest  $\sigma$ -algebra of subsets of  $\mathbb{R}^n$  containing all the open subsets of  $\mathbb{R}^n$  is called the Borel  $\sigma$ -algebra  $\mathfrak{B}$  of  $\mathbb{R}^n$ , and the sets in  $\mathfrak{B}$  are called Borel subsets of  $\mathbb{R}^n$ . Sets of type  $F_\sigma, G_\delta, F_{\sigma\delta}, G_{\delta\sigma}$  (see p.6 in Section 1.3), etc., are Borel sets.<sup>a</sup>

<sup>a</sup>Ibid., p. 49.

**Problem 1. Zygmund p59 exercise 08**

Show that the Borel  $\sigma$ -algebra  $\mathfrak{B}$  in  $\mathbb{R}^n$  is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ .

*Solution.*

To show that the Borel  $\sigma$ -algebra  $\mathfrak{B}$  in  $\mathbb{R}^n$  is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ , we need to show that

- (i)  $\mathfrak{B}$  contains the closed sets in  $\mathbb{R}^n$ .
- (ii)  $\mathfrak{B} \subseteq A$  if  $A$  is a  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ .

(i)

For every  $E$  closed subset in  $\mathbb{R}^n$ , there is a  $\setminus E$  open subset  $\mathbb{R}^n$ . By definition of the Borel  $\sigma$ -algebra  $\mathfrak{B}$ , we have  $\setminus E \in \mathfrak{B}$ , implying that  $E \in \mathfrak{B}$ . Thus,  $\mathfrak{B}$  contains the closed sets in  $\mathbb{R}^n$ .

(ii)

Let  $A$  be a  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ , the definition of  $\sigma$ -algebra implies that  $A$  is a  $\sigma$ -algebra containing all the open sets in  $\mathbb{R}^n$ .

By definition of the Borel  $\sigma$ -algebra  $\mathfrak{B}$ , it is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ . Thus, we have  $\mathfrak{B} \subseteq A$ .

Therefore, we can conclude that the Borel  $\sigma$ -algebra  $\mathfrak{B}$  in  $\mathbb{R}^n$  is the smallest  $\sigma$ -algebra containing the closed sets in  $\mathbb{R}^n$ .

■

**Problem 2. Zygmund p59 exercise 09**

If  $\{E_k\}_{k=1}^\infty$  is a sequence of sets with  $\sum_{k=1}^\infty |E_k|_e < +\infty$ , show that  $\limsup_{k \rightarrow \infty} E_k$  (and so also  $\liminf_{k \rightarrow \infty} E_k$ ) has measure zero.

**Definition of  $\sigma$ -algebra**

<sup>a</sup>Suppose  $\{E_k\}_{k=1}^\infty$  is a sequence of subsets:

$$\limsup_{k \rightarrow \infty} E_k := \lim_{n \rightarrow \infty} V_n = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k,$$

$$\liminf_{k \rightarrow \infty} E_k := \lim_{n \rightarrow \infty} B_n = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k.$$

In other words,

$$V_n = \bigcup_{k=n}^\infty E_k \searrow V = \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k =: \limsup_{k \rightarrow \infty} E_k,$$

$$B_n = \bigcap_{k=n}^\infty E_k \nearrow B = \bigcup_{n=1}^\infty \bigcap_{k=n}^\infty E_k =: \liminf_{k \rightarrow \infty} E_k.$$

<sup>a</sup>Wheeden and Zygmund, see n. a, p. 49.

*Solution.* By the definition of the limit of a sequence, let  $\epsilon > 0$ , there exists  $N$  s.t.  $\forall n \geq N$ ,

$$\sum_{k=n+1}^\infty |E_k|_e = \sum_{k=1}^\infty |E_k|_e - \sum_{k=1}^n |E_k|_e < \epsilon.$$

Since

$$\begin{aligned} \limsup_{k \rightarrow \infty} E_k &:= \bigcap_{n=1}^\infty \bigcup_{k=n}^\infty E_k \\ &\subseteq \bigcup_{k=N}^\infty E_k, \end{aligned}$$

we have

$$\begin{aligned} |\limsup_{k \rightarrow \infty} E_k|_e &\leq \left| \bigcup_{k=N}^\infty E_k \right|_e \\ &= \sum_{k=N}^\infty |E_k|_e < \epsilon. \end{aligned}$$

Let  $\epsilon \rightarrow 0$ , we have

$$|\limsup_{k \rightarrow \infty} E_k|_e = 0.$$

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By the definition, we have

$$\liminf_{k \rightarrow \infty} E_k \subseteq \limsup_{k \rightarrow \infty} E_k.$$

Thus,

$$|\liminf_{k \rightarrow \infty} E_k|_e \leq |\limsup_{k \rightarrow \infty} E_k|_e = 0.$$

Therefore,  $|\liminf_{k \rightarrow \infty} E_k|_e = 0$ .

■

**Problem 3. Zygmund p59 exercise 09**

Show that there exist sets  $\{E_k\}_{k=1}^\infty$  such that  $E_k \searrow E$ ,  $|E_k|_e < +\infty$ , and  $\lim_{k \rightarrow \infty} |E_k|_e > |E|_e$  with strict inequality.