

**Problem 1.**

- (a) Suppose that  $\{E_k\}_{k=1}^{\infty}$  is a countable family of measurable subsets of  $\mathbb{R}^n$  and that

$$\sum_{k=1}^{\infty} |E_k| < +\infty.$$

Let  $E = \limsup_{k \rightarrow \infty} E_k$ . Prove that  $|E| = 0$ .

- (b) Given an irrational  $x$ , one can show (using the pigeonhole principle, for example) that there exist infinitely many fractions  $\frac{p}{q}$ , with relatively prime integers  $p$  and  $q$  such that

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}.$$

However, prove that the set of those  $x \in \mathbb{R}$  such that there exist infinitely many fractions  $\frac{p}{q}$ , with relatively prime integers  $p$  and  $q$  such that  $\forall \epsilon > 0$ ,

$$\left| x - \frac{p}{q} \right| \leq \frac{1}{q^{2+\epsilon}}$$

is a set of measure zero.

*Solution.*

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**Problem 2.**

- (a) Let  $E$  be a subset of  $\mathbb{R}$  with  $|E|_e > 0$ . Prove that for each  $0 < \alpha < 1$ , there exists an open interval  $I$  so that

$$|E \cap I|_e \geq \alpha |I|_e.$$

Loosely speaking, this estimate shows that  $E$  contains almost a whole interval.

- (b) Suppose  $E$  is a measurable subset of  $\mathbb{R}$  with  $|E| > 0$ . Prove that the difference set of  $E$ , which is defined by

$$|E - E| = \{x - y \in \mathbb{R} \mid x, y \in E\}$$

contains an open interval centered at the origin.

*Solution.*

(a)

For any  $\alpha \in (0, 1)$ , let  $I \subseteq \mathbb{R}$  be an open set s.t.  $E \subseteq I$  and  $|E|_e \geq \alpha |I|_e$ , implying that

$$\alpha |I|_e \leq |E|_e \leq |I|_e.$$

Write the open set  $I$  as a countable union of disjoint open intervals:

$$I = \bigsqcup_{k=1}^{\infty} I_k.$$

Thus,

$$E = E \cap I = E \cap \left( \bigsqcup_{k=1}^{\infty} I_k \right) = \bigsqcup_{k=1}^{\infty} (E \cap I_k).$$

By the countable subadditivity of Lebesgue Outer Measure (Theorem 3.4), we have

$$|E|_e \leq \sum_{k=1}^{\infty} |E \cap I_k|_e.$$

Suppose, by way of contradiction, that  $\forall I_k$ ,

$$|E \cap I_k|_e < \alpha |I_k|_e,$$

then we have

$$|E|_e \leq \sum_{k=1}^{\infty} |E \cap I_k|_e < \sum_{k=1}^{\infty} \alpha |I_k|_e = \alpha \sum_{k=1}^{\infty} |I_k|_e = \alpha |I|_e \leq |E|_e.$$

The second equality holds since  $I_k$  are disjoint and open.

Thus, it is implied that

$$|E|_e < |E|_e,$$

which is a contradiction. Therefore, our assumption that  $\forall I_k, |E \cap I_k|_e < \alpha |I_k|_e$  must be false, which means that there exists an open interval  $I$  so that

$$|E \cap I|_e \geq \alpha |I|_e.$$

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(b)

$\exists G$  open s.t.  $E \subseteq G$  and  $|G| < |E|(1 + \epsilon)$ .

Since  $G$  is open,  $G$  can be written as a countable union of disjoint open intervals

$$G = \bigcup_{k=1}^{\infty} \overset{\circ}{I}_k.$$

Let  $E_k = \overset{\circ}{I}_k \cap E$ ,  $\{E_k\}_{k=1}^{\infty}$  is a sequence of disjoint measurable sets.

$$\begin{aligned} |G| &= \sum_{k=1}^{\infty} |\overset{\circ}{I}_k|, \\ |E| &= \sum_{k=1}^{\infty} |E_k|. \\ |G| &= \sum_{k=1}^{\infty} |\overset{\circ}{I}_k| < |E|(1 + \epsilon) \\ &= \left( \sum_{k=1}^{\infty} |E_k| \right) (1 + \epsilon). \end{aligned} \tag{1}$$

$\exists k_0$  s.t.  $|\overset{\circ}{I}_{k_0}| < |E_{k_0}|(1 + \epsilon)$ .

Suppose not; in other words,  $|\overset{\circ}{I}_k| \geq |E_k|(1 + \epsilon), \forall k$ ,

$$\sum_{k=1}^{\infty} |\overset{\circ}{I}_k| \geq \left( \sum_{k=1}^{\infty} |E_k| \right) (1 + \epsilon),$$

contradicting (1).

Let  $\epsilon = \frac{1}{3}$ , we have

$$\begin{aligned} |\overset{\circ}{I}_{k_0}| &< |E_{k_0}|(1 + \frac{1}{3}) = \frac{4}{3}|E_{k_0}|, \\ \frac{3}{4}|\overset{\circ}{I}_{k_0}| &< |E_{k_0}|. \end{aligned}$$

Let  $E_{k_0} + d = \{x + d \mid x \in E_{k_0}\}$ .

Claim: (to be proved by contradiction)

If  $|d| \leq \frac{1}{2}|\overset{\circ}{I}_{k_0}|$ ,

$$(E_{k_0} + d) \cap E_{k_0} \neq \emptyset.$$

$\Rightarrow$

$$\begin{aligned} \left( -\frac{1}{2}|\overset{\circ}{I}_{k_0}|, \frac{1}{2}|\overset{\circ}{I}_{k_0}| \right) &\subseteq \{x - y \mid x, y \in E_{k_0}\} \\ &\subseteq \{x - y \mid x, y \in E\} \end{aligned}$$

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## Proof of Claim

Claim:

If  $|d| \leq \frac{1}{2}|\mathring{I}_{k0}|$ ,

$$(E_{k0} + d) \cap E_{k0} \neq \emptyset.$$

*Proof.* Suppose not; in other words,  $E_{k0} + d$  and  $E_{k0}$  are disjoint measurable sets.

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| &= |(E_{k0} + d)| + |E_{k0}| \\ &= 2|E_{k0}|. \end{aligned}$$

Since  $E_k = \mathring{I}_k \cap E$ , we have

$$(E_{k0} + d) \cup E_{k0} \subseteq (\mathring{I}_{k0} + d) \cup \mathring{I}_{k0},$$

and

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| &\leq |(\mathring{I}_{k0} + d) \cup \mathring{I}_{k0}| \\ &= |\mathring{I}_{k0}| + |d| \\ &< \frac{3}{2}|\mathring{I}_{k0}|. \end{aligned}$$

Thus, we have

$$\begin{aligned} |(E_{k0} + d) \cup E_{k0}| &= 2|E_{k0}| < \frac{3}{2}|\mathring{I}_{k0}| < \frac{3}{2}|E_{k0}|(1 + \frac{1}{3}) \\ &= 2|E_{k0}|. \end{aligned}$$

This leads to a contradiction:  $2|E_{k0}| < 2|E_{k0}|$ .

Therefore, if  $|d| \leq \frac{1}{2}|\mathring{I}_{k0}|$ ,

$$(E_{k0} + d) \cap E_{k0} \neq \emptyset.$$

□

## Countable Subadditivity of Lebesgue Outer Measure

**Theorem 3.4.** If  $E = \bigcup_k E_k$  is a countable union of sets, then  $|E|_e \leq \sum_k |E_k|_e$ .

*Proof.* We may assume that  $|E_k|_e < +\infty$  for each  $k = 1, 2, \dots$ , since otherwise, the conclusion is obvious. Fix  $\varepsilon > 0$ . Given  $k$ , choose intervals  $I_j^{(k)}$  such that  $E_k \subset \bigcup_j I_j^{(k)}$  and  $\sum_j v(I_j^{(k)}) < |E_k|_e + \varepsilon 2^{-k}$ .

Since  $E \subset \bigcup_{j,k} I_j^{(k)}$ , we have  $|E|_e \leq \sum_{j,k} v(I_j^{(k)}) = \sum_k \sum_j v(I_j^{(k)})$ . Therefore,

$$|E|_e \leq \sum_k (|E_k|_e + \varepsilon 2^{-k}) = \sum_k |E_k|_e + \varepsilon,$$

and the result follows by letting  $\varepsilon \rightarrow 0$ .<sup>a</sup>

□

<sup>a</sup>Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis*. CRC, 2015, p. 42.