Math 6337 : Real Analysis I

Mid-term Exam 1
23 September 2011

Instructions: Answer all of the problems.

1. Recall that the outer measure of $E \subset \mathbb{R}^n$ is defined to be

$$|E|_e = \inf\left\{ \sum_j \mathrm{vol}(I_j) \;:\; \{I_j\} \text{ is a cover of } E \text{ by closed intervals in } \mathbb{R}^n \right\}$$

Define a closely related notion as follows:

$$|E|_{e,\mathrm{open}} = \inf\left\{ \sum_j \mathrm{vol}(I_j) \;:\; \{I_j\} \text{ is a cover of } E \text{ by } \textit{open} \text{ intervals in } \mathbb{R}^n \right\}$$

Show that for all $E \subset \mathbb{R}^n$ we have $|E|_e = |E|_{e,\mathrm{open}}$.

Solution: If $\{I_j\}$ is a cover of E by open intervals, then by passing to the closed sets $\{\overline{I_j}\}$, we have a cover of E by closed intervals, and $\mathrm{vol}(I_j) = \mathrm{vol}(\overline{I_j})$ so that we see that $|E|_{e,\mathrm{open}} \geq |E|_e$.

Conversely, let $\{I_j\}$ be a cover of E by closed intervals, and $\epsilon > 0$. To each I_j we associate *open* interval $\widetilde{I}_j$ which strictly contains I_j, and satisfies $\mathrm{vol}(\widetilde{I}_j) \leq \mathrm{vol}(I_j) + \epsilon 2^{-j}$. It follows that

$$\begin{aligned}
|E|_{e,\mathrm{open}} &\leq \sum_{j=1}^{\infty} \mathrm{vol}(\widetilde{I}_j) \\
&\leq \sum_{j=1}^{\infty} \mathrm{vol}(I_j) + \epsilon 2^{-j} \\
&\leq \epsilon + \sum_{j=1}^{\infty} \mathrm{vol}(I_j)
\end{aligned}$$

Taking the infimum over $\{I_j\}$, we see that $|E|_{e,\mathrm{open}} \leq \epsilon + |E|_e$. As $\epsilon > 0$ is arbitrary, this proves $|E|_{e,\mathrm{open}} \leq |E|_e$.

2. Let $Z \subset \mathbb{R}^1$ have measure zero. Show that $Z \times \mathbb{R}$ has measure zero in $\mathbb{R}^2$.

Solution: Set $Z_n = Z \times [n, n+1)$. Then Z is the countable union of the Z_n for $n \in \mathbb{Z}$, so it suffices to show that $|Z_n| = 0$ for all $n \in \mathbb{Z}$.

Given $\epsilon > 0$, select cover of Z by closed intervals $\{I_j\}$ with $\sum_j |I_j| < \epsilon$. Then, $\{I_j \times [n, n+1) \,:\, j \in \mathbb{N}\}$ is a cover of Z_n by closed intervals in $\mathbb{R}^2$, and

$$\sum_j \operatorname{vol}(I_j \times [n, n+1)) = \sum_j |I_j| < \epsilon\,.$$

3. Let C denote the circle of circumference 1, and let α be an irrational number. Let all points of C that can be obtained from each other by rotating C through and angle $n\alpha\pi$, $n \in \mathbb{Z}$, be assigned to the same class (which clearly contains countably many points). Let Φ_0 be any set containing one point from each class. Prove that Φ_0 is non-measurable. Hint: Let Φ_n denote the set Φ_0 obtained by rotating through an angle of $n\alpha\pi$. Then $\Phi_m \cap \Phi_n = \emptyset$ when $m \neq n$ and $C = \bigcup_{n\in\mathbb{Z}} \Phi_n$. Proceed by contradiction.

Solution: Suppose, to reach a contradiction, that the set Φ_0 is measurable. Then the sets Φ_n would be measurable as well. Also note that the measure of the set Φ_0 and the measure of the set Φ_n would coincide. Then, we have that

$$\begin{aligned} 1 &= |C| \\ &= \sum_k |\Phi_k|\,. \end{aligned}$$

Here we used the disjointness of the sets Φ_k. But, this yields a contradiction since the sets Φ_n have the same measure as Φ_0, and there is no way that the sum of the measures of the set Φ_n can add up to 1.

4. Let $A \triangle B = A \cup B \setminus (A \cap B)$. Suppose that $|A|_e\,, |B|_e < \infty$, and then show that for any two sets A and B that

$$\left||A|_e - |B|_e\right| \leq |A \triangle B|_e$$

Solution: Note that $A \subset B \cup (A \triangle B)$. Then use sub-additivity of outer measure to conclude that

$$|A|_e \leq |B|_e + |A \triangle B|_e\,.$$

Re-arrangement gives that

$$|A|_e - |B|_e \leq |A \triangle B|_e\,.$$

To deduce the inequality

$$|B|_e - |A|_e \leq |A \triangle B|_e$$

use the observation that $B \subset A \cup (A \triangle B)$ and repeat the argument from above. This then proves the result.

5. Prove that the function

$$f(x) := \begin{cases} \frac{1}{q} & : \quad x = \frac{p}{q} \text{ is rational} \\ 0 & : \quad x \text{ is irrational} \end{cases}$$

is measurable on every interval $[a,b]$.

Solution: There are lots of ways to approach this problem. Here is the easiest. Note that the function $g(x) = 0$ is clearly measurable on every interval $[a,b]$. This follows since

$$\{x \in [a,b] : g(x) > \lambda\} = \begin{cases} [a,b] & : \quad \lambda < 0 \\ \emptyset & : \quad \lambda \geq 0. \end{cases}$$

The emptyset is of course measurable, and the interval $[a,b]$ is measurable since it is a closed set in $\mathbb{R}$. Thus, $g(x)$ is measurable. Now the function $f(x)$ differs from $g(x)$ on a set of measure zero, namely the set $\mathbb{Q} \cap [a,b]$. So, $f(x) = g(x)$ almost everywhere on $[a,b]$. Then simply apply Theorem 4.5 from the text to conclude the result.

One can of course show this function is measurable by checking from the definition, but this is more work.

6. Let f and $\{f_k\}$ be measurable on E, and define $\omega_f(a) = |\{x \in E : f(x) > a\}|$ for finite $a \in \mathbb{R}$. If $f_k \nearrow f$ show that $\omega_{f_k} \nearrow \omega_f$.

Solution: Fix a, and consider the sets

$$\{x \in E : f_k(x) > a\}$$

and

$$\{x \in E : f_{k+1}(x) > a\}.$$

Note that since the sequence $f_k \nearrow f$, we have that $f_k \leq f_{k+1}$. This inequality them implies that

$$\{x \in E : f_k(x) > a\} \subset \{x \in E : f_{k+1}(x) > a\}.$$

So in particular we have that

$$\omega_{f_k}(a) = |\{x \in E : f_k(x) > a\}| \leq |\{x \in E : f_{k+1}(x) > a\}| = \omega_{f_{k+1}}(a).$$

Similarly, we have that for all k

$$\{x \in E : f_k(x) > a\} \subset \{x \in E : f(x) > a\},$$

and in particular

$$\bigcup_k \{x \in E : f_k(x) > a\} \subset \{x \in E : f(x) > a\}.$$

Next, observe that

$$\{x \in E : f(x) > a\} \subset \bigcup_k \{x \in E : f_k(x) > a\}.$$

This is easy to deduce from the fact that f_k is increasing to f. Suppose that $x \in \{x \in E : f(x) > a\}$, and let $\epsilon = f(x) - a$. Then there exists an integer K such that for all $k \geq K$ we have

$$f(x) - f_k(x) < \epsilon = f(x) - a.$$

Note here we have used that f_k is increasing. Re-arrangement gives that $f_k(x) > a$ for $k > K$, and so in fact we have

$$x \in \bigcup_k \{x \in E : f_k(x) > a\}.$$

$$\omega_{f_k}(a) = |\{x \in E : f_k(x) > a\}| \leq |\{x \in E : f(x) > a\}| = \omega_f(a).$$

So we have $\omega_{f_k} \leq \omega_{f_{k+1}}$. We also have a sequence of increasing measurable sets converging to a fixed set,

$$\bigcup_k \{x \in E : f_k(x) > a\} = \{x \in E : f(x) > a\}.$$

So taking limits (Theorem 3.26) we have that

$$\omega_{f_k}(a) \nearrow \omega_f(a).$$

7. Let f be measurable and finite almost everywhere on $[a, b]$. Show that given $\epsilon > 0$, there is a continuous function g on $[a, b]$ such that

$$|\{x : f(x) \neq g(x)\}| < \epsilon.$$

Solution: The idea is to apply Lusin's Theorem. Since f is finite and measurable, the function satisfies condition $\mathcal{C}$. Then given the parameter ϵ, we can find a closed set $F \subset [a, b]$ such that

(i) $|[a, b] \setminus F| < \epsilon$;

(ii) f is continuous relative to F.

Now as the hint indicates, we should use the Tietze Extension Theorem. Recall that the the Tietze Extension Theorem is the following:

Theorem 1. *Let f be continuous on a closed set $F \subset \mathbb{R}$. Then there exists a continuous function g on all of $\mathbb{R}$ such that $g|_F = f$. Moreover, if $|f(x)| \leq M$ then the extension can be chosen so that $|g(x)| \leq M$.*

Proof. The idea behind the theorem is very simple. Look at the complement $\mathbb{R} \setminus F$, which is open. Since it is open, it is a disjoint union of intervals, i.e. $\mathbb{R} \setminus F = \bigcup (a_k, b_k)$. So, we then define $g(x)$ on each of these intervals to be a linear function such that $g(a_k) = f(a_k)$ and $g(b_k) = f(b_k)$.

Then simply define the function

$$g(x) = \left\{ \begin{array}{ccc} f(x) & : & x \in F \\ f(a_k)\frac{b_k - x}{b_k - a_k} + f(b_k)\frac{x - a_k}{b_k - a_k} & : & x \in (a_k, b_k). \end{array} \right.$$

Note that if one of the intervals in $\mathbb{R} \setminus F$ is of the form (a_k, ∞) then the above representation of g on this interval simply gives that $g(x) = f(a_k)$ for all $x \in (a_k, \infty)$. A similar argument applies to an interval of the form $(-\infty, b_k)$.

We then have that $g(x)$ is continuous since f is continuous on F and we have "matched" up the endpoints of the interval (a_k, b_k). Also, note that if f is bounded, then we will have that g is bounded too. $\square$

Note that the function f satisfies the hypotheses of the Tietze Extension Theorem, namely f is continuous relative to F. So, by the Tietze Extension Theorem , we have a continuous function g defined on $[a, b]$ such that $g|_F = f$ (Actually, we only need the proof of the Tietze Extension Theorem since we are working within the interval $[a, b]$ and when we consider $[a, b] \setminus F$ this will still be a collection of intervals to which we can apply the above argument.). Note that then we have

$$\{x \in [a, b] : f(x) \neq g(x)\} \subset [a, b] \setminus F.$$

(We don't know that the sets are equal since it could be possible that f and g coincide on points outside of F). Then we have that

$$|\{x \in [a, b] : f(x) \neq g(x)\}| \leq |[a, b] \setminus F| < \epsilon.$$