Problem 1. Zygmund p111 exercise 28

Let E be a measurable set in $\mathbb{R}^n$ with $|E| < \infty$. Suppose that $f > 0$ a.e. in E and $f, \log f \in L^1(E)$. Prove that

$$\lim_{p\to 0^+}\left(\frac{1}{|E|}\int_E f^p\right)^{1/p} = \exp\left(\frac{1}{|E|}\int_E \log f\right).$$

(Start by using Theorem 5.36 to show that $\int_E f^p \to |E|$ as $p \to 0^+$. Note that $\int_E (f^p - 1)^{1/p} \to \log f$.)

Theorem 5.36 (Lebesgue's Dominated Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on E such that $f_k \to f$ a.e. in E. If there exists $\phi \in L(E)$ such that $|f_k| \le \phi$ a.e. in E for all k, then $\int_E f_k \to \int_E f$.

Theorem 5.32 (Monotone Convergence Theorem)

Let $\{f_k\}$ be a sequence of measurable functions on E:

1. If $f_k \nearrow f$ a.e. on E and there exists $\phi \in L(E)$ such that $f_k \ge \phi$ a.e. on E for all k, then
$$\int_E f_k \to \int_E f.$$
2. If $f_k \searrow f$ a.e. on E and there exists $\phi \in L(E)$ such that $f_k \le \phi$ a.e. on E for all k, then
$$\int_E f_k \to \int_E f.$$

Since $f_p = f^p \searrow 1$ in E as $p \to 0^+$, and let $|f_p| \le \phi = \max\{f, f_p\}$ in E for $p < 1$, thus we have

$$\int_E f_p \to \int_E 1 = |E|,$$

as $p \to 0^+$.

Let

$$h_p = \frac{f^p - 1}{p}$$

$$h_p = \chi_{(0,1]}h_p + \chi_{(1,\infty)}h_p \searrow \log f$$

By Monotone Convergence Theorem, as $p \to 0^+$,

$$\int_E h_p \to \int_E \log f,$$

Thus,

$$g_p = \frac{1}{|E|}\int_E \frac{f^p - 1}{p} \to \frac{1}{|E|}\int_E \log f.$$

$$\begin{aligned} p \cdot g_p &= p \cdot \frac{1}{|E|}\int_E \frac{f^p - 1}{p} \\ &= \frac{1}{|E|}\int_E (f^p - 1) \end{aligned}$$

$$
\begin{aligned}
&= \underbrace{\frac{1}{|E|}\int_E f^p}_{\to 1} - \underbrace{\frac{1}{|E|}\int_E 1}_{\to 1} \\
&\to 0.
\end{aligned}
$$

$$
\begin{aligned}
\lim_{p\to 0^+}\left(\frac{1}{|E|}\int_E f^p\right)^{1/p} &= \lim_{p\to 0^+}(p\cdot g_k+1)^{1/p} \\
&= \lim_{p\to 0^+}\exp\left\{g_p\log\left[(p\cdot g_p+1)^{\dfrac{1}{p\cdot g_p}}\right]\right\} \\
&= \exp\left\{\lim_{p\to 0^+} g_p\log\left[(p\cdot g_p+1)^{\dfrac{1}{p\cdot g_p}}\right]\right\} \\
&= \exp\left(\frac{1}{|E|}\int_E \log f\right)
\end{aligned}
$$

as required.

The last equation is followed by

$$
\begin{aligned}
\lim_{p\to 0^+}\log\left[(p\cdot g_p+1)^{\dfrac{1}{p\cdot g_p}}\right] &= \log[e] \qquad\qquad \because e := \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \\
&= 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\lim_{p\to 0^+} g_p &= \lim_{p\to 0^+}\frac{1}{|E|}\int_E \frac{f^p-1}{p} \\
&= \frac{1}{|E|}\int_E \log f. \quad \text{(By MCT}
\end{aligned}
$$

Problem 2. Zygmund p111 exercise 29

Let f be measurable, nonnegative, and finite a.e. in a set E. Prove that for any nonnegative constant c,

$$\int_E e^{cf(x)} = |E| + c\int_0^\infty e^{c\alpha}\omega f(\alpha)\,d\alpha.$$

Deduce that $e^{cf} \in L^1(E)$ if $|E| < \infty$ and there exist constants C_1 and c_1 such that $c_1 > c$ and $\omega f(\alpha) \le C_1 e^{-c_1\alpha}$ for all $\alpha > 0$. We will study such an exponential integrability property in Section 14.5.

Zygmund p97 Distribution function

$$\omega(\alpha) = \omega_{f,E}(\alpha) = |\{x \in E : f(x) > \alpha\}|,$$

where f is a measurable function on E and $-\infty < \alpha < +\infty$. We call $\omega_{f,E}$ the distribution function of f on E.

Let ϕ be an arbitrary once continuously differentiable function s.t. $\phi(0) = 1$.

It is to be shown that

$$\int_E \phi(f(x)) = |E| + \int_0^\infty \phi'(f(x))\omega f(\alpha)\,d\alpha.$$

$$\begin{aligned}
\int_0^\infty \phi'(f(x))\omega f(\alpha)\,d\alpha &= \int_0^\infty \phi'(f(x)) \int_E \chi_{\{x\in E: f(x)>\alpha\}}\,dx\,d\alpha \\
&= \int_E \int_0^{f(x)} \phi'(\alpha)\,d\alpha\,dx \\
&= \int_E [\phi(f(x)) - \phi(0)]\,dx \\
&= \int_E [\phi(f(x)) - 1]\,dx \\
&= \int_E \phi(f(x))\,dx - |E|
\end{aligned}$$

Thus,

$$\int_E \phi(f(x)) = |E| + \int_0^\infty \phi'(f(x))\omega f(\alpha)\,d\alpha.$$

Let

$$\begin{gathered}
\phi(\alpha) := e^{c\alpha}, \\
\phi(f(x)) = e^{cf(x)}, \\
\phi'(f(x)) = c\cdot e^{cf(x)}
\end{gathered}$$

We have

$$\int_E e^{cf(x)} = |E| + c\int_0^\infty e^{c\alpha}\omega f(\alpha)\,d\alpha,$$

as required.

—

Under the assumption that $|E| < \infty$ and there exist constants C_1 and c_1 such that $c_1 > c$ and $\omega f(\alpha) \leq C_1 e^{-c_1 \alpha}$ for all $\alpha > 0$,

We have

$$
\begin{aligned}
e^{c_1 \alpha} \omega f(\alpha) &\leq C_1 \\
e^{c \alpha} \omega f(\alpha) &\leq C_1 e^{(c - c_1)\alpha}.
\end{aligned}
$$

Let $k = c_1 - c > 0$,

$$
\begin{aligned}
\int_0^\infty C_1 e^{-k\alpha}\, d\alpha &= C_1 \int_0^\infty e^{-k\alpha}\, d\alpha \\
&= C_1 \left(-\frac{1}{k}\right) \left(\lim_{\beta \to \infty} e^{-k\beta} - \lim_{\alpha \to 0} e^{-k\alpha}\right) \\
&= C_1 \left(-\frac{1}{k}\right) (0 - 1) \\
&= \frac{C_1}{c_1 - c} < \infty.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\int_E e^{cf(x)} &= \underbrace{|E|}_{<\infty} + c \underbrace{\int_0^\infty e^{c\alpha} \omega f(\alpha)\, d\alpha}_{<\infty} \\
&< \infty.
\end{aligned}
$$

We can conclude that $e^{cf} \in L(E)$, as required.