# Math 5051 : Real Analysis I

## Mid-term Exam 1
## 03 October 2016

Instructions: Answer all of the problems.

1. Let $X$ be a metric space with metric $\rho$. For any nonempty set $E \subset X$ define:

$$\rho_E(x) = \inf\{\rho(x,y) : y \in E\}.$$

Prove that $\rho_E$ is a uniformly continuous function on $X$.

2. Let $f$ be a continuous function on interval $[a,b]$, and $\phi$ a monotone increasing function on $[a,b]$. Given partition $\Gamma = \{a = x_0 < \cdots < x_n = b\}$, define

$$U_\Gamma = \sum_{m=1}^{n} \sup_{x_{m-1} \le x \le x_m} f(x)\{\phi(x_m) - \phi(x_{m-1})\} \; L_\Gamma = \sum_{m=1}^{n} \inf_{x_{m-1} \le x \le x_m} f(x)\{\phi(x_m) - \phi(x_{m-1})\}.$$

Show that $\lim_{|\Gamma| \to 0} U_\Gamma - L_\Gamma = 0$.

3. Suppose that $E$ is a measurable subset of $\mathbb{R}^d$:

(a) Prove that for any $\epsilon$ there exists a closed set $F$ and an open set $G$ such that $F \subset E \subset G$ and $|G \setminus F| < \epsilon$.

(b) Prove that there exists a $F_\sigma$ set $F$ and a $G_\delta$ set $G$ such that $F \subset E \subset G$ and $|G \setminus F| = 0$.

4. Suppose that $\mathbb{R}^n = A \cup B$ with $A$ and $B$ measurable sets. Prove that $f$ is measurable on $\mathbb{R}^n$ if and only if $f$ is measurable on $A$ and $B$.

5. A family of measurable functions $\{f_n\}$ is said to converge almost uniformly to $f$ on a measurable set $E$ with $|E| < \infty$ if for every $\epsilon > 0$ there exists a $R$ with $R \subset E$ and $|R| < \epsilon$ and $f_n \to f$ uniformly on $E \setminus R$. Show that if $\{f_n\}$ converges to $f$ almost uniformly then $f_n \to f$ almost everywhere and $f_n \overset{m}{\to} f$.

6. Suppose that $\{f_n\}$ is a sequence of non-negative measurable functions on $\mathbb{R}^d$, $f_n \to f$ pointwise, and $\int_{\mathbb{R}^d} f dx = \lim \int_{\mathbb{R}^d} f_n dx < \infty$. Prove that for every measurable set $E \subset \mathbb{R}^d$ that $\int_E f dx = \lim \int_E f_n dx$.

7. Suppose $f : \mathbb{R}^d \to [0, \infty)$, that $\int_{\mathbb{R}^d} f(x)dx = c$ with $0 < c < \infty$ and $\alpha$ is a positive contant. Prove that:

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} n \log\left(1 + \left(\frac{f(x)}{n}\right)^\alpha\right) dx = \begin{cases} \infty & \text{if} \quad 0 < \alpha < 1 \\ c & \text{if} \quad \alpha = 1 \\ 0 & \text{if} \quad 1 < \alpha < \infty. \end{cases}$$

8. Let $f$ be measurable, nonnegative, and finite almost everywhere in a set $E$. Prove that for any nonnegative constant $c$

$$\int_E e^{cf(x)} dx = |E| + c \int_0^\infty e^{c\lambda} \omega_f(\lambda) d\lambda.$$

Deduce that $e^{cf} \in L(E)$ if $|E| < \infty$ and there exists constants $C_1$ and $c_1$ such that $c_1 > c$ and $\omega_f(\lambda) \le C_1 e^{-c_1 \lambda}$ for all $\lambda > 0$. Here $\omega_f(\lambda)$ is the distribution function associated to $f$.