MATH 6337: Homework 1

1.1. Prove the following facts, which were left as exercises.

(a) For a sequence of sets $\{E_k\}$, $\limsup E_k$ consists of those points which belong to infinitely many E_k, and $\liminf E_k$ consists of those points which belong to all E_k from some k on.

Solution. Suppose $x \in \limsup E_k$. Then $x \in \bigcup_{k\geq j} E_k$ for all j, so for every $j \geq 1$, there exists $k \geq j$ such that $x \in E_i$; hence $x \in E_k$ for infinitely many k. All of these implications are reversible.

Suppose $x \in \liminf E_k$. Then $x \in \bigcap_{k\geq j} E_k$ for some k. Thus, for some $j \geq 1$, $x \in E_k$ for all $k \geq j$; hence $x \in E_k$ for all but finitely many k. All of these implications are reversible. □

(b) The De Morgan laws:

$$\left(\bigcup_{E\in\mathcal{F}} E\right)^c = \bigcap_{E\in\mathcal{F}} E^c \quad \text{and} \quad \left(\bigcap_{E\in\mathcal{F}} E\right)^c = \bigcup_{E\in\mathcal{F}} E^c.$$

Solution. Suppose $x \in \left(\bigcup_{E\in\mathcal{F}} E\right)^c$; then $x \notin \bigcup_{E\in\mathcal{F}} E$, so x fails to be in every $E \in \mathcal{F}$; hence $x \in E^c$ for every $E \in \mathcal{F}$, so $x \in \bigcap_{E\in\mathcal{F}} E^c$. All of these implications are reversible.

Suppose $x \in \left(\bigcap_{E\in\mathcal{F}} E\right)^c$; then $x \notin \bigcap_{E\in\mathcal{F}} E$, so x fails to be some $E \in \mathcal{F}$; hence $x \in E^c$ for some $E \in \mathcal{F}$, so $x \in \bigcup_{E\in\mathcal{F}} E^c$. All of these implications are reversible.

[Note: $\mathcal{F}$ is an arbitrary family of sets and may have infinite size, so a proof by induction cannot work here.] □

(d) Theorem 1.4:

(a) $L = \limsup_{k\to\infty} a_k$ if and only if (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ which converges to L and (ii) if $L' > L$, there is an integer K such that $a_k < L'$ for $k \geq K$.

(b) $\ell = \liminf_{k\to\infty} a_k$ if and only if (i) there is a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ which converges to ℓ and (ii) if $\ell' < \ell$, there is an integer K such that $a_k > \ell'$ for $k \geq K$.

Solution.

(a) First, the forward direction. Let $L_j = \sup_{k\geq n} a_k$, so $L = \lim_{n\to\infty} L_n$. Given $j \geq 1$, choose $n_j \geq j$ such that $\left|L_{n_j} - L\right| < \frac{1}{2j}$ and choose $k_j \geq n_j$ such that $\left|a_{k_j} - L_{n_j}\right| < \frac{1}{2j}$. Then

$$\left|a_{k_j} - L\right| \leq \left|a_{k_j} - L_{n_j}\right| + \left|L_{n_j} - L\right| < \frac{1}{j},$$

so $a_{k_j} \to L$. Now, given $L' > L$, choose K such that $|L_K - L| < L' - L$; then $\sup_{k\geq K} a_k < L'$, so $a_k < L'$ for all $k \geq K$.

Now, the reverse direction. Let $\varepsilon > 0$; then there exists J such that for every $j \geq J$, $\left|a_{k_j} - L\right| < \varepsilon$. Thus, $\sup_{k\geq J} a_k = L_J \geq L - \varepsilon$; thus, $\lim_{j\to\infty} L_j \geq L$. On the other hand, there exists K such that for every $k \geq K$, $a_k < L + \varepsilon$; thus, $\sup_{k\geq K} a_k = L_K \leq L + \varepsilon$, so $\lim_{j\to\infty} L_j \leq L$. Thus, $\limsup a_k = \lim_{j\to\infty} L_j = L$.

(b) Recall that $\liminf a_k = -\limsup(-a_k)$. Using the result of part (a), if $\ell = \liminf a_k$, then $-\ell = \limsup(-a_k)$, so there is a subsequence $\{-a_{k_j}\}$ of $\{-a_k\}$ converging to $-\ell$ (thus a subsequence $\{a_{k_j}\}$ of $\{a_k\}$ converging to ℓ), and if $\ell' < \ell$, then $-\ell' > \ell$, so there exists K such that $-a_k < -\ell'$ (hence $a_k > \ell'$).

On the other hand, given $\{a_k\}$, if there is a subsequence $\{a_{k_j}\}$ converging to ℓ and for all $\ell' < \ell$ there exists K such that $a_k > \ell'$ whenever $k \geq K$, then the hypothesis of the reverse implication of part (a) is satisfied: there is a subsequence $\{-a_{k_j}\}$ of $\{-a_k\}$ which converges to $-\ell$ and, if $-\ell' > -\ell$, there is an integer K such that $-a_k < -\ell'$ for $k \geq K$. Thus $-\ell = \limsup(-a_k)$, so $\ell = \liminf a_k$.

$\square$

(n) Theorem 1.14:

(a) $M = \limsup_{x \to x_0; x \in E} f(x)$ if and only if (i) there exists $\{x_k\}$ in E such that $x_k \to x_0$ and $f(x_k) \to M$, and (ii) if $M' > M$, there exists $\delta > 0$ such that $f(x) < M'$ for $x \in B'(x_0; \delta) \cap E$.

(b) $m = \liminf_{x \to x_0; x \in E} f(x)$ if and only if (i) there exists $\{x_k\}$ in E such that $x_k \to x_0$ and $f(x_k) \to m$, and (ii) if $m' < m$, there exists $\delta > 0$ such that $f(x) > m'$ for $x \in B'(x_0; \delta) \cap E$.

Solution. For simplicity, let B_δ denote the open ball of radius δ centered at x_0 intersected with E.

(a) First, the forward direction. Let $M_\delta = \sup_{B_\delta} f(x)$, so $M = \lim_{\delta \to 0} M_\delta$. Given $k \geq 1$, choose $\delta_k \leq \frac{1}{j}$ such that $|M_{\delta_k} - M| < \frac{1}{2k}$, and choose $x_k \in B_{\delta_k}$ such that $|f(x_k) - M_{\delta_k}| < \frac{1}{2k}$. Then

$$|f(x_k) - M| \leq |f(x_k) - M_{\delta_k}| + |M_{\delta_k} - M| < \frac{1}{k},$$

so $f(x_k) \to M$ (and $x_k \to x_0$ since $|x_k - x_0| \leq \frac{1}{k}$). Now, given $M' > M$, choose δ such that $|M_\delta - M| < M' - M$; then $\sup_{B_\delta} f(x) < M'$, so $f(x) < M'$ for all $x \in B_\delta$.

Now, the reverse direction. Let $\varepsilon > 0$; then there exists K such that for every $k \geq K$, $|f(x_k) - M| < \varepsilon$. Thus, letting $\delta_k = |x_k - x_0|$, we have $\sup_{B_{\delta_k}} f(x) = M_{\delta_k} \geq M - \varepsilon$; thus, $\lim_{\delta \to 0} M_\delta \geq M$. On the other hand, there exists δ such that for every $x \in B_\delta$, $f(x) < M + \varepsilon$; thus, $\sup_{B_\delta} f(x) = M_\delta \leq M + \varepsilon$, so $\lim_{\delta \to 0} M_\delta \leq M$. Thus, $\limsup f(x) = \lim_{\delta \to 0} M_\delta = M$.

(b) Recall that $\liminf f(x) = -\limsup(-f(x))$. Using the result of part (a), if $m = \liminf f(x)$, then $-m = \limsup(-f(x))$, so there is a sequence $\{-f(x_k)\}$ (with $x_k \to x_0$) converging to $-m$ (thus a sequence $\{f(x_k)\}$ with $x_k \to x_0$ and $f(x_k) \to m$), and if $m' < m$, then $-m' > m$, so there exists δ such that $-f(x) < -m'$ (hence $f(x) > m'$) for all $x \in B_\delta$.

On the other hand, if there is a sequence $\{f(x_k)\}$ with $x_k \to x_0$ and $f(x_k) \to m$ and for all $m' < m$ there exists δ such that $f(x) > m'$ whenever $x \in B_\delta$, then the hypothesis of the reverse implication of part (a) is satisfied: there is a sequence with $x_k \to x_0$ and $-f(x_k) \to -m$ and, if $-m' > -m$, there exists δ such that $-f(x) < -m'$ for $x \in B_\delta$. Thus $-m = \limsup(-f(x))$, so $m = \liminf f(x)$.

$\square$

1.2. Find $\limsup E_k$ and $\liminf E_k$ if $E_k = [-1/k, 1]$ for k odd and $E_k = [-1, 1/k]$ for k even.

Solution. Recall that $\limsup E_k$ is the set of points which appear in infinitely many E_k, and $\liminf E_k$ is the set of points which appear in all but finitely many E_k. If $x \in [-1, 0]$, then $x \in E_k$ for all even k, and if $x \in [0, 1]$, then $x \in E_k$ for all odd k, so $\limsup E_k = [-1, 1]$. If $x \in [-1, 0)$, then $x \notin E_k$ for odd k when $x < -1/k$, which must eventually happen for large enough k; similarly, if $x \in (0, 1]$, then $x \notin E_k$ for even k when $x > 1/k$. Thus, every point but $x = 0$ fails to appear in infinitely many E_k, so $\liminf E_k = \{0\}$. $\square$

1.3.

(a) Show that $(\limsup E_k)^c = \liminf E_k^c$.
(b) Show that if $E_k \nearrow E$ or $E_k \searrow E$, then $\limsup E_k = \liminf E_k = E$.

Solution. (a) This follows from the De Morgan laws:

$$(\limsup E_k)^c = \left(\bigcap_{j=1}^{\infty}\left(\bigcup_{k\geq j} E_k\right)\right)^c = \bigcup_{j=1}^{\infty}\left(\bigcup_{k\geq j} E_k\right)^c = \bigcup_{j=1}^{\infty}\left(\bigcap_{k\geq j} E_k^c\right) = \liminf E_k^c.$$

(b) $E_k \nearrow E$ means that $E_k \subseteq E_{k+1}$ and $\bigcup E_k = E$; in this case,

$$\liminf E_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k\geq j} E_k\right) = \bigcup_{j=1}^{\infty} E_j = E \quad \text{and} \quad \limsup E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k\geq j} E_k\right) = \bigcap_{j=1}^{\infty} E = E.$$

On the other hand, $E_k \searrow E$ means that $E_k \supseteq E_{k+1}$ and $\bigcap E_k = E$; in this case,

$$\liminf E_k = \bigcup_{j=1}^{\infty}\left(\bigcap_{k\geq j} E_k\right) = \bigcup_{j=1}^{\infty} E = E \quad \text{and} \quad \limsup E_k = \bigcap_{j=1}^{\infty}\left(\bigcup_{k\geq j} E_k\right) = \bigcap_{j=1}^{\infty} E_j = E.$$

$\square$

1.9. Prove that any closed subset of a compact set is compact.

Solution. Let X be a compact space, let $F \subseteq X$ be closed in X, and let $\mathcal{C} = \{G_\alpha\}$ be an open cover of F; then each $G_\alpha = G'_\alpha \cap F$ for a subset $G'_\alpha \subseteq X$ which is open in X. Since $X = F \cap F^c$ and $\bigcup G'_\alpha \supseteq F$, $F^c \cup \bigcup G'_\alpha = X$; moreover, since F is closed in X, F^c is open in X. Thus, $\mathcal{C}' = \{G'_\alpha\} \cup \{F^c\}$ is an open cover of X.

Since X is compact, there exists a finite subcover $\mathcal{C}'' = \{G_1', ..., G_n'\} \subseteq \mathcal{C}'$ such that $\bigcup_{k=1}^n G_k' = X$; hence $\bigcup_{k=1}^n G_k' \supseteq F$. Now, each G_k' has the property that either $G_k' \cap F = G_k \in \mathcal{C}$ or $G_k \cap F = \varnothing$ (if $G_k = F^c$). Thus, we can take $\mathcal{C}''' = \{G_1, G_2, ..., G_n\} \setminus \{\varnothing\}$; then $\bigcup_{G \in \mathcal{C}'''} G = F$ and $\mathcal{C}''' \subset \mathcal{C}$, so $\mathcal{C}'''$ is a finite open subcover of $\mathcal{C}$. Hence F is a compact subspace of X.

[Note that X need not have a norm, so arguments involving boundedness would fail in such a case.] $\square$

1.16. If $\{f_k\}$ is a sequence of bounded, Riemann-integrable functions on an interval I which converges uniformly on I to f, show that f is Riemann integrable on I and that

$$(R) \int_I f_k(x)\, dx \to (R) \int_I f(x)\, dx.$$

Solution. Given $\varepsilon > 0$, choose k such that for every $x \in I$, we have

$$|f_k(x) - f(x)| < \varepsilon;$$

it follows that

$$\left| \sup_I f_k(x) - \sup_I f(x) \right| < \varepsilon \quad \text{and} \quad \left| \inf_I f_k(x) - \inf_I f(x) \right| < \varepsilon.$$

Since f_k is Riemann-integrable, there exists a partition Γ such that

$$U_\Gamma(f_k) - L_\Gamma(f_k) < \varepsilon.$$

Then

$$|U_\Gamma(f) - L_\Gamma(f)| \leq |U_\Gamma(f) - U_\Gamma(f_k)| + |U_\Gamma(f_k) - L_\Gamma(f_k)| + |L_\Gamma(f_k) - L_\Gamma(f)| \leq 3\varepsilon\, |I| \, .$$

Hence f is also Riemann-integrable. Moreover,

$$\left| (R) \int_I f_k(x)\, dx - (R) \int_I f(x)\, dx \right| \leq (R) \int_I |f_k(x) - f(x)| \; dx < \varepsilon\, |I| \, .$$

Hence the integral of f is the limit of the integrals of the f_k. $\square$