Math 6337 : Real Analysis I

Final Exam
14 December 2011

Instructions: Answer any 5 of the problems.

1. Prove that if $0 < \epsilon < 1$, there is no measurable subset E of $\mathbb{R}$ that satisfies

$$\epsilon < \frac{|E \cap I|}{|I|} < 1 - \epsilon$$

for every interval I in $\mathbb{R}$.

Solution: We proceed by contradiction, so suppose that there exists a measurable set E such that

$$\epsilon < \frac{|E \cap I|}{|I|} < 1 - \epsilon.$$

Observe that we have

$$\frac{1}{|I|} \int_I \chi_E(y)\, dy = \frac{|E \cap I|}{|I|}$$

By the Lebesgue Differentiation Theorem, for almost every $x \in \mathbb{R}$ we have that

$$\chi_E(x) = \lim_{|I| \searrow x} \frac{1}{|I|} \int_I \chi_E(y)\, dy = \lim_{|I| \searrow x} \frac{|E \cap I|}{|I|}$$

We then have by our supposition that

$$\epsilon \leq \chi_E(x) \leq 1 - \epsilon.$$

But, this will present a problem since if $x \in E$, then we have $1 \leq 1 - \epsilon$, which is a contradiction. Similarly, if $x \in E^c$, then $\epsilon \leq 0$, which is again a contradiction. So, our supposition is wrong, and there can not exist a measurable subset E of $\mathbb{R}$ with the property that

$$\epsilon < \frac{|E \cap I|}{|I|} < 1 - \epsilon$$

for every interval I in $\mathbb{R}$.

2. Let E be a measurable subset of $\mathbb{R}$ with $|E| < \infty$. Let $\{f_n\}$ be a sequence of measurable function on E, and let f be a measurable functions on E. Show that $f_k \to f$ in measure on E if and only if

$$\lim_{n \to \infty} \int_E \frac{|f(x) - f_n(x)|}{1 + |f(x) - f_n(x)|}\, dx = 0.$$

Solution: First, suppose that $f_n \to f$ in measure. Choose any $\epsilon > 0$. Then we have

$$\begin{aligned}
\int_E \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx &= \int_{|\{f_n(x)-f(x)|>\epsilon\}} \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx \\
&\quad + \int_{\{|f_n(x)-f(x)|\le\epsilon\}} \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx \\
&\le \int_{\{|f_n(x)-f(x)|>\epsilon\}} dx + \epsilon \int_{\{|f_n(x)-f(x)|\le\epsilon\}} dx \\
&\le |\{x \in E : |f_n(x)-f(x)| > \epsilon\}| + \epsilon\,|E|\,.
\end{aligned}$$

This then implies that

$$\begin{aligned}
\overline{\lim_{n\to\infty}} \int_E \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx &\le \epsilon\,|E| + \overline{\lim_{n\to\infty}}\,|\{x\in E : |f_n(x)-f(x)|>\epsilon\}| \\
&= \epsilon\,|E|\,,
\end{aligned}$$

since we have that $f_n \to f$ in measure. But since $|E| < \infty$ and $\epsilon > 0$ was arbitrary, we have that

$$\lim_{n\to\infty} \int_E \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx = 0.$$

Conversely, now suppose that

$$\lim_{n\to\infty} \int_E \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx = 0.$$

Chose any $\epsilon > 0$ and note that

$$x \ge \epsilon \Rightarrow \frac{x}{1+x} \ge \frac{\epsilon}{1+\epsilon}.$$

Then we have that

$$\begin{aligned}
|\{x \in E : |f(x)-f_n(x)| > \epsilon\}| &= \frac{1+\epsilon}{\epsilon} \int_{|\{|f_n(x)-f(x)|>\epsilon\}|} \frac{\epsilon}{1+\epsilon}\,dx \\
&\le \frac{1+\epsilon}{\epsilon} \int_{|\{|f_n(x)-f(x)|>\epsilon\}|} \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx \\
&\le \frac{1+\epsilon}{\epsilon} \int_E \frac{|f(x)-f_n(x)|}{1+|f(x)-f_n(x)|}\,dx.
\end{aligned}$$

But since right hand side goes to zero as $n \to \infty$, we clearly have that $f_n \to f$ in measure.

3. Suppose that $p > 0$, $E \subset \mathbb{R}^n$ with $|E| < \infty$ and that f is measurable on E. Show that if

$$|\{x \in E : |f(x)| > t\}| = O(t^{-p})$$

as $t \to +\infty$ that $|f|^{p-\epsilon} \in L^1(E)$ for any $\epsilon \in (0,p)$.

Solution: The idea is to use the distribution function to compute the integral in question. Note that for $q > 0$ we have

$$\int_E |f(x)|^q \, dx = q \int_0^\infty t^{q-1} \omega(t) \, dt$$

where $\omega(t) = |\{x \in E : |f(x)| > t\}|$. Now let $q = p - \epsilon$ in the above, and so we have

$$\begin{aligned}
\int_E |f(x)|^{p-\epsilon} \, dx &= (p-\epsilon) \int_0^\infty t^{p-\epsilon-1} \omega(t) \, dt \\
&= (p-\epsilon) \int_0^L t^{p-\epsilon-1} \omega(t) \, dt + (p-\epsilon) \int_L^\infty t^{p-\epsilon-1} \omega(t) \, dt \\
&\leq (p-\epsilon) \int_0^L t^{p-\epsilon-1} \omega(t) \, dt + C(p-\epsilon) \int_L^\infty t^{-\epsilon-1} \, dt.
\end{aligned}$$

Here we have chosen the number $L > 0$ so that $\omega(t) \leq Ct^{-p}$ for $t > L$, which is guaranteed by the hypothesis on the function f. We can now evaluate each of these integrals directly to find

$$(p-\epsilon) \int_0^L t^{p-\epsilon-1} \omega(t) \, dt \leq |E| \, L^{p-\epsilon}$$

and

$$C(p-\epsilon) \int_L^\infty t^{-\epsilon-1} \, dt = C \frac{(p-\epsilon)}{\epsilon L^\epsilon}.$$

Thus we have

$$\int_E |f(x)|^{p-\epsilon} \, dx \leq |E| \, L^{p-\epsilon} + C \frac{(p-\epsilon)}{\epsilon L^\epsilon} < \infty.$$

4. A Banach space X is said to be *uniformly convex* if for any $\epsilon > 0$ there exists a $\delta > 0$ such that if $\|f\|_X < 1 + \delta$, $\|g\|_X < 1 + \delta$ and $\left\|\frac{f+g}{2}\right\|_X \geq 1$ implies that $\|f - g\|_X < \epsilon$. Show that any Hilbert space is uniformly convex.

Solution: Recall that the Parallelogram Identity in a Hilbert space says:

$$\left\|\frac{f+g}{2}\right\|_H^2 + \left\|\frac{f-g}{2}\right\|_H^2 = \frac{1}{2} \|f\|_H^2 + \frac{1}{2} \|g\|_H^2 .$$

Let $\epsilon > 0$ be given, and choose $0 < \delta$ so that $2\sqrt{\delta + \delta^2} < \epsilon$ ($\delta < \frac{\epsilon^2}{12}$ will work below). Suppose that we have $\|f\|_H < 1 + \delta$ and $\|g\|_H < 1 + \delta$ and $\left\|\frac{f+g}{2}\right\|_H \geq 1$, then by the Parallelogram Identity we have

$$\begin{aligned}
\left\|\frac{f-g}{2}\right\|_H^2 &= \frac{1}{2} \|f\|_H^2 + \frac{1}{2} \|g\|_H^2 - \left\|\frac{f+g}{2}\right\|_H^2 \\
&\leq (1+\delta)^2 - 1 \\
&= 2\delta + \delta^2.
\end{aligned}$$

This last inequality then implies

$$\|f-g\|_H \leq 2\sqrt{2\delta+\delta^2} < \epsilon$$

by choice of δ

5. Let Q be the unit square in $\mathbb{R}^2$, and suppose that $\{f_n\}$ is a sequence of non-negative measurable functions in $L^1(Q)$ such that

$$\lim_{n\to\infty}\int_Q f_n(x)\,dx = \int_Q f(x)\,dx < \infty,$$

and $f_n(x) \to f(x)$ pointwise. Show that for any measurable subset $A \subset Q$ that

$$\lim_{n\to\infty}\int_A f_n(x)\,dx = \int_A f(x)\,dx.$$

Hint: Apply Fatou's Lemma in a clever way.

Solution: By Fatou's Lemma, for every measureable set $A \subset Q$ we have

$$\int_A f(x)\,dx \leq \varliminf_{n\to\infty}\int_E f_n(x)\,dx.$$

Use this inequality for the set A and $Q \setminus A$ we obtain:

$$\begin{aligned}
\int_A f(x)\,dx &\leq \varliminf_{n\to\infty}\int_A f_n(x)\,dx \\
&\leq \varlimsup_{n\to\infty}\int_A f_n(x)\,dx \\
&= \varlimsup_{n\to\infty}\left(\int_Q f_n(x)\,dx - \int_{Q\setminus A} f_n(x)\,dx\right) \\
&= \lim_{n\to\infty}\int_Q f_n(x)\,dx - \varliminf_{n\to\infty}\int_{Q\setminus A} f_n(x)\,dx \\
&\leq \int_Q f(x)\,dx - \int_{Q\setminus A} f(x)\,dx \\
&= \int_A f(x)\,dx.
\end{aligned}$$

This implies that all the inequalities are actually equalities, and so we have

$$\lim_{n\to\infty}\int_A f_n(x)\,dx = \int_A f(x)\,dx.$$

6. Suppose that $f(x_1,x_2)$ is a function on $\mathbb{R}^2$ such that the functions $\partial_{x_1}\partial_{x_2}f(x_1,x_2)$ and $\partial_{x_2}\partial_{x_1}f(x_1,x_2)$ are continuous. Use Fubini's Theorem to give a proof that $\partial_{x_1}\partial_{x_2}f(x_1,x_2) = \partial_{x_2}\partial_{x_1}f(x_1,x_2)$.

Solution: Suppose that equality doesn't hold at some point (a_1, a_2). Then we have $\partial_{x_1}\partial_{x_2} f(a_1, a_2) - \partial_{x_2}\partial_{x_1} f(a_1, a_2)$ is not zero there, and without loss of generality, we can assume that this quantity is positive (else, we replace the function f by $-f$). Since these expressions are continuous, there exists a whole rectangle about the point (a, b) for which the $\partial_{x_1}\partial_{x_2} f(x_1, x_2) - \partial_{x_2}\partial_{x_1} f(x_1, x_2)$ is positive. Let $R = [s_1, t_1] \times [s_2, t_2]$ denote this rectangle for which $\partial_{x_1}\partial_{x_2} f(x_1, x_2) - \partial_{x_2}\partial_{x_1} f(x_1, x_2) > 0$ when $(x_1, x_2) \in R$. This implies that

$$\int_R \left(\partial_{x_1}\partial_{x_2} f(x_1, x_2) - \partial_{x_2}\partial_{x_1} f(x_1, x_2)\right)\, dx_1 dx_2 > 0$$

since the integrand is positive there, and the area of R is positive.
Now, we compute via Fubini's Theorem to see that

$$\begin{aligned}
\int_R \partial_{x_2}\partial_{x_1} f(x_1, x_2)\, dx_1 dx_2 &= \int_{s_1}^{t_1} \int_{s_2}^{t_2} \partial_{x_2}\partial_{x_1} f(x_1, x_2)\, dx_2\, dx_1 \\
&= \int_{s_1}^{t_1} \partial_{x_1} f(x_1, t_2) - \partial_{x_1} f(x_1, s_2)\, dx_1 \\
&= f(t_1, t_2) - f(t_1, s_2) - f(s_1, t_2) + f(s_1, s_2).
\end{aligned}$$

A similar computation, but first integrating over x_1 and x_2 gives

$$\int_R \partial_{x_1}\partial_{x_2} f(x_1, x_2)\, dx_1 dx_2 = f(t_1, t_2) - f(t_1, s_2) - f(s_1, t_2) + f(s_1, s_2).$$

Thus, we have that

$$\int_R \left(\partial_{x_1}\partial_{x_2} f(x_1, x_2) - \partial_{x_2}\partial_{x_1} f(x_1, x_2)\right)\, dx_1 dx_2 = 0.$$

However this is a contradiction, and so we must have that $\partial_{x_1}\partial_{x_2} f(x_1, x_2) = \partial_{x_2}\partial_{x_1} f(x_1, x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$.

7. Suppose that $f \in L^p(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Show

$$\text{supp } (f * g) \subset \overline{\text{supp}\, f + \text{supp}\, g}.$$

Solution: Suppose that

$$z \notin \overline{\text{supp}\, f + \text{supp}\, g}.$$

Then for any $y \in \text{supp}\, g$ we have that $z - y \notin \text{supp}\, f$. Thus,

$$g(y) f(z - y) = 0$$

for all $y \in \mathbb{R}^n$, so $f * g(z) = 0$. This implies that

$$\mathbb{R}^n \setminus \overline{\text{supp}\, f + \text{supp}\, g} \subset \mathbb{R}^n \setminus \text{supp } (f * g)\,,$$

which is equivalent to the desired result by taking compliments.

8. Suppose that $E_j \subset (0,1)$ for $j = 1, \ldots, N$ are measurable and such that

$$\sum_{j=1}^{N} |E_j| > N - 1.$$

Show that $\left|\bigcap_{j=1}^{N} E_j\right| > 0$.

Solution: Note that by de Morgan's Laws, we have that

$$\left(\bigcap_{j=1}^{N} E_j\right)^c = \bigcup_{j=1}^{N} E_j^c.$$

So it suffices to show that $\left|\bigcup_{j=1}^{N} E_j^c\right| < 1$ since we then have that

$$1 = |(0,1)| = \left|\bigcup_{j=1}^{N} E_j^c\right| + \left|\bigcap_{j=1}^{N} E_j\right|,$$

which will give

$$\left|\bigcap_{j=1}^{N} E_j\right| > 0.$$

However, we have

$$\begin{aligned}
\left|\bigcup_{j=1}^{N} E_j^c\right| &\leq \sum_{j=1}^{N} |E_j^c| \\
&= \sum_{j=1}^{N} (1 - |E_j|) \\
&= N - \sum_{j=1}^{N} |E_j| \\
&< N - (N-1) = 1.
\end{aligned}$$

Here the last inequality follows from the hypothesis of the problem.