Definition of σ-algebra

A nonempty collection Σ of subsets E is called a σ-algebra if it satisfies the following conditions[a]:

(i) $\backslash E \in \Sigma$ if $E \in \Sigma$

(ii) $\cup_k E_k \in \Sigma$ if $E_k \in \Sigma, k = 1, 2, \ldots$

[a]Richard L. Wheeden and Antoni Zygmund. *Measure and integral: An introduction to real analysis.* CRC, 2015, p. 49.

Borel σ-algebra

The smallest σ-algebra of subsets of $\mathbb{R}^n$ containing all the open subsets of $\mathbb{R}^n$ is called the Borel σ-algebra $\mathfrak{B}$ of $\mathbb{R}^n$, and the sets in $\mathfrak{B}$ are called Borel subsets of $\mathbb{R}^n$. Sets of type $F_\sigma, G_\delta, F_{\sigma\delta}, G_{\delta\sigma}$ (see p.6 in Section 1.3), etc., are Borel sets.[a]

[a]Ibid., p. 49.

Problem 1. Zygmund p59 exercise 08

Show that the Borel σ-algebra $\mathfrak{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

Solution.

To show that the Borel σ-algebra $\mathfrak{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$, we need to show that

(i) $\mathfrak{B}$ contains the closed sets in $\mathbb{R}^n$.

(ii) $\mathfrak{B} \subseteq A$ if A is a σ-algebra containing the closed sets in $\mathbb{R}^n$.

(i)

For every closed subset $E \in \mathbb{R}^n$, there is a open subset $\backslash E \in \mathbb{R}^n$. By definition of the Borel σ-algebra $\mathfrak{B}$, we have $\backslash E \in \mathfrak{B}$, implying that $E \in \mathfrak{B}$. Thus, $\mathfrak{B}$ contains the closed sets in $\mathbb{R}^n$.

(ii)

Let A be a σ-algebra containing the closed sets in $\mathbb{R}^n$, the definition of σ-algebra implies that A is a σ-algebra containing all the open sets in $\mathbb{R}^n$

By definition of the Borel σ-algebra $\mathfrak{B}$, it is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$. Thus, we have $\mathfrak{B} \subseteq A$.

Therefore, we can conclude that the Borel σ-algebra $\mathfrak{B}$ in $\mathbb{R}^n$ is the smallest σ-algebra containing the closed sets in $\mathbb{R}^n$.

$\blacksquare$

Problem 2. Zygmund p59 exercise 09

If $\{E_k\}_{k=1}^{\infty}$ is a sequence of sets with $\sum_{k=1}^{\infty} |E_k|_e < +\infty$, show that $\limsup_{k\to\infty} E_k$ (and so also $\liminf_{k\to\infty} E_k$) has measure zero.

[a]Suppose $\{E_k\}_{k=1}^{\infty}$ is a sequence of subsets:

$$\limsup_{k\to\infty} E_k := \lim_{n\to\infty} V_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k,$$
$$\liminf_{k\to\infty} E_k := \lim_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k.$$

In other words,

$$V_n = \bigcup_{k=n}^{\infty} E_k \searrow V = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k =: \limsup_{k\to\infty} E_k,$$
$$B_n = \bigcap_{k=n}^{\infty} E_k \nearrow B = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} E_k =: \liminf_{k\to\infty} E_k.$$

[a]Wheeden and Zygmund, see n. a, p. 49.

Solution. By the definition of the limit of a sequence, let $\epsilon > 0$, there exists $N \in \mathbb{N}$ s.t. $\forall n \geq N$,

$$\sum_{k=n}^{\infty} |E_k|_e = \sum_{k=1}^{\infty} |E_k|_e - \sum_{k=1}^{n-1} |E_k|_e < \epsilon.$$

Since

$$\begin{aligned}\limsup_{k\to\infty} E_k &:= \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k \\ &\subseteq \bigcup_{k=N}^{\infty} E_k,\end{aligned}$$

we have

$$\begin{aligned}|\limsup_{k\to\infty} E_k|_e &\leq |\bigcup_{k=N}^{\infty} E_k|_e \\ &= \sum_{k=N}^{\infty} |E_k|_e < \epsilon.\end{aligned}$$

Let $\epsilon \to 0$, we have

$$|\limsup_{k\to\infty} E_k|_e = 0.$$

. . . .

By the definition, we have

$$\liminf_{k\to\infty} E_k \subseteq \limsup_{k\to\infty} E_k.$$

Thus,

$$|\liminf_{k\to\infty} E_k|_e \leq |\limsup_{k\to\infty} E_k|_e = 0.$$

Therefore, $|\liminf_{k\to\infty} E_k|_e = 0$.

■

Problem 3. Zygmund p59 exercise 09

Show that there exist sets $\{E_k\}_{k=1}^{\infty}$ such that $E_k \searrow E$, $|E_k|_e < +\infty$, and $\lim_{k\to\infty} |E_k|_e > |E|_e$ with strict inequality.

Problem 4. Zygmund p76 exercise 02

Let f be a simple function, taking distinct values on disjoint sets $E_1, \ldots, E_N$. Show that f is measurable if and only if E_i is measurable for $1 \leq i \leq N$.

Problem 5. Zygmund p76 exercise 05

Give an example to show that $\phi(f(x))$ may not be measurable if ϕ and f are measurable and finite.
(Let F be the Cantor-Lebesgue function and let f be its inverse, suitably defined. Let ϕ be the characteristic function of a set of measure zero whose image under F is not measurable.)
—
Show that the same may be true even if f is continuous.
(Let $g(x) = x + F(x)$, where F is the Cantor-Lebesgue function, and consider $f = g^{-1}$.) Cf. Exercise 22.