

MATH 6337: HOMEWORK 4 SOLUTIONS

4.1. Prove the following

- (i) Corollary 4.2: If f is measurable, then $\{f > -\infty\}, \{f < +\infty\}, \{f = +\infty\}, \{a \leq f \leq b\}, \{f = a\}$, etc. are all measurable. Moreover, f is measurable if and only if $\{a < f < +\infty\}$ is measurable for every finite a .
- (ii) Theorem 4.8: If f is measurable and λ is any real number, then $f + \lambda$ and λf are measurable.

Solution.

- (i) All of these sets are elements of the σ -algebra generated by sets of the form $\{f > a\}$ (i.e., taking some combination of countable unions and complements of sets like $\{f > a\}$ will produce each of the sets above). For example,

- $\{f > -\infty\} = \bigcup_{k=1}^{\infty} \{f > -k\}$
- $\{f < +\infty\} = \bigcup_{k=1}^{\infty} \{f < k\}$
- $\{f = +\infty\} = \{f < +\infty\}^c$
- $\{a \leq f \leq b\} = (\bigcap_{k=1}^{\infty} \{f > a - 1/k\}) \cap \{f > b\}^c$
- $\{f = a\} = \{a \leq f \leq a\}$

Since the sets $\{f > a\}$ are measurable (by definition), and since measurability is preserved under countable unions and complements, the first statement is true.

As for the second statement, the forward implication is implied by the above, for

$$\{a < f < +\infty\} = \{f > a\} \cap \{f < +\infty\}.$$

On the other hand, suppose all the sets $\{a < f < +\infty\}$ are measurable: then

$$\bigcup_{k=1}^{\infty} \{-k < f < +\infty\} = \{f < +\infty\}$$

is measurable, so

$$\{f < +\infty\}^c = \{f = +\infty\}$$

is measurable, so

$$\{a < f < +\infty\} \cup \{f = +\infty\} = \{f > a\}$$

is measurable. Thus, f is measurable.

- (ii) If f is measurable, then $\{f > a - \lambda\} = \{f + \lambda > a\}$ is a measurable set for all a . If $\lambda > 0$, then $\{f > a/\lambda\} = \{\lambda f > a\}$ is measurable for all a . If $\lambda < 0$, then $\{f < a/\lambda\} = \{\lambda f > a\}$ is measurable for all a . If $\lambda = 0$, then $\lambda f \equiv 0$ identically, which is measurable. Thus, $f + \lambda$ and λf are measurable functions for all $\lambda \in \mathbb{R}$.

□

4.2. Let f be a simple function, taking its distinct values on disjoint sets E_1, \dots, E_N . Show that f is measurable if and only if E_1, \dots, E_N are measurable.

Solution. Write $f = \sum_{i=1}^N a_i \chi_{E_i}$. We may assume none of the $a_i = 0$. If f is measurable, then the set $\{f = a_i\} = E_i$ is measurable for all i . If the E_i are measurable, then let $g_i = |a_i| \chi_{E_i}$ for each i . Then

$$\{g_i > x\} = \begin{cases} \emptyset, & x > a_i \\ E_i, & 0 < x \leq a_i \\ \mathbb{R}^n, & x \leq 0, \end{cases}$$

all of which are measurable sets; therefore, $a_i \chi_{E_i} = \pm g_i$ is measurable for each i . Thus, f is a finite sum of measurable functions which take on finite values, so f is measurable. \square

4.3. Theorem 4.3* can be used to define measurability for vector-valued (e.g. complex-valued) functions. Suppose, for example, that f and g are real-valued and defined in \mathbb{R}^n , and let $F(x) = (f(x), g(x))$. Then F is said to be measurable if $F^{-1}(G)$ is measurable for every open $G \subseteq \mathbb{R}^2$. Prove that F is measurable if and only if both f and g are measurable in \mathbb{R}^n .

Solution. Suppose that F is measurable. Then $F^{-1}((a, +\infty) \times \mathbb{R}) = \{f > a\}$ and $F^{-1}(\mathbb{R} \times (a, +\infty)) = \{g > a\}$ are measurable for all $a \in \mathbb{R}$, so f and g are measurable. Now suppose that f and g are measurable; then $\{a \leq f \leq b\}$ and $\{c \leq g \leq d\}$ are measurable for all $a, b, c, d \in \mathbb{R}$. Every open set in \mathbb{R}^2 can be written as the countable union of closed rectangles. So, if G is an open subset of \mathbb{R}^2 , then

$$F^{-1}(G) = F^{-1} \left(\bigcup_k [a_k, b_k] \times [c_k, d_k] \right) = \bigcup_k F^{-1}([a_k, b_k] \times [c_k, d_k]) = \bigcup_k (\{a_k \leq f \leq b_k\} \cap \{c_k \leq g \leq d_k\}),$$

which is a countable union of measurable sets and is therefore measurable; thus, F is measurable. \square

* f is measurable if and only if for every open set G in \mathbb{R}^1 , the inverse image $f^{-1}(G)$ is a measurable subset of \mathbb{R}^n

4.8.

- (a) Let f and g be two functions which are usc at x_0 . Show that $f + g$ is usc at x_0 . Is $f - g$ usc at x_0 ? When is fg usc at x_0 ?
- (b) If $\{f_k\}$ is a sequence of functions which are usc at x_0 , show that $\inf_k f_k(x)$ is usc at x_0 .
- (c) If $\{f_k\}$ is a sequence of functions which are usc at x_0 and which converge uniformly near x_0 , show that $\lim f_k$ is usc at x_0 .

Solution.

- (a) Let $L_f = \limsup_{x \rightarrow x_0} f(x)$, and define L_g and L_{f+g} analogously. We'll suppose that $f(x_0), g(x_0) \neq +\infty$, so that neither L_f nor L_g can be $+\infty$. Then $L_{f+g} \leq L_f + L_g \leq f(x_0) + g(x_0) = (f + g)(x_0)$, so $f + g$ is usc.[†] If $f(x_0)$ or $g(x_0)$ is $+\infty$ and the other is not $-\infty$, then the result is trivial. (If one is $+\infty$ and the other is $-\infty$, then it's not clear what to make of $f(x_0) + g(x_0)$, so we'll ignore that case.) $f - g$ is not in general usc at x_0 : take $f = 0$ and g to be any usc function which is not continuous at x_0 . Then $-g$ is lsc but not usc. Since $\sup fg \leq (\sup f)(\sup g)$ whenever $f, g \geq 0$, it follows that $L_{fg} \leq L_f L_g \leq f(x_0)g(x_0)$ under the same conditions.
- (b) Let $f(x) = \inf_k f_k(x)$, and let L_f be the limit superior of f . Then

$$L_f \leq L_{f_k} \leq f_k(x_0)$$

for all k , so $L_f \leq \inf_k f_k(x_0) = f(x_0)$; thus f is usc at x_0 .

- (c) Let $f(x) = \lim f_k(x)$, and let L_f be the limit superior of f . Then, for every $\varepsilon > 0$ there exists $k > 0$ such that $\sup |f - f_k| < \varepsilon$. Then

$$L_f < L_{f_k + \varepsilon} \leq f_k(x_0) + \varepsilon < f(x_0) + 2\varepsilon$$

for every $\varepsilon > 0$; hence $L_f \leq f(x_0)$, so f is usc.

□

[†]To see that $L_{f+g} \leq L_f + L_g$, observe the standard fact that $\sup_{x \in E} f(x) + g(x) \leq \sup_{x \in E} f(x) + \sup_{x \in E} g(x)$, take $E = B_\delta(x_0)$ and let $\delta \rightarrow 0^+$.

4.13. One difficulty encountered in trying to extend the proof of Egorov's theorem to the continuous parameter case $f_y(x) \rightarrow f(x)$ as $y \rightarrow y_0$ is show that the analogues of the sets E_m in Lemma 4.18[†] are measurable. This difficulty can often be overcome in individual cases. Suppose, for example, that $f(x, y)$ is defined and continuous in the square $0 \leq x \leq 1$, $0 < y \leq 1$, and that $f(x) = \lim_{y \rightarrow 0} f(x, y)$ exists and is finite for x in a measurable subset E of $[0, 1]$. Show that if ε and δ satisfy $0 < \varepsilon, \delta < 1$, the set

$$E_{\varepsilon\delta} = \{x \in E : |f(x, y) - f(x)| \leq \varepsilon \text{ for all } y < \delta\}$$

is measurable. [Hint: If $\{y_k\}_{k=1}^\infty$ is a dense subset of $(0, \delta)$, show that

$$E_{\varepsilon\delta} = \bigcap_k \{x \in E : |f(x, y_k) - f(x)| \leq \varepsilon\} .]$$

Solution. Let $\{y_k\}_{k=1}^\infty$ be a countable dense subset of $(0, \delta)$. Certainly if $x \in E_{\varepsilon\delta}$, then $|f(x, y_k) - f(x)| \leq \varepsilon$ for all k . Now suppose $x \in E$ such that $|f(x, y_k) - f(x)| \leq \varepsilon$ for all k , and let $0 < y < \delta$. Choose a subsequence $y_{k_j} \rightarrow y$. Then, since f is a continuous function at (x, y) , we have

$$|f(x, y) - f(x)| = \lim_{j \rightarrow \infty} |f(x, y_{k_j}) - f(x)| \leq \varepsilon.$$

Now, since $f(x, y_k)$ and $f(x)$ are measurable functions of x (the former is continuous, the latter is the limit of the continuous-therefore-measurable functions $f(x, 1/k)$), $|f(x, y_k) - f(x)|$ is measurable, so the sets $\{x \in E : |f(x, y_k) - f(x)| \leq \varepsilon\}$ are measurable; hence $E_{\varepsilon\delta}$ is measurable. \square

[†]If $\{f_k\}$ is a sequence of measurable functions which converge almost everywhere to the finite limit f in a set E of finite measure, then given $\varepsilon, \eta > 0$ there is a closed subset $F \subseteq E$ and an integer K such that $|E \setminus F| < \eta$ and $|f(x) - f_k(x)| < \varepsilon$ for all $x \in F$ and all $k > K$.