Problem 1. Zygmund p58 exercise 01

(a) There is an analogue for bases different from 10 of usual decimal expansion of number. If b is an integer larger than 1 and $0 < x < 1$, show that there exist integral coefficient c_k, $0 \le c_k < b$, such that $x = \sum_{k=1}^{\infty} c_k b^{-k}$. Furthermore, show that expansion is unique unless $x = cb^{-k}$, in which case there are two expansions.
(b) When $b = 3$, the expansion is called the triadic or ternary expansion of x. Show that Canter set consist of point in $[0, 1]$ which has triadic representation such that c_k is either 0 or 2, namely,

$$\mathcal{C} = \{x \in [0, 1] : x = \sum_{k=1}^{\infty} c_k 3^{-k}, c_k \in \{0, 2\}\}$$

Random Walks on Graphs

We define the following terms[a]:

Definition 0.1 (Hitting time)**.** The hitting time h_{uv} (sometimes called the mean first passage time) is the expected number of steps in a random walk that starts at u and ends upon first reaching v.

Definition 0.2 (Commute time)**.** We define C_{uv}, the commute time between u and v, to be $C_{uv} = h_{uv} + h_{vu} = C_{vu}$. This is the expected time for a random walk starting at u to return to u after at least one visit to v.

Definition 0.3 (Cover time)**.** Let $C_u(G)$ denote the expected length of a walk that starts at u and ends upon visiting every vertex in G at least once. The cover time of G, denoted $C(G)$, is defined by $C(G) = \max_u C_u(G)$.

[a] **motwani95**.

Solution.

$$mR(G) \le C(G) \le 32mR(G) \log_2 n + 1.$$

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Problem 2. Zygmund p58 exercise 03

Construct a two-dimensional Cantor set in the unit square $\{(x, y) : 0 \le x, y \le 1\}$ as follows. Subdivide the square into nine equal parts and keep only the four closed corner squares, removing the remaining region (which forms a cross). Then repeat this process in a suitably scaled version for the remaining squares, ad infinitum. Show that the resulting set is perfect, has plane measure zero, and equals $\mathcal{C} \times \mathcal{C}$.

Solution.
Let D_0 be the unit square $\{(x, y) : 0 \le (x, y) \le 1\}$. Let D_k be the set remaining after i steps. Let $D = \bigcap_{k=1}^{\infty} D_k$ be the resulting set.

(a) To prove that it is a perfect set, we need to show that every point in the set is a limit point of the set.

Theorem 1.7

(i) The intersection of any number of closed sets is closed.

(ii) The union of any number of open sets is open.

Since each D_k is closed, it follows from Theorem 1.7 that D is closed. Note that D_k consists of 4^k closed disjoint intervals, each of which

(b)
Since D is covered by the intervals in any D_k, we have

$$|D|_e \le |D_k|_e = \left(\frac{4}{9}\right)^k.$$

Let $k \to \infty$, we have $|D|_e = 0$.

(c)

$$D := \bigcap_{k=1}^{\infty} D_k := \bigcap_{k=1}^{\infty} C_k \times C_k = \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right) = C \times C.$$

To prove that $\bigcap_{k=1}^{\infty} C_k \times C_k = (\bigcap_{k=1}^{\infty} C_k) \times (\bigcap_{k=1}^{\infty} C_k)$, start with the inclusion from left to right:

For all

$$(x, y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$$

we have

$$(x, y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, we have

$$x \in C_k, \forall k \in \mathbb{N},$$
$$y \in C_k, \forall k \in \mathbb{N}.$$

Thus,

$$x \in \bigcap_{k=1}^{\infty} C_k,$$
$$y \in \bigcap_{k=1}^{\infty} C_k.$$

Therefore,

$$(x,y) \in \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right), \forall (x,y) \in D.$$

Next, prove the inclusion from right to left. For all

$$(x,y) \in \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right)$$

we have

$$x \in C_k, \forall k \in \mathbb{N},$$

$$y \in C_k, \forall k \in \mathbb{N}.$$

By the definition of Cartesian product, this implies that

$$(x,y) \in C_k \times C_k, \forall k \in \mathbb{N}.$$

Thus, $(x,y) \in \bigcap_{k=1}^{\infty} C_k \times C_k$ since it satisfies the definition of intersection.

Therefore, we can conclude that the two sets are equal:

$$\bigcap_{k=1}^{\infty} C_k \times C_k = \left(\bigcap_{k=1}^{\infty} C_k\right) \times \left(\bigcap_{k=1}^{\infty} C_k\right).$$

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Problem 3. Zygmund p59 exercise 04

Construct a subset of $[0, 1]$ in the same manner as the Cantor set by removing from each remaining interval a subinterval of relative length $\theta, 0 < \theta < 1$. Show that the resulting set is perfect and has measure zero.