Math 5051 : Real Analysis I

Mid-term Exam 2
07 November 2016

Instructions: Answer all of the problems.

1. Let H be a Hilbert space over the complex numbers. The form $\mathfrak{a} : H \times H \to \mathbb{C}$ is sesquilinear if for all $\alpha \in \mathbb{C}$ and $u, v, w \in H$:

$$\mathfrak{a}(\alpha u + v, w) = \alpha \mathfrak{a}(u, w) + \mathfrak{a}(v, w) \quad \mathfrak{a}(w, \alpha u + v) = \overline{\alpha}\mathfrak{a}(w, u) + \mathfrak{a}(w, v)$$

Suppose that $|\mathfrak{a}(u, v)| \leq M \|u\|_H \|v\|_H$. Prove that there exists a unique bounded linear operator $T : H \to H$ such that

$$\mathfrak{a}(u, v) = \langle Tu, v\rangle_H .$$

2. Use Fubini's Theorem to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

3. Suppose that $f \geq 0$ on $(0, \infty)$, $f \in L^p(0, \infty)$ and

$$F(x) = \frac{1}{x} \int_0^x f(t)dt.$$

Prove that for $1 < p < \infty$

$$\int_0^\infty F(x)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(t)^p dt$$

Hint: Write $xF(x) = \int_0^x f(t)t^a t^{-a} dt$ and use Hölder's Inequality.

4. Fix a positive integer N, and put $\xi = e^{\frac{2\pi i}{N}}$.

   (a) Prove the orthogonality relation

$$\frac{1}{N}\sum_{n=1}^{N} \xi^{nk} = \begin{cases} 1 & : \ k = 0 \\ 0 & : \ 1 \leq k \leq N-1. \end{cases}$$

   (b) Use this identity to prove that in a Hilbert space H, when $N \geq 3$ we have

$$\langle x, y\rangle_H = \frac{1}{N}\sum_{n=1}^{N} \|x + \xi^n y\|_H^2 \xi^n$$

   (c) Show that more generally we have

$$\langle x, y\rangle_H = \frac{1}{2\pi}\int_{-\pi}^{\pi} \|x + e^{i\theta} y\|_H^2 e^{i\theta}\, d\theta$$

5. A sequence $\{f_k\}$ in a Hilbert space H converges weakly to f if and only if for any $h \in H$ we have

$$\lim_{k\to\infty} \langle f_k, h\rangle_H = \langle f, h\rangle_H.$$

Suppose that $\{f_k\}$ converges weakly to f and

$$\lim_{k\to\infty} \|f_k\|_H = \|f\|_H.$$

Show that

$$\lim_{k\to\infty} \|f_k - f\|_H = 0.$$

6. Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then x_0 is in the closure $\overline{M}$ if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$. Hint: Use the Hahn-Banach Theorem.

7. Suppose that $1 \leq p \leq \infty$ that $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$. Show that $f * g(x)$ exists for almost all x and that $f * g \in L^p(\mathbb{R})$ and

$$\|f * g\|_{L^p(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})} \|g\|_{L^p(\mathbb{R})}.$$

8. Let $f \in L(-\infty, \infty)$ and $h > 0$ and fixed. Prove that:

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} \left(\frac{1}{2h} \int_{x-h}^{x+h} f(y)dy \right) dx$$