5/9/2022

Real Analysis

Thm If $|E| < \infty$ $\|f\|_\infty = \lim_{p \to \infty} \|f\|_p$

case $\|f\|_\infty = \infty$

$\forall M > 0$

$$|\{f > M\}| > 0$$

$$\int |f|^p dx \geq \int_{\{f > M\}} |f|^p dx \geq \int_{\{f > M\}} M^p dx$$

$$= |\{f > M\}| M^p$$

$$\|f\|_p \geq M \mu(\{f > M\})^{\frac{1}{p}}$$

$$\liminf_{p \to \infty} \|f\|_p \geq \lim_{p \to \infty} M \mu(\{f > M\})^{\frac{1}{p}}$$

$$\geq M$$

$$\Rightarrow \liminf_{p \to \infty} \|f\|_p = \infty$$

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Example $|E| = \infty$

$\cancel{\|f} \; f = 1$ on $\mathbb{R}$

$$\|f\|_\infty = 1$$

$$\|f\|_p = \infty$$

$$\lim_{p \to \infty} \|f\|_p \neq \|f\|_\infty$$

Thm If $f, g \in L^p(E)$ $p > 0$, and $c \in \mathbb{R}$, then

$$f + g \in L^p(E)$$

$$cf \in L^p(E)$$

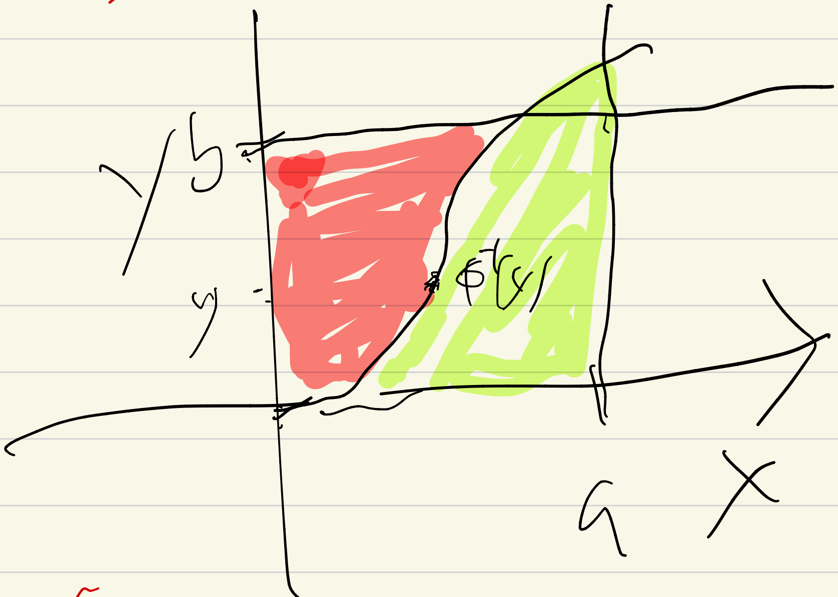
② Thm Young's inequality

Let $y=\phi$ be a strictly increasing continuous function on $[0,\infty)$ and $\phi(0)=0$.

Then, if $a,b\geq 0$.

$$ab \leq \int_0^a \phi(x)\,dx + \int_0^b \phi^{-1}(y)\,dy$$

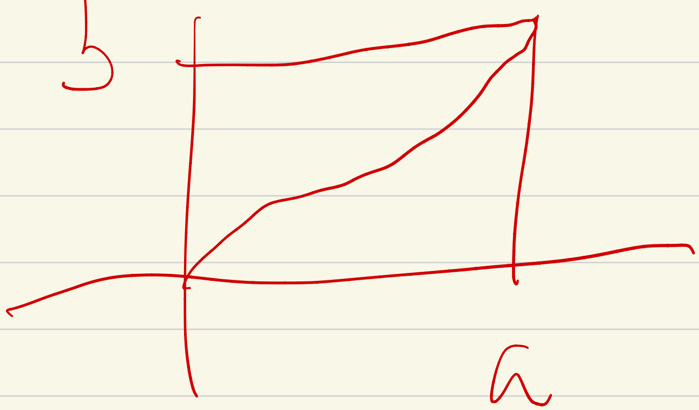
case 1



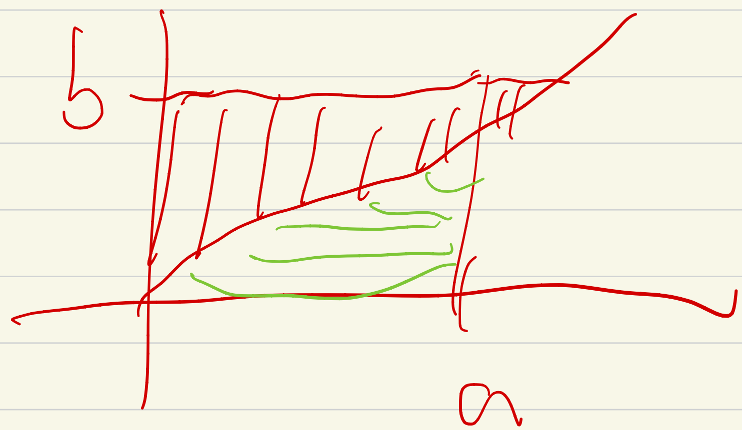
$$\int_0^a \phi(x)\,dx$$

$$\int_0^b \phi^{-1}(y)\,dy$$

case 2



case 3



$$\phi(x) = x^{\alpha} \qquad \phi^{-1}(x) = x^{\frac{1}{\alpha}}$$

$$\int_0^a \phi(x)\,dy = \frac{x^{\alpha+1}}{\alpha+1}\Big|_0^a = \frac{a^{\alpha+1}}{\alpha+1} \qquad \int_0^b y^{\frac{1}{\alpha}}\,dy = \frac{b^{1+\frac{1}{\alpha}}}{1+\frac{1}{\alpha}}$$

$$= \frac{b^{\frac{\alpha+1}{\alpha}}}{\frac{\alpha+1}{\alpha}}$$

By Young's inequality

$$ab \le \frac{a^{\alpha+1}}{\alpha+1} + \frac{b^{\frac{\alpha+1}{\alpha}}}{\frac{\alpha+1}{\alpha}}$$

$$= \frac{a^p}{p} + \frac{b^q}{q}$$

Def.:

$p = \alpha+1 \qquad \frac{1}{p} = \frac{1}{\alpha+1}$

$q = \frac{\alpha+1}{\alpha} \qquad \frac{1}{q} = \frac{\alpha}{\alpha+1}$

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{\alpha+1} + \frac{\alpha}{\alpha+1} = 1$$

Def We say $1 \leq p, q < \infty$ are Hölder's conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$

Def $1, \infty$ are Hölder's conjugate exponents.

Thm (Hölder's inequality). Suppose p, q are Hölder's conjugate exponents. Then,

$$\int |fg| dx \leq \|f\|_p \|g\|_q$$

proof

when $1 < p, q < \infty$ $\frac{1}{p} + \frac{1}{q} = 1$

If $\|f\|_p = 0$ $\Rightarrow \int |f|^p dx = 0$ $|f| = 0$ a.e.

$\Rightarrow$ $|fg| = 0$ a.e.

$\Rightarrow \int |fg| dy = 0 = \|f\|_p \|g\|_q$

similarly, for $\|g\|_q = 0$

$\int |fg| dx = 0 = \|f\|_p \|g\|_q$

If $\|f\|_p > 0$ $\|g\|_q > 0$.

Let $\tilde{f} = \dfrac{f}{\|f\|_p}$ $\tilde{g} = \dfrac{g}{\|g\|_q}$

$$\|\tilde{f}\|_p = \left\| \frac{f}{\|f\|_p} \right\|_p = \frac{\|f\|_p}{\|f\|_p} = 1$$

$$\|\tilde{g}\|_q = 1$$

Recall $ab \le \dfrac{a^p}{p} + \dfrac{b^q}{q}$

$$\int |\tilde{f}\tilde{g}|\,dx \le \int |\tilde{f}||\tilde{g}|\,dx$$

$$\le \int \frac{|\tilde{f}|^p}{p} + \frac{|\tilde{g}|^q}{q}\,dx$$

$$\le \frac{1}{p}\|\tilde{f}\|_p^p + \frac{1}{q}\|\tilde{g}\|_q^q$$

$$= \frac{1}{p}1 + \frac{1}{q}1 = 1$$

$$\int \left| \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \right| dx \le 1$$

$$\frac{1}{\|f\|_p \|g\|_q} \int |fg| \, dx \le 1$$

$$\int |fg| \, dx \le \|f\|_p \|g\|_q$$

when $p=1$ $q=\infty$

$$\|g\|_{L^\infty} = M$$

$$\{|g| \le M\} = E \setminus Z$$

$$Z = \{|g| > M\} \quad |Z| = 0$$

$$\int_E |fg|\,dx = \int_{E\setminus Z} |fg|\,dx \le \int_{E\setminus Z} M|f|\,dx$$

$$\le M\int_{E\setminus Z} |f|\,dx = M\int_E |f|\,dx$$

$$= \|f\|_1 \|g\|_\infty$$

$$\frac{1}{2} + \frac{1}{2} = 1$$

$$\int |fg| \, dx \le \left(\int |f|^2 \, dx \right)^{\frac{1}{2}} \left(\int |g|^2 \, dx \right)^{\frac{1}{2}}$$

Thm Let f be realvalued and measurable on E. Let $1 \leq p \leq \infty$, p, q are Hölders conjugate exponent ($\frac{1}{p} + \frac{1}{q} = 1$)

Then,

$$\|f\|_p = \sup \int_E fg$$

where supremum is taten over all realvalued g s.t $\|g\|_q \leq 1$.

proof

$$\int_E fg\,dx \le \int_E |fg|\,dx$$

$$\le \|f\|_p \|g\|_q \le \|f\|_p$$

$$\sup \int_E fg\,dx \le \|f\|_p$$

If $\|f\|_p = 0 \Rightarrow |f| = 0$ a.e.

$$\sup \int fg\,dx = 0 = \|f\|_p.$$

If $0 < \|f\|_p < \infty$ $\quad 0 < p < \infty$

$$g = \begin{cases} |f|^{\frac{p}{q}} \dfrac{f}{|f|} & \text{if } |f| \neq 0 \\ |f| = 0 \end{cases}$$

$$\int_E fg\,dx = \int_{\{|f| \neq 0\}} f\,|f|^{\frac{p}{p}} \frac{f}{|f|}\,dx$$

$$= \int_{|f| \neq 0} |f|\,|f|^{\frac{p}{q}}\, dx = \int_E |f|^{\frac{p}{q}+1}$$

$$= \int_E |f|^p\, dx$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$1 + \frac{p}{q} = p$$

$$= \|f\|_p^p$$

$$\|g\|_q = \left(\int_E \left(|f|^{\frac{p}{q}}\right)^q\right)^{\frac{1}{q}} = \left(\int_E |f|^p\right)^{\frac{1}{q}}$$

$$= \left(\|f\|_p^p\right)^{\frac{1}{q}} = \|f\|_p^{\frac{p}{q}}$$

$g \in L_q(E)$

$$\tilde{g} = \frac{g}{\|g\|_q} \qquad \|\tilde{g}\|_q = 1$$

$$\int f \tilde{g}\,dx = \frac{}{\|\tilde{g}\|_q}\int f g\,dx = \frac{\|f\|_p^p}{\|f\|_p^{\frac{p}{q}}}$$

$$= \|f\|_p^{p-\frac{p}{q}} = \|f\|_p$$

$$\cancel{\|f\|_p} \int f \tilde{g}\,dx \le \sup \int f g\,dx \le \|f\|_p$$

when $\|f\|_p = \infty$

$$\|f\|_p^p = \int_E |f|^p \, dx = \int_E |f^+|^p \, dx + \int_E |f^-|^p \, dx$$

$= \infty$. $\Rightarrow$ Either $\int_E |f^+|^p \, dx = \infty$

or $\int_E |f^-| \, dx = \infty$.

W. LOG WMA $\int |f^+|^p \, dx = \infty$

$$\int |f^+|^p dx = \infty$$

Let $f_k(x) = 0$ if $|x| > k$

$f_k(x) = \max\{k, f^+(x)\}$

$\|f_k(x)\|_p < \infty$

$\exists\, g_k$ s.t. $\|g_k\|_q = 1$ $\int f_k g_k dx = \|f_k\|_p$

$f_k \nearrow f^+$ By L.M.C.T

$\|f_k\|_p \nearrow \|f^+\|_p = \infty$

$\operatorname{supp} f_k = \operatorname{supp} g_k$ (by construction)

$$\int f g_k = \int f^+ g_k \cancel{+} - \overset{0}{\overset{\shortparallel}{\int f^- g_k}}$$

$$= \int f^+ g_k \geq \int f_k g_k \to \infty \text{ as } k \to \infty$$

$$\sup \int f g \geq \int f g_k$$

$$\Rightarrow \sup \int f g = \infty$$

1, a case is for homework.

Recall $\quad k \in L^1 \quad g \in L^p$

$$\|k * g\|_p \le \|k\|_1 \|g\|_p.$$

$$\frac{1}{p} + \frac{1}{q} = 1$$

$$\frac{1}{q} = \frac{p-1}{p} \qquad q = \frac{p}{p-1}$$

proof

$$\int \left| \int k(x-y)\, g(y)\, dy \right|^p dx$$

$$\int |k(x-y)\, g(y)|\, dy \le \left(\int \quad dx \right)^{\frac{1}{q}} \left(\int \quad '' \quad |g(y)|^p\, dy \right)^{\frac{1}{p}}$$

$$\left(|k(x-y)|^{\alpha} \right)^{q} = 1 \qquad \alpha q = 1 \qquad \alpha = \frac{1}{q} = \frac{p-1}{p}$$

$$\int |k(x-y)\, g(y)|\,dy = \int \left(|k(x-y)|^{\frac{p-1}{p}}\right)\left(|k(x-y)|^{\frac{1}{p}} g(y)\right)dy$$

$$\leq \left(\int \left(|k(x-y)|^{\frac{p-1}{p}}\right)^{q} dy\right)^{\frac{1}{q}} \left(\int |k(x-y)|\,|g(y)|^{p}\,dy\right)^{\frac{1}{p}}$$

$$\leq \left(\int |k(x-y)|\,dx\right)^{\frac{1}{q}} \left(\int |k(x-y)|\,|g(y)|^{p}\,dy\right)^{\frac{1}{p}}$$

$$\|k\|_{1}^{\frac{1}{q}}$$

$$\int |k * g|^p \, dx$$

$$\leq \int \left(\|k\|_1^{\frac{1}{q}}\right)^p \left(\int |k(x-y)||g(y)|^p \, dy\right) dx$$

$$\leq \|k\|_1^{\frac{p}{q}} \int\int |k(x-y)| \, dx \, |g(y)|^p \, dy$$

$$\leq \|k\|_1^{\frac{p}{q}} \int \|k\|_1 |g(y)|^p \, dy = \|k\|_1^{\frac{p}{q}+1} \|g\|_p^p$$

$$\|k*g\|_p^p \le \|k\|_1^{\frac{p}{q}+1} \|g\|_p^p = \|k\|_1^p \|g\|_p^p$$

$$\frac{1}{p}+\frac{1}{q}=1$$

$$1+\frac{p}{q}=p$$

$$\|k*g\| \le \|k\|_1 \|g\|_p$$

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Minkowski's inequality

If $1 \le p \le \infty$, then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$