Real Analysis

II

5/27

Thm If $f \in L^p(E)$ $1 \le p < \infty$, then

$$\lim_{|h| \to 0} \| f(x+h) - f(x) \|_p = 0$$

proof. Let C_p denote the class $f \in L^p$ s.t

$\| f(x+h) - f(x) \|_p \to 0$ as $|h| \to 0$.

Then,

(a) a finite linear combination of functions in C_p is in C_p.

(b) If $f_k \in C_p$ and

$$\lim_{k\to\infty} \|f_k - f\|_p = 0$$

, then $f \in C_p$.

proof of a.

$$f = \sum_{i=1}^{N} a_i f_i \qquad f_i \in C_p$$

$$\|f(x+h) - f(x)\|_p = \left\| \sum_{i=1}^{N} a_i f_i(x+h) - \sum_{i=1}^{N} a_i f_i(x) \right\|_p$$

$$\leq \sum_{i=1}^{N} \| a_i f_i(x+h) - a_i f(x) \|_p$$

$$\leq \sum_{i=1}^{N} |a_i| \, \| f_i(x+h) - f(x) \|_p \to 0 \text{ as } |h| \to 0$$

(b) $\lim_{k\to\infty} \| f_k - f \|_p = 0$

For $\frac{\varepsilon}{3} > 0$ $\exists$ $N_1 > 0$ s.t.

$$\| f_k - f \|_p \leq \frac{\varepsilon}{3} \text{ whenever } k \geq N_1$$

Choose $k_0 \geq N_1$

$$\| f_{k_0}(x+h) - f_{k_0}(x) \|_p \to 0 \quad \text{as } h \to 0$$

For $\frac{\epsilon}{3} > 0 \quad \exists \ \delta \quad$ s.t

$$\| f_{k_0}(x+h) - f_{k_0}(x) \|_p \leq \frac{\epsilon}{3} \quad \text{whenever}$$

$$|h| < \delta$$

For $\epsilon > 0$,

$$\| f(x+h) - f(x) \|_p = \| f(x+h) - f_{k_0}(x+h) + f_{k_0}(x+h) - f_{k_0}(x)$$
$$+ f_{k_0}(x) - f(x) \|$$

$$\leq \|f(x+h) - f_{k_0}(x+h)\|_p + \|f_{k_0}(x+h) - f_{k_0}(x)\|_p$$

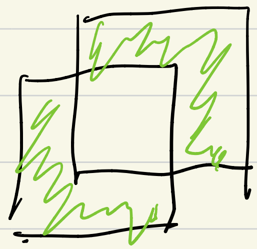
$$+ \|f_{k_0}(x) - f(x)\|_p$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \leq \varepsilon.$$

From the definition of limit,

$$\lim_{|h| \to 0} \|f(x+h) - f(x)\|_p = 0$$

Suppose $A \subseteq I$ is a cube with sidelength ℓ.



$$\int_I |\chi_A(x) - \chi_A(x+h)|^p \, dx$$

$$= \int_I \chi_{(A \setminus A+h)} + \chi_{(A+h) \setminus A} \, dx = |A \setminus A+h| + |(A+h) \setminus A|$$

$$\leq n \ell^{n-1} h \to 0 \quad \text{as } h \to 0$$

If A is a open set with finite measure.

$$A = \bigcup_{i=1}^{\infty} C_i$$ where C_i are nonoverlap cubes.

$$f_k = \sum_{i=1}^{k} \chi_{C_i} \qquad f_k \nearrow \chi_A$$

$$0 \leq \chi_A - f_k \leq \chi_A$$

$$0 \leq |\chi_A - f_k|^p \leq |\chi_A|^p$$

$$\int |\chi_A|^p dx = |A| < \infty$$

By L.D.C.T

$$\lim_{k\to\infty}\int |\chi_A - f_k|^p \, dx \to 0$$

$$f_k \to \chi_A \text{ in } L_p$$

By (a) $f_k \in C_p$ By (b) $\chi_A \in C_p$

If A is of G_δ Type with finite measure

$\exists$ H open st. $A \subseteq H$ $|H| \leq |A| + 1$

$\exists\ G_k$ s.t. G_k are open

$$A = \bigcap_{k=1}^{\infty} G_k$$

Let $\widetilde{G}_k = H \cap G_k$, $\widetilde{G}_k$ is open

Let $H_k = \bigcap_{i=1}^{k} \widetilde{G}_i$ H_k is open

$$H \supseteq H_k \searrow A$$

$$\chi_H \geq \chi_{H_k} \searrow \chi_A$$

$$0 \leq \chi_{H_k} - \chi_A \leq \chi_H$$

$$0 \leq |\chi_{H_k} - \chi_A|^p \leq |\chi_H|^p$$

$$\|\chi_H\|^p \leq (|A|+1)^{\frac{1}{p}} < \infty$$

By L. D. C. T

$$\int |\chi_{H_k} - \chi_A|^p \, dx \to 0 \quad \text{as } k \to \infty$$

$\chi_{H_k} \to \chi_A$ in $L_p(E)$

By b, $\chi_A \in L_p$ when A is of G_δ type with finite measure

If A is measurable with finit measure

$\exists\ H \in G_\delta$ s.t

$A = H \setminus Z$ where $|Z| = 0$

$\chi_A \sim \chi_H$ $\chi_A \in L_p$

By a. measurable simple function with finite measure support. is in C_p

Recall, for every $f \in L_p(E)$ $1 \le p < \infty$

$\exists \{f_k\}$ $k=1,2,\cdots\infty$ are measurable simple functions with finite measure support. s.t

$f_k \to f$ in $L_p(E)$.

Thus, by b. f is in C_p if

$f \in L_p(E)$

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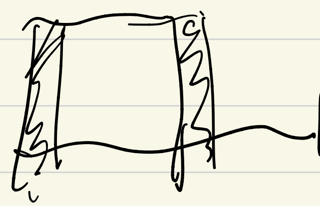
Example L_∞ is not in C_∞

$L_\infty((0,1))$



$\chi_{(0,\frac{1}{2})} \subseteq L_\infty(0,1)$

$$\|\chi_{(0,\frac{1}{2})}(x+h) - \chi_{(0,\frac{1}{2})}(x)\|_\infty = 1 \quad \forall\, h>0$$

§ Weak convergence.

Let $p \in [1, \infty)$ $E \subseteq \mathbb{R}^n$, q, p are Hölder's conjugate exponent $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$

We say a sequence in $L^p(E)$ converges weakly to u if

$$\lim_{k \to \infty} \int_E u_k g \, dx = \int_E u g \, dx$$

Thm If $u_k \to u$ in L_p, then

$$u_k \rightharpoonup u \text{ in } L_p.$$

We us $u_k \rightharpoonup u$ to denote

u_k converges to u weakly in $L_p(U)$

proof If $g \in L_q$, by Hölder's inequality

$$\left|\int (u_k - u) g \, dx\right| \le \|u_k - u\|_p \|g\|_q \to 0$$

as $k \to \infty$

Therefore $\lim_{k \to \infty} \int u_k g \, dx = \int u g \, dx$ #

Example. $E(0,1)$

$$g_k = \sin kx$$

$g_k \rightharpoonup 0$ in $L^p(E)$ $1 < p < \infty$ (Riemann Lebesgue lemma)

proof we say f is in the class of A_p

if $f \in L^q(E)$

$$\lim_{k\to\infty} \int g_k f \, dx = 0 = \int 0 f \, dx$$

(a) a finite linear combination of functions in A_p is in A_p

(b) If $\{f_k\}_{k=1}^{\infty} \subseteq A_p \subseteq L_q(E)$ s.t

$f_k \to f$ in L_q, then $f \in A_p$.

① If U is an open interval in $(0,1)$

$$\int_0^1 \sin kx \, \chi_U \, dx = \int_0^1 \sin(kx) \, \chi_{(c,d)} \, dx$$

$$= \int_c^d \sin kx \, dx = \left. \frac{-\cos kx}{k} \right|_c^d = \frac{\cos ck - \cos dk}{k}$$

$$\left| \int_0^1 \sin kx \, \chi_U \, dx \right| \leq \frac{2}{k} \to 0 \text{ as } k \to \infty$$

If U is a open set.

$$U = \bigcup_{k=1}^{\infty} (a_k, b_k) \quad \text{where } (a_k, b_k) \text{ are}$$

disjoint open intervals.

$$f_k = \sum_{i=1}^{k} \chi_{(a_i, b_i)} \nearrow \chi_U$$

$$0 \le |\chi_U - f_k|^p \le |\chi_U|^p \le 1$$

By B.C.T

$$f_k \to \chi_U \text{ in } L^q(E),$$

By a., $f_k \in A_p$, By b, $\chi_U \in A_p$.

$\Rightarrow U$ is of G_δ type $^{|U|<\infty}$ $\Rightarrow U$ is measurable
$|U|<\infty$

$\Rightarrow$ simple measurable function with finite support $\Rightarrow$ $f \in L^p(E)$,

$\sin kx \rightharpoonup 0$

$\sin kx \not\to 0$

Thm Suppose $\{u_k\}_{k=1}^{\infty}$ is a bounded sequence in $L^p(E)$, where $1<p<\infty$ and $E \subseteq \mathbb{R}^n$.

Then, $\exists$ a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and $u \in L^p$ s.t

$$u_{k_j} \rightharpoonup u \text{ in } L^p(E).$$

Notice In $L^p(E)$: bdd and closed

$\not\Rightarrow$ sequentially compact

Example. $A = f \in L^p(E)$

$\|f\|_p \leq 1$.

in $L^2([0, 2\pi])$.

$u_k = \sin kx$

$\|u_k\|_2 = \left(\int_0^{2\pi} (\sin kx)^2 \, dx \right)^{\frac{1}{2}} = \left(\int_0^{2\pi} \frac{1 - \cos 2\theta}{2} \, dx \right)^{\frac{1}{2}}$

$= \sqrt{\pi}$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$= 1 - 2\sin^2\theta$

$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$

$$\|u_k\|_2 \leq \sqrt{\pi} \qquad B = \{ f \in L_2([0,2\pi]) \mid \|f\|_2 \leq \sqrt{\pi} \}$$

$$u_k \subseteq B$$

$$\| u_k - u_j \|^2 = \int_0^{2\pi} (u_k - u_j)(u_k - u_j)$$

$$= \int_0^{2\pi} (\sin kx)^2 + (\sin jx)^2 - 2\sin kx \sin jx \, dx$$

$$= \pi + \pi - 2\int_0^{2\pi} \frac{\cos(\frac{k+j}{2}x) - \cos(\frac{k-j}{2}x)}{2} dx$$

$$= 2\pi$$

Suppose B is sequentially compact.

$\Rightarrow \exists$ subsequence of u_k, $u_{k_j} \to u$ in $L^2(E)$

$\Rightarrow \|u_{k_j} - u_{k_l}\|_2 \to 0$ as $j, l \to \infty$ $\rightarrow\leftarrow$

Duel space suppose $\{X, \|\|\}$ is a Banach space

We say T is a bdd linear function of X if $T: X \to \mathbb{R}$

$$T(x+y) = T(x) + T(y)$$

$$T(ax) = aT(x)$$

$$|T(x)| \leq c\|x\|$$

$X^* = \{ T \mid T$ is a bdd linear functional on $\{X, \|\,\|\}\}$

in X^*, $\quad \|\,\|_*$

$$\inf_{\|x\| \neq 0} \frac{|T(x)|}{\|x\|}$$

Recall, $\|f\|_p = \inf\limits_{\|g\|_q \le 1} \int fg\,dx = \inf\limits_{\|g\|_q \neq 0} \dfrac{\int fg\,dx}{\|g\|_q}$

Thm (Riesz representation Thm),

For $1 \le p < \infty$ $\quad E \subseteq \mathbb{R}^n$

$$(L^p(E))^* = L^q(E) \quad \text{i.e.}$$

$\forall \ T \in L^p(E) \quad \exists \quad g \in L^q(E)$ s.t

$$T(f) = \int_E fg\,dx,$$

Furthermore, $\|T\|_* = \|g\|_q$