Problem 1.

Prove that $L^\infty(E)$ is not separable for any E with $|E| > 0$.

Solution.

Take $E = [0, 1]$ for example.

Suppose $L^\infty(E)$ is separable.

Then there exists a dense subset A consisting of countable elements, i.e., $A \subset L^\infty(E)$.

Define $f_\alpha = \chi_{[0,\alpha]}$ for $\alpha \in [0, 1]$, where $\chi_{[0,\alpha]}$ is the indicator function for the interval $[0, \alpha]$.

If $\alpha \neq \beta$, then $\|f_\alpha - f_\beta\|_\infty = 1$.

Since A is dense, for every $\alpha \in [0, 1]$, and for every $\epsilon > 0$,

there exists $g_\alpha \in A$ such that

$$\|g_\alpha - f_\alpha\|_\infty < \epsilon.$$

By density, for $\beta \neq \alpha$, there is also $g_\beta \in A$ such that

$$\|g_\beta - f_\beta\|_\infty < \epsilon.$$

Using the triangle inequality, we get

$$\|f_\alpha - f_\beta\|_\infty \leq \|f_\alpha - g_\alpha\|_\infty + \|g_\alpha - g_\beta\|_\infty + \|g_\beta - f_\beta\|_\infty$$

Rearranging, we have

$$\begin{aligned}\|g_\alpha - g_\beta\|_\infty &\geq \|f_\alpha - f_\beta\|_\infty - \|f_\alpha - g_\alpha\|_\infty - \|f_\beta - g_\beta\|_\infty \\ &> 1 - \epsilon - \epsilon = 1 - 2\epsilon\end{aligned}$$

Since ϵ can be arbitrarily small, set $\epsilon = \frac{1}{4}$.

This implies $\|g_\alpha - g_\beta\|_\infty > \frac{1}{2}$. Therefore, $g_\alpha \neq g_\beta$ if $\alpha \neq \beta$.

This implies that the function mapping $\alpha \mapsto g_\alpha$ from $[0, 1]$ to A is injective, i.e., "one-one."

Since $[0, 1]$ is uncountably infinite, this implies that A is also uncountably infinite.

However, A is assumed to be a countable subset of $L^\infty[0, 1]$.

The existence of an uncountable subset $\{g_\alpha \mid \alpha \in [0, 1]\}$ within A contradicts the countability of A.

Therefore, $L^\infty(E)$ is not separable.

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Problem 2.

Let $1 \le p < \infty$ and $f \in L^p(\mathbb{R}^n)$. Show that the function g defined by

$$g_f(h) = \|f(x+h) - f(x)\|_p$$

is a uniformly continuous function on $\mathbb{R}^n$. Is the same statement true when $0 < p < 1$?

The statement only holds for $1 \le p < \infty$.

Theorem: Continuous functions with compact support are dense in L^p

Let $f \in L^p(\mathbb{R}^n)$. For every $\epsilon > 0$, there exists a function k that is continuous with compact support such that:

$$\|f - k\|_p < \epsilon.$$

....

We want to show $g_f(h)$ is uniformly continuous: i.e. For every $\epsilon > 0$, there exists δ such that:

$$|g_f(h_1) - g_f(h_2)| < \epsilon, \quad \text{whenever } |h_1 - h_2| < \delta.$$

We know that:

$$\begin{aligned} |g_f(h_1) - g_f(h_2)| &:= |\|f(x+h_1) - f(x)\|_p - \|f(x+h_2) - f(x)\|_p| \\ \text{(By Reverse Triangle Inequality)} &\le \|f(x+h_1) - f(x+h_2)\|_p \\ \text{(By Minkowski's Inequality)} &\le \|f(x+h_1) - k(x+h_1)\|_p + \|k(x+h_1) - k(x+h_2)\|_p \\ &\quad + \|f(x+h_2) - k(x+h_2)\|_p, \end{aligned}$$

where $\forall \epsilon > 0$:

1. $$\|f(x+h_1) - k(x+h_1)\|_p < \epsilon,$$ $$\|f(x+h_2) - k(x+h_2)\|_p < \epsilon,$$
   by the existence of a continuous function k with compact support dense in L^p;

2. $$\|k(x+h_1) - k(x+h_2)\|_p < \epsilon,$$
   since k is uniformly continuous with compact support E.

Thus, the estimations above lead to:

$$\begin{aligned} \|f(x+h_1) - f(x+h_2)\|_p &\le \|f(x+h_1) - k(x+h_1)\|_p + \|k(x+h_1) - k(x+h_2)\|_p \\ &\quad + \|f(x+h_2) - k(x+h_2)\|_p \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon. \end{aligned}$$

Therefore, choosing δ small enough to ensure the inequality above, whenever $|h_1 - h_2| < \delta$, guarantees that:

$$|g_f(h_1) - g_f(h_2)| < 3\epsilon.$$

Hence, we have shown that $g_f(h)$ is uniformly continuous. This confirms that any L^p function with $1 \le p < \infty$ behaves such that the mapping $h \mapsto \|f(x+h) - f(x)\|_p$ is uniformly continuous on $\mathbb{R}^n$.

For $0 < p < 1$**:**

Minkowski's inequality fails for $0 < p < 1$.

To see this, take $E = (0, 1)$, $f = \chi_{(0,\frac{1}{2})}$, and $g = \chi_{(\frac{1}{2},1)}$.

Then $\|f + g\|_p = 1$, while $\|f\|_p + \|g\|_p = 2^{-\frac{1}{p}} + 2^{-\frac{1}{p}} = 2 \cdot 2^{-\frac{1}{p}} = 2^{1-\frac{1}{p}} < 1$.

This demonstrates that the statement does not hold for $0 < p < 1$.