Exercise 1.

Proof. (a) The proof is rutin. (i) By exercise 16. chapter II, it yields that $\int_{\mathbb{R}^d} K_\delta(x)dx = 1$. Similarly, we also have

$$\int_{\mathbb{R}^d} |K_\delta(x)|dx = \int_{\mathbb{R}^d} |\phi(x)|dx.$$

Finally, for every $\eta > 0$, since ϕ is integrable, as $\delta \longrightarrow 0$,

$$\int_{|x|\geq\eta} |K_\delta(x)|dx = \frac{1}{\delta^d}\int_{|x|\geq\eta} |\phi(x/\delta)|dx = \int_{|y|\geq\eta\delta} |\phi(y)|dy \longrightarrow 0$$

Thus, $\{K_\delta\}_{\delta>0}$ is a family of good kernels.
(b) We assume without loss of generality that ϕ is bounded by $M > 0$ and supported on $[-1,1]$. Since $\phi(\delta^{-1}x) = 0$ for all $x \in [-\delta,\delta]^c$, and $|\phi(\delta^{-1}x)| \leq M$ for all $x \in [-\delta,\delta]$, then for every $x \in \mathbb{R}^d$,

$$|K_\delta(x)| = \frac{1}{\delta^d}|\phi(\delta^{-1}x)| = \frac{\delta}{\delta^{d+1}}|\phi(\delta^{-1}x)| \leq \frac{\delta}{|x|^{d+1}}|\phi(\delta^{-1}x)| \leq \frac{\delta M}{|x|^{d+1}}$$

On the other hand, it is obvious that $|K_\delta(x)| \leq \frac{M}{\delta^d}$, for every $x \in \mathbb{R}^d$. Hence, $\{K_\delta\}_{\delta>0}$ is an approximation to the identity.
(c) Suppose that $\{K_\delta\}_{\delta>0}$ is a family of good kernels. Let f be an integrable function. For any $\epsilon > 0$, by proposition 2.5, chapter II, there exists some $\eta > 0$ such that $\|f_y - f\| < \epsilon$ for every $|y| < \eta$, and thus,

$$\int_{|y|<\eta} \|f_y - f\|_1 K_\delta(y)dy < \epsilon.$$

On the other hand, by the property of good kernels, it follows that as $\delta \longrightarrow 0$,

$$\int_{|y|\geq\eta} \|f_y - f\|_1 K_\delta(y)dy \leq 2\|f\|_1 \int_{|y|\geq\eta} K_\delta(y)dy \longrightarrow 0.$$

Finally,

$$\begin{aligned}\lim_{\delta\to0} \|f * K_\delta - f\|_1 &= \lim_{\delta\to0} \int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} [f(x-y) - f(x)]\, K_\delta(y)dy \right| dx \\ &\leq \lim_{\delta\to0} \int_{\mathbb{R}^d} \|f_y - f\|_1 K_\delta(y)dy \leq \epsilon.\end{aligned}$$

Hence, we are done. ∎

Exercise 2.

Proof. We have seen that by if K'_δ is a family of kernels satisfying (i), (ii), and $\int K'_\delta = 1$ for all $\delta > 0$ then the conclusion holds indeed. Now, set

$$H_\delta(x) = \frac{1}{(4\pi)^{d/2}\delta^d} \exp\left\{-\frac{|x|^2}{4\delta^2}\right\}.$$

Then H_δ satisfies (i) and (ii) since $|x|^n H_\delta(x) \to 0$ as $|x| \to \infty$ for all $n \in \mathbb{N}$. Define $K'_\delta \triangleq K_\delta + H_\delta$. Then we have a.s. both $f * K'_\delta(x)$ and $f * H_\delta(x)$ tends to $f(x)$. Now, observe that $f * K_\delta = f * K_\delta - f * H_\delta$. □

Exercise 4.

Proof. Assume that f is integrable and not identically zero, then there exists some open ball $B_0 = B(0, r)$ such that

$$\int_{B_0} |f(x)|dx > 0$$

Now, we can see that for every $|x| \geq 1$,

$$\begin{aligned}
f^*(x) = &\sup_{x\in B} \frac{1}{m(B)} \int_B |f(x)|dx \\
\geq &\frac{1}{m(B(x, |x| + r))} \int_{B(x,|x|+r)} |f(x)|dx \\
\geq &\frac{1}{v_d(|x| + r)^d} \int_{B_0} |f(x)|dx \\
= &\frac{1}{|x|^d} \frac{|x|^d}{v_d(|x| + r)^d} \int_{B_0} |f(x)|dx \\
\geq &\frac{c}{|x|^d},
\end{aligned}$$

where $c = \frac{1}{v_d(1+r)^d} \int_{B_0} |f(x)|dx$. Since $|x|^{-d}\chi_{|x|\geq 1}$ is not integrable, then so is f^*.
Next, we want to show that if f is supported in the unit ball with $\int |f| = 1$, then

$$m(\{x : f^*(x) > \alpha\}) \geq c'/\alpha$$

for some $c' > 0$ and all sufficiently small α.
It is easy to see that $f^*(x) \geq 1$ for every $|x| < 1$. ¿From our previous discussion we have that $f^*(x) \geq \frac{c}{|x|^d}$ for some c and all $|x| \geq 1$. Now, if $1 \leq |x| \leq \left(\frac{c}{\alpha}\right)^{1/d}$, it follows that $f^*(x) \geq \alpha$. Therefore, if $0 < \alpha < 1 \wedge c$,

$$m\left(\{x : f^*(x) > \alpha\}\right) \geq m\left(|x| < 1\right) + m\left(1 \leq |x| \leq \left(\frac{c}{\alpha}\right)^{1/d}\right) = m\left(|x| \leq \left(\frac{c}{\alpha}\right)^{1/d}\right) = \frac{cv_d}{\alpha}.$$

Indeed we can take $c = \frac{1}{v_d 2^d}$, and for all $\alpha < 1 \wedge c$, it yields

$$\frac{1}{2^d\alpha} \leq m(\{x : f^*(x) > \alpha\}) \leq \frac{3^d}{\alpha}.$$

□

Exercise 5.

Proof. (a) Observe that the antiderivative of $f(x)$ for $0 < x \leq 1/2$ is

$$\frac{1}{\log(1/x)},$$

which has a limit as x tends to $0+$. Observe that $f(x)$ is a symmetric function about the origin to conclude the desired result.

(b) If suffices to consider $0 < x \leq 1/2$. Now, by the fundamental theorem of calculus,

$$f^*(x) \geq \lim_{m(B)\to 0, x\in B} \frac{1}{m(B)} \int_B f(y)dy = \frac{1}{x(\log(1/x))^2}, \qquad \forall x \in (0, 1/2]. \quad (1)$$

On the other hand,

$$\lim_{x\downarrow 0+} \frac{\frac{1}{x\log(1/x)}}{\frac{1}{2x\log(1/2x)}} = 2.$$

Therefore, there exists $1/2 > \delta > 0$ such that for all $0 < x < \delta$,

$$\frac{\frac{1}{x\log(1/x)}}{\frac{1}{2x\log(1/2x)}} \leq 3.$$

Observe that for all $\delta \leq x \leq 1/2$, by (1), there is a constant c such that

$$f^*(x) \geq \frac{c}{x\log(1/x)}$$

For all $x \in (0, \delta)$, by taking $r = x$, we see the average integral on $B(0, x)$ is:

$$\frac{1}{2x}\int_0^{2x} \frac{1}{y(\log(1/y))^2}dy = \frac{1}{2x\log(1/2x)} \geq \frac{1}{3x\log(1/x)}.$$

This completes the proof. $\square$