Real Analysis

5/11

2022

- Minkowski's inequality

If $1 \le p \le \infty$, then

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

proof $p=1$

$$\int_E |f+g|\,dx \le \int_E |f|+|g|\,dx \le \int |f|\,dx + \int |g|\,dx$$

$1 < p < \infty$

$$\int_E |f+g|^p \, dx \leq \int_E (|f|+|g|)\,|f+g|^{p-1} \, dx$$

$$\leq \int |f|\,|f+g|^{p-1} dx + \int |g|\,|f+g|^{p-1} dx$$

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad \frac{1}{q} = \frac{p-1}{p} \qquad q = \frac{p}{p-1}$$

$$\leq \|f\|_p \left(\int \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \|g\|_p \left(\int \left(|f+g|^{p-1}\right)^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}}$$

$$\leq \|f\|_p \|f+g\|_p^{p-1} + \|g\|_p \|f+g\|_p^{p-1}$$

$$\|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{p-1}$$

If $\|f+g\|_p \neq 0$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p .$$

If $\|f+g\|_p = 0$

$$0 = \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$p = \infty$ excersis

Vector space. A vector space over a field is a set V together with

$$+ : V \times V \to V$$

$$\cdot : F \times V \to V \quad \text{s.t.}$$

① If u $v \in V$, then

$$u + v \in V$$

② if $c \in F$ $u \in W$, then

$cu \in W$

③ if $u, v, w \in W$

$(u+v)+w = u+(v+w)$

④ ~~if~~

$u+v = v+u$

⑤ $\exists\, 0 \in W$ s.t. $0+v = v$ $\forall\, v \in W$.

6 $\forall u \in W \ \exists \ -u \in W$ s.t.

$u + -u = 0$

7 $a(bv) = (ab)v$

8 $1v = v$

9 $a(u+v) = au + av$

10 $(a+b)u = au + bu$.

Claim $L_p(E)$ is a vector space over $\mathbb{R}$.

$f \in L_p(E)$ $g \in L_p(E)$

$f+g \in L_p(E)$

θ $cf \in L_p(E)$.

Let X be a vector space over $\mathbb{R}$.

$\| \ \| : X \to [0, \infty)$ is called a norm

(i) $\|u+v\| \le \|u\| + \|v\|$

(ii) $\|\alpha u\| = |\alpha| \, \|u\|$

(iii) $\|u\| = 0$ iff $u = 0$

(in $\mathbb{R}$) (in X)

If only i, iii holds, $\| \ \|$ is called a seminorm.

Observation: $\| \ \|_p$ is a seminorm in $L_p(E)$.

If X is a vectorspace over F with a seminorm $\| \ \|$. Then we define $a \sim b$ if $\|a-b\|=0$

$\tilde{X} = X/\sim$ is a normed space.

Def. A complete normed space is called a Banach space.

Def $(\|\ \|, X)$ is said to be complete iff

if $\|f_k - f_j\| \to 0$ as $k, j \to \infty$

then $\exists f \in X$ s.t $\lim_{k\to\infty} \|f_k - f\| = 0$ i.e.

Every Cauchy sequence converges in X inside X.

$\|f-g\|_p = 0 \quad 1 \le p < \infty$

$$\int |f-g|^p dx = 0 \iff |f-g|^p = 0 \text{ a.e}$$

$$\iff |f-g| = 0 \text{ a.e.} \iff f = g \text{ a.e}$$

Def

$f \sim g$ if $f = g$ a.e.

$$\|f-g\|_\infty = 0 \iff |\{|f-g|>0\}| = 0$$

$f = g$ a.e.

Let $\tilde{L}_p(E) = L_p(E)/\sim$. Then

$\tilde{L}_p(E)$ is a Banach space.

proof $1 \le p < \infty$

$\|f_k - f_j\|_p \to 0$ as $k, j \to \infty$ i.e.

$\forall \varepsilon > 0 \ \exists \ N > 0$ s.t

$\|f_k - f_j\|_p < \varepsilon$ whenever $k, j \ge N$.

$$\int |f_k - f_j|^p \, dx = \|f_k - f_j\|_p^p$$

By the Chebyshev's inequality

$$\left|\{|f_k - f_j|^p > \delta\}\right| \le \frac{\int |f_k - f_j|^p \, dx}{\delta}$$

$$\left|\{|f_k - f_j|^p > \delta\}\right| \le \frac{\varepsilon^p}{\delta} \quad \text{whenever}$$

$$k, j \ge N.$$

$$\left|\left\{ |f_k - f_j| > \delta^{\frac{1}{p}} \right\}\right| \leq \frac{\varepsilon^p}{\delta} \quad \text{whenever}$$

$$k, j \geq N.$$

$\forall \ \delta' > 0$

$$\left|\left\{ |f_k - f_j| > \delta' \right\}\right| \to 0 \quad \text{as} \quad k, j \to \infty$$

$\forall j \ \exists N_j$ s.t.

$\left| \{ |f_k - f_\ell| > 2^{-j} \} \right| \leq 2^{-j}$ whenever

$k, \ell \geq N_j$

W.L.O.G.W.M.A.,

$N_j < N_{j+1}$ and $N_j \in \mathbb{N}$.

$\{f_{N_j}\}$ is a subsequenof $\{f_k\}$

$$\{|f_{N_j} - f_{N_{j+1}}| > 2^{-j}\} = E_j$$

$$|E_j| \lesssim 2^{-j}$$

Let $F_k = \bigcup_{j=k}^{\infty} E_j$

If $x \notin F_k$

when $j \geq k$

$$|f_{N_j}(x) - f_{N_{j+1}}(x)| < 2^{-j}$$

if $k \le j < l$

$$|f_{N_j}(x) - f_{N_l}(x)| \le \sum_{k=j}^{l-1} |f_{N_k}(x) - f_{N_{k+1}}(x)|$$

$$\le \sum_{k=j}^{l-1} 2^{-k} \le \frac{2^{-j}}{1-\frac{1}{2}} = 2^{-j+1}$$

$\to 0$ as $j, l \to \infty$

f_{N_j} converges pointwise outside F_k $\forall k$

Let $F = \bigcap_{k=1}^{\infty} F_k$

If $x \notin F$

$$x \in E \setminus F = E \setminus \bigcap_{k=1}^{\infty} F_k$$

$$= E \cap \left(\setminus \bigcap_{k=1}^{\infty} F_j \right)$$

$$= E \cap \left(\cup \setminus F_j \right)$$

$$= \cup \left(E \setminus F_j \right)$$

$x \in E \setminus F_j$ for some F_j

$\Rightarrow f_{n_j}$ converges at x

f_{N_j} converges on $E \setminus F$

$$|F_k| \leq \sum_{j=k}^{\infty} |E_j| \leq \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1}.$$

$$F \subseteq F_k \quad \forall k$$

$$|F| \leq 2^{-k+1} \quad \forall k$$

$$|F| = 0.$$

$f_{N_j} \to f$ in $E \setminus F$

$f_{N_j} \to f$ a.e. in E.

$\{f_j\}$ Cauchy in L^p

$\forall \varepsilon > 0 \quad \exists \quad N > 0$ s.t

$$\int |f_{N_j} - f_{N_k}|^p \, dx < \varepsilon^p$$

whenever $j, k \geq N$

By Fatou's lemma,

$$\int |f_{N_j} - f|^p dx \le \varepsilon$$

$$\Rightarrow \lim_{i \to \infty} \|f_{N_j} - f\|_p \to 0.$$

$$\|f\|_p \le \|f_{N_j} - f\|_p + \|f_{N_j}\|_p$$

$f \in L_p(E)$.

$\forall \frac{\epsilon}{2} > 0 \quad \exists N > 0$ s.t

$$\|f_k - f_j\|_p \leq \frac{\epsilon}{2} \text{ whenever}$$

$$k, j \geq N$$

$f_{N_j} \to f$ in L_p

$\exists M > 0$ s.t

$$\|f_{N_j} - f\|_p \leq \frac{\epsilon}{2} \text{ whenever}$$

$$j \geq M$$

Choose $N' = \max(N, N_M)$.

$$\|f_k - f\|_p \le \|f_k - f_{N_j}\|_p + \|f_{N_j} - f\|_p$$

for some $j \ge M$

$$\le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon$$

#

Def A Banach space is called separable if $\exists$ a densed set consists of countable elements.

$L_p(E)$ $1 \leq p < \infty$ is separatable.

$L_\infty(E)$ is not separable!