Problem 1.

Define $\log^+(x) = \max\{0, \log(x)\}$. Show that if $|f|\left(1+\log^+|f|\right) \in L^1(E)$ then $f^* \in L^1(E)$, where E is a measurable set with $|E| < \infty$.

Define

$$d_{f*}(\alpha) = |\{x \in \mathbb{R}^n : |f^*(x)| > \alpha\}|$$

By Fubini's theorem, we know

$$\int_E |f^*(x)|\,dx = \int_E \int_0^{|f*(x)|} d\alpha\,dx = \int_0^\infty \int_E \chi_{\{x\in\mathbb{R}^n:|f^*(x)|>\alpha\}}\,dx\,d\alpha = \int_0^\infty d_{f*}(\alpha)\,d\alpha$$

Hence, for every $\gamma > 0$, we have

$$\begin{aligned}
\int_E |f^*(x)|\,dx &= \int_0^\gamma d_{f*}(\alpha)\,d\alpha + \int_\gamma^\infty d_{f*}(\alpha)\,d\alpha \\
&\le \gamma|E| + \int_\gamma^\infty d_{f*}(\alpha)\,d\alpha \\
&\le \gamma|E| + \int_\gamma^\infty \frac{c_n}{\alpha}\int_{\mathbb{R}^n} |f(x)|\,dx\,d\alpha \\
&\le \gamma|E| + \int_\gamma^\infty \frac{C_n}{\alpha}\int_{\{x\in\mathbb{R}^n:|f(x)|>\alpha\}} |f(x)|\,dx\,d\alpha \\
&\le \gamma|E| + C_n\int_{\mathbb{R}^n} |f(x)| \int_\gamma^{|f(x)|} \frac{1}{\alpha}\,d\alpha\,dx \\
&= \gamma|E| + C_n\int_{\mathbb{R}^n} |f(x)|\left(\log^+|f(x)| + \log\left(\frac{1}{\gamma}\right)\right)\,dx
\end{aligned}$$

Let

$$\gamma = \frac{1}{e}$$

Then we obtain

$$\int_E |f^*(x)|\,dx < \infty$$

Problem 2.

Prove that if $f \in L^1(\mathbb{R}^n)$, and f is not identically zero, then there exists some $C > 0$ such that for all $\|x\| \geq 1$,

$$f^*(x) \geq \frac{C}{\|x\|^n}.$$

Let $f \neq 0$ on E, where $|E| > 0$.

Let $Q_k(x)$ denote the cube centered at x, with edge length k.

Let $E_k = E \cap Q_k(0)$, which is measurable.

Since $E_k \nearrow E$, by the Monotone Convergence Theorem (MCT) for measure, we have

$$\lim_{k\to\infty} |E_k| = |E| > 0.$$

In particular, there exists $K > 0$ such that $|E_K| > 0$.

Thus,

$$b := \int_{Q_K(0)} |f(y)|\, dy \geq \int_{E_K} |f(y)|\, dy > 0,$$

since $|f| > 0$ on $E_K \subseteq E$.

Then,

$$\begin{aligned}
f^*(x) &= \sup_Q \frac{1}{|Q|} \int_Q |f(y)|\, dy \\
&\geq \frac{1}{|Q_{K+2\|x\|_\infty}(x)|} \int_{Q_{K+2\|x\|_\infty}(x)} |f(y)|\, dy \\
&\geq \frac{1}{(K + 2\|x\|_\infty)^n} \int_{Q_K(0)} |f(y)|\, dy,
\end{aligned}$$

where $\|x\|_\infty := \max(|x_1|, \ldots, |x_n|)$.

The last inequality follows since $Q_K(0) \subseteq Q_{K+2\|x\|_\infty}(x)$.

Since $\|x\|_\infty \leq |x|$ for all $x \in \mathbb{R}^n$, thus for $|x| \geq 1$,

$$\begin{aligned}
f^*(x) &\geq \frac{b}{(K + 2|x|)^n} \\
&\geq \frac{b}{(K|x| + 2|x|)^n} \quad \text{(since } |x| \geq 1\text{)} \\
&= \frac{b}{(K+2)^n |x|^n}.
\end{aligned}$$

Problem 3.

Let $A \subseteq [a,b]$ with $|A| > 0$. Show that for every $n \in \mathbb{N}$, there exists $(x,d) \in A \times \mathbb{R}^\times$ such that

$$\{x+d, x+2d, \dots, a+nd\} \subseteq A.$$

Lebesgue Density Theorem: Let A be a Lebesgue measurable subset of $\mathbb{R}^n$. Then, almost every point p in A is a point of density of A. In other words,

$$\lim_{Q \searrow p} \frac{|A \cap Q|}{|Q|} = 1,$$

where Q denotes any cube centered at x.

Let $S = [p, p+h)$ be the set that shrink to p. For almost every point p in A,

$$\lim_{S \searrow p} \frac{|A \cap S|}{|S|} = 1.$$

That is, for every $n \in \mathbb{N}$, there exists a density point $p \in A$ such that

$$1 - \frac{|[p, p+h) \cap A|}{|[p, p+h)|} < \frac{1}{n},$$

i.e.

$$\frac{|[p, p+h) \cap A|}{|[p, p+h)|} > \frac{n-1}{n}$$

for some $h > 0$.

Define

$$A_p = [p, p+h) \cap A \subseteq p + \mathbb{R}/h\mathbb{Z}$$

Let

$$\pi : p + \mathbb{R} \to p + \mathbb{R}/h\mathbb{Z}$$

be the natural projection. Then for every $0 \le k < n$, we know

$$|\pi(A_p + \frac{k}{n} \cdot h)| = |\pi(A_p)| = |A_p| > \frac{n-1}{n} \cdot h$$

In other words, for every $0 \le k < n$, we have

$$\left| \pi \left(A_p + \frac{k}{n} \cdot h \right)^C \right| \cap (p + \mathbb{R}/h\mathbb{Z}) \setminus \pi \left(A_p + \frac{k}{n} \cdot h \right) < \frac{1}{n} \cdot h$$

Hence, we obtain

$$\left| \bigcap_{0 \le k < n} \pi \left(A_p + \frac{k}{n} \cdot h \right) \right| = \left| (p + \mathbb{R}/h\mathbb{Z}) \setminus \bigcup_{0 \le k < n} \pi \left(A_p + \frac{k}{n} \cdot h \right)^C \right| > 0$$

That is, there exists $x_0 \in p + \mathbb{R}/h\mathbb{Z}$ such that

$$x_0 \in \bigcap_{0 \le k < n} \pi \left(A_p + \frac{k}{n} \cdot h \right)$$

Therefore, for every $0 \leq k < n$, we get

$$\pi\left(x_0 - \frac{k}{n} \cdot h\right) \in \pi(A_p)$$

Since we know

$$\left\{\pi\left(x_0 - \frac{k}{n} \cdot h\right) \in p + \mathbb{R}/h\mathbb{Z} : 0 \leq k < n\right\} \cong \mathbb{Z}/n\mathbb{Z}$$

forms a cyclic subgroup of $p + \mathbb{R}/h\mathbb{Z}$, it follows that A_p contains an arithmetic sequence with n numbers if we order the above elements increasingly.

Problem 4. Bonus:

Show that

$$\int_{\mathbb{R}} \Gamma(1 + ix)\, dx = \frac{2\pi}{e}$$

where Γ is the Gamma function.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

Residue theorem