Real Analysis

II

5/22/2022

Thm Let $1<p<\infty$. Suppose $\{u_k\}_{k\geq 1}^{\infty}$ is a bdd sequence in $L^p(E)$. Then, $\exists$ a subsequence $\{u_{k_j}\}_{j=1}^{\infty}$ and $u \in L^p(E)$ s.t

$$u_{k_j} \rightharpoonup u \quad \text{as } j \to \infty \quad \text{i.e.}$$

$$\lim_{j\to\infty} \int u_{k_j}\, g\, dx = \int u\, g\, dx$$

$$\forall\, g \in L^q(E).$$

proof. Since $L^q(E)$ is separable, $\exists$

$\{g_k\}_{k=1}^{\infty} \subseteq L^q(E)$ $\qquad \overline{\{g_k\}_{k=1}^{\infty}} = L^q(E)$.

$$\|u_k\|_{L^p(E)} \leq M$$

$$\left| \int_E u_k(x) g_1(x)\, dx \right| \leq \|u_k(x)\|_{L^p(E)} \|g_1(x)\|_{L^q(E)}$$

$$\leq M \|g_1(x)\|_{L^q(E)}$$

$\left\{ \int_E u_k(x) g_1(x)\, dx \right\}_{k=1}^{\infty}$ is a bdd sequence on $\mathbb{R}$

$\exists \{u_j^1\}_{j=1}^{\infty}$, a subsequence of $\{u_k\}_{k=1}^{\infty}$ s.t

$\langle u_j^1, g_1 \rangle := \int_E u_j^1(x) g_1(x) dx$ converges to

a_1

Suppose we have $\{u_j^{k-1}\}_{j=1}^{\infty}$

$$|\langle u_j^{k-1}, g_k \rangle| \leq \|u_j^{k-1}\|_{L^p} \|g_k\|_{L^q}$$
$$\leq M \|g_k\|_{L^q}$$

$\exists$ a subsequence $\{u_j^k\}_{j=1}^{\infty}$ s.t

$$\langle u_j^k , g_k \rangle \to a_k \quad \text{as } j \to \infty$$

We constract $\{u_j^1\}$ $\{u_j^2\}$ $\cdots$ $\{u_j^k\}$ s.t

$\{u_j^k\}$ is a subsequence of $\{u_j^{k-1}\}$ $\Rightarrow$ ~~$\{u_j^k\}$~~

$\{u_j^k\}$ is a subsequence of $\{u_j^l\}$, $1 \le l \le k$.

$$\begin{array}{lllll}
u_1 & u_2 & u_3 & \cdots & \\
u_1^1 & u_2^1 & u_3^1 & & \\
u_1^2 & u_2^2 & u_3^2 & & \\
 & & u_3^3 & &
\end{array}$$

$\{u_k^k\}_{k=1}^{\infty}$

$\{u_k^k\}$ is a subsequence of $\{u_k^l\}$

$\forall \ l \in \mathbb{N}$

Therefore

$$\lim_{k\to\infty} \langle u_k^k, g_l \rangle = a_l \quad \forall\, l \in \mathbb{N}$$

Define

$$T : L^2(E) \to \mathbb{R}.$$

$$T(g) = \lim_{k\to\infty} \langle u_k^k, g \rangle$$

(We need to prove the limit exist for all $g \in L^q(E)$.)

Since $\{g_k\}_{k=1}^{\infty}$ is densed in $L^q(E)$,

$\exists \{g_{k_j}\}_{k=1}^{\infty} \subseteq \{g_k\}_{k=1}^{\infty}$ s.t

$$\lim_{j\to\infty} \| g_{k_j} - g \|_{L^q(E)} = 0$$

For, $\frac{\varepsilon}{4M} > 0$ $\exists\, N_1 > 0$ s.t

$$\| g_{k_j} - g \|_{L^2(E)} \le \varepsilon \text{ whenever}$$

$$j \ge N_1$$

$$\langle u_j^j, g_{k_j} \rangle - \langle u_i^i, g_{k_i} \rangle =$$

$$\langle u_j^j, g_{k_j} - g_{k_{N_1}} \rangle + \langle u_j^j, g_{k_{N_1}} \rangle$$

$$- \langle u_i^i, g_{k_i} - g_{k_{N_1}} \rangle - \langle u_i^i, g_{k_{N_1}} \rangle$$

$$|\langle u_j^i, g_{k_j} - g_{k_{N_1}}\rangle| \leq \|u_j^i\|_{L^p(E)} \|g_{k_j} - g_{k_{N_1}}\|_{L^q(E)}$$

$$\leq M \frac{\varepsilon}{4M} = \frac{\varepsilon}{4} \text{ whenever } j \geq N_1$$

$$\langle u_i^i, g_{k_{N_1}}\rangle \to a_{k_{N_1}}$$

$$\langle u_j^j, g_{k_{N_1}}\rangle \to a_{k_{N_1}}$$

For $\frac{\varepsilon}{4} > 0$ $\exists$ N_2 s.t.

$$|\langle u_i^i, g_{k_{N_1}}\rangle - a_{k_{N_1}}| \leq \frac{\varepsilon}{4} \text{ whenever } i \geq N_2$$

If $i, j \geq N = \max\{N_1, N_2\}$

$$\left|\langle u_i^i, g\rangle - \langle u_j^j, g\rangle\right| \leq \left|\langle u_i^i, g - g_{K_{N_1}}\rangle\right.$$

$$\left. + \langle u_i^i, g_{K_{N_1}}\rangle - \langle u_j^j, g - g_{K_{N_1}}\rangle - \langle u_j^j, g_{K_{N_1}}\rangle\right|$$

$$\leq \frac{\varepsilon M}{4M} + \frac{\varepsilon}{4} + \frac{\varepsilon M}{4M} + \frac{\varepsilon}{4} \leq \varepsilon$$

i.e.

$$\lim_{i,j\to\infty} \left|\langle u_i^i, g\rangle - \langle u_j^j, g\rangle\right| \to 0$$

$\Rightarrow \langle u_i^i, g \rangle$ is a Cauchy sequence in $\mathbb{R}$

$\Rightarrow T(g) = \lim_{i \to \infty} \langle u_i^i, g \rangle$ exists.

Observe T is linear

$$T(g) = \lim_{i \to \infty} \langle u_i^i, g \rangle$$

$$\langle u_i^i, g \rangle \leq \|u_i^i\|_{L^p(\Xi)} \|g\|_{L^q(\Xi)}.$$

$$\left|\langle u_i^1, g\rangle\right| \leq M\|g\|_{L^q(E)}$$

$$\Rightarrow |T(g)| = \left|\lim_{i\to\infty}\langle u_i^1, g\rangle\right| \leq M\|g\|_{L^q(E)}$$

T is bdd $\quad T \in L^q(E)^*$

By Riezs representation Thm $\exists\, u \in L^p$ s.t.

$$T(g) = \langle u, g\rangle$$

$$\Rightarrow \lim_{i\to\infty}\langle u_i^1, g\rangle = \langle u, g\rangle \quad \#$$

$\{\mathbb{X}, \|\ \|\}$ is a Banach space

$\mathbb{X}^*$ is a Banach space

$\mathbb{X}^{**}$ is a Banach space

$\mathbb{X} \subseteq \mathbb{X}^{**}$

$$\mathbb{X} \qquad \mathbb{X}^*$$

$$g \mapsto T(g)$$

$$\|T(g)\| \leq \|T\|\,\|g\|$$

Thm We say $\{\mathbb{X}, \|\ \|\}$ is reflective if $\mathbb{X} = \mathbb{X}^{**}$

Thm suppose $\{\mathbb{X}\ \|\ \|\}$ is reflective Banach space. Then, bdd sequence $\{u_k\}$ in $\mathbb{X}$ has a weakly convergent subsequence in $\mathbb{X}$,

$$\lim_{j\to\infty} T(u_{k_j}) = T(u) \quad \forall\ T \in \mathbb{X}^*.$$

Def. We say a complex-valued function f on $\mathbb{R}^n$ is measurable if

$$f(x) = g_1(x) + i g_2(x)$$

where g_1 and g_2 are real-valued measurable functions.

We say $f \in L^2(E)$ if

$$\|f\|_{L^2(E)}^2 = \int_E |g_1|^2 \, dx + \int_E |g_2|^2 \, dx < \infty.$$

Def Let $\mathbb{X}$ be a vector space.

$\langle \, , \, \rangle : \mathbb{X} \times \mathbb{X} \to \mathbb{C}$ is called an inner product if

(i) $\langle x, x \rangle \geq 0$ (ii) $\langle x, y \rangle = 0 \ \forall y \in \mathbb{X}$

$\iff x = 0$

(iii) $\langle x+y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$

(iv) $\langle cx, z \rangle = c \langle x, z \rangle$

(v) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

$\langle\, , \rangle$ induces a norm

$\|x\| = (\langle x, x\rangle)^{\frac{1}{2}}$

Def If a inner produce space $\{X, \langle\, , \rangle\}$ is complete under the induced norm $\|\ \|$, then we say $\{X, \langle\, , \rangle\}$ is a Hilbert space.

if $\langle x, y \rangle = 0$ we say $x \perp y$.

Thm (Riesz representation Thm for Hilbert space

Suppose $\{H, \langle\ ,\ \rangle\}$ is a Hilbert space,

then, $\forall\ T \in H^*$ $\exists!\ f \in H$ s.t

$$T(g) = \langle g, f \rangle.$$

Lemma. Suppose L is a closed subspace of a Hilbert space H. Then, $\forall\ x \in H\ \exists!\ d \in L$ s.t

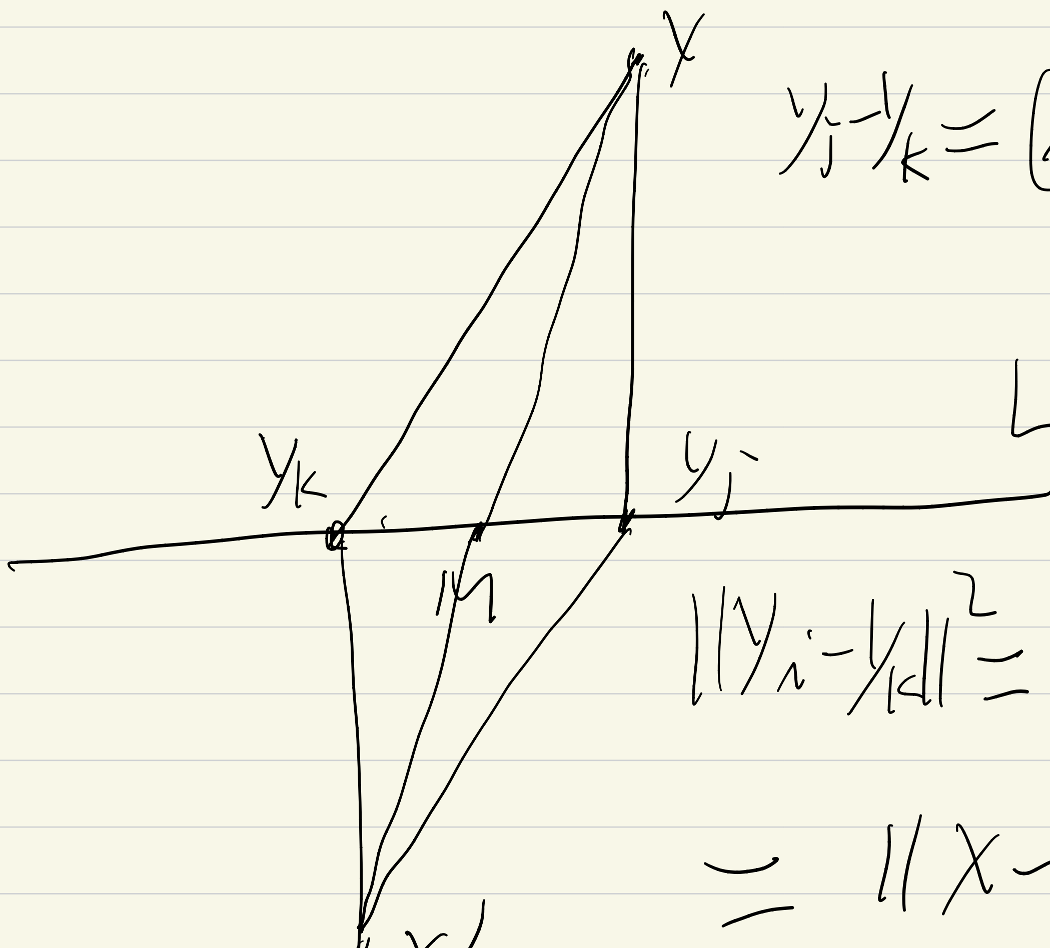
$$\|x-d\| = \inf_{y \in L} \|x-y\| = \text{dist}(x, L) \underset{=d}{}$$

Furthermore

$$\langle x-d, y \rangle = 0 \quad \forall\ y \in L.$$

proof $\exists \{y_k\}_{k=1}^{\infty} \subseteq L$ s.t

$$\lim_{k\to\infty} \|y_k - x\| = d$$



$$y_j - y_k = (x - y_k) - (x - y_j)$$

$$L \quad \langle y_j - y_k, y_j - y_k \rangle$$

$$\|y_i - y_k\|^2 = \langle (x-y_k) - (x-y_j), (x-y_k) - (x-y_j) \rangle$$

$$= \|x - y_k\|^2 + \|x - y_j\|^2 - \langle x - y_j, x - y_k \rangle - \langle x - y_k, x - y_j \rangle$$

$$2(X-M) = (X-y_j) + (X-y_k)$$

$$4\|X-M\|^2 = \|X-y_j\|^2 + \|X-y_k\|^2$$
$$+ \langle X-y_j, X-y_k\rangle + \langle X-y_k, X-y_j\rangle$$

$$\Rightarrow \quad 4\|X-M\|^2 + \|y_j - y_k\|^2 = 2\|X-y_j\|^2$$
$$+ 2\|X-y_k\|^2$$

$$4d^2 \leq 4\|x-M\|^2 = 2\|x-y_j\|^2 + 2\|x-y_k\|^2 - \|y_j-y_k\|^2$$

$$0 \leq \|y_j - y_k\|^2 \leq 2\|x-y_j\|^2 + 2\|x-y_k\|^2 - 4d^2$$

$$\lim_{k,j\to\infty} 2\|x-y_j\|^2 + 2\|x-y_k\|^2 - 4d^2 = 0$$

By squeeze lemma, $\lim_{j,k\to\infty} \|y_j - y_k\| = 0$

$\therefore \{y_j\}$ is a Cauchy sequence.

and L is closed $\Rightarrow$

$\exists\, y \in L$ s.t

$$\lim_{j\to\infty} y_j = y \text{ in } H.$$

$$d \leq \|x-y\| \leq \|x-y_j\| + \|y_j\|$$

$$\lim_{j\to\infty} \|x-y_j\| + \|y_j\| = d$$

By the squeeze thm.

$$\|x-y\| = d.$$

If $v \in L$.

$$y + tv \in L.$$

$$\|x-(y+tv)\|^2 = \langle x-(y+tv), x-(y+tv)\rangle$$ has minimum

at $t=0$

$$\|x-y+tv\|^2 = \|x-y\|^2 - t(\langle v, x-y\rangle + \langle x-y, v\rangle) + t^2\|v\|^2$$

By Fermat thm

$$\langle v, x-y \rangle + \langle x-y, v \rangle = 0$$

$$\mathrm{Re}\langle v, x-y \rangle = 0$$

similary we consider

$$\langle x-(y+itv), x-(y+itv) \rangle$$

$$\Rightarrow \quad \mathrm{Im}\langle v, x-y \rangle = 0$$

$$\Rightarrow \langle v, x-y \rangle = 0$$

proof of Riezs representation Thm
for Hilbert space.

Suppose $T \in H^*$

case 1 if $T(f) = 0 \ \forall f \in H$

$$T(f) = \langle f, 0 \rangle = 0$$

case 2. if $\exists h$ s.t $T(h) \neq 0$

$T(h) = a \neq 0$

T is bdd $\Leftrightarrow$ T is continuous.

$T^{-1}(0)$ is a closed linear space.

$\exists\ \tilde{h} \in T^{-1}(0)$ s.t

$\|h - \tilde{h}\| = \operatorname{dist}(h, T^{-1}(0))$

$\langle h - \tilde{h}, v \rangle = 0 \quad \forall\ v \in T^{-1}(0)$

$$g = \frac{h - \tilde{h}}{\|h - \tilde{h}\|} \qquad u_f = T(g) f - T(f) g$$

$$T(u_f) = T(g) T(f) - T(f) T(g)$$
$$= 0$$

$$u_f \in T^{-1}(0)$$

$$\langle u_f , g \rangle = 0$$

$$\langle T(g)f - T(f)g , g \rangle = 0$$

$$T(g)\langle f, g\rangle - T(f)\langle g, g\rangle = 0$$

$$T(g)\langle f, g\rangle = T(f)$$

$$\langle f, \overline{T(g)}\, g\rangle = T(f) = \langle f,$$