Real Analysis II Homework 1

Due Date: March 20

Solve the following problems except bonus.

Problem 1 Define

$$\log^+(x) = \max\{0, \log(x)\}$$

Show that if $|f|\left(1 + \log^+|f|\right) \in L^1(\mathbb{R}^n)$ then $f^* \in L^1(E)$, where E is a measurable set with $|E| < \infty$.

Problem 2 Prove that if $f \in L^1(\mathbb{R}^d)$, and f is not identically zero, then

$$f^*(x) \geq \frac{C}{\|x\|^d}$$

for some $C > 0$ and all $\|x\| \geq 1$.

Problem 3 Let $A \subseteq [a, b]$ with $|A| > 0$. Show that for every $n \in \mathbb{N}$, there exists $(x, d) \in A \times \mathbb{R}^\times$ such that

$$\{x + d, x + 2d, \ldots, x + nd\} \subseteq A$$

Bonus Show that

$$\int_{\mathbb{R}} \Gamma(1 + ix)dx = \frac{2\pi}{e}$$

where Γ is the Gamma function.