Def We say a Banach space. is separable if and only if it has a densed subset consists of countable many elements.

Ex $f: (0,1) \to \mathbb{R}$.

$L_\infty((0,1))$ is not separable.

proof

$f_\alpha = \chi_{[0,\alpha)} \qquad \alpha \in (0,1)$

if $\alpha \neq \beta$

$\|f_\alpha - f_\beta\|_\infty = 1$

Suppose $\exists$ countable densed subset $A \subseteq L_\infty(0,1)$

For $\frac{1}{4} > 0 \quad \forall f_\alpha \quad \exists \quad g_\alpha \in A$ s.t

$$\|g_\alpha - f_\alpha\|_\infty \leq \frac{1}{4}$$

$$1 = \|f_\alpha - f_\beta\|_\infty \leq \|f_\alpha - g_\alpha\|_\infty + \|g_\alpha - g_\beta\|_\infty + \|g_\beta - f_\beta\|$$

$$\|f_\alpha - f_\beta\|_\infty - \|f_\alpha - g_\alpha\|_\infty - \|f_\beta - g_\beta\|_\infty \leq \|g_\alpha - g_\beta\|_\infty$$

$$\frac{1}{2} \leq \|g_\alpha - g_\beta\|_\infty$$

$$g_\alpha \neq g_\beta$$

$$\{g_\alpha\}_{\alpha \in (0,1)} \subseteq A$$

$\alpha \to g_\alpha$ is one-one and onto

$\{g_\alpha\}$ is uncountable $\rightarrow\leftarrow$

Thm For $1 \le p < \infty$, $L_p(E)$ is separable.

proof Recall, every open set in $\mathbb{R}^n$ can be written as a countable nonoverlap union of cubes.

$$A_k = \{ [z_1 2^{-k}, (z_1+1) 2^{-k}] \times [z_2 2^{-k}, (z_2+1) 2^{-k}] \times \cdots [z_n 2^{-k}, (z_n+1) 2^{-k}]$$

$$\mid \quad z_1, z_2 \cdots z_n \in \mathbb{Z} \}$$

$$A = \bigcup_{k=0}^{\infty} A_k$$

if E is open.

$\exists \{C_k\}_{k=1}^{\infty} \subseteq A$ s.t

$\bigcup_{k=1}^{\infty} C_k = E$

if $|E| < \infty$

$f_N = \sum_{k=1}^{N} \chi_{C_k} \nearrow \chi_E$

$0 \leq f_N \leq \chi_E$

$\|\chi_E\|_p^p = \int_E 1^p dx$

$= |E|$

$\|\chi_E\|_p = |E|^{\frac{1}{p}} < \infty$

$0 \leq \chi_E - f_N \leq \chi_E$

$0 \leq |\chi_E - f_N|^p \leq |\chi_E|^p$

$\chi_E(x) - f_N(x) \to 0$ as $N \to \infty$

By L.D.C.T

$\lim_{N\to\infty} \int |\chi_E - f_N|^p dx = 0$

$$\lim_{n\to\infty} \|f_k - \chi_E\|_p \to 0$$

$$B_k = \left\{ \sum_{n=1}^{k} \chi_{c_k} \;\middle|\; c_k \in A \right\}$$

$$B = \bigcup_{k=1}^{\infty} B_k$$ B is countable.

$f_k \in B$

If E is measurable and $|E| < \infty$.

$E = G \setminus Z$ where G is of G_δ type

$|Z| = 0$ $\quad \chi_E \sim \chi_G$

$\exists\ G_1$ s.t $G_1 \supseteq G$ and

$$|G_1| \le |G| + 1 = |E| + 1 < \infty$$

G is of G_δ type $\Rightarrow$

$$\exists\ \{G_k\}_2^\infty \text{ s.t } G = \bigcap_{k=2}^{\infty} G_k$$

$$H_k = \bigcap_{i=1}^{k} G_k \qquad H_k \searrow G \qquad H_k \text{ is open}$$

$$0 \le \chi_{H_k} \le \chi_{G_1}$$

$$0 \le |\chi_{H_k} - \chi_G|^p \le |\chi_{G_1}|^p$$

$|\chi_{H_k}(x) - \chi_G(x)|^p \to 0$ as $k \to \infty$

By L.D.C.T,

$$\lim_{k\to\infty} \int |\chi_{H_k}(x) - \chi_G(x)|^p = 0$$

$$\lim_{k\to\infty} \|\chi_{H_k} - \chi_E\|_p \to 0. \quad — ①$$

Since H_k $k=1 \dots \infty$ are open and $|H_k| < \infty$.

$\forall\ H_k\ \exists\ \{f_\ell^k\}_{\ell=1}^{\infty} \subseteq B$ s.t.

$$\lim_{\ell\to\infty} \|f_\ell^k - \chi_{H_k}\|_p \to 0. \quad — ②$$

① $\Rightarrow$ For $2^{-j} > 0$ $\exists$ $N_j > 0$ s.t $\|\chi_{H_k} - \chi_E\|_p \le 2^{-j}$ whenever $k \geq N$

② $\Rightarrow$ For $2^{-j} > 0$ $\exists$ $N_{kj} > 0$ s.t $\|f_\ell^k - \chi_E\|_p \le 2^{-j}$ whenever $\ell \geq N_{kj}$.

$$g_j = f^{N_j}_{N_{N_j} j}$$

$$\|g_j - \chi_E\|_p \leq \|f^{N_j}_{N_{N_j} j} - \chi_{H_{N_j}}\|_p +$$

$$\|\chi_{H_{N_j}} - \chi_E\|_p$$

$$\leq 2^{-j} + 2^{-j} = 2^{-j+1} \to 0 \text{ as } j \to \infty$$

$$g_j \in B.$$

$f = f^+ - f^-$ $\qquad$ $f \in L_p(E) \iff f^+; f^- \in L_p(E)$

$f^+ \in L_p(E)$.

$$f_k = \begin{cases} j2^{-k} & \text{if } j2^{-k} \leq f^+(x) < (j+1)2^{-k} \quad 0 \leq j \leq 2^{2k} - 1 \\ 2^k & \text{if } 2^k \leq f^+(x) \end{cases}$$

f_k is a simple function

$f_k \nearrow f^+$

$$|\mathrm{supp}\{f_k\}| = |\{f_k \geq 2^{-k}\}| \leq \frac{\int f^{+p}\,dx}{(2^{-k})^p} < \infty$$

f_k is a measurable simple function

$|\{\operatorname{supp} f_k\}| < \infty$

$0 \le f_k \le f^+$

$0 \le |f^+ - f_k|^p \le |f^+|^p$

$\lim_{k \to \infty} |f_k(x) - f^+(x)|^p$

By the L.D.C.T

$$\lim_{k\to\infty} \|f_k - f^+\|_p = 0$$

$$D_k = \{ j 2^{-k} g \mid g \in B \; j \in \mathbb{Z} \}$$

$$D = \bigcup_{k=0}^{\infty} D_k$$

$$L_k = \{ \sum_{j=1}^{k} h_i \mid h_k \in D \}$$

$$L = \bigcup_{k=1}^{\infty} L_k$$

Claim If f is a measurable simple function taking values at $k 2^{-j}$ for $j \in \mathbb{N}$, $k \in \mathbb{Z}$, $|\operatorname{supp} f| < \infty$

then $\exists\ f_k \in L$ s.t.

$$\lim_{k \to \infty} \| f_k - f \|_p = 0$$

$$f = \sum_{i=1}^{N} 2^{-j_i} k_i \chi_{E_i}$$

since E_i measurable and $|E_i| < \infty$

$\exists$ $g_\ell^i \in$ $\quad \|g_\ell^i - \chi_{E_i}\|_p \to 0$ as $\ell \to \infty$

$$\left\| f - \sum_{i=1}^{N} 2^{-j_i} k_i g_\ell^i \right\|_p \leq \sum_{i=1}^{N} 2^{-j_i} |k_i| \, \|\chi_{E_i} - g_\ell^i\|_p$$

$\to 0$ as $\ell \to \infty$

By a diagonal argument.

$\exists\ f_k \in L$ s.t

$$\lim_{k\to\infty} \|f^+ - f_k\|_p = 0. \quad \text{similarly for } f^-$$

and for f.