Hölder's Inequality

If $1 \le p \le \infty$ *and* $\frac{1}{p} + \frac{1}{p'} = 1$, *then* $\|fg\|_1 \le \|f\|_p \|g\|_{p'}$; *that is,*

$$\int_E |fg| \le \left(\int_E |f|^p\right)^{1/p} \left(\int_E |g|^{p'}\right)^{1/p'}, \quad 1 < p < \infty;$$
$$\int_E |fg| \le (\text{ess sup}_E |f|) \int_E |g|.$$

$$\|fg\|_r = \left(\int |fg|^r\right)^{1/r}$$
$$\|fg\|_r^r = \left(\int |fg|^r\right)^{1} = \|f^r g^r\|_1$$

Problem 1.

(a) Let $1 \le p_i, r \le \infty$ and $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$. Prove the following generalization of Hölder's inequality:

$$\|f_1 \cdot f_2 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdot \|f_2\|_{p_2} \cdots \|f_k\|_{p_k}$$

(b) Let $1 \le p < r < q \le \infty$ and define $\theta \in (0,1)$ by

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

Prove the interpolation estimate:

$$\|f\|_r \le \|f\|_p^\theta \|f\|_q^{1-\theta}$$

Solution.

(a) Generalization of Hölder's Inequality

We prove this by induction.

The $k = 2$ case is a consequence of Hölder's inequality:

If $\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{r}$, then $\frac{r}{p_1} + \frac{r}{p_2} = 1$, so

$$\|fg\|_r^r = \|f^r g^r\|_1 \le \|f^r\|_{p_1/r} \|g^r\|_{p_2/r} = \|f\|_{p_1}^r \|g\|_{p_2}^r.$$

It is implied that

$$\|fg\|_r \le \|f\|_{p_1} \|g\|_{p_2}.$$

Now if $\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$ for $k > 2$, we have

$$\begin{aligned}\|f_1 \cdots f_k\|_r &\le \|f_1 \cdots f_{k-1}\|_s \|f_k\|_{p_k} \\ &\le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k},\end{aligned}$$

where $\frac{1}{s} = \frac{1}{r} - \frac{1}{p_k} = \frac{1}{p_1} + \cdots + \frac{1}{p_{k-1}}$.

(b) Interpolation Estimate

Let $1 \leq p < r < q \leq \infty$ and define $\theta \in (0,1)$ by

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$

In other words, $1/r$ is the convex interpolation between $1/p$ and $1/q$.

$$\begin{aligned} \frac{1}{r} &= \frac{\theta}{p} + \frac{1-\theta}{q} \\ \frac{1}{r} &= \frac{1}{p/\theta} + \frac{1}{q/(1-\theta)} \end{aligned}$$

Apply the Hölder's Inequality,

$$\begin{aligned} \|f\|_r &= \|f^\theta f^{1-\theta}\|_r \\ \text{(Apply the Hölder's Inequality)} \quad &\leq \|f^\theta\|_{p/\theta} \|f^{1-\theta}\|_{q/(1-\theta)}. \\ &= \|f\|_p^\theta \|f\|_q^{1-\theta} \end{aligned}$$

The last equation is due to

$$\|f^\theta\|_{p/\theta} = \left(\int |f^\theta|^{p/\theta}\right)^{\theta/p} = \left(\int |f|^p\right)^{\theta/p} = \|f\|_p^\theta$$

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Lemma. (used in the proof of Problem 2)

For $a, b \in \mathbb{R}$, $|a+b|^p \leq 2^p(|a|^p + |b|^p)$, where $0 < p < \infty$.

Proof.

$$\begin{aligned} |a+b|^p &\leq (|a| + |b|)^p \\ &\leq (2\max\{|a|, |b|\})^p \\ &= 2^p(\max\{|a|, |b|\})^p \\ &\leq 2^p(|a|^p + |b|^p). \end{aligned}$$

Problem 2.

Let $f \in L^p(\mathbb{R}^n)$, where $0 < p < \infty$. Show that

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p \, dy = 0 \quad \text{a.e.}$$

Let $\{r_k\}$ be the rational numbers. First note that for any Q, x, and r_k,

$$|f(y) - f(x)|^p \leq |f(y) - r_k|^p + |r_k - f(x)|^p$$

$$\begin{aligned}
\frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy &\leq 2^p \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy + 2^p \frac{1}{|Q|} \int_Q |r_k - f(x)|^p dy \\
&= 2^p \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy + 2^p |r_k - f(x)|^p.
\end{aligned}$$

For every r_k, let Z_k be the set in which the formula

$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - r_k|^p dy = |f(x) - r_k|^p$$

is not valid.

Since

$$|f(y) - r_k|^p \leq 2^p (|f(y)|^p + |r_k|^p)$$

is locally integrable, by Lebesgue's Differentiation Theorem, $|Z_k| = 0$. Let $Z = \bigcup Z_k$, then $|Z| = 0$.

Thus, if $x \notin Z$, for every r_k,

$$\begin{aligned}
\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy &\leq 2^p |f(x) - r_k|^p + |r_k - f(x)|^p \\
&= 2^{p+1} |f(x) - r_k|^p.
\end{aligned}$$

For an x at which $f(x)$ is finite (in particular, almost everywhere since $f \in L^p(\mathbb{R}^n)$), by the density of rationals in $\mathbb{R}^n$ we can choose r_k such that $|f(x) - r_k|^p$ is arbitrarily small.

Thus

$$\limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy = 0 \quad \text{a.e.}$$

and this completes the proof. Since

$$\liminf_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy \leq \limsup_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p dy = 0 \quad \text{a.e.}$$

Problem 3.

Show that every subset Λ of a separable metric space (M, d) is separable.

separable

Def. A *metric space* (X, d) is said to be **separable** if there exists a countable subset $A \subseteq X$ that is dense in X, i.e., $\overline{A} = X$.

That is, (X, d) is **separable** if and only if there exists a countable subset $A \subseteq X$ such that for every $x \in X$ and every $\epsilon > 0$, there exists $a \in A$ with $d(x, a) < \epsilon$.

Let $D = \{f_k\}$ be a countable dense set in M.

i.e.,

$$\forall \lambda \in \Lambda, \forall n \geq 1, \exists f_k \in D \text{ such that } d(\lambda, f_k) < \frac{1}{n}.$$

For $n \geq 1$, define

$$D_n = \{f \in D : \inf_{\lambda \in \Lambda} d(\lambda, f) < \frac{1}{n}\}.$$

If $f_k \in D_n$, pick $\lambda_{k,n} \in \Lambda$ with

$$d(\lambda_{k,n}, f_k) < \frac{1}{n}.$$

Claim 0.1. *The subset $\{\lambda_{k,n}\}$ is dense in Λ.*

Proof. Consider any $\lambda \in \Lambda$ and any $n \geq 1$. There exists $\lambda_{k,n}$ such that

$$d(\lambda, \lambda_{k,n}) \leq d(\lambda, f_k) + d(f_k, \lambda_{k,n}) = \frac{1}{n} + \frac{1}{n}.$$

This simplifies to

$$d(\lambda, \lambda_{k,n}) \to 0 \text{ as } n \to \infty.$$

This implies that every point in Λ can be approximated arbitrarily closely by elements of $\{\lambda_{k,n}\}$, thereby proving that $\{\lambda_{k,n}\}$ is dense in Λ, which establishes the claim. $\square$