Problem 1.

Let $f : \mathbb{R} \to \mathbb{C}$. Prove that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \le M|x - y|$$

for some $M > 0$ and for all $x, y \in \mathbb{R}$, if and only if f satisfies the following two properties:

(i) f is absolutely continuous.

(ii) $|f'(x)| \le M$ for a.e. x

$\Rightarrow$

Suppose f is Lipschitz continuous. For $\epsilon > 0$, let $\delta = \frac{\epsilon}{M}$, whenever $\sum |b_i - a_i| < \delta$, we have

$$\sum |f(b_i) - f(a_i)| \le M \sum |b_i - a_i| < \epsilon,$$

implying that f is absolutely continuous.

By Theorem 7.27 and Corollary 7.23, we can see that f' exists (f is differentiable) almost everywhere.

For x where $f'(x)$ exists, the Lipschitz condition implies that, for all h,

$$\left| \frac{f(x+h) - f(x)}{h} \right| \le M$$

Taking the limit as $h \to 0$, we have

$$|f'(x)| = \lim_{h \to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \le M$$

$\Leftarrow$

Suppose that f is absolutely continuous. By Theorem 7.27 and Corollary 7.23, we have that f is of bounded variation and thus f' exists almost everywhere.

By Theorem 7.29, for all $x < y \in \mathbb{R}$,

$$|f(x) - f(y)| = \left| \int_x^y f'(t)\, dt \right| \le \int_x^y |f'(t)|\, dt \le \int_x^y M\, dt = M\, |x - y|\, .$$

We can conclude that f is Lipschitz continuous.

Corollary 7.23

If f is of bounded variation on $[a, b]$, then f' exists almost everywhere in $[a, b]$, and $f' \in L[a, b]$.

Theorem 7.27

If f is absolutely continuous on $[a, b]$, then it is of bounded variation on $[a, b]$.

Theorem 7.29

A function f is absolutely continuous on $[a, b]$ if and only if f' exists almost everywhere in $[a, b]$, f' is integrable on $[a, b]$, and for $a \leq x \leq b$,

$$f(x) - f(a) = \int_a^x f'(t)\, dt.$$

A function ϕ is convex in (a, b) if and only if

$$\phi(\theta x_1 + (1-\theta)x_2) \le \theta\phi(x_1) + (1-\theta)\phi(x_2)$$

for $x_1, x_2 \in (a, b)$ and $0 \le \theta \le 1$.

Theorem 7.40

If ϕ is convex in (a, b), then ϕ is continuous in (a, b). Moreover, ϕ exists except at most in a countable set and is monotone increasing.

Problem 2.

Prove that f is convex on (a, b) if and only if it is continuous and

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$.

$\Rightarrow$

Suppose that f is convex. Following the definition of convexity, let $\theta = \frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$.

By Theorem 7.40, f is continuous.

$\Leftarrow$

Suppose that f is continuous and satisfies the midpoint inequality for all $x, y \in (a, b)$.

To prove convexity, consider any $x, y \in (a, b)$ and any $\theta \in [0, 1]$. We need to show that

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta)f(y).$$

This can be proved using an induction approach on the dyadic rationals (i.e., numbers of the form $\frac{m}{2^n}$, where $n \in \mathbb{N}$, and $m = 0, 1, \ldots, 2^n$), then generalizing to all θ using the continuity of f.

Base Case: For $n = 1$, the midpoint inequality ensures that the convexity condition holds for $\theta = \frac{1}{2}$.

Inductive Step: Assume the condition holds for n. Consider $\theta = \frac{m}{2^{n+1}}$.

Case 1: If m is even, θ is a dyadic rational of the form $\frac{k}{2^n}$, so the hypothesis applies.

Case 2: If m is odd, write θ as

$$\theta = \frac{1}{2}\left(\frac{(m-1)/2}{2^n}\right) + \frac{1}{2}\left(\frac{(m+1)/2}{2^n}\right),$$

noting that $\frac{m-1}{2}$ and $\frac{m+1}{2}$ are integers.

Using the midpoint inequality:

$$f\left(\theta x+(1-\theta)y\right) \leq \frac{1}{2} f\left(\frac{(m-1)/2}{2^n} x+\left(1-\frac{(m-1)/2}{2^n}\right) y\right)+\frac{1}{2} f\left(\frac{(m+1)/2}{2^n} x+\left(1-\frac{(m+1)/2}{2^n}\right) y\right).$$

By applying the convexity condition assumed for $\frac{(m-1)/2}{2^n}$ and $\frac{(m+1)/2}{2^n}$:

$$\begin{aligned} f\left(\theta x+(1-\theta)y\right) &\leq \frac{1}{2}\left(\frac{(m-1)/2}{2^n} f(x)+\left(1-\frac{(m-1)/2}{2^n}\right) f(y)\right)+\frac{1}{2}\left(\frac{(m+1)/2}{2^n} f(x)+\left(1-\frac{(m+1)/2}{2^n}\right) f(y)\right) \\ &= \theta f(x)+(1-\theta) f(y). \end{aligned}$$

Thus, the convexity condition is preserved for $n+1$.

The continuity of f implies that since the inequality holds for all dyadic rationals, it also holds by the limit for any $\theta \in [0,1]$ as dyadic rationals are dense in $[0,1]$.

Thus, f is convex on (a,b).