Real Analysis II Homework 6

Submission is NOT mandatory

Solve the following problems.

Problem 1 Prove that for $1 \leq p < \infty$ and a measurable function $f(x, y)$,

$$\left(\int \left(\int |f(x,y)| dx \right)^p dy \right)^{\frac{1}{p}} \leq \int \left(\int |f(x,y)|^p dy \right)^{\frac{1}{p}} dx$$

Problem 2 Suppose that the sequence $\{f_k\}$ converges weakly in L^2 to f. If $\|f_k\|_2 \to \|f\|_2$, show that $f_k \to f$ in L^2 norm.

Problem 3 Let $\{\phi_k\}$ be a complete orthonormal system in L^2 and let $m = \{m_k\}$ be a fixed bounded sequence of numbers.

$$\text{If } f \in L^2 \text{ and } f \sim \sum c_k \phi_k, \text{ define } T(f) \text{ by } T(f) \sim \sum m_k c_k \phi_k.$$

Such an operator is called a **Fourier multiplier operator**. Show that

$$\|T\| = \sup_{f \in L^2} \frac{\|T(f)\|_2}{\|f\|_2} \leq \|m\|_{l^\infty}$$

Problem 4 Let f be a continuous real-valued function on $\mathbb{R}$ that has derivatives of all orders. Suppose that for every real number x, there is an index $n = n(x)$ for which $f^{(n)}(x) = 0$. Show that f is a polynomial. (Hint: Apply the Baire Category Theorem twice.)