Problem 1. The Prékopa-Leindler Inequality

Let $0 < \lambda < 1$ and let f, g and h be non-negative integrable functions on $\mathbb{R}^n$ satisfying for every $x, y \in \mathbb{R}^n$, one has

$$h\left((1-\lambda)x + \lambda y\right) \geq f(x)^{1-\lambda} g(y)^{\lambda}.$$

Show that

$$\int_{\mathbb{R}^n} h(x)\,dx \geq \left(\int_{\mathbb{R}^n} f(x)\,dx\right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(x)\,dx\right)^{\lambda}.$$

First, consider the case $n = 1$.

Without loss of generality, we may assume that

$$\int_{\mathbb{R}} f(x)\,dx = F > 0 \quad \text{and} \quad \int_{\mathbb{R}} g(x)\,dx = G > 0. \tag{1}$$

Define $u, v : (0,1) \to \mathbb{R}$ such that $u(t)$ and $v(t)$ are the smallest numbers satisfying

$$\frac{1}{F}\int_{-\infty}^{u(t)} f(x)\,dx = \frac{1}{G}\int_{-\infty}^{v(t)} g(x)\,dx = t. \tag{2}$$

Then u and v may be discontinuous, but they are strictly increasing functions and so are differentiable almost everywhere. Let

$$w(t) = (1-\lambda)u(t) + \lambda v(t). \tag{3}$$

Take the derivative of (2) with respect to t to obtain

$$\frac{f(u(t))u'(t)}{F} = \frac{g(v(t))v'(t)}{G} = 1. \tag{4}$$

Using this and the arithmetic-geometric mean inequality, we obtain (when $f'(u(t)) \neq 0$ and $g'(v(t)) \neq 0$)

$$\begin{aligned} w'(t) = (1-\lambda)u'(t) + \lambda v'(t) &\geq \left(\frac{u'(t)}{1-\lambda}\right)^{1-\lambda} \left(\frac{v'(t)}{\lambda}\right)^{\lambda} \\ &= \left(\frac{F}{f'(u(t))}\right)^{1-\lambda} \left(\frac{G}{g'(v(t))}\right)^{\lambda}. \end{aligned} \tag{5}$$

Therefore

$$\begin{aligned} \int_{\mathbb{R}} h(x)\,dx &= \int_0^1 h(w(t))w'(t)\,dt \\ &\geq \int_0^1 \left(f'(u(t))\right)^{1-\lambda} \left(g'(v(t))\right)^{\lambda} \left(\frac{F}{f'(u(t))}\right)^{1-\lambda} \left(\frac{G}{g'(v(t))}\right)^{\lambda} dt \\ &= F^{1-\lambda} G^{\lambda}. \end{aligned} \tag{6}$$

Next, we perform the induction on n.

Let's indicate the variables as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and define the functions

$$\begin{aligned} F(x_1) &= \int_{\mathbb{R}^{n-1}} f(x_1, x')\,dx', \\ G(y_1) &= \int_{\mathbb{R}^{n-1}} g(y_1, y')\,dy', \\ H(z_1) &= \int_{\mathbb{R}^{n-1}} f(z_1, z')\,dz'. \end{aligned}$$

Fix $x_1, y_1, z_1 \in \mathbb{R}$ such that $z_1 = (1-\lambda)x_1 + \lambda y_1$. Then

$$h(z_1, z') \geq f(x_1, x')^{1-\lambda} g(y_1, y')^{\lambda}$$

for all $x', y', z' \in \mathbb{R}^{n-1}$ with $z' = (1-\lambda)x' + \lambda y'$. By induction, Prékopa-Leindler holds in dimension $n-1$, thus it follows that

$$H(z_1) \geq F(x_1)^{1-\lambda} G(y_1)^{\lambda}.$$

So now we can apply again the 1-dimensional Prékopa-Leindler inequality to obtain the desired conclusion.

Problem 2. The Brunn-Minkowski Inequality

Let $0 < \lambda < 1$ and let A and B be non-empty bounded measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable. Show that

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

Let

$$\begin{aligned} f(x) &= \mathbf{I}_A, \\ g(x) &= \mathbf{I}_B, \\ h(x) &= \mathbf{I}_{(1-\lambda)A+\lambda B}\,. \end{aligned}$$

Applying the Prékopa-Leindler Inequality, we have that

$$|(1-\lambda)A + \lambda B| \geq |A|^{1-\lambda}|B|^{\lambda}.$$

Now show that this inequality (multiplicative BM) implies the additive version$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}$:

We assume that both A, B have positive volume, as otherwise the inequality is trivial, and normalize them to have volume 1 by setting

$$A' = \frac{A}{|A|^{1/n}}, \quad B' = \frac{B}{|B|^{1/n}}.$$

We define λ' by

$$\lambda' = \frac{\lambda|B|^{1/n}}{(1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}}, \quad 1-\lambda' = \frac{(1-\lambda)|A|^{1/n}}{(1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}}.$$

With these definitions, and using that $|A'| = |B'| = 1$, we calculate using the multiplicative Brunn–Minkowski inequality that:

$$\left|\frac{(1-\lambda)A + \lambda B}{(1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}}\right| = |(1-\lambda')A' + \lambda' B'| \geq |A'|^{1-\lambda'}|B'|^{\lambda'} = 1.$$

The additive form of Brunn–Minkowski now follows by pulling the scaling out of the leftmost volume calculation and rearranging:

$$|(1-\lambda)A + \lambda B| \geq \left\{(1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}\right\}^n.$$

$$|(1-\lambda)A + \lambda B|^{1/n} \geq (1-\lambda)|A|^{1/n} + \lambda|B|^{1/n}$$

Set $\lambda = 1/2$,

$$\begin{aligned} \left|\frac{1}{2}A + \frac{1}{2}B\right|^{1/n} &= \frac{1}{2}|A+B|^{1/n} \\ &\geq \frac{1}{2}|A|^{1/n} + \frac{1}{2}|B|^{1/n} \end{aligned}$$

$\Longrightarrow$

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$