If f is infinitely differentiable then f coincides with a polynomial

Ask Question

Asked 13 years, 10 months agoModified 1 year agoViewed 42k times

235

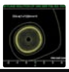
Let f be an *infinitely differentiable function* on $[0,1]$ and suppose that for each $x \in [0,1]$ there is an integer $n \in \mathbb{N}$ such that $f^{(n)}(x)=0$. Then does f coincide on $[0,1]$ with some polynomial? If yes then how.

I thought of using Weierstrass approximation theorem, but couldn't succeed.

real-analysispolynomials

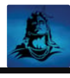
ShareCiteImprove this questionFollow

edited Jul 23, 2018 at 11:06



Ali Taghavi

asked Jul 31, 2010 at 21:37



C.S.

11 This seems like a homework problem in a 1st year course on calculus. – Ryan Budney Jul 31, 2010 at 21:51

21 This is a jewel, I will try to recall the solution. – Andrey Gogolev Jul 31, 2010 at 22:05

27 @Ryn: no, this is a classic little problem. @Michael: the problem is correct as stated. – Qiaochu Yuan Jul 31, 2010 at 22:39

8 This is basically a double-starred exercise in the book "Linear Analysis" by Bela Bollobas (second edition), and presumably uses the Baire Category Theorem. Since it is double-starred, it is probably very hard!! Solutions are not given, and even single starred questions in that book can be close to research level. However, the version in that book has f on the whole real line, and $f^{(m)}(x)=0$ for ALL $m>n$. So are you sure your question is correct, since it's assuming a lot less but coming to roughly the same conclusion? – Zen Harper Jul 31, 2010 at 23:32

35 I agree with Andrew L.'s opinion(but not the more extreme part of it). If such hard questions are given as homework for a first year calculus course, then there will be complaints about the instructor, and indeed about the department. It is my modest contention that anyone who criticizes a question as homework should be able to substantiate it by giving a short solution in the comments. This doesn't take much effort. What I am preaching is just a variant of "All right, but let the one who has never sinned throw the first stone!". Before closing a question as homework, first solve it. – Anweshi Aug 1, 2010 at 13:16

Show **24** more comments

10 Answers

Sorted by: Highest score (defa

183

The proof is by contradiction. Assume f is not a polynomial.

Consider the following closed sets:

$$S_n = \{x : f^{(n)}(x) = 0\}$$

and

$$X = \{x : \forall (a,b) \ni x : f\restriction_{(a,b)} \text{ is not a polynomial}\}.$$

It is clear that X is a non-empty closed set without isolated points. Applying Baire category theorem to the covering $\{X \cap S_n\}$ of X we get that there exists an interval (a,b) such that $(a,b) \cap X$ is non-empty and

$$(a,b) \cap X \subset S_n$$

for some n. Since every $x \in (a,b) \cap X$ is an accumulation point we also have that $x \in S_m$ for all $m \geq n$ and $x \in (a,b) \cap X$.

Now consider any maximal interval $(c,e) \subset ((a,b) - X)$. Recall that f is a polynomial of some degree d on (c,e). Therefore $f^{(d)} = \text{const} \neq 0$ on $[c,e]$. Hence $d < n$. (Since either c or e is in X.)

So we get that $f^{(n)} = 0$ on (a,b) which is in contradiction with $(a,b) \cap X$ being non-empty.

ShareCiteImprove this answerFollow

edited Nov 1, 2011 at 11:19
Emil Jeřábek

answered Jul 31, 2010 at 23:20
Andrey Gogolev

---

32 Thank you! Filling in all the details to this outline is a fantastic exercise in basic real analysis and topology. It strikes me as a great "capstone" to a relevant course. It went through at least 20 relevant topics/ideas: (in roughly decreasing order of complexity) Baire Category Theorem, Heine-Borel, infs/sups (so LUB property of R), compactness, Cauchy/convergent sequences/completeness, (infinite) differentiability, continuity, connectedness, perfect sets, limit points (from the sides), induction, isolated points, open/closed sets, interiors, derivatives of polynomials, and boundedness. – Joshua P. Swanson May 8, 2011 at 22:48

3 Is a similar statement true for functions of n>1 variables? – Alex W Jan 14, 2019 at 17:40

@AlexW In that case, the natural hypothesis would be that for each $x_0 \in \mathbb{R}^n$ there exists a multi-index $\alpha = \alpha(x_0)$ such that $\frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}\Big|_{x_0} = 0$. Essentially the same argument works, by appropriately replacing intervals (a,b) as in this answer with open balls. In particular there are only countably many multi-indices so Baire's theorem applies. – MathematicsStudent1122 May 1, 2020 at 16:02

2 @MathematicsStudent1122 Let $f(x_1,x_2) = e^{x_2}$, $\alpha = (1,0)$. Then $f^{(\alpha)}(x) = 0$ for all $x \in \mathbb{R}^2$ but f is not a polynomial. – Alex W May 2, 2020 at 12:04

@AlexW You're right, thank you! – MathematicsStudent1122 May 2, 2020 at 16:44

Show **1** more comment

38

Note that [The Fabius function] is nowhere analytic but admits a *dense* set of points where all but finitely many derivatives vanish.

ShareCiteImprove this answerFollow

edited Apr 13, 2017 at 12:58
CommunityBot

answered May 8, 2011 at 2:12
Gerald Edgar

Add a comment

32

The theorem:

Theorem: Let $f(x)$ be C^∞ on (c,d) such that for every point x in the interval there exists an integer N_x for which $f^{(N_x)}(x) = 0$; then $f(x)$ is a polynomial.

is due to two Catalan mathematicians:

F. Sunyer i Balaguer, E. Corominas, Sur des conditions pour qu'une fonction infiniment dérivable soit un polynôme. Comptes Rendues Acad. Sci. Paris, 238 (1954), 558-559.

F. Sunyer i Balaguer, E. Corominas, Condiciones para que una función infinitamente derivable sea un polinomio. Rev. Mat. Hispano Americana, (4), 14 (1954).

The proof can also be found in the book (p. 53):

W. F. Donoghue, Distributions and Fourier Transforms, Academic Press, New York, 1969.

I will never forget it because in an "Exercise" of the "Opposition" to became "Full Professor" I was posed the following problem:

What are the real functions indefinitely differentiable on an interval such that a derivative vanish at each point?

31

Let me add one more solution. **It is not really different from the accepted one, but it includes all details.** The problem is that a student without sufficient experience will not even see necessity to fill details.

> **Theorem.** *If $f \in C^\infty(\mathbb{R})$ and for every $x \in \mathbb{R}$ there is a nonnegative integer n such that $f^{(n)}(x) = 0$, then f is a polynomial.*

The following exercise shows that the result cannot be to easy.

> **Exercise.** *Prove that there is a function $f \in C^{1000}(\mathbb{R})$ which is not a polynomial, but has the property described in the above theorem.*

Proof of the theorem. Let $\Omega \subset \mathbb{R}$ be the union of all open intervals $(a,b) \subset \mathbb{R}$ such that $f|_{(a,b)}$ is a polynomial. The set Ω is open, so

$$\Omega = \bigcup_{i=1}^{N} (a_i, b_i), \qquad (1)$$

$$\Omega = \bigcup_{i=1}^{N} (a_i, b_i), \qquad (1)$$

where $a_i < b_i$ and $(a_i,b_i) \cap (a_j,b_j) = \emptyset$ for $i \neq j$, $1 \leq N \leq \infty$. Observe that $f|_{(a_i,b_i)}$ is a polynomial (Why?)*. We want to prove that $\Omega = \mathbb{R}$. First we will prove that $\bar{\Omega} = \mathbb{R}$. To this end it suffices to prove that for any interval $[a,b]$, $a < b$ we have $[a,b] \cap \Omega \neq \emptyset$. Let

$$E_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}.$$

$$E_n = \{x \in \mathbb{R} : f^{(n)}(x) = 0\}.$$

The sets $E_n \cap [a,b]$ are closed and

$$[a,b] = \bigcup_{n=0}^{\infty} E_n \cap [a,b]\,.$$

$$[a,b] = \bigcup_{n=0}^{\infty} E_n \cap [a,b]\,.$$

Since $[a,b]$ is complete, it follows from the Baire theorem that for some n the set $E_n \cap [a,b]$ has nonempty interior (in the topology of $[a,b]$), so there is $(c,d) \subset E_n \cap [a,b]$ such that $f^{(n)} = 0$ on (c,d). Accordingly f is a polynomial on (c,d) and hence

$$(c,d) \subset \Omega \cap [a,b] \neq \emptyset.$$

$$(c,d) \subset \Omega \cap [a,b] \neq \emptyset\,.$$

The set $X = \mathbb{R} \setminus \Omega$ is closed and hence complete. It remains to prove that $X = \emptyset$. Suppose not. Observe that every point $x \in X$ is an accumulation point of the set, i.e. there is a sequence $x_i \in X$, $x_i \neq x$, $x_i \to x$. Indeed, otherwise x would be an isolated point, i.e. there would be two intervals

$$(a,x),\ (x,b) \subset \Omega,\ \ x \notin \Omega\,. \tag{2}$$

$$(a,x),(x,b) \subset \Omega,\ x \notin \Omega\,. \qquad (2)$$

The function f restricted to each of the two intervals is a polynomial, say of degrees n_1 and n_2. If $n > \max\{n_1, n_2\}$, then $f^{(n)} = 0$ on $(a,x) \cup (x,b)$. Since $f^{(n)}$ is continuous on (a,b), it must be zero on the entire interval and hence f is a polynomal of degree $\leq n-1$ on (a,b), so $(a,b) \subset \Omega$ which contradicts (2).

The space $X = \mathbb{R} \setminus \Omega$ is complete. Since

$$X = \bigcup_{n=1}^{\infty} X \cap E_n\,,$$

$$X = \bigcup_{n=1}^{\infty} X \cap E_n,$$

the second application of the Baire theorem gives that $X \cap E_n$ has a nonempty interior in the topology of X, i.e. there is an interval (a,b) such that

$$X \cap (a,b) \subset X \cap E_n \neq \emptyset\,. \tag{3}$$

$$X \cap (a,b) \subset X \cap E_n \neq \emptyset\,. \qquad (3)$$

Accordingly $f^{(n)}(x) = 0$ for all $x \in X \cap (a,b)$. Since for every $x \in X \cap (a,b)$ there is a sequence $x_i \to x$, $x_i \neq x$ such that $f^{(n)}(x_i) = 0$ it follows from the definition of the derivative that $f^{(n+1)}(x) = 0$ for every $x \in X \cap (a,b)$, and by induction $f^{(m)}(x) = 0$ for all $m \geq n$ and all $x \in X \cap (a,b)$.

We will prove that $f^{(n)} = 0$ on (a,b). This will imply that $(a,b) \subset \Omega$ which is a contradiction with (3). Since $f^{(n)} = 0$ on $X \cap (a,b) = (a,b) \setminus \Omega$ it remains to prove that $f^{(n)} = 0$ on $(a,b) \cap \Omega$. To this end it suffices to prove that for any interval (a_i, b_i) that appears in (1) such that $(a_i, b_i) \cap (a,b) \neq \emptyset$, $f^{(n)} = 0$ on (a_i, b_i). Since (a,b) is not contained in (a_i, b_i) one of the endpoints belongs to (a,b), say $a_i \in (a,b)$. Clearly $a_i \in X \cap (a,b)$ and hence

$f^{(m)}(a_i) = 0$ for all $m \geq n$. If f is a polynomial of degree k on (a_i, b_i), then $f^{(k)}$ is a nonzero constant on (a_i, b_i), so $f^{(k)}(a_i) \neq 0$ by continuity of the derivative. Thus $k < n$ and hence $f^{(n)} = 0$ on (a_i, b_i). $\square$

> **Exercise.** *As the previous exercise shows the theorem is not true if we only assume that $f \in C^{1000}$. Where did we use in the proof the assumption $f \in C^\infty(\mathbb{R})$?*

*It suffices to prove that f is a polynomial on every compact subinterval $[c, d] \subset (a_i, b_i)$. This subinterval has a finite covering by open intervals on which f is a polynomial. Taking an integer n larger than the maximum of the degrees of these polynomials, we see that $f^{(n)} = 0$ on $[c, d]$ and hence f is a polynomial of degree $< n$ on $[c, d]$.

ShareCiteImprove this answerFollow edited Jun 1, 2019 at 12:39 answered Mar 30, 2018 at 14:34

Piotr Hajlasz

I like this answer a lot, but I am confused about the sentence "The set Ω is open so $\Omega = \bigcup_{i=1}^\infty (a_i, b_i)$ where $a_i < b_i$ and $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$}. Is the last part (empty intersections of intervals) correct? It seems a bit at odds with the claim further down that $\Omega = \mathbb{R}$. – Vincent May 31, 2019 at 8:37

@Vincent I edited my proof, see formula (1). Is it okay now? – Piotr Hajlasz Jun 1, 2019 at 12:40

Huh no, now it is even more confusing. I understand that open sets are a union of intervals, but not that they are a union of *disjoint* intervals. Take the case $\Omega = \mathbb{R}$, I don't see any way of writing this as a union of non-overlapping intervals (let alone a finite number of such intervals as the new formula (1) suggests). But on closer inspection I think the condition $(a_i, b_i) \cap (a_j, b_j) = \emptyset$ for $i \neq j$ which is causing my concern is not being used further down the proof, is it? – Vincent Jun 2, 2019 at 17:42

2 @Vincent $\mathbb{R} = (-\infty, \infty)$. I never said that the intervals are finite. – Piotr Hajlasz Jun 3, 2019 at 0:50

If we assume $f \in C^{1000}$, then the sets E_n for $n > 1000$ are not guaranteed to be closed. For the example, consider $f(x) = 0$ if $x < 0$, $f(x) = x^{1001}$ if $x \geq 0$. – Sungjin Kim Mar 5, 2021 at 21:18

Show **2** more comments

16

For what it's worth, I post my solution. I assume $f: \mathbb{R} \to \mathbb{R}$, which makes no difference but lets me use one less symbol.

1. Let $A_n = \{x \in R \mid f^{(n)}(x) = 0\}$, E_n the interior of A_n. Clearly $E_n \subset E_m$ for $n < m$, and by Baire E_n is eventually not empty.
2. Each E_n is a countable union of open segments. It is easy to see that in passing from E_n to E_{n+1} new segments can appear, but those already in E_n remain unchanged. Moreover two such segments are never adiacent.
3. By this remark is it enough to prove that $\bigcup E_n = \mathbb{R}$. Indeed if this holds and $E_n \neq \emptyset$, then $E_n = \mathbb{R}$, which implies the thesis. Otherwise the points in the boundary of E_n don't appear in the union.

4. Let $E = \bigcup E_n$, B its complementary set, and assume by contradiction $B \neq \emptyset$. B is itself a complete metric space, hence can apply Baire to it. So for some k we find that $A_k \cap B$ has non-empty interior in B. This means that there is an interval I such that $B \cap I \subset A_k$ (and $B \cap I \neq \emptyset$).
5. From remark 2, B has no isolated points. The contradiction that we want to find is that $I \setminus B \subset A_k$. Indeed from this it follows that $I \subset A_k$, hence $E_k \cap B \neq \emptyset$.
6. By construction $I \setminus B$ is a union of intervals which appear in some E_n. Take such an interval J, say $J \subset E_N$ (where N is minimal), and let x be one end point of J (which is not on the boundary of I). Then $x \in I \cap B \subset A_k$, so $f^{(k)}(x) = 0$. Moreover x is not isolated in B, so it is the limit of a sequence x_i of points in B.
7. By the same argument $f^{(k)}(x_i) = 0$. Between two point where the k-th derivative vanish lies a point where the $k+1$-th does, so by continuity we find $f^{(k+1)}(x) = 0$. Similarly we find $f^{(m)}(x) = 0$ for all $m \geq k$. On J f is a polynomial of degree N; it follows that $N \leq k$, and we conclude that $J \subset E_k$. Since J was arbitrary we conclude that $I \setminus B \subset E_k$, which we have shown to be a contradiction.

ShareCiteImprove this answerFollow

answered Aug 1, 2010 at 1:55

Andrea Ferretti

Hi-- Thanks a lot. Now, does this remain true if we replace $[0,1]$ by $\mathbb{R}$ or $[a,b]$ – C.S. Aug 1, 2010 at 13:21

3 Yes, of course. The proof is the same. – Andrea Ferretti Aug 1, 2010 at 16:21

In step 3, what about functions of the form $e^{-1/x}$. They can have a derivative 0 on an interval and all future ones zero on the boundary. – Will Sawin Nov 1, 2011 at 5:38

1 Those functions have all derivatives 0 in a point, not on a whole interval – Andrea Ferretti Nov 3, 2011 at 18:54

1 E_n is the interior of A_n. For a point in E_n you have a whole interval where the nth derivative vanishes identically, hence all subsequent derivatives vanish – Andrea Ferretti Sep 4, 2016 at 10:09

Show **2** more comments

Maybe unuseful, but it remains true if you consider $f \in C^\infty(\mathbb{R},\mathbb{R})$.

7 Try showing that

Lemma. Let $I \subseteq \mathbb{R}$ be a nonempty interval and $f \in C^\infty(I)$. If f is not a polynomial on I, then there exists a compact subset $J \Subset I$ in which f is not a polynomial. Moreover, $f(x) \neq 0 \ \forall x \in J$.

ShareCiteImprove this answerFollow

answered Jul 31, 2010 at 22:35

fosco

Add a comment