Real Analysis II Homework 5

Due Date: May 29

Solve the following problems.

Problem 1

(a) Let $1 \leq p_i, r \leq \infty$ and
$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} = \frac{1}{r}$$
Prove the following generalization of Hölder's inequality:
$$\|f_1 \cdots f_k\|_r \leq \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}$$

(b) Let $1 \leq p < r < q \leq \infty$ and define $\theta \in (0,1)$ by
$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q}$$
Prove the interpolation estimate
$$\|f\|_r \leq \|f\|_p^{\theta} \|f\|_q^{1-\theta}$$

Problem 2 If $f \in L^p\left(\mathbb{R}^n\right), 0 < p < \infty$, show that
$$\lim_{Q \searrow x} \frac{1}{|Q|} \int_Q |f(y) - f(x)|^p \, dy = 0 \quad \text{a.e.}$$

Problem 3 Show that every subset Λ of a separable metric space (M, d) is separable.