Real Analysis II Homework Solution 1

Jian-An Wang

https://jiananwang1007.wordpress.com · jiananwang1007@gmail.com

Problem 1

Define

$$\log^+(x) = \max\{0, \log(x)\}$$

Show that if $|f|\left(1 + \log^+|f|\right) \in L^1(\mathbb{R}^n)$ then $f^* \in L^1(E)$, where E is a measurable set with $|E| < \infty$.

Proof. Define

$$d_{f^*}(\alpha) = |\{x \in \mathbb{R}^n : |f^*(x)| > \alpha\}|$$

By Fubini's theorem, we know

$$\int_E |f^*(x)|dx = \int_E \int_0^{|f^*(x)|} d\alpha dx = \int_0^\infty \int_E \chi_{\{x\in\mathbb{R}^n:|f^*(x)|>\alpha\}} dx d\alpha = \int_0^\infty d_{f^*}(\alpha)d\alpha$$

Hence, for every $\gamma > 0$, we have

$$\begin{aligned}
\int_E |f^*(x)|dx &= \int_0^\gamma d_{f^*}(\alpha)d\alpha + \int_\gamma^\infty d_{f^*}(\alpha)d\alpha \\
&\leq \gamma\,|E| + \int_\gamma^\infty d_{f^*}(\alpha)d\alpha \\
&\leq \gamma\,|E| + \int_\gamma^\infty \frac{C_n}{\alpha} \int_{\left\{x\in\mathbb{R}^n:|f(x)|>\frac{\alpha}{2}\right\}} |f(x)|dxd\alpha \\
&\leq \gamma\,|E| + C_n \int_{\mathbb{R}^n} |f(x)| \int_\gamma^{2|f(x)|} \frac{1}{\alpha} d\alpha dx \\
&= \gamma\,|E| + C_n \int_{\mathbb{R}^n} |f(x)| \left(\log^+|f(x)| + \log\left(\frac{2}{\gamma}\right)\right) dx
\end{aligned}$$

Let

$$\gamma = \frac{2}{e}$$

Then we obtain

$$\int_E |f^*(x)|dx < \infty$$

□

Problem 2

Let $A \subsetneq [a,b]$ with $|A| > 0$. Show that for every $n \in \mathbb{N}$, there exists $(x,d) \in A \times \mathbb{R}^\times$ such that

$$\{x+d, x+2d, \ldots, x+nd\} \subseteq A$$

Proof. By Lebesgue's density theorem, for every $n \in \mathbb{N}$, there exists a density point $p \in A$ such that

$$\frac{|[p,p+h) \cap A|}{|[p,p+h)|} > \frac{n-1}{n}$$

for some $h > 0$. Define

$$A_p = [p,p+h) \cap A \subseteq p + \mathbb{R}/h\mathbb{Z}$$

Let

$$\pi \colon p + \mathbb{R} \longrightarrow p + \mathbb{R}/h\mathbb{Z}$$

be the natural projection. Then for every $0 \le k < n$, we know

$$\left|\pi\left(A_p + \frac{k}{n} \cdot h\right)\right| = |\pi(A_p)| = |A_p| > \frac{n-1}{n} \cdot h$$

In other words, for every $0 \le k < n$, we have

$$\left|\pi\left(A_p + \frac{k}{n} \cdot h\right)^\complement\right| = \left|\left(p + \mathbb{R}/h\mathbb{Z}\right) \setminus \pi\left(A_p + \frac{k}{n} \cdot h\right)\right| < \frac{1}{n} \cdot h$$

Hence, we obtain

$$\left|\bigcap_{0 \le k < n} \pi\left(A_p + \frac{k}{n} \cdot h\right)\right| = \left|\left(p + \mathbb{R}/h\mathbb{Z}\right) \setminus \bigcup_{0 \le k < n} \pi\left(A_p + \frac{k}{n} \cdot h\right)^\complement\right| > 0$$

That is, there exists $x_0 \in p + \mathbb{R}/h\mathbb{Z}$ such that

$$x_0 \in \bigcap_{0 \le k < n} \pi\left(A_p + \frac{k}{n} \cdot h\right)$$

Therefore, for every $0 \le k < n$, we get

$$\pi\left(x_0 - \frac{k}{n} \cdot h\right) \in \pi(A_p)$$

Since we know

$$\left\{\pi\left(x_0 - \frac{k}{n} \cdot h\right) \in p + \mathbb{R}/h\mathbb{Z} : 0 \le k < n\right\} \cong \mathbb{Z}/n\mathbb{Z}$$

forms a cyclic subgroup of $p + \mathbb{R}/h\mathbb{Z}$, it follows that A_p contains an arithmetic sequence with n numbers if we order the above elements increasingly. $\square$