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Log-concave measures

Recent results and open problems

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Chapter 1

Introduction

In this thesis we will discuss some results related to a special class of probability measures. In order to study a problem in stochastic programming, in [Pré71] Prékopa was led to consider probabilities satisfying the following inequality regarding the convex combination of sets:

$$\mathbb{P}\big((1-\lambda)A + \lambda B\big) \geq \mathbb{P}(A)^{1-\lambda}\mathbb{P}(B)^{\lambda}$$

for every convex sets $A, B \subseteq \mathbb{R}^n$ and $0 < \lambda < 1$. Such probabilities are called logarithmically concave, or *log-concave* for short, because we can equivalently say that $\log \mathbb{P}$ is concave.

In that same article, he introduced an inequality for functions on the real line that represents a sort of a converse to Hölder's inequality. This inequality was then proved in a more general setting by Leindler in [Lei72] and now is known as Prékopa-Leindler inequality.

Later, Borell began to develop a theory surrounding log-concave measures and a generalization thereof: instead of confining himself to the geometric mean on the right hand side, he introduces a notion of concavity based on other means, with a real parameter $s \in [-\infty, \infty]$,

$$\mathbb{P}\big((1-\lambda)A + \lambda B\big) \geq M_s^{\lambda}\big(\mathbb{P}(A), \mathbb{P}(B)\big).$$

Many result can then be stated and proved for this larger collection of measures with almost no additional effort. His most relevant articles on this subject are [Bor74] and [Bor75].

Of course, from his work some new tools emerge: for example, the Prékopa-Leindler inequality evolves into one that covers other means and sheds light on the duality between the concavity of a measure and the concavity of its density (in the finite dimensional case). Borell's contribution includes the characterization of such measures in $\mathbb{R}^n$, the characterization by the finite projections in the infinite dimensional case and the proof that many measure theoretic operations preserve the concavity.

Furthermore, he generalizes some theorems previously known to be be true for the Gaussian measures: for instance, the integrability of a norm, properties of the support and a zero-one law for subgroups.

His studies have been carried on by Brascamp and Lieb, who have developed a refinement of the Borell's inequality by considering the essential supremum of functions and the essential addition of sets. In their article [BL76b] they also provide sharper estimates of the moments of a log-concave measure.

Chapter 2 is devoted to the study of the properties of s-concave measures, including the finite dimensional characterization, and some operations that can be performed on them, such as projection, convolution, disintegration.

In chapter 3 we prove the integrability of seminorms with respect to concave measures, a result which generalizes Fernique's one, and we discuss the estimates of the moments found by Brascamp and Lieb.

Finally, in chapter 4 we talk about differentiability of measures in an infinite dimensional context, in the sense of Fomin and Skorohod. Then we review some know facts about Gaussian measures and we present a result of Krugova which states a dichotomy property of log-concave measures, in a fashion similar to the one for Gaussian measures regarding the Cameron-Martin space.

1.1 Notation

Here is a short table of common notation.

Symbol	**Meaning**
$\mathscr{B}(E)$	Borel σ-algebra of the topological space E
$\mathcal{M}_+(E)$	positive Radon measures on E
$\mathcal{P}(E)$	Radon probability measures on E
$\mathfrak{M}_s(E)$	concave measures on E
$\mathfrak{P}_s(E)$	concave probabilities on E
$\mathscr{L}^n$	Lebesgue measure in $\mathbb{R}^n$
$f_\# \mu$	pushforward of a measure
M_p^λ	p-mean
ω_n	volume of the ball of radius 1 in $\mathbb{R}^n$: $\mathscr{L}^n(B(0,1))$

Chapter **2**

Log-concave measures and their properties

Let's begin by recalling the definition of Radon measure.

Definition 2.1. A *Radon measure* on a topological Hausdorff space E is a positive measure μ defined on the σ-algebra $\mathscr{B}(E)$ of Borel sets which is locally finite (every point has a neighbourhood with finite measure) and inner regular:

$$\mu(B) = \sup \{ \mu(K) \mid K \subseteq B,\, K \text{ compact} \} \qquad \text{for all } B \in \mathscr{B}(E).$$

If the space is locally compact or the Radon measure μ is finite, then it is also outer regular:

$$\mu(B) = \inf \{ \mu(U) \mid U \supseteq B,\, U \text{ open} \} \qquad \text{for all } B \in \mathscr{B}(E).$$

Radon measures will be denoted by $\mathcal{M}_+(E)$. Radon probabilities are those measures $\mu \in \mathcal{M}_+(E)$ whose total mass equals one, i.e. $\mu(E) = 1$, and we shall indicate them by $\mathcal{P}(E)$.

For arbitrary subsets $A \subseteq E$, not necessarily Borel, we can define the inner measure

$$\mu_*(A) = \sup \{ \mu(K) \mid K \subseteq A,\, K \text{ compact} \}$$

and the outer measure

$$\mu^*(A) = \inf \{ \mu(U) \mid U \supseteq A,\, U \text{ open} \}.$$

If μ is Radon, μ_* coincides with μ on $\mathscr{B}(E)$.

In order to present the theory in the greatest possible generality, in the following treatment E will denote a locally convex Hausdorff vector space on the real field. It will be explicitly specified when E is required to be separable, complete, Banach, Hilbert and so on, so that one can easily keep track of the assumptions that are really needed for any particular theorem.

Given two subsets $A, B \subseteq E$, $A + B$ denotes their Minkowski sum

$$A + B = \{ a + b \mid a \in A,\ b \in B \}.$$

Note that the sum of two Borel sets need not be Borel. See for example [ES70] where the authors construct a compact set and a G_δ set whose sum is not Borel. It is however analytic, hence universally measurable, if E is a Polish space (a separable completely metrizable topological space).

2.1 Log-concave measures

Now we encounter the measures that are at the core of this thesis.

Definition 2.2. A Radon measure $\mu \in \mathcal{M}_+(E)$ is said to be *log-concave* if

$$\mu_*\big((1-\lambda)A + \lambda B\big) \geq \mu(A)^{1-\lambda}\mu(B)^\lambda \tag{2.1}$$

for every $A, B \in \mathscr{B}(E)$ and $0 < \lambda < 1$.

To see a first example, let's show that the Lebesgue measure $\mathscr{L}^n$ in $\mathbb{R}^n$ is log-concave. The starting point is the fact that $\mathscr{L}^n$ satisfies the Brunn-Minkowski inequality.

Theorem 2.3 (Brunn-Minkowski inequality)**.** *Let $0 < \lambda < 1$ and let A and B be non-empty measurable sets in $\mathbb{R}^n$ such that $(1-\lambda)A + \lambda B$ is also measurable. Then*

$$\mathscr{L}^n\big((1-\lambda)A + \lambda B\big)^{1/n} \geq (1-\lambda)\mathscr{L}^n(A)^{1/n} + \lambda \mathscr{L}^n(B)^{1/n}. \tag{2.2}$$

This is one of many equivalent formulations of the Brunn-Minkowski inequality. See for example [Gar02] for a detailed survey. Note that if $(1-\lambda)A + \lambda B$ is not measurable, then by approximating A and B from the inside with compact sets we get that

$$\mathscr{L}^n_*\big((1-\lambda)A + \lambda B\big)^{1/n} \geq (1-\lambda)\mathscr{L}^n(A)^{1/n} + \lambda \mathscr{L}^n(B)^{1/n}.$$

Now, the arithmetic-geometric mean inequality implies that

$$\begin{aligned} \mathscr{L}^n_*\big((1-\lambda)A + \lambda B\big) &\geq \big[(1-\lambda)\mathscr{L}^n(A)^{1/n} + \lambda \mathscr{L}^n(B)^{1/n}\big]^n \geq \\ &\geq \mathscr{L}^n(A)^{1-\lambda}\mathscr{L}^n(B)^\lambda, \end{aligned} \tag{2.3}$$

so the Lebesgue measure is actually log-concave. An advantage of this last inequality is that it is independent of the dimension.

It is interesting to observe that we can recover the Brunn-Minkowski inequality from the log-concavity of the Lebesgue measure. Indeed, (2.3) implies the even weaker

$$\mathscr{L}^n_*\big((1-\lambda)A + \lambda B\big) \geq \min\{\mathscr{L}^n(A), \mathscr{L}^n(B)\}. \tag{2.4}$$

This inequality may seem much weaker than Brunn-Minkowski (and it is, for general measures), but here we can exploit the homogeneity of the Lebesgue measure to rediscover the latter. Take two non-empty sets $A, B \subseteq \mathbb{R}^n$. We may assume $\mathscr{L}^n(A), \mathscr{L}^n(B) > 0$, otherwise (2.2) is trivial. Consider the sets

$$A' = \mathscr{L}^n(A)^{-1/n} A \qquad\qquad B' = \mathscr{L}^n(B)^{-1/n} B$$

and put

$$\lambda = \frac{\mathscr{L}^n(B)^{1/n}}{\mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}}.$$

Then inequality (2.4) for the sets A' and B' reads

$$\mathscr{L}^n_* \left(\frac{A}{\mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}} + \frac{B}{\mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}} \right) \geq 1.$$

Bringing out the denominator we finally get

$$\mathscr{L}^n_*(A + B)^{1/n} \geq \mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}.$$

Using again the homogeneity, this inequality is equivalent to (2.2).[1]

Later we will see a complete characterization of log-concave measure that will allow us to determine more easily whether a measure is log-concave or not. For the moment, let's start with a powerful inequality which is the functional counterpart of (2.3) and can be used to construct log-concave measures in $\mathbb{R}^n$. The inequality is traditionally known as Prékopa-Leindler inequality, as Prékopa first stated it in [Pré71] for $\mathbb{R}$ and with $\lambda = 1/2$, while Leindler generalized it in [Lei72] to $\mathbb{R}^n$ and other exponents.

Theorem 2.4 (Prékopa-Leindler inequality)**.** *Let $0 < \lambda < 1$ and let f, g, h be non-negative measurable functions on $\mathbb{R}^n$ satisfying*

$$h\big((1-\lambda)x + \lambda y\big) \geq f(x)^{1-\lambda} g(y)^{\lambda}$$

for all $x, y \in \mathbb{R}^n$. Then

$$\int_{\mathbb{R}^n} h(z)\, dz \geq \left(\int_{\mathbb{R}^n} f(x)\, dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)\, dy \right)^{\lambda}.$$

Proof. First, consider the case $n = 1$. If $f(x) > t$ and $g(y) > t$, then

$$h\big((1-\lambda)x + \lambda y\big) > t,$$

therefore

$$\{h > t\} \supseteq (1-\lambda)\{f > t\} + \lambda\{g > t\}.$$

[1] To see more details about this, see the already cited [Gar02].

By Fubini's theorem, the one-dimensional version of Brunn-Minkowski and the arithmetic-geometric mean inequality, this translates into

$$\begin{aligned}\int_{\mathbb{R}} h(z)\,dz = \int_0^\infty \mathscr{L}^1(\{h>t\})\,dt \geq\\ \geq (1-\lambda)\int_0^\infty \mathscr{L}^1(\{f>t\})\,dt + \lambda\int_0^\infty \mathscr{L}^1(\{g>t\})\,dt =\\ = (1-\lambda)\int_{\mathbb{R}} f(x)\,dx + \lambda\int_{\mathbb{R}} g(y)\,dy \geq\\ \geq \left(\int_{\mathbb{R}} f(x)\,dx\right)^{1-\lambda}\left(\int_{\mathbb{R}} g(y)\,dy\right)^{\lambda}.\end{aligned}$$

Next we perform the induction on n. Let's indicate the variables as $x = (x_1, x') \in \mathbb{R}\times\mathbb{R}^{n-1}$ and define the functions

$$F(x_1) = \int_{\mathbb{R}^{n-1}} f(x_1, x')\,dx', \qquad G(y_1) = \int_{\mathbb{R}^{n-1}} g(y_1, y')\,dy',$$
$$H(z_1) = \int_{\mathbb{R}^{n-1}} f(z_1, z')\,dz'.$$

Fix $x_1, y_1, z_1 \in \mathbb{R}$ such that $z_1 = (1-\lambda)x_1 + \lambda y_1$. Then

$$h(z_1, z') \geq f(x_1, x')^{1-\lambda} g(y_1, y')^{\lambda}$$

for all $x', y', z' \in \mathbb{R}^{n-1}$ with $z' = (1-\lambda)x' + \lambda y'$. By induction, Prékopa-Leindler holds in dimension $n-1$, thus it follows that

$$H(z_1) \geq F(x_1)^{1-\lambda} G(y_1)^{\lambda}.$$

So now we can apply again the 1-dimensional Prékopa-Leindler inequality to arrive at the desired conclusion. □

It's interesting to see how quickly Prékopa-Leindler implies (2.3). Indeed, take $A, B \in \mathscr{B}(\mathbb{R}^n)$, $0 < \lambda < 1$ and define $C = (1-\lambda)A + \lambda B$. Then

$$\chi_C(z) \geq \chi_A(x)^{1-\lambda}\chi_B(y)^{\lambda} \qquad \text{if } z = (1-\lambda)x + \lambda y.$$

Therefore we conclude that $\mathscr{L}^n(C) \geq \mathscr{L}^n(A)^{1-\lambda}\mathscr{L}^n(B)^{\lambda}$. This is actually a practical way of proving Brunn-Minkowski. In fact, in the proof of Prékopa-Leindler we have only used the 1-dimensional version of Brunn-Minkowski, an inequality which can be verified much easily. This is the reasoning followed in [Bal97].

An interesting situation arises when Theorem 2.4 can be applied to a single function f, i.e. $g = h = f$. In order to do so, the function f must be of the following type.

Definition 2.5. A function $f\colon E \to [0,\infty]$ is said to be *log-concave* if

$$f\big((1-\lambda)x+\lambda y\big) \geq f(x)^{1-\lambda} f(y)^{\lambda}$$

for every $x, y \in E$ and $0 < \lambda < 1$.

Log-concave functions are of the form $f(x) = e^{-V(x)}$ with $V\colon E \to [-\infty,\infty]$ convex. As anticipated, the Prékopa-Leindler inequality can be used to prove that measures with log-concave density with respect to Lebesgue measure are log-concave themselves.

Theorem 2.6. *Let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ be a non-negative log-concave function. Then the measure $f\mathscr{L}^n$ is log-concave.*

Proof. Let $A, B \Subset \mathbb{R}^n$ be compact sets, fix $0 < \lambda < 1$ and define $C = (1-\lambda)A + \lambda B$. For every $z = (1-\lambda)x + \lambda y$ we have

$$(f\chi_C)(z) \geq [(f\chi_A)(x)]^{1-\lambda}[(f\chi_B)(y)]^{\lambda}.$$

Then Theorem 2.4 implies that

$$\int_C f(z)\,dz \geq \left(\int_A f(x)\,dx\right)^{1-\lambda}\left(\int_B f(y)\,dy\right)^{\lambda}.$$

The inequality (2.1) for all $A, B \in \mathscr{B}(\mathbb{R}^n)$ can be obtained by an approximation argument. □

As a consequence, the Gaussian probability on $\mathbb{R}^n$ with mean $m \in \mathbb{R}^n$ and symmetric positive matrix $\Sigma \in \mathbb{R}^{n\times n}$ given by

$$\gamma^n_{m,\Sigma}(A) = \sqrt{\frac{\det\Sigma}{(2\pi)^n}}\int_A \exp\left(-\frac{1}{2}(x-m)^T\Sigma(x-m)\right)dx$$

is log-concave.

2.2 p-concavity

Log-concave measures are part of a wider class of measures, parametrized by a real number, satisfying a similar inequality regarding the convex sum of sets. The theory about these measures has been developed primarily by Borell in [Bor75][2] and [Bor74].

We will need a definition of p-means that is slightly different from the usual one[3] in the case where some numbers are zero. The reason for this is that otherwise in the subsequent sections we would have to pay attention to negligible sets, which in some cases could cause some troubles.

[2]This article was originally published in 1973 on the Uppsala Univ. Dept. of Math. Report No. 8.

[3]See [HLP52].

Definition 2.7. Let $p \in [-\infty, \infty]$, let k be an integer greater than 1, $\lambda_1, \ldots, \lambda_k > 0$ real numbers such that $\sum_{i=1}^k \lambda_i = 1$, and $a_1, \ldots, a_k \in [0, \infty]$. If some of the a_i's are 0 we define the *p-mean* as

$$M_p^{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k) = 0,$$

otherwise ($a_i > 0$ for every $i = 1, \ldots, k$) we define

$$M_p^{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k) = \begin{cases} \min\{a_1, \ldots, a_k\} & p = -\infty, \\ \max\{a_1, \ldots, a_k\} & p = \infty, \\ a_1^{\lambda_1} \cdots a_k^{\lambda_k} & p = 0, \\ (\lambda_1 a_1^p + \cdots + \lambda_k a_k^p)^{1/p} & p \in \mathbb{R} \setminus \{0\}. \end{cases}$$

In case where $k = 2$, we set $\lambda_1 = 1 - \lambda$, $\lambda_2 = \lambda$ and write M_p^λ instead of $M_p^{1-\lambda,\lambda}$.

Since the mean is independent of $\{\lambda_i\}$ when $p = \pm\infty$, the superscript will often be left out in such cases.

The following proposition is a well known fact and its proof, based on Jensen's inequality, is very standard. We'll just say a few words about the continuity in 0.

Proposition 2.8. *Given p, k, λ_i and a_i as before, the function*

$$\begin{aligned} [-\infty, \infty] &\to \mathbb{R} \\ p &\mapsto M_p^{\lambda_1,\ldots,\lambda_k}(a_1, \ldots, a_k) \end{aligned}$$

is continuous and monotonically increasing. It is strictly increasing if the a_i are positive and not all equal.

Proof of the continuity in 0. We may assume $a_i > 0$ for every i. By applying l'Hôpital's rule, one can see that

$$\begin{aligned} \lim_{p \to 0} \log M_p^\lambda(a_1, \ldots, a_k) &= \lim_{p \to 0} \frac{1}{p} \log \left(\sum_{i=1}^k \lambda_i a_i^p \right) = \\ &= \lim_{p \to 0} \frac{\sum_{i=1}^k \lambda_i a_i^p \log a_i}{\sum_{i=1}^k \lambda_i a_i^p} = \\ &= \sum_{i=1}^k \lambda_i \log a_i, \end{aligned}$$

from which it follows that

$$\lim_{p \to 0} M_p^\lambda(a_1, \ldots, a_k) = \exp\left(\sum_{i=1}^k \lambda_i \log a_i \right) = a_1^{\lambda_1} \cdots a_k^{\lambda_k}. \qquad \square$$

In terms of p-means, the Brunn-Minkowski inequality can be restated by saying that for *any* two measurable sets $A, B \subseteq \mathbb{R}^n$ (no longer necessarily non-empty) one has

$$\mathscr{L}^n_*\big((1-\lambda)A + \lambda B\big) \geq M^\lambda_{1/n}\big(\mathscr{L}^n(A), \mathscr{L}^n(B)\big).$$

You see that the benefit of our definition is that as soon as one of the two sets is negligible, the mean is zero, so the inequality trivially holds and one doesn't have to treat separately the empty case.

Incidentally, the inequality that we used to prove the log-concavity from Brunn-Minkowski is nothing but $M^\lambda_{1/n} \geq M^\lambda_0$, which is a consequence of the monotonicity stated in Proposition 2.8.

In [Bor75], Borell introduces and studies the following classes of measures.

Definition 2.9. Let $s \in [-\infty, \infty]$. A Radon measure $\mu \in \mathcal{M}_+(E)$ is said to be s-*concave* if it satisfies

$$\mu_*\big((1-\lambda)A + \lambda B\big) \geq M^\lambda_s\big(\mu(A), \mu(B)\big)$$

for every $A, B \in \mathscr{B}(E)$ and $0 < \lambda < 1$. The family of s-concave measures is denoted by $\mathfrak{M}_s(E)$. The family of s-concave probability measures is denoted by $\mathfrak{P}_s(E)$.

Comparing with Definition 2.2, we see that $\mathfrak{M}_0(E)$ corresponds to log-concave measures. By the monotonicity of p-means, Proposition 2.8, we have that $\mathfrak{M}_{s_1}(E) \supseteq \mathfrak{M}_{s_2}(E)$ if $-\infty \leq s_1 \leq s_2 \leq \infty$. The members of the largest class $\mathfrak{M}_{-\infty}(E)$ will be called simply concave measures.[4]

A simple observation is that if $\mu \in \mathfrak{M}_s(E)$ and $C \subseteq E$ is a convex Borel set, then $\mu \llcorner C \in \mathfrak{M}_s(E)$ too. In fact,

$$\big((1-\lambda)A + \lambda B\big) \cap C \supseteq (1-\lambda)(A \cap C) + \lambda(B \cap C)$$

therefore

$$\begin{aligned}(\mu \llcorner C)_*\big((1-\lambda)A + \lambda B\big) &\geq \mu_*\big((1-\lambda)(A \cap C) + \lambda(B \cap C)\big) \geq \\ &\geq M^\lambda_s\big((\mu \llcorner C)(A), (\mu \llcorner C)(B)\big).\end{aligned}$$

There is also a corresponding concept of concavity for functions.

Definition 2.10. Let $p \in [-\infty, \infty]$. We say that a function $f\colon E \to [0, +\infty]$ is p-concave if

$$f\big((1-\lambda)x + \lambda y\big) \geq M^\lambda_p\big(f(x), f(y)\big)$$

for every $x, y \in E$ and $0 < \lambda < 1$.

[4] Borell and some others call them convex measures.

The two concepts of concavity, for measures and for functions, turn out to be dual one of each other once we consider functions as densities. More precisely, we have the following theorem.

Theorem 2.11. *Let $k \geq 2$ be an integer, $\lambda_1, \dots, \lambda_k > 0$ such that $\sum_{i=0}^{k} \lambda_i = 1$ and let*

$$\mu_i = f_i \mathscr{L}^n \qquad \textit{with} \qquad f_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \qquad \textit{for } i = 0, \dots, k.$$

Assume $s \in [-\infty, 1/n]$. Then

$$\mu_{0*}(A_0) \geq M_s^{\lambda_1,\dots,\lambda_k}\big(\mu_{1*}(A_1), \dots, \mu_{k*}(A_k)\big), \qquad A_0 = \sum_{i=1}^{k} \lambda_i A_i,$$

for all $A_1, \dots, A_k \in \mathscr{B}(\mathbb{R}^n)$ if and only if there exist representatives of the f_i's such that

$$f_0(x_0) \geq M_p^{\lambda_1,\dots,\lambda_k}\big(f_1(x_1), \dots, f_k(x_k)\big), \qquad x_0 = \sum_{i=1}^{k} f(x_i),$$

for all $(x_1, \dots, x_k) \in \mathbb{R}^k$, with

$$p = \begin{cases} s/(1-ns) & -\infty < s < 1/n, \\ -1/n & s = -\infty, \\ \infty & s = 1/n. \end{cases}$$

Note that the relation between s and p can also be written more concisely as

$$\frac{1}{s} - \frac{1}{p} = n, \qquad s \in [-\infty, 1/n], \qquad p \in [-1/n, \infty].$$

See Figure 2.1.

Borell gave the first proof of this theorem in [Bor75, Theorem 3.1]. The argument is based on a fairly intricate functional inequality and his approach appears to have been abandoned in the subsequent literature. In the later article [Bor74], he states a particular case of the theorem and briefly sketches a different proof, based on Hölder's inequality and a careful sectioning procedure.

A more elegant proof has been found later by Brascamp and Lieb and is based on a powerful functional inequality that extends Theorem 2.4. In the next section we build the necessary tools following the work of Brascamp and Lieb and at the end we prove Theorem 2.11.

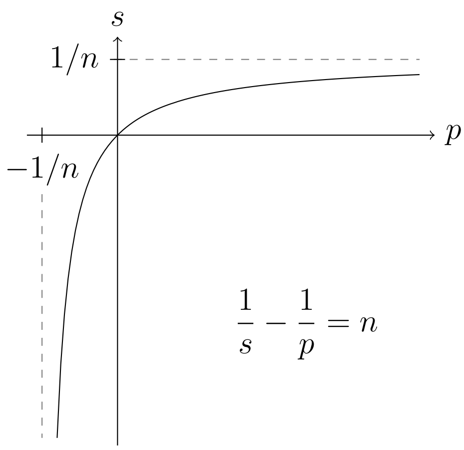

Figure 2.1: The relation between s and p.

2.3 Borell-Brascamp-Lieb inequality

After the initial contribution to the subject by Borell, Brascamp and Lieb provided in [BL76b] a strengthening of the Prékopa-Leindler inequality introducing the following idea. Instead of taking the ordinary Minkowski addition of sets in $\mathbb{R}^n$

$$A + B = \{ x \in \mathbb{R}^n \mid (x - A) \cap B \neq \emptyset \},$$

they consider the *essential addition*

$$A +_{\text{ess}} B = \left\{ x \in \mathbb{R}^n \;\middle|\; \mathscr{L}^n\big((x - A) \cap B\big) > 0 \right\}.$$

The essential addition has several advantages over the set-theoretic one. For one thing, it is invariant under Lebesgue equivalence, that is, if A and A' differ by a negligible set and the same is true for B and B', then $A +_{\text{ess}} B = A' +_{\text{ess}} B'$. This is a feature that will come in handy when we will deal with the functional counterpart. Secondly, one has the obvious inclusion $(1 - \lambda)A +_{\text{ess}} \lambda B \subseteq (1 - \lambda)A + \lambda B$, therefore the modified Brunn-Minkowski inequality stated by the authors as

$$\mathscr{L}^n(A +_{\text{ess}} B)^{1/n} \geq \mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}$$

is actually stronger than the classical one. Nevertheless, it can still be deduced from Theorem 2.3. The idea behind it is that of density points.

Proposition 2.12. *Let A and B be measurable sets in $\mathbb{R}^n$ with positive Lebesgue measure. Then $A +_{\text{ess}} B$ is open and*

$$\mathscr{L}^n(A +_{\text{ess}} B)^{1/n} \geq \mathscr{L}^n(A)^{1/n} + \mathscr{L}^n(B)^{1/n}.$$

Proof. Given $A \subseteq \mathbb{R}^n$, let A^* be the set of points of Lebesgue density one, that is

$$A^* = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r\to 0} \frac{\mathscr{L}^n\big(A \cap B(x,r)\big)}{\omega_n r^n} = 1 \right\}.$$

It is well known that A^* is equivalent to A, therefore

$$A +_{\text{ess}} B = A^* +_{\text{ess}} B^*.$$

Let $x_0 \in A^* + B^*$, i.e. there is a point $y \in (x_0 - A^*) \cap B^*$. By definition of density point, there exists $\bar{r} > 0$ such that

$$\begin{aligned} \mathscr{L}^n\big((x_0 - A^*) \cap B(y,r)\big) &> \frac{3}{4}\omega_n r^n \qquad && \forall r \in (0,\bar{r}), \\ \mathscr{L}^n\big(B^* \cap B(y,r)\big) &> \frac{3}{4}\omega_n r^n \qquad && \forall r \in (0,\bar{r}). \end{aligned}$$

This implies that

$$\mathscr{L}^n\big((x - A^*) \cap B^*\big) > \frac{1}{2}\omega_n r^n > 0$$

for all x in a neighbourhood of x_0, from which we deduce that $A^* + B^* = A^* +_{\text{ess}} B^*$ and it is open. Finally,

$$\begin{aligned} \mathscr{L}^n(A +_{\text{ess}} B)^{1/n} &= \mathscr{L}^n(A^* +_{\text{ess}} B^*)^{1/n} = \\ &= \mathscr{L}^n(A^* + B^*)^{1/n} \geq \\ &\geq \mathscr{L}^n(A^*)^{1/n} + \mathscr{L}(B^*)^{1/n} = \\ &= \mathscr{L}^n(A^*)^{1/n} + \mathscr{L}(B^*)^{1/n}. \end{aligned}$$

□

Corollary 2.13. *If $A, B \subseteq \mathbb{R}^n$ are (possibly negligible) measurable sets and $0 < \lambda < 1$, then*

$$\mathscr{L}^n\big((1-\lambda)A +_{\text{ess}} \lambda B\big) \geq M^\lambda_{1/n}\big(\mathscr{L}^n(A), \mathscr{L}^n(B)\big). \tag{2.5}$$

The true power of the idea of essential addition emerges when one applies it to functions. Recall that in Theorem 2.4 we require

$$h\big((1-\lambda)x + \lambda y\big) \geq M^\lambda_0\big(f(x), g(y)\big), \qquad \text{for all } x, y \in \mathbb{R}^n.$$

Given the two functions f and g, it seems natural to consider

$$h(z) = \sup\left\{ M^\lambda_0\big(f(x), g(y)\big) \;\middle|\; z = (1-\lambda)x + \lambda y \right\},$$

this function being the smallest that satisfies the above condition. This function, however, might be non-measurable. Indeed, if $f = \chi_A$ and $g = \chi_B$, then with the above definition we have $h = \chi_{(1-\lambda)A+\lambda B}$, and this function is not necessarily measurable.

So, instead of taking the pointwise supremum, Brascamp and Lieb define

$$h(z) = \operatorname{ess\,sup} \left\{ M_0^\lambda\big(f(x), g(y)\big) \;\middle|\; z = (1-\lambda)x + \lambda y \right\} = \\ = \operatorname*{ess\,sup}_{w \in \mathbb{R}^n} M_0^\lambda\left(f\Big(\frac{w}{1-\lambda}\Big), g\Big(\frac{z-w}{\lambda}\Big)\right).$$

The measurability of this function can then be proved with ideas similar to those of the previous proposition. Moreover, the result holds for other means too.

Proposition 2.14. *If f and g are non-negative measurable functions on $\mathbb{R}^n$, $\alpha \in [-\infty, \infty]$, $0 < \lambda < 1$ and*

$$h_\alpha^\lambda(z; f, g) = \operatorname{ess\,sup} \left\{ M_\alpha^\lambda\big(f(x), g(y)\big) \;\middle|\; z = (1-\lambda)x + \lambda y \right\},$$

then $h_\alpha^\lambda(z; f, g)$ is lower semicontinuous in z.

Proof. Given a non-negative measurable function f, let

$$A_f = \left\{ (x,t) \in \mathbb{R}^{n+1} \;\middle|\; 0 < t < f(x) \right\}$$

and define

$$f^*(x) = \sup \left\{ t \;\middle|\; (x,t) \in A_f^* \right\} \vee 0.$$

Then clearly $(A_f)^* \supseteq A_{f^*}$ and

$$(A_f)^* \setminus A_{f^*} \subseteq G = \left\{ \big(x, f^*(x)\big) \;\middle|\; x \in \mathbb{R}^n \right\}.$$

Since $\mathscr{L}^{n+1}(G) = 0$, it follows that $\mathscr{L}^{n+1}\big((A_f)^* \setminus A_{f^*}\big) = 0$. Therefore

$$\int_{\mathbb{R}^n} |f^* - f| \, dx = \mathscr{L}^{n+1}(A_{f^*} \triangle A_f) = \mathscr{L}^{n+1}\big((A_f)^* \triangle A_f\big) = 0.$$

From the equivalence of f and f^* we deduce that $h_\alpha^\lambda(z; f, g) = h_\alpha^\lambda(z; f^*, g^*)$. Now consider the function

$$k_\alpha^\lambda(z; f, g) = \sup \left\{ M_\alpha^\lambda\big(f(x), g(y)\big) \;\middle|\; z = (1-\lambda)x + \lambda y \right\}.$$

We want to prove that for any $t \geq 0$ the set

$$D(t) = \left\{ z \in \mathbb{R}^n \;\middle|\; k_\alpha^\lambda(z; f^*, g^*) > t \right\}$$

is open, as this would simultaneously imply that $k_\alpha^\lambda(f^*, g^*)$ is lower semicontinuous and

$$k_\alpha^\lambda(z; f^*, g^*) = h_\alpha^\lambda(z; f^*, g^*).$$

To this end, take $t \geq 0$ and $z_0 \in D(t)$. Now chose $x, y \in \mathbb{R}^n$, $z_0 = (1-\lambda)x + \lambda y$, and $u, v > 0$ such that

$$f^*(x) > u, \qquad g^*(y) > v, \qquad M_\alpha^\lambda(u,v) \geq t.$$

Then $(x,u) \in A_{f^*}$ and $(y,v) \in A_{g^*}$. Since $A_{f^*} \subseteq (A_f)^*$ and the two sets are equivalent, for any $\delta > 0$ there exists $\bar{r} > 0$ such that

$$\begin{aligned} \mathscr{L}^{n+1}\left(A_{f^*} \cap B\big((x,u),r\big)\right) &> (1-\delta)\omega_{n+1} r^{n+1} && \forall r \in (0,\bar{r}), \\ \mathscr{L}^{n+1}\left(A_{g^*} \cap B\big((y,v),r\big)\right) &> (1-\delta)\omega_{n+1} r^{n+1} && \forall r \in (0,\bar{r}). \end{aligned}$$

Taking δ sufficiently small we can guarantee that

$$\begin{aligned} \mathscr{L}^{n}\left(\{f^* > u\} \cap B(x,r)\right) &> \frac{3}{4}\omega_n r^n, \\ \mathscr{L}^{n}\left(\{g^* > v\} \cap B(y,r)\right) &> \frac{3}{4}\omega_n r^n, \end{aligned}$$

for some $r > 0$, from which we get that $h_\alpha^\lambda(z_0; f^*, g^*) > t$, hence $k_\alpha^\lambda(z; f^*, g^*) = h_\alpha^\lambda(z; f^*, g^*)$. Moreover, $h_\alpha^\lambda(z; f^*, g^*) > t$ holds for all z in a neighbourhood of z_0, thus $D(t)$ is open and $h_\alpha^\lambda(z; f, g)$ is lower semicontinuous. $\square$

Apart from the measurability issue, the definition of h_α^λ given by Brascamp and Lieb has other benefits: inequalities à la Prekopa-Leindler involving lower estimates on h_α^λ are stronger because the essential supremum is smaller, and such inequalities can be more easily extended to more than two functions.

Definition 2.15. Given non-negative measurable functions $f_1, \dots, f_k$ on $\mathbb{R}^n$, positive numbers $\lambda_1, \dots, \lambda_k$ such that $\sum_{i=1}^k \lambda_i = 1$ and $\alpha \in [-\infty, \infty]$, they define

$$h_\alpha^{\{\lambda_i\}}(y; f_1, \dots, f_k) = \operatorname{ess\,sup} \left\{ M_\alpha^{\{\lambda_i\}}\big(f_1(x_1), \dots, f_k(x_k)\big) \;\middle|\; y = \sum_{i=0}^k \lambda_i x_i \right\},$$

where this slightly ambiguous notation is to be understood as the essential supremum over the affine plane

$$\left\{ (x_1, \dots, x_k) \in \mathbb{R}^{n \times k} \;\middle|\; y = \sum_{i=0}^k \lambda_i x_i \right\}$$

with respect to the $(k-1) \times n$ dimensional Hausdorff measure.

The first basic inequality, used by the authors to prove the subsequent ones, is this.

Theorem 2.16. *Let $0 < \lambda < 1$, let f, g be non-negative measurable functions on $\mathbb{R}$ and assume $\|f\|_\infty = \|g\|_\infty = m$. Then*

$$\left\|h_{-\infty}^\lambda(f,g)\right\|_1 \geq (1-\lambda)\|f\|_1 + \lambda \|g\|_1 .$$

Proof. For convenience, let's call $h^\lambda_{-\infty} = h^\lambda_{-\infty}(f,g)$. Note that also $\left\|h^\lambda_{-\infty}\right\| \le m$. For $0 \le t < m$ we have $\mathscr{L}^1(\{f > t\}) > 0$, $\mathscr{L}^1(\{g > t\}) > 0$ and

$$\{h^\lambda_{-\infty} > t\} \supseteq (1-\lambda)\{f > t\} +_{\text{ess}} \lambda\{g > t\},$$

therefore by (2.5)

$$\begin{aligned}\left\|h^\lambda_{-\infty}\right\|_1 &= \int_0^m \mathscr{L}^1(\{h^\lambda_{-\infty} > t\})\,dt \ge \\ &\ge (1-\lambda)\int_0^m \mathscr{L}^1(\{f > t\})\,dt + \lambda\int_0^m \mathscr{L}^1(\{g > t\})\,dt = \\ &= (1-\lambda)\left\|f\right\|_1 + \lambda\left\|g\right\|_1.\end{aligned}$$

□

The next step is generalizing the result to other means.

Theorem 2.17. *Let $0 < \lambda < 1$, let $\alpha \in [-1, \infty]$ and let f and g be non-negative measurable functions on $\mathbb{R}$. Then*

$$\left\|h^\lambda_\alpha(f,g)\right\|_1 \ge M^\lambda_\beta(\left\|f\right\|_1, \left\|g\right\|_1),$$

where $\beta = \alpha/(1+\alpha)$. (With the obvious convention that $\beta = -\infty$ if $\alpha = -1$).

Proof. We may assume $\left\|f\right\|_1 > 0$ and $\left\|g\right\|_1 > 0$, otherwise the inequality is trivial. We can also restrict ourselves to bounded functions, as the others can be approximated from below in L^1 by these. Define the renormalized functions

$$F(x) = f(x)\left\|f\right\|_\infty, \qquad G(x) = g(x)/\left\|g\right\|_\infty.$$

Let's deal with the case $\alpha \neq 0$ first. We have

$$\begin{aligned}h^\lambda_\alpha(z; f, g) &= \operatorname*{ess\,sup}_{z=(1-\lambda)x+\lambda y} M^\lambda_\alpha\big(\left\|f\right\|_\infty F(x), \left\|g\right\|_\infty G(y)\big) = \\ &= M^\lambda_\alpha(\left\|f\right\|_\infty, \left\|g\right\|_\infty) \operatorname*{ess\,sup}_{z=(1-\lambda)x+\lambda y} M^\theta_\alpha\big(F(x), G(y)\big) \ge \\ &\ge M^\lambda_\alpha(\left\|f\right\|_\infty, \left\|g\right\|_\infty) \operatorname*{ess\,sup}_{z=(1-\lambda)x+\lambda y} M_{-\infty}\big(F(x), G(y)\big) = \\ &= M^\lambda_\alpha(\left\|f\right\|_\infty, \left\|g\right\|_\infty)\, h^\lambda_{-\infty}(z; F, G),\end{aligned}$$

where

$$\theta = \frac{(1-\lambda)\left\|g\right\|^\alpha_\infty}{\lambda\left\|f\right\|^\alpha_\infty + (1-\lambda)\left\|g\right\|^\alpha_\infty}.$$

By the previous theorem we deduce that

$$\begin{aligned}\left\|h^\lambda_\alpha(f,g)\right\|_1 &\ge M^\lambda_\alpha(\left\|f\right\|_\infty, \left\|g\right\|_\infty)\left\|h^\lambda_{-\infty}(F,G)\right\|_1 \ge \\ &\ge M^\lambda_\alpha(\left\|f\right\|_\infty, \left\|g\right\|_\infty) M^\lambda_1\left(\frac{\left\|f\right\|_1}{\left\|f\right\|_\infty}, \frac{\left\|g\right\|_1}{\left\|g\right\|_\infty}\right) \ge \\ &\ge M^\lambda_\beta(\left\|f\right\|_1, \left\|g\right\|_1),\end{aligned}$$

where the last inequality follows from Lemma 2.18.

Now consider the case $\alpha = 0$. We have

$$h_0^\lambda(f,g) = M_0^\lambda(\|f\|_\infty, \|g\|_\infty) h_0^\lambda(F,G) \geq M_0^\lambda(\|f\|_\infty, \|g\|_\infty) h_{-\infty}^\lambda(F,G),$$

therefore the previous theorem implies that

$$\begin{aligned}\left\|h_0^\lambda(f,g)\right\|_1 &\geq M_0^\lambda(\|f\|_\infty, \|g\|_\infty) M_1^\lambda\left(\frac{\|f\|_1}{\|f\|_\infty}, \frac{\|g\|_1}{\|g\|_\infty}\right) \geq \\ &\geq M_0^\lambda(\|f\|_\infty, \|g\|_\infty) M_0^\lambda\left(\frac{\|f\|_1}{\|f\|_\infty}, \frac{\|g\|_1}{\|g\|_\infty}\right) = \\ &= M_0^\lambda(\|f\|_1, \|g\|_1).\end{aligned}$$

□

Lemma 2.18. *Let $\alpha \geq -1$, $\beta = \alpha/(1+\alpha)$, $0 < \lambda < 1$ and $x_1, x_2, y_1, y_2 > 0$. Then*

$$M_\beta^\lambda(x_1 y_1, x_2 y_2) \leq M_\alpha^\lambda(x_1, x_2) M_1^\lambda(y_1, y_2).$$

Proof. The case $\alpha = \beta = 0$ follows immediately from the arithmetic-geometric mean inequality.

If $\alpha > 0$, then also $\beta > 0$ and

$$1 = \frac{\beta}{\alpha} + \beta.$$

The Hölder's inequality implies that

$$\begin{aligned}M_\beta^\lambda(x_1 y_1, x_2 y_2)^\beta &= M_1^\lambda(x_1^\beta y_1^\beta, x_2^\beta y_2^\beta) \leq \\ &\leq M_{\alpha/\beta}^\lambda(x_1^\beta, x_2^\beta) M_{1/\beta}^\lambda(y_1^\beta, y_2^\beta) = \\ &= M_\alpha^\lambda(x_1, x_2)^\beta M_1^\lambda(y_1, y_2)^\beta\end{aligned}$$

and the thesis follows from the fact that the function $t \mapsto t^{1/\beta}$ is increasing on $(0, \infty)$.

If $-1 < a < 0$, then also $\beta < 0$ and

$$1 = \frac{\alpha}{\beta} + (-\alpha).$$

Therefore the standard Hölder's inequality says that

$$\begin{aligned}M_\alpha^\lambda(x_1, x_2)^\alpha &= M_1^\lambda(x_1^\alpha, x_2^\alpha) \leq \\ &\leq M_{\beta/\alpha}^\lambda\big((x_1 y_1)^\alpha, (x_2 y_2)^\alpha\big) M_{-1/\alpha}^\lambda(y_1^{-\alpha}, y_2^{-\alpha}) = \\ &= M_\beta^\lambda(x_1 y_1, x_2 y_2)^\alpha M_1^\lambda(y_1, y_2)^{-\alpha}.\end{aligned}$$

From this we get

$$M_\beta^\lambda(x_1 y_1, x_2 y_2)^\alpha \leq M_\alpha^\lambda(x_1, x_2)^\alpha M_1^\lambda(y_1, y_2)^\alpha$$

and the thesis follows because the function $t \mapsto t^{1/\alpha}$ is decreasing on $(0, \infty)$.

Finally, the case $\alpha = -1$, $\beta = -\infty$ is obtained with a limiting argument. □

The final step performed by Brascamp and Lieb is the extension of the previous result to $\mathbb{R}^n$.

Theorem 2.19 (Borell-Brascamp-Lieb inequality). *Let $0 < \lambda < 1$, let n be a positive integer, let $\alpha \in [-1/n, \infty]$ and let f and g be non-negative measurable functions on $\mathbb{R}^n$. Then*

$$\left\| h_\alpha^\lambda(f,g) \right\|_1 \geq M_\beta^\lambda(\|f\|_1, \|g\|_1),$$

where $\beta = \alpha/(1+n\alpha)$,and $\beta = -\infty$ if $\alpha = -1/n$, as usual.

In particular, we have the essential form of the Prékopa-Leindler inequality

$$\left\| h_0^\lambda(f,g) \right\|_1 \geq \|f\|_1^{1-\lambda} \|g\|_1^\lambda .$$

Proof. The proof is by induction on n, with the previous theorem being the basis. Split the space as $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ and define

$$F(x_1) = \int_{\mathbb{R}^{n-1}} f(x_1, x')\, dx', \qquad G(y_1) = \int_{\mathbb{R}^{n-1}} g(y_1, y')\, dy'.$$

We have

$$\begin{aligned} h_\alpha^\lambda(z; f, g) &= \operatorname*{ess\,sup}_{z_1=(1-\lambda)x_1+\lambda y_1} \ \operatorname*{ess\,sup}_{z'=(1-\lambda)x'+\lambda y'} M_\alpha^\lambda\big(f(x_1,x'), g(x_1,x')\big) = \\ &= \operatorname*{ess\,sup}_{z_1=(1-\lambda)x_1+\lambda y_1} h_\alpha^\lambda\big(z'; f(x_1,\cdot), g(y_1,\cdot)\big) \end{aligned}$$

therefore

$$\begin{aligned} \int_{\mathbb{R}^{n-1}} h_\alpha^\lambda(z_1, z'; f, g)\, dz' &\geq \operatorname*{ess\,sup}_{z_1=(1-\lambda)x_1+\lambda y_1} \int_{\mathbb{R}^{n-1}} h_\alpha^\lambda\big(z'; f(x_1,\cdot), g(y_1,\cdot)\big)\, dz' \geq \\ &\geq \operatorname*{ess\,sup}_{z_1=(1-\lambda)x_1+\lambda y_1} M_\gamma^\lambda\big(F(x_1), G(y_1)\big) = \\ &= h_\gamma^\lambda(z_1; F, G), \end{aligned}$$

where $\gamma = \alpha/\big(1 + (n-1)\alpha\big)$ by the inductive hypothesis.

Then, integrating with respect to z_1 and using the base case yields

$$\begin{aligned} \left\| h_\alpha^\lambda(f,g) \right\|_1 &\geq \left\| h_\gamma^\lambda(F,G) \right\|_1 \geq \\ &\geq M_\beta^\lambda(\|F\|_1, \|G\|_1) = M_\beta^\lambda(\|f\|_1, \|g\|_1), \end{aligned}$$

where $\beta = \gamma/(1+\gamma) = \alpha/(1+n\alpha)$. □

We conclude the list of functional inequalities with a generalization to more than two functions. This is the theorem that will be used in the proof of Theorem 2.11.

Theorem 2.20. *Let $k \geq 2$ be an integer, $\lambda_1, \dots, \lambda_k > 0$ such that $\sum_{i=1}^k \lambda_i = 1$, let n be a positive integer, $\alpha \in [-1/n, \infty]$ and let $f_1, \dots, f_k$ be non-negative measurable functions on $\mathbb{R}^n$. Then*

$$\left\| h_\alpha^{\{\lambda_i\}}(f_1, \dots, f_k) \right\|_1 \geq M_\beta^{\{\lambda_i\}}(\|f_1\|_1, \dots, \|f_k\|_1),$$

where $\beta = \alpha/(1 + n\alpha)$.

Proof. We have that

$$\begin{aligned} h_\alpha^{\lambda_1,\dots,\lambda_k}(y; f_1, \dots, f_k) &= \sup_{y=\sum_{i=1}^k \lambda_i x_i} M_\alpha^{\lambda_1,\dots,\lambda_k}\big(f_1(x_1), \dots, f_k(x_k)\big) = \\ &= \sup_{y=(1-\lambda_k)z+\lambda_k x_k} M_\alpha^{\lambda_k}\left(h_\alpha^{\frac{\lambda_1}{1-\lambda_k},\dots,\frac{\lambda_{k-1}}{1-\lambda_k}}(z; f_1, \dots, f_{k-1}), f_k(x_k) \right) = \\ &= h_\alpha^{\lambda_k}\left(y; h_\alpha^{\frac{\lambda_1}{1-\lambda_k},\dots,\frac{\lambda_{k-1}}{1-\lambda_k}}(f_1, \dots, f_{k-1}), f_k \right), \end{aligned}$$

therefore, by Theorem 2.19 and induction,

$$\begin{aligned} \left\| h_\alpha^{\lambda_1,\dots,\lambda_k}(f_1, \dots, f_k) \right\|_1 &\geq M_\beta^{\lambda_k}\left(\left\| h_\alpha^{\frac{\lambda_1}{1-\lambda_k},\dots,\frac{\lambda_{k-1}}{1-\lambda_k}}(f_1, \dots, f_{k-1}) \right\|_1, \|f_k\|_1 \right) \geq \\ &\geq M_\beta^{\lambda_k}\left(M_\beta^{\frac{\lambda_1}{1-\lambda_k},\dots,\frac{\lambda_{k-1}}{1-\lambda_k}}(\|f_1\|_1, \dots, \|f_{k-1}\|_1), \|f_k\|_1 \right) = \\ &= M_\beta^{\lambda_1,\dots,\lambda_k}(\|f_1\|_1, \dots, \|f_k\|_1). \end{aligned}$$

□

We can finally go back to the proof Theorem 2.11, which has been skipped.

Proof of Theorem 2.11. Let's start with the "if" part, which is an immediate consequence of Theorem 2.20. For $i = 0, \dots, k$, define the functions $g_i = f_i \chi_{A_i}$. Then $g_0 \geq h_p^{\lambda_1,\dots,\lambda_k}(g_1, \dots, g_k)$, therefore

$$\begin{aligned} \mu_0(A_0) = \|g_0\|_1 &\geq \left\| h_p^{\lambda_1,\dots,\lambda_k}(g_1, \dots, g_k) \right\|_1 \geq \\ &\geq M_s^{\lambda_1,\dots,\lambda_k}(\|g_1\|_1, \dots, \|g_k\|_1) = \\ &= M_s^{\lambda_1,\dots,\lambda_k}\big(\mu_1(A_1), \dots, \mu_k(A_k)\big). \end{aligned}$$

We used the fact that the relation between p and s is the same as that between α and β in Theorem 2.20: indeed $s = p/(1 + np)$.

The "only if" part is a little trickier. For simplicity, assume $s \neq -\infty, 0, 1/n$. Substitute the f_i's with their Lebesgue limits, that is redefine

$$f_i(x) = \lim_{\epsilon \to 0} \frac{1}{\omega_n \varepsilon^n} \int_{B(x,\varepsilon)} f_i(y)\, dy \qquad \text{if the limit exists,}$$

or $f_i(x) = 0$ if the limit doesn't exist, with the exception of f_0 which is defined to be $f_0(x) = \infty$ in such case. Fix $x_1, \dots, x_k$ such that $f_i(x_i) > 0$, or else the thesis is trivial, and take $\rho_0, \dots, \rho_k > 0$. We may also assume that $x_0 = \sum_{i=1}^k \lambda_i x_i$ is a Lebesgue point for f_0 for the same reason. From the previous limit we have that

$$\mu_i\big(B(x_i, \rho_i \varepsilon)\big) = \omega_n f_i(x_i) \rho_i^n \varepsilon^n + o(\varepsilon^n).$$

By hypothesis we have

$$\mu_0\left(B\left(x_0, \sum_{i=1}^k \lambda_i \rho_i \varepsilon\right)\right) \geq M_s^{\lambda_1,\dots,\lambda_k}\big(\mu_1(B(x_1,\rho_1\varepsilon)), \dots, \mu_k(B(x_k,\rho_k\varepsilon))\big),$$

which, using the previous expansion and dividing by $\omega_n \varepsilon^n$, translates into

$$f_0(x_0)\left(\sum_{i=1}^k \lambda_i \rho_i\right)^n + o(1) \geq M_s^{\lambda_1,\dots,\lambda_k}\big(f_1(x_1)\rho_1^n + o(1), \dots, f_k(x_k)\rho_k^n + o(1)\big).$$

Taking the limit $\varepsilon \to 0$ leads to

$$\begin{aligned} f_0(x_0)\left(\sum_{i=1}^k \lambda_i \rho_i\right)^n &\geq M_s^{\lambda_1,\dots,\lambda_k}\big(f_1(x_1)\rho_1^n, \dots, f_k(x_k)\rho_k^n\big) = \\ &= \left(\sum_{i=1}^k \lambda_i f_i(x_i)^s \rho_i^{ns}\right)^{1/s}. \end{aligned}$$

Now we can choose $\rho_i = f_i(x_i)^{s/(1-ns)}$ so that $\rho_i = f_i(x_i)^s \rho_i^{ns}$ and doing so yields

$$f_0(x_0) \geq \left(\sum_{i=1}^k \lambda_i f_i(x_i)^{s/(1-ns)}\right)^{1/s-n} = M_{s/(1-ns)}^{\lambda_1,\dots,\lambda_k}\big(f_1(x_1), \dots, f_k(x_k)\big).$$

The remaining cases for s are dealt with similarly:

- if $s = -\infty$, choose $\rho_i = f_i(x_i)^{-1/n}$ for all i;
- if $s = 0$, choose $\rho_i = 1$ for all i;
- if $s = 1/n$, set $\rho_{i_0} = 1$ for i_0 such that $f_{i_0}(x_{i_0}) = \max\{f_1(x_1), \dots, f_k(x_k)\}$ and $\rho_i = 0$ for $i \neq i_0$. □

Corollary 2.21. *Let $F\colon \mathbb{R}^{m+n} \to [0,\infty]$ be p-concave with $p \in [-1/n, \infty]$. Then*

$$G(x) = \int_{\mathbb{R}^n} F(x,y)\, dy$$

is q-concave with $q = p/(1+np)$. In particular, G is log-concave if so is F.

2.4 Operations on s-concave measures

In this section we study how concave measures can be manipulated in several ways that are common in measure theory and probability and we give a complete characterization of s-concave measures in $\mathbb{R}^n$. First, we need a lemma about the pushforward of a Radon measure.

Lemma 2.22. *Let E and F be two Hausdorff spaces and let $h\colon E \to F$ be a continuous function. If $\mu \in \mathcal{M}_+(E)$ is finite, then $h_\#\mu \in \mathcal{M}_+(E)$. In particular, we have that $(h_\#\mu)_*(A) = \mu_*\big(h^{-1}(A)\big)$ if $A \in \mathscr{B}(F)$.*

Proof. Take a set $A \in \mathscr{B}(F)$. If $H \Subset h^{-1}(A)$, then $K := h(H) \Subset A$ and $h^{-1}(K) \supset H$, therefore $\mu\big(h^{-1}(K)\big) \geq \mu(H)$. This implies that

$$(h_\#\mu)_*(A) = \sup\big\{\, \mu\big(h^{-1}(K)\big) \mid K \Subset A \,\big\}$$

is greater than or equal to

$$\mu_*\big(h^{-1}(A)\big) = \sup\big\{\, \mu(H) \mid H \Subset h^{-1}(A) \,\big\}\,.$$

Thus

$$h_\#\mu(A) \geq (h_\#\mu)_*(A) \geq \mu_*\big(h^{-1}(A)\big) \geq \mu\big(h^{-1}(A)\big) = h_\#\mu(A)$$

and all inequalities must be equalities. □

Remark 2.23. The assumption that μ be finite is essential because in general $h_\#\mu$ is not locally finite even if μ is. A simple example is $(\pi_1)_\#\mathscr{L}^2$, where $\pi_1\colon \mathbb{R}^2 \to \mathbb{R}$ is the projection. However, the result remains true if we explicitly require $h_\#\mu$ to be locally finite.

Now we see that s-concavity is preserved by many measure theoretic operations. For instance, we have the following result, which is proved in [Bor74] for probabilities, but can be slightly extended.

Theorem 2.24. *Let E and F be two locally convex Hausdorff vector spaces and let $h\colon E \to F$ be a continuous linear mapping. If $\mu \in \mathfrak{M}_s(E)$ is finite, then $h_\#\mu \in \mathfrak{M}_s(F)$.*

Moreover, a finite Radon measure μ on E belongs to $\mathfrak{M}_s(E)$ if and only if $\mu_{\xi_1,\dots,\xi_n} \in \mathfrak{M}_s(\mathbb{R}^n)$ for all $n \in \mathbb{N}_+$ and all $\xi_i \in E'$, where $\mu_{\xi_1,\dots,\xi_n}$ denotes the pushforward $(\xi_1,\dots,\xi_n)_\#\mu$.

Proof. For all sets $A, B \in \mathscr{B}(F)$ and $0 < \lambda < 1$ we have

$$(1-\lambda)h^{-1}(A) + \lambda h^{-1}(B) = h^{-1}\big((1-\lambda)[A \cap h(E)] + \lambda[B \cap h(E)]\big).$$

In fact, if $x \in h^{-1}(A)$ and $y \in h^{-1}(B)$, then

$$h\big((1-\lambda)x + \lambda y\big) = (1-\lambda)h(x) + \lambda h(y) \in (1-\lambda)[A \cap h(E)] + \lambda[B \cap h(E)].$$

Conversely, if z belongs to the right hand side, then

$$h(z) = (1-\lambda)h(x) + \lambda h(y)$$

with $h(x) \in A$ and $h(y) \in B$. Consequently,

$$z = (1-\lambda)x + \lambda y + w = (1-\lambda)(x+w) + \lambda(y+w)$$

where $w \in \ker h$, hence z belongs to the left hand side.

If A and B are compact, then $(1-\lambda)A + \lambda B \in \mathscr{B}(F)$, therefore thanks to Lemma 2.22 we deduce that

$$\begin{aligned} (h_{\#}\mu)_*\big((1-\lambda)A + \lambda B\big) &= \mu_*\Big(h^{-1}\big((1-\lambda)A + \lambda B\big)\Big) \geq \\ &\geq \mu_*\big((1-\lambda)h^{-1}(A) + \lambda h^{-1}(B)\big) \geq \\ &\geq M_s^\lambda\big(h_{\#}\mu(A), h_{\#}\mu(B)\big). \end{aligned}$$

The same inequality for arbitrary $A, B \in \mathscr{B}(F)$ can be recovered, as usual, by inner approximation. This proves the first part of the theorem.

The "only if" of the second statement is an immediate consequence of the first part.

Now we prove the "if" part. Fix $0 < \lambda < 1$, choose two compact sets $A, B \subset E$ and take an open set $U \supset (1-\lambda)A + \lambda B$. Since $(1-\lambda)A + \lambda B$ is compact we can find an open convex set $V \ni 0$ such that

$$(1-\lambda)A + \lambda B + 2V \subseteq U.$$

Moreover, we can find points $x_1, \ldots, x_m \in A$ and $y_1, \ldots, y_n \in B$ such that

$$A \subset \bigcup_{i=1}^m (x_i + V), \qquad B \subset \bigcup_{j=1}^n (y_j + V).$$

Consider the closed set

$$F = \bigcup_{i=1}^m \bigcup_{j=1}^m \big((1-\lambda)x_i + \lambda y_i + \overline{V}\big) \subset U.$$

For each $z \notin F$, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$, the Hahn-Banach theorem allows us to separate z from $(1-\lambda)x_i + \lambda y_i + \overline{V}$: there exists $k_{ijz} \in \mathbb{R}$ and $\xi_{ijz} \in E'$ such that

$$\overline{V} \subseteq \xi_{ijz}^{-1}\big([k_{ijz}, \infty)\big), \qquad z \notin (1-\lambda)x_i + \lambda y_i + \xi_{ijz}^{-1}\big([k_{ijz}, \infty)\big).$$

If we set

$$F_z = \bigcup_{i=1}^m \bigcup_{j=1}^n \left[(1-\lambda)x_i + \lambda y_i + \xi_{ijz}^{-1}\big([k_{ijz}, \infty)\big)\right],$$

then clearly $F = \bigcap_{z \notin F} F_z$. Furthermore, since μ is Radon and finite, given $\varepsilon > 0$ there are $z_1, \dots, z_p \notin F$ such that $\mu(F_{z_1} \cap \cdots \cap F_{z_p}) \leq \mu(F) + \varepsilon$. But we have

$$\begin{aligned}
F_{z_1} \cap \cdots \cap F_{z_p} &= \bigcap_{l=1}^{p} \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \left[(1-\lambda)x_i + \lambda y_i + \xi_{ijz_l}^{-1}\big([k_{ijz_l}, \infty)\big)\right] \supseteq \\
&\supseteq \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \left[(1-\lambda)x_i + \lambda y_i + \bigcap_{l=1}^{p} \xi_{ijz_l}^{-1}\big([k_{ijz_l}, \infty)\big)\right] = \\
&= (1-\lambda) \bigcup_{i=1}^{m} \left[x_i + \bigcap_{l=1}^{p} \xi_{ijz_l}^{-1}\big([k_{ijz_l}, \infty)\big)\right] + \\
&\qquad + \lambda \bigcup_{j=1}^{n} \left[y_j + \bigcap_{l=1}^{p} \xi_{ijz_l}^{-1}\big([k_{ijz_l}, \infty)\big)\right] = \\
&= (1-\lambda)h^{-1}(C) + \lambda h^{-1}(D)
\end{aligned}$$

for suitable $C, D \subseteq \mathbb{R}^{mnp}$ and $h \in (E')^{mnp}$, thus

$$\begin{aligned}
\mu(F_{z_1} \cap \cdots \cap F_{z_p}) &\geq \mu\big((1-\lambda)h^{-1}(C) + \lambda h^{-1}(D)\big) = \\
&= h_{\#}\mu\big((1-\lambda)C + \lambda D\big) \geq \\
&\geq M_s^{\lambda}\big(h_{\#}\mu(C), h_{\#}\mu(D)\big) \geq \\
&\geq M_s^{\lambda}\big(\mu(A), \mu(B)\big)
\end{aligned}$$

because $A \subseteq h^{-1}(C)$ and $B \subseteq h^{-1}(D)$. By the arbitrariness of ε we deduce

$$\mu(U) \geq \mu(F) \geq M_s^{\lambda}\big(\mu(A), \mu(B)\big)$$

and finally

$$\mu\big((1-\lambda)A + \lambda B\big) \geq M_s^{\lambda}\big(\mu(A), \mu(B)\big)$$

thanks to the outer regularity of the finite Radon measure μ. Once we have obtained the inequality for compact sets A and B, the usual inner approximation argument proves the s-concavity of the measure. □

Remark 2.25. As anticipated, the result can be slightly extended to cover some infinite measures too. The main difficulty is the one pointed out in Remark 2.23.

The first part of the theorem can be stated like this: if $\mu \in \mathfrak{M}_s(E)$ and $h_{\#}\mu$ is locally finite, then $h_{\#}\mu \in \mathfrak{M}_s(F)$. The proof remains unaltered.

The second part becomes: if $\mu \in \mathcal{M}_+(E)$ is such that $\mu_{\xi_1,\dots,\xi_n}$ is locally finite for all $\xi_i \in E'$, then $\mu \in \mathfrak{M}_s(E)$ if and only if $\mu_{\xi_1,\dots,\xi_n} \in \mathfrak{M}_s(\mathbb{R}^n)$ for all $\xi_i \in E'$. That is, under the assumption of the local finiteness of all the measures involved, the s-concavity is characterized by the finite projections.

Let's see why this is true. Let A and B be compact sets in E. The set $K = \operatorname{conv}(A \cup B)$ is compact. Indeed it is the image of the compact set

$(A \cup B) \times (A \cup B) \times [0,1]$ under the continuous map $(x,y,t) \mapsto (1-t)x + ty$. Fix a linear functional $\xi_0 \in E'$. The set $H = \xi_0(K)$ is convex and compact, so it has finite μ_{ξ_0} measure. This implies that $\nu = \mu \llcorner C$, where $C = \xi_0^{-1}(H)$, is finite. To conclude, we just need to verify that $\nu_{\xi_1,\ldots,\xi_n} \in \mathfrak{M}_s(\mathbb{R}^n)$ for all $\xi_i \in E'$, because in this case ν is s-concave thanks to the previous theorem, hence

$$\begin{aligned} \mu\big((1-\lambda)A + \lambda B\big) &= \nu\big((1-\lambda)A + \lambda B\big) \geq \\ &\geq M_s^\lambda\big(\nu(A), \nu(B)\big) = M_s^\lambda\big(\mu(A), \mu(B)\big), \end{aligned}$$

since A, B and $(1-\lambda)A + \lambda B$ are all contained inside C. Fix $\xi_1, \ldots, \xi_n \in E'$ and take $A \subseteq \mathbb{R}^n$. We have

$$\begin{aligned} \nu_{\xi_1,\ldots,\xi_n}(A) &= \big(\mu \llcorner \xi_0^{-1}(H)\big)\big((\xi_1,\ldots,\xi_n)^{-1}(A)\big) = \\ &= \mu\big((\xi_1,\ldots,\xi_n)^{-1}(A) \cap \xi_0^{-1}(H)\big) = \\ &= \mu\big((\xi_0,\xi_1,\ldots,\xi_n)^{-1}(H \times A)\big) = \\ &= \mu_{\xi_0,\ldots,\xi_n}(H \times A) = \\ &= \mu_{\xi_0,\ldots,\xi_n}(\mathbb{R} \times A \cap H \times \mathbb{R}^n) = \\ &= \big(\mu_{\xi_0,\ldots,\xi_n} \llcorner (H \times \mathbb{R}^n)\big)(\mathbb{R} \times A), \end{aligned}$$

but the measure $\sigma = \mu_{\xi_0,\ldots,\xi_n} \llcorner (H \times \mathbb{R}^n)$ is s-concave, so

$$\begin{aligned} \nu_{\xi_1,\ldots,\xi_n}\big((1-\lambda)A + \lambda B\big) &= \sigma\big(\mathbb{R} \times [(1-\lambda)A + \lambda B]\big) = \\ &= \sigma\big((1-\lambda)(\mathbb{R} \times A) + \lambda(\mathbb{R} \times B)\big) \geq \\ &\geq M_s^\lambda\big(\sigma(\mathbb{R} \times A), \sigma(\mathbb{R} \times B)\big) = \\ &= M_s^\lambda\big(\nu_{\xi_1,\ldots,\xi_n}(A), \nu_{\xi_1,\ldots,\xi_n}(B)\big) \end{aligned}$$

and $\nu_{\xi_1,\ldots,\xi_n} \in \mathfrak{M}_s(\mathbb{R}^n)$.

The following proposition, in conjunction with Theorem 2.11, provides a complete characterization of s-concave measures on a finite dimensional vector space. The proof is taken from [AGS08, Theorem 9.4.10].

Proposition 2.26. *If $\mu \in \mathfrak{M}_s(\mathbb{R}^n)$ for some $s \in [-\infty, \infty]$, $H = \operatorname{aff}\big((\operatorname{supp}(\mu)\big)$ is the least affine subspace containing $\operatorname{supp}(\mu)$ and $d = \dim H$, then μ is absolutely continuous with respect to $\mathscr{H}^d \llcorner H$.*

Proof. Observe that $\mu \in \mathfrak{M}_s(H)$, therefore we can reduce to the case $H = \mathbb{R}^n$, $d = n$. Consider the set

$$D = \left\{ x \in \mathbb{R}^n \;\middle|\; \liminf_{r \to 0} \frac{\mu\big(B(x,r)\big)}{\omega_n r^n} > 0 \right\}.$$

Because of the concavity of μ the set D is convex and, clearly, $D \subseteq \operatorname{supp}(\mu)$. General results on the derivation of measures (see for example [AFP00, Theorem

2.56]) provide

$$\limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\omega_n r^n} < \infty \qquad\qquad \text{for } \mathscr{L}^n\text{-a.e. } x \in \mathbb{R}^n,$$
$$\liminf_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\omega_n r^n} > 0 \qquad\qquad \text{for } \mu\text{-a.e. } x \in \mathbb{R}^n,$$

therefore $\mu(D^c) = 0$, that is μ is concentrated on D, hence $\operatorname{supp}(\mu) \subseteq \overline{D}$. From this it also follows that $\operatorname{aff}(D) = \mathbb{R}^n$, hence in particular $\operatorname{Int}(D) \neq \emptyset$.

If there is a point $\bar{x} \in \mathbb{R}^n$ such that

$$\limsup_{r\to 0} \frac{\mu\big(B(\bar{x},r)\big)}{\omega_n r^n} = \infty,$$

then the concavity of μ and the definition of D imply that every point of $\operatorname{Int}(D)$ has the same property, but this is impossible because $\mu\big(\operatorname{Int}(D)\big) > 0$. Therefore

$$\limsup_{r\to 0} \frac{\mu\big(B(x,r)\big)}{\omega_n r^n} < \infty \qquad \text{for every } x \in \mathbb{R}^n$$

and the same theorem [AFP00, Theorem 2.56] proves that $\mu \ll \mathscr{L}^n$. ☐

Here is the statement of the characterization, based on the one given in [Bor75].

Theorem 2.27. *Let $\mu \in \mathcal{M}_+(\mathbb{R}^n)$, let $H = \operatorname{aff}\big(\operatorname{supp}(\mu)\big)$ be the least affine subspace containing $\operatorname{supp}(\mu)$ and let $d = \dim(H)$. Then $\mu \in \mathfrak{M}_s(\mathbb{R}^n)$ if and only if $\mu = f\mathscr{H}^d \llcorner H$ and*

- *f is p-concave with $p = s/(1-ds)$, if $s \in [-\infty, 1/d)$;*
- *$f = 0$, if $s \in [1/d, \infty]$.*

Proof. The "if" part is a direct consequence of Theorem 2.11 and Theorem 2.24.

The "only if" part follows from Proposition 2.26, which proves the absolute continuity, and Theorem 2.11 again. ☐

Remark 2.28. Note that a non-negative function f is p-concave if and only if the set $D = \{f > 0\}$ is convex and

- f^p is convex in D, if $p < 0$;
- $\log f$ is concave in D, if $p = 0$;
- f^p is concave in D, if $p > 0$.

Theorem 2.24 has an important consequence regarding the convolution of two measures.

Theorem 2.29. *If $\mu \in \mathfrak{P}_0(E)$ and $\nu \in \mathfrak{P}_0(F)$, then $\mu \otimes \nu \in \mathfrak{P}_0(E \times F)$. In the case $E = F$, we have also $\mu * \nu \in \mathfrak{P}_0(E)$.*

Proof. Thanks to Theorem 2.24, it is possible to consider only the finite dimensional case. Then $\mu = f\mathscr{L}^m$ and $\nu = g\mathscr{L}^n$ with f and g log-concave. If $0 < \lambda < 1$, $z_1 = (1-\lambda)x_1 + \lambda y_1 \in \mathbb{R}^m$ and $z_2 = (1-\lambda)x_2 + \lambda y_2 \in \mathbb{R}^n$, we have

$$f(z_1)g(z_2) \geq f(x_1)^{1-\lambda} f(y_1) g(x_2)^{1-\lambda} g(y_2)^{\lambda},$$

therefore the measure $\mu \otimes \nu$ is log-concave because its density $f \times g$ is log-concave. Alternatively, one can verify the log-concavity on rectangular sets and from this deduce the log-concavity on $\mathscr{B}(E \times F)$ with an approximation argument; see for instance [Bor75, Corollary 3.1].

For the second part, it is sufficient to observe that $\mu * \nu = h_{\#}(\mu \otimes \nu)$, where $h(x, y) = x + y$, therefore $\mu * \nu$ is log-concave too. $\square$

Corollary 2.30. *If $\mu \in \mathfrak{P}_0(E)$ and $f, g \colon E \to [0, \infty]$ are log-concave, then $f *_\mu g$ is log-concave.*

Next, we study the concavity property of a common construction used in measure theory called *disintegration*, or *conditional probabilities* by probabilists. But first we recall an existence and uniqueness theorem about the disintegration of a measure. For the proofs, the reader may refer to [Bog07, Section 10.6]; see also [Bog10, Section 1.3].

Theorem 2.31. *Let X be an Hausdorff space and Y a complete separable metric space (also know as a Polish space in the literature), let $\mu \in \mathcal{P}(X \times Y)$, let $\pi \colon X \times Y \to X$ be the projection and $\sigma = \pi_{\#}\mu \in \mathcal{P}(X)$. Then there exists a family of probabilities $(\mu_x)_{x \in X} \subset \mathcal{P}(Y)$ such that for every set $A \in \mathscr{B}(X \times Y)$ we have*

$$\mu(A) = \int_X \mu_x(A_x)\, d\sigma(x),$$

where $A_x = \{\, y \in Y \mid (x, y) \in A \,\}$. The measures μ_x are called conditional probabilities and are determined σ-a.e.

If the space X is Polish too, the conditional measures μ_x can be chosen in such a way that the function $x \mapsto \mu_x(A_x)$ is Borel measurable for every $A \in \mathscr{B}(X \times Y)$.

Before we can present the concavity result, we need an inequality which generalizes Hölder's one.

Lemma 2.32. *Let $\mu \in \mathcal{M}_+(E)$, let $f, g \in L^1(\mu)$ be non-negative functions, let $0 < \lambda < 1$ and $s \in [-\infty, 1]$. Then*

$$\left\| M_s^\lambda(f, g) \right\|_1 \leq M_s^\lambda(\|f\|_1, \|g\|_1).$$

For $s \in [1, \infty]$ we have the reverse inequality. Note that $s = 0$ corresponds to the Hölder's inequality.

Proof. Let's consider the case $s \in [-\infty, 1]$. We may assume $\|f\|_1, \|g\|_1 > 0$. Observe that for $s = 1$ equality holds. In fact, the linearity of the integral yields

$$\int_E [(1-\lambda)f + \lambda g] d\mu = (1-\lambda)\int_E f\, d\mu + \lambda \int_E g\, d\mu.$$

Then, with $\theta = \lambda \|g\|_1 / M_s^\lambda(\|f\|_1, \|g\|_1)$ if $s \neq 0$, or $\theta = \lambda$ if $s = 0$, we have

$$\begin{aligned}
\left\|M_s^\lambda(f,g)\right\|_1 &= \left\|M_s^\lambda(\|f\|_1, \|g\|_1) M_s^\theta\left(\frac{f}{\|f\|_1}, \frac{g}{\|g\|_1}\right)\right\|_1 = \\
&= M_s^\lambda(\|f\|_1, \|g\|_1)\left\|M_s^\theta\left(\frac{f}{\|f\|_1}, \frac{g}{\|g\|_1}\right)\right\|_1 \leq \\
&\leq M_s^\lambda(\|f\|_1, \|g\|_1)\left\|M_1^\theta\left(\frac{f}{\|f\|_1}, \frac{g}{\|g\|_1}\right)\right\|_1 = \\
&= M_s^\lambda(\|f\|_1, \|g\|_1) M_1^\theta\left(\frac{\|f\|_1}{\|f\|_1}, \frac{\|g\|_1}{\|g\|_1}\right) = \\
&= M_s^\lambda(\|f\|_1, \|g\|_1).
\end{aligned}$$ □

The case $s \in [1, \infty]$ is identical, with just the inequality reversed.

Corollary 2.33. *Let $\mu \in \mathcal{M}_+(E)$, $0 < p < \infty$, let $f, g \in L^p(\mu)$ be non-negative functions, let $0 < \lambda < 1$ and $s \in [-\infty, p]$. Then*

$$\left\|M_s^\lambda(f,g)\right\|_p \leq M_s^\lambda(\|f\|_p, \|g\|_p).$$

For $s \in [p, \infty]$ we have the reverse inequality.

Proof. Consider $s \in [-\infty, p]$. Then $s/p \leq 1$, therefore

$$\begin{aligned}
\left\|M_s^\lambda(f,g)\right\|_p &= \left\|M_{s/p}^\lambda(f^p, g^p)\right\|_1^{1/p} \leq \\
&\leq M_{s/p}^\lambda(\|f^p\|_1, \|g^p\|_1)^{1/p} = M_s^\lambda(\|f\|_p, \|g\|_p).
\end{aligned}$$ □

It is interesting to observe that this last inequality, involving a mean and a norm, can be seen as a particular case of a more general one regarding composite Lebesgue spaces L^pL^q. If μ is a measure on X and $f\colon X \to [-\infty, \infty]$ is measurable, we use the notation $\|f\|_{L^p_\mu(X)}$ to denote

$$\|f\|_{L^p_\mu(X)} = \left(\int_X |f|^p\, d\mu\right)^{1/p} \qquad \text{for any } p \in \mathbb{R} \setminus \{0\}.$$

If μ is a probability, we can extend

$$\|f\|_{L^0_\mu(X)} = \exp\left(\int_X \log|f|\, d\mu\right)$$

and $\|f\|_{L^p(\mu)}$ is an increasing continuous function of p. Be aware that it is not a norm for $p < 1$.

Given two measure spaces $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$, the composite Lebesgue space $L^p_\mu L^q_\nu = L^p_\mu(X; L^q_\nu(Y))$ consists of the functions $f\colon X \times Y \to [-\infty, \infty]$ for which the quantity

$$\Big\| \|f(x,y)\|_{L^q_\nu(y)} \Big\|_{L^p_\mu(x)} = \left(\int_X \left(\int_Y |f(x,y)|^q \, d\nu(y) \right)^{p/q} d\mu(x) \right)^{1/p}$$

is finite. If $p, q \geq 1$, $L^p_\mu L^q_\nu$ is a normed space.

Proposition 2.34. *Let $(X, \mathscr{X}, \mu)$ and $(Y, \mathscr{Y}, \nu)$ be two measure spaces, let $f\colon X \times Y \to [-\infty, \infty]$ be measurable and let $p \leq q$. Then*

$$\Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^q_\nu(y)} \leq \Big\| \|f(x,y)\|_{L^q_\nu(y)} \Big\|_{L^p_\mu(x)} .$$

In particular, $L^p_\mu L^q_\nu \subseteq L^q_\nu L^p_\mu$.

Proof. With the same trick we used in the proof of Corollary 2.33, we can reduce ourselves to the case $q = 1$. For simplicity, assume also $p \neq 0$. If the right hand side is infinite, there is nothing to prove. If not, we have

$$\|f(x,y)\|_{L^p_\mu(x)} = \left[\int_X \left(\frac{|f(x,y)|}{\|f(x,y)\|_{L^1_\nu(y)}} \right)^p \|f(x,y)\|^p_{L^1_\nu(y)} \, d\mu(x) \right]^{1/p} =$$

$$= \Big\| \|f(x,y)\|_{L^1_\nu(y)} \Big\|_{L^p_\mu(x)} \left[\int_X \left(\frac{|f(x,y)|}{\|f(x,y)\|_{L^1_\nu(y)}} \right)^p \frac{\|f(x,y)\|^p_{L^1_\nu(y)} \, d\mu(x)}{\Big\| \|f(x,y)\|_{L^1_\nu(y)} \Big\|^p_{L^p_\mu(x)}} \right]^{1/p} =$$

$$= \Big\| \|f(x,y)\|_{L^1_\nu(y)} \Big\|_{L^p_\mu(x)} \left\| \frac{f(x,y)}{\|f(x,y)\|_{L^1_\nu(y)}} \right\|_{L^p_\theta(x)} \leq$$

$$\leq \Big\| \|f(x,y)\|_{L^1_\nu(y)} \Big\|_{L^p_\mu(x)} \left\| \frac{f(x,y)}{\|f(x,y)\|_{L^1_\nu(y)}} \right\|_{L^1_\theta(x)} ,$$

where θ is the probability on X given by

$$d\theta(x) = \frac{\|f(x,y)\|^p_{L^1_\nu(y)} \, d\mu(x)}{\Big\| \|f(x,y)\|_{L^1_\nu(y)} \Big\|^p_{L^p_\mu(x)}} .$$

Therefore

$$
\begin{aligned}
\Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^1_\nu(y)} &\leq \Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^1_\nu(y)} \left\| \left\| \frac{f(x,y)}{\|f(x,y)\|_{L^1_\nu(y)}} \right\|_{L^1_\theta(x)} \right\|_{L^1_\nu(y)} = \\
&= \Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^1_\nu(y)} \left\| \left\| \frac{f(x,y)}{\|f(x,y)\|_{L^1_\nu(y)}} \right\|_{L^1_\nu(y)} \right\|_{L^1_\theta(x)} = \\
&= \Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^1_\nu(y)} \|1\|_{L^1_\theta(x)} = \\
&= \Big\| \|f(x,y)\|_{L^p_\mu(x)} \Big\|_{L^1_\nu(y)} .
\end{aligned}
$$

□

Corollary 2.33 can be recovered by applying Proposition 2.34 to the space $X = \{0,1\}$ with the probability measure $\mu = (1-\lambda)\delta_0 + \lambda\delta_1$ and the function $\varphi\colon \{0,1\} \times E \to [0,\infty]$ given by $\varphi(0,x) = f(x)$, $\varphi(1,x) = g(x)$.

Finally, here is the theorem which shows that in some cases the concavity of a measure is preserved under disintegration. For the log-concave case, see [APG12, Lemma 4.1] or [Bog10, Theorem 4.3.6].

Theorem 2.35. *Let X be a locally convex Hausdorff vector space and Y a separable Fréchet space, let $s \in [-\infty, 1]$ and $\mu \in \mathfrak{P}_s(X \times Y)$. Consider the projection $\pi\colon X \times Y \to X$, $\sigma = \pi_\# \mu \in \mathfrak{P}_s(X)$ and the family of conditional probabilities $\mu_x \in \mathcal{P}(Y)$ given by Theorem 2.31. Then $\mu_x \in \mathfrak{P}_s(Y)$ for σ-a.e. $x \in X$.*

Proof. Let $A, B \in \mathscr{B}(Y)$, $0 < \lambda < 1$, $C = (1-\lambda)A + \lambda B$ and let $K \in \mathscr{B}(X)$ be convex. Then, thanks to Lemma 2.32 (here we use $s \leq 1$), we have

$$
\begin{aligned}
\int_K \mu_x(C)\, d\sigma(x) = \mu(K \times C) \geq \\
&\geq \mu\big((1-\lambda)(K \times A) + \lambda(K \times B)\big) \geq \\
&\geq M_s^\lambda\big(\mu(K \times A), \mu(K \times B)\big) = \\
&= M_s^\lambda \left(\int_K \mu_x(A)\, d\sigma(x), \int_K \mu_x(B)\, d\sigma(x) \right) \geq \\
&\geq \int_K M_s^\lambda\big(\mu_x(A), \mu_x(B)\big)\, d\sigma(x).
\end{aligned}
$$

From the arbitrariness of K we deduce that

$$
\mu_x(C_x) \geq M_s^\lambda\big(\mu_x(A), \mu_x(B)\big) \qquad \text{for } \sigma\text{-a.e. } x \in X;
$$

Indeed the class of convex sets is a π-system.

Moreover, there exists a countable family $\mathcal{F} \subset \mathscr{B}(Y)$ such that the $\mathfrak{M}_s$ inequality for all $A, B \in \mathcal{F}$ implies the s-concavity of a measure. The countability of $\mathcal{F}$ and the result above enable us to find a set $X_0 \subset X$ of full σ-measure such that μ_x satisfies the $\mathfrak{M}_s$ inequality for all $A, B \in \mathcal{F}$ and $x \in X_0$. Therefore, for all $x \in X_0$ the measure μ_s is s-concave.

Let's see why such a family $\mathcal{F}$ exists. In $\mathbb{R}^n$, it is sufficient to verify the s-concavity on sets of the form $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with $a_i, b_i \in \mathbb{Q}$. We can then choose a dense family $(\xi_i)_{i \in \mathbb{N}} \subset Y'$ and it is sufficient to verify the s-concavity on the cylindrical sets $(\xi_1, \dots, \xi_n)^{-1}([a_1, b_1] \times \cdots \times [a_n, b_n])$ with $a_i, b_i \in \mathbb{Q}$, which are countable. □

2.5 Support of a concave measure and zero-one law

In this section we are going to examine some issues regarding the support of concave measures. Recall that the support of a (positive) measure μ, denoted by $\operatorname{supp}(\mu)$, is defined as the set of points x such that $\mu(U) > 0$ for every neighbourhood U of x. Equivalently, $\operatorname{supp}(\mu)^c$ is the set of points with a negligible neighbourhood. It is immediate to observe that $\operatorname{supp}(\mu)$ is a closed set.

We also say that a measure is concentrated on a set if its complement has measure zero. Thus, $\operatorname{supp}(\mu)$ is the (uncountable) intersection of the closed sets on which μ is concentrated. It is not true in general that a measure is concentrated on its support. However, this is correct if the space is second-countable or the measure is Radon. The former is a simple exercise, but we are more interested in the latter.

Let μ be a Radon measure, consider the open set $A = \operatorname{supp}(\mu)^c$ and take a compact subset $K \Subset A$. By definition, for every $x \in A$ there exists an open set $U_x \ni x$ with $\mu(U_x) = 0$. Because K is compact, we can find points $x_1, \dots, x_m \in K$ such that $K \subset U_{x_1} \cup \cdots \cup U_{x_m}$. Therefore

$$\mu(K) \leq \mu(U_{x_1}) + \cdots + \mu(U_{x_m}) = 0.$$

By the arbitrariness of K and the inner regularity of μ, we deduce that $\mu(A) = 0$, which proves that μ is concentrated on its support.

When dealing with infinite measures, it is often useful to be able to reduce ourselves to the finite case. For this reason, the following proposition is relevant.

Proposition 2.36. *If $s \in [0, \infty]$ and $\mu \in \mathfrak{M}_s(E)$, then μ is σ-finite.*

Proof. Take $\mu \in \mathfrak{M}_s(E) \setminus \{0\}$. Then $\operatorname{supp}(\mu) \neq \emptyset$ and we may assume that $0 \in \operatorname{supp}(\mu)$. There exists an open convex set $U \ni 0$ such that $0 < \mu(U) < \infty$.

For every integer $n \geq 2$ we have

$$0 < \mu\left(\frac{1}{2}U\right) \leq \mu\left(\frac{n}{2(n-1)}U\right) \leq \mu(U) < \infty,$$
$$0 < \mu(U) \leq \mu\left(\frac{n}{2}U\right),$$

and the inclusion

$$U \supseteq \left(1 - \frac{1}{n}\right)\frac{n}{2(n-1)}U + \frac{1}{n}\left(\frac{n}{2}U\right),$$

therefore

$$\mu(U) \geq M_s^{1/n}\left(\mu\left(\frac{n}{2(n-1)}U\right), \mu\left(\frac{n}{2}U\right)\right) \geq M_s^{1/n}\left(\mu\left(\frac{1}{2}U\right), \mu\left(\frac{n}{2}U\right)\right).$$

From this inequality one deduces interesting estimates on the growth of the dilates of U:

$$\mu\left(\frac{n}{2}U\right) \leq \mu(U), \qquad \text{if } s = \infty,$$
$$\mu\left(\frac{n}{2}U\right) \leq \mu(U)^n \mu\left(\frac{1}{2}U\right)^{1-n}, \qquad \text{if } s = 0,$$
$$\mu\left(\frac{n}{2}U\right) \leq \left\{\left[\mu(U)^s - \mu\left(\frac{1}{2}U\right)^s\right]n - \mu\left(\frac{1}{2}U\right)^s\right\}^{1/s}, \qquad \text{if } 0 < s < \infty.$$

Note that the last inequality is a sort of a converse to (3.2). In any case, it follows that $\mu\left(\frac{n}{2}U\right)$ must be finite, therefore μ is σ-finite.

To our knowledge, an s-concave measure with $s < 0$ could be not σ-finite. □

It is well known that the support of a Gaussian measure is a closed affine subspace. It was proved by Itō in [Itō70] for the Hilbert case and then subsequent refinements have been found; see for example [Kal71, KN72]. Borell further generalized this in [Bor74, Theorem 5.1] and the following theorem is based on his result.

Theorem 2.37. *Let E be a locally convex Hausdorff vector space and let μ be a Radon measure on E with convex support and such that*

$$\operatorname{supp}(\mu_\xi) = \mathbb{R} \quad \textit{or} \quad \operatorname{supp}(\mu_\xi) = \textit{singleton set} \qquad \textit{for all } \xi \in E'.$$

Then $\operatorname{supp}(\mu)$ *is a closed affine subspace of E. More precisely,*

$$\operatorname{supp}(\mu) = \bigcap \{\, H \subseteq E \mid H \textit{ closed affine subspace, } \mu(H^c) = 0 \,\}.$$

Proof. Let $\mathcal{F}$ denote the family of closed affine subspaces with negligible complement. We have the inclusion $\operatorname{supp}(\mu) \subseteq \bigcap \mathcal{F}$. Suppose there is a point $x_o \in \bigcap \mathcal{F} \setminus \operatorname{supp}(\mu)$. Since $\operatorname{supp}(\mu)$ is a closed convex set, we can separate it from the point x_0 with the Hahn-Banach theorem, which provides us with a linear functional $\xi \in E'$ such that

$$a := \sup_{\operatorname{supp}(\mu)} \xi < \xi(x_0).$$

By definition of a, we have $\mu_\xi\big((a, \infty)\big) = 0$ so that, because of the hypothesis, it must be $\operatorname{supp}(\mu_\xi) = \{b\}$, with $b \leq a$. Then $H = \xi^{-1}(b)$ is a closed affine subspace with

$$\mu(H^c) = \mu\big(\xi^{-1}(\mathbb{R} \setminus \{b\})\big) = \mu_\xi(\mathbb{R} \setminus \{b\}) = 0,$$

which means that $H \in \mathcal{F}$ and as a consequence $x_0 \in H$. From this we deduce that

$$b = \xi(x_0) > a \geq b,$$

a contradiction which concludes the proof. □

Remark 2.38. The convexity of $\operatorname{supp}(\mu)$ is guaranteed if $\mu \in \mathfrak{M}_s(E)$ for some $s \in [-\infty, \infty]$. Indeed, in this case μ satisfies the $\mathfrak{M}_{-\infty}$ inequality. Suppose there exist $x, y \in \operatorname{supp}(\mu)$ and $z \notin \operatorname{supp}(\mu)$ with $z = (1-\lambda)x + \lambda y$ for some $\lambda \in (0,1)$. By hypothesis, there exists an open convex set $U \ni 0$ such that $\mu(z + U) = 0$. But then we get a contradiction

$$\mu(z+U) = \mu\big((1-\lambda)(x+U) + \lambda(y+U)\big) \geq \min\{\mu(x+U), \mu(y+U)\} > 0.$$

We still have to require separately the condition about $\operatorname{supp}(\mu_\xi)$.

Next, we want to show a zero-one law for additive subgroups of E. The first result of this type is due to Kallianpur who proved in [Kal70] that an additive subgroup is trivial (has measure zero or one) with respect to any Gaussian measure. In [Bor74, Theorem 4.1] Borell gives the following extension to concave measures (he considers only probabilities, but the results holds for infinite measures too).

Theorem 2.39. *Let E be a locally convex Hausdorff vector space, G an additive subgroup of E and μ a concave Radon measure on E. Then*

$$\mu_*(G) = 0 \quad \text{or} \quad \mu^*(G^c) = 0.$$

Remark 2.40. The motivation for the appearance of inner and outer measures is that G need not be measurable. An example can be found with a Hamel basis.

Proof. Suppose $\mu_*(G) > 0$. Then we can find a compact subset $K_0 \Subset G$ such that $\mu(K_0) > 0$. Set $K = K_0 \cup (-K_0)$ and let $H < G$ be the additive subgroup generated by K:

$$H = \bigcup_{n\in\mathbb{N}} \underbrace{(K + \cdots + K)}_{n \text{ terms}}.$$

We will prove $\mu(H^c) = 0$, so that $\mu^*(G^c) \le \mu(H^c)$ will also be zero.

We can restrict ourselves to finite measures in the following way. By assumption, every point $x \in E$ has an open convex neighbourhood U_x with finite μ-measure. The measures $\mu_x = \mu \llcorner U_x$ are still concave (see right after Definition 2.9) and are finite. Suppose we have proved that $\mu_x(H^c) = 0$ for every $x \in E$ and take $C \Subset H^c$. Then there exist $x_1, \ldots, x_l \in E$ such that $C \subseteq U_{x_1} \cup \cdots \cup U_{x_l}$, therefore

$$\mu(C) \le \mu_{x_1}(C) + \cdots + \mu_{x_l}(C) \le \mu_{x_1}(H^c) + \cdots + \mu_{x_l}(H^c) = 0.$$

Since this is true for every $C \Subset H^c$, it must be $\mu(H^c) = \mu_*(H^c) = 0$.

So, let μ be finite now. Suppose by way of contradiction that $\mu(H^c) > 0$ and choose $\varepsilon > 0$ so that

$$\varepsilon < \mu(K) \quad \text{and} \quad 2\varepsilon < \mu(H^c).$$

Thanks to the inner regularity of μ, we can find a set $L \Subset H^c$ such that

$$\mu(H^c \setminus L) < \varepsilon.$$

For all $n \in \mathbb{N}_+$ we have the inclusion

$$E \setminus (H \cup L) \supseteq \frac{1}{n} E \setminus \left[H \cup \big((n-1)K + nL\big)\right] + \left(1 - \frac{1}{n}\right) K.$$

Indeed, assume we have $x = \frac{1}{n}y + \left(1 - \frac{1}{n}\right) z$ with $x \in H \cup L$ and $z \in K$. If $x \in H$, then $y \in H$ too. If $x \in L$, then $y = nx + (n-1)(-z) \in (n-1)K + nL$. Applying the $\mathfrak{M}_{-\infty}$ inequality we find

$$\begin{aligned} \varepsilon > \mu(H^c \setminus L) = \mu\big(E \setminus (H \cup L)\big) \ge \\ \ge \min\left\{\mu\left(E \setminus \left[H \cup \big((n-1)K + nL\big)\right]\right), \mu(K)\right\}. \end{aligned}$$

Because it can't be $\varepsilon > \mu(K)$, we must have

$$\varepsilon > \mu\Big(H^c \setminus \big((n-1)K + nL\big)\Big) \qquad \text{for every } n \in \mathbb{N}_+.$$

Note that for $n = 1$ we recover $\mu(H^c \setminus L) < \varepsilon$, which is true by assumption.

We want to prove that there exists $n \in \mathbb{N}_+$ such that

$$L \cap \big((n-1)K + nL\big) = \emptyset.$$

To begin with, let $I = [0, 1]$ and consider the closed set

$$F = \{ (r, k, l) \in I \times K \times L \mid rk + l = 0 \} .$$

Because $K \times L$ is compact, the projection $\pi_I(F)$ is closed. Furthermore, $1 \notin \pi_I(F)$ because $K \cap L = \emptyset$. Therefore there exists $0 < \delta \leq 1$ such that

$$[1 - \delta, 1] \times K \times L = \emptyset.$$

This is the same as saying that 0 is not an element of the set $[1 - \delta, 1]K + L$, which is compact and hence closed. Then we have found an open neighbourhood U of the origin

$$0 \in ([1 - \delta, 1]K + L)^c =: U$$

such that

$$U \subseteq \left(\left(1 - \frac{1}{n}\right) K + L\right)^c \qquad \text{definitely.}$$

Since L is compact, there is n as large as desired such that

$$L \subset nU \subset \big((n - 1)K + nL\big)^c.$$

To conclude, it is sufficient to observe that

$$\mu(H^c) \leq \mu(H^c \setminus L) + \mu\Big(H^c \setminus \big((n - 1)K + nL\big)\Big) < 2\varepsilon,$$

in contradiction with the previous choice of ε. □

Remark 2.41. The last part of the proof would have been slightly more intuitive if we knew for example that E is metrizable. Indeed, in this case, compactness implies sequential compactness, so that we can reason as follows. To prove

$$L \cap \big((n - 1)K + nL\big) = \emptyset$$

for some $n \in \mathbb{N}_+$, assume the contrary. Then there exist sequences $(k_n) \subset K$ and $(l_n), (h_n) \subset L$ such that

$$\left(1 - \frac{1}{n}\right) k_n + l_n = \frac{h_n}{n}.$$

Possibly extracting a subsequence, we may assume $k_n \to k \in K$, $l_n \to l \in L$ and $h_n \to h \in L$. In the limit we get $k + l = 0$, which is impossible because $K \cap L = \emptyset$.

This argument actually applies in more generality than it may seem. In fact, for compactness to imply sequential compactness it is sufficient that E is first-countable (so we cover the locally metrizable case too), and the Eberlein-Šmulian theorem[5] says that the same is true also for a Banach space with the weak topology.

[5] See R. Whitley, *An elementary proof of the Eberlein-Šmulian theorem*, Mathematische Annalen **172** (1967), 116-118.

Chapter 3

Moments estimates

3.1 Integrability of seminorms

We are interested in the L^p-integrability of seminorms in a locally convex Hausdorff vector space with respect to concave measures. The problem originated in the setting of a separable Banach space equipped with a Gaussian measure and remained open for a long time. After some partial solutions (for example, in the space l^q, $1 \leq q < \infty$), a strong result was found by Fernique in [Fer70]. Given a Gaussian measure μ and a μ-measurable seminorm φ which is finite μ-a.e., Fernique proved that $\exp(\varepsilon\varphi^2) \in L^1(\mu)$, for all $\varepsilon > 0$ small enough. Borell generalizes this kind of result to concave measures in [Bor75]. It is immediate to observe that, for the integrability of a seminorm, it is crucial that the measure μ be finite, so we will restrict ourselves to probability measures. We recall the following definition.

Definition 3.1. A (possibly infinite) *seminorm* on a real vector space E is a function $\varphi\colon E \to [0, \infty]$ such that

$$\begin{aligned} \varphi(tx) &= |t|\,\varphi(x) && \text{for all } x \in E,\ t \in \mathbb{R}, \\ \varphi(x+y) &\leq \varphi(x) + \varphi(y) && \text{for all } x, y \in E. \end{aligned}$$

Theorem 3.2. *Let $\mu \in \mathfrak{P}_s(E)$ and assume that φ is a μ-measurable seminorm on E which is finite μ-a.e. Then*

- *if $s = 0$, the function $\exp(\varepsilon\varphi) \in L^1(\mu)$ for any $\varepsilon > 0$ sufficiently small;*
- *if $-\infty < s < 0$, the function $\varphi \in L^p(\mu)$ for every $p \in (0, -1/s)$.*

Remark 3.3. Of course if $s = 0$ the theorem implies that $\varphi \in L^p(\mu)$ for all $p \in (0, \infty]$. Estimates about the moments of a log-concave measure will be studied later.

With regard to Fernique's theorem, the statement about the case $s = 0$ is optimal in the sense that one cannot have, in general, the integrability of

$\exp(\varepsilon\varphi^\alpha)$ for any $\alpha > 1$. Indeed, consider the log-concave probability $\mu \in \mathfrak{P}_0(\mathbb{R})$ given by

$$\mu = C \exp\left(-|x|^\beta\right) \mathscr{L}^1,$$

where $1 < \beta < \alpha$ and C is the normalization constant. Then

$$\int_{\mathbb{R}} \exp\left(\varepsilon |x|^\alpha\right) d\mu(x) = C \int_{\mathbb{R}} \exp\left(\varepsilon |x|^\alpha - |x|^\beta\right) dx$$

diverges for all $\varepsilon > 0$.

Similarly, one cannot always have $\exp(\varepsilon\varphi) \in L^1(\mu)$ for all $\varepsilon > 0$. In fact, if $\mu = \frac{1}{2}e^{-|x|}\mathscr{L}^1$ and $\varphi(x) = |x|$, it is immediate to verify that $\exp(\varepsilon\varphi) \in L^1(\mu)$ if and only if $0 < \varepsilon < 1$.

To prove the theorem we need a lemma which tells us how fast the mass is absorbed by dilating a sufficiently big set.

Lemma 3.4 (Borell's lemma). *Let $\mu \in \mathfrak{P}_s(E)$ and let $A \subset E$ be convex, μ-measurable and symmetric about the origin. Assume $\theta = \mu(A) > 1/2$. Then*

- *if $s = 0$, for every $t \geq 1$*

$$\mu(E \setminus tA) \leq \theta \left(\frac{1-\theta}{\theta}\right)^{\frac{t+1}{2}}; \tag{3.1}$$

- *if $-\infty < s < 0$, for every $t \geq 1$*

$$\mu(E \setminus tA) \leq \left\{\frac{t+1}{2}[(1-\theta)^s - \theta^s] + \theta^2\right\}^{\frac{1}{s}}. \tag{3.2}$$

Proof. For every $t \geq 1$ we have

$$E \setminus A \supseteq \frac{2}{t+1}(E \setminus tA) + \frac{t-1}{t+1}A.$$

In fact, if

$$a' = \frac{2}{t+1}x + \frac{t-1}{t+1}a''$$

with $a', a'' \in A$, then

$$x = t\left(\frac{t+1}{2t}a' + \frac{t-1}{2t}(-a'')\right) \in tA.$$

Suppose now $s = 0$. The $\mathfrak{M}_0$ inequality says that

$$\mu(E \setminus A) \geq \mu(E \setminus tA)^{\frac{2}{t+1}} \mu(A)^{\frac{t-1}{t+1}},$$

that is

$$(1-\theta)^{\frac{t+1}{2}} \geq \mu(E \setminus tA)\theta^{\frac{t-1}{2}},$$

from which (3.1) follows.

Similarly, when $-\infty < s < 0$, the $\mathfrak{M}_s$ inequality implies that

$$\mu(E \setminus A)^s \leq \frac{2}{t+1}\mu(E \setminus tA)^s + \frac{t-1}{t+1}\mu(A)^s$$

and a simple computation leads to (3.2). $\square$

Proof of Theorem 3.2. Let's first consider the case $-\infty < s < 0$. Since $\varphi < \infty$ μ-a.e., there exists $0 < m < \infty$ such that $\mu(\{\varphi < m\}) > 1/2$. Because φ is a seminorm, $\{\varphi < tm\} = t\{\varphi < m\}$ and Lemma 3.4 implies that

$$\mu(\{\varphi \geq tm\}) = O(t^{1/s}) \qquad \text{as } t \to \infty.$$

By Fubini's theorem,

$$\int_E \varphi(x)^p \, d\mu(x) = p \int_0^\infty t^{p-1} \mu(\{\varphi \geq t\}) \, dt$$

and the integrand is definitely dominated by $t^{p-1+1/s}$, which is integrable when $p - 1 + 1/s < -1$.

Let's now deal with the case $s = 0$. As before, we can find $0 < m < \infty$ such that $\theta := \mu(\{\varphi < m\}) > 1/2$. Fubini's theorem enables us to compute

$$\int_E \exp\big(\varepsilon\varphi(x)\big) \, d\mu(x) = \frac{m}{\varepsilon} \int_0^\infty \exp(\varepsilon m t) \, \mu(\{\varphi \geq tm\}) \, dt.$$

The integrand is asymptotically smaller than

$$\exp\left[\varepsilon m t + \log\left(\frac{1-\theta}{\theta}\right) t\right],$$

which is integrable if

$$\varepsilon < \frac{1}{m} \log\left(\frac{\theta}{1-\theta}\right). \qquad \square$$

3.2 Brascamp-Lieb moment inequality

Consider a function $f\colon \mathbb{R}^{m+n} \to [-\infty, \infty]$ and define $F(x) = \exp[-f(x)]$. If we write the coordinates $x = (y, z)$ with $y \in \mathbb{R}^M$ and $z \in \mathbb{R}^n$, then the matrix of second derivatives can be partitioned as

$$f_{xx} = \begin{pmatrix} f_{yy} & f_{yz} \\ f_{zy} & f_{zz} \end{pmatrix}.$$

The function F is log-concave iff f is convex, iff f_{xx} is non-negative. Define also the functions

$$G(y) = \exp[-g(y)] = \int_{\mathbb{R}^n} F(y, z)\, dz.$$

By Corollary 2.21, G is log-concave if F is so. If f and g are C^2, this means that $g_{yy} \geq 0$ if $f_{xx} \geq 0$. This will be refined in Theorem 3.5. If we have real functions $h(x) = h(y, z)$ and $k(y)$, we define the averages with densities F and G

$$\begin{aligned}
\langle h \rangle &= \int_{\mathbb{R}^{m+n}} h(x) F(x)\, dx \Bigg/ \int_{\mathbb{R}^{m+n}} F(x)\, dx, \\
\langle h \rangle_z (y) &= \int_{\mathbb{R}^n} h(y, z) F(y, z)\, dz \Bigg/ \int_{\mathbb{R}^n} F(z)\, dz, \\
\langle k \rangle_y &= \int_{\mathbb{R}^m} k(y) G(y)\, dy \Bigg/ \int_{\mathbb{R}^m} G(y)\, dy.
\end{aligned}$$

We have $\langle h \rangle = \langle \langle h \rangle_z \rangle_y$. We define also the variance and covariance

$$\begin{aligned}
\operatorname{var}(h) &= \left\langle |h - \langle h \rangle|^2 \right\rangle, \\
\operatorname{cov}(h_1, h_2) &= \big\langle (h_1 - \langle h_1 \rangle)(h_2 - \langle h_2 \rangle) \big\rangle,
\end{aligned}$$

and similarly var_y, cov_y, var_z, cov_z.

The next theorem is a refinement of Corollary 2.21 because it provides a stricter bound for the log-concavity of the section.

Theorem 3.5. *Let $F(x) = F(y, z) = \exp[-f(y, z)]$, with $f \in C^2(\mathbb{R}^{m+n})$ strictly convex. Assume also that, for all vectors $\phi \in \mathbb{R}^m$, the integrals*

$$\int_{\mathbb{R}^n} \phi^T f_{yy} \phi F\, dz, \qquad \int_{\mathbb{R}^n} (\phi^T f_y)^2 F\, dz$$

converge uniformly in y in a neighbourhood of a point y_0. Then, with the definitions above, g is twice continuously differentiable near y_0 and

$$g_{yy} \geq \left\langle f_{yy} - f_{yz} (f_{zz})^{-1} f_{zy} \right\rangle_z . \tag{3.3}$$

The proof of Theorem 3.5 is not particularly difficult, but relies on another theorem by Brascamp and Lieb ([BL76b, Theorem 4.1]) whose proof is quite longer.

Theorem 3.6. *Let $F(x) = \exp[-f(x)] \in L^1(\mathbb{R}^n; [0, \infty])$ with $f \in C^2$ strictly convex. Let $h \in C^1(\mathbb{R}^n)$ with $\operatorname{var} h < \infty$. Then*

$$\operatorname{var}(h) \leq \left\langle h_x^T (f_{xx})^{-1} h_x \right\rangle = \left\langle (\nabla h)^T (D^2 f)^{-1} \nabla h \right\rangle .$$

Corollary 3.7. *Taking* $h(x) = \phi^T x$ *for* $\phi \in \mathbb{R}^N$*, we get*

$$M \leq \left\langle (f_{xx})^{-1} \right\rangle,$$

where $M_{ij} = \operatorname{cov}(x_i, x_j)$*.*

The estimates of the previous theorems become equalities is F is a Gaussian density. Indeed, if $F(x) = \exp -x^T Ax$, then $M = (2A)^{-1}$, and if

$$F(y,z) = \Phi(y,z) := \exp\left[-(y^T, z^T)\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}\right],$$

then $G(y) = \exp(-x^T Dx)$ with $D = A - BC^{-1}B^T$. Note also that $D > 0$ if

$$\begin{pmatrix} A & B \\ B^T & C \end{pmatrix} > 0,$$

so that (3.3) is meaningful because it provides a lower bound stricter than 0. In fact, a straightforward computation shows that

$$\begin{pmatrix} I_m & -A^{-1}B \\ -C^{-1}B^T & I_n \end{pmatrix}^T \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \begin{pmatrix} I_m & -A^{-1}B \\ -C^{-1}B^T & I_n \end{pmatrix} = \\ = \begin{pmatrix} A - BC^{-1}B^T & -B + BC^{-1}B^TA^{-1}B \\ -B^T + B^TA^{-1}BC^{-1}B^T & C - B^TA^{-1}B \end{pmatrix}.$$

This last calculation enables us to prove the following corollary.

Corollary 3.8. *Let* Φ *and* D *be as above. If* F *and* G *are related by*

$$\int_{\mathbb{R}^n} \varphi(y,z)F(y,z)\,dz = G(y)\exp(-x^T Dx),$$

and F *is log-concave, then* G *is log-concave too.*

Proof. We have $\Phi(y,z) = \exp(-y^T Dy - w^T Cw)$ where $w = z + C^{-1}B^T y$. Therefore

$$G(y) = \int_{\mathbb{R}^n} \exp(-z^T Cz)F(y, z - C^{-1}B^T y)\,dy$$

is log-concave by Corollary 2.21 because so is the integrand. □

Now we arrive at the moment inequality, which can be found in [BL76b, Theorem 5.1]. Let A be a real positive definite $n \times n$ matrix. If $F\colon \mathbb{R}^n \to [0,\infty]$ is such that $\exp(-x^T Ax)F(x)$ is integrable, we define the average

$$\langle h \rangle_{A,F} = \int_{\mathbb{R}^n} h(x)\exp(-x^T Ax)F(x)\,dx \Bigg/ \int_{\mathbb{R}^n} \exp(-x^T Ax)F(x)\,dx.$$

Theorem 3.9. *Let A and F be as above and let $\phi \in \mathbb{R}^n$. If F is log-concave and $\alpha \geq 1$, then*

$$\left\langle \left| \phi^T x - \left\langle \phi^T x \right\rangle_{A,F} \right|^\alpha \right\rangle_{A,F} \leq \left\langle \left| \phi^T x \right|^\alpha \right\rangle_{A,1}.$$

Proof. We can choose coordinates such that $\phi^T x = x_1$ and, thanks to Corollary 3.8, we can reduce ourselves to the one dimensional case by integrating over the hyperplanes perpendicular to ϕ.

Now we have $F\colon \mathbb{R} \to [0,\infty]$ log-concave and we want to prove that

$$\left\langle \left| x - \langle x \rangle_{1,F} \right|^\alpha \right\rangle_{1,F} \leq \langle |x|^\alpha \rangle_{1,1}.$$

The trick is to rewrite

$$\left\langle \left| x - \langle x \rangle_{1,F} \right|^\alpha \right\rangle_{1,F} = \langle |x|^\alpha \rangle_{1,G},$$

where

$$G(x) = F(x + \langle x \rangle_{1,F}) \exp(-2x \langle x \rangle_{1,F}).$$

Note that $\langle x \rangle_{1,G} = 0$, therefore

$$\int_{\mathbb{R}} \exp(-x^2) G'(x)\, dx = 2 \int_{\mathbb{R}} x \exp(-x^2) G(x)\, dx = 0.$$

Since G is log-concave, there exists x_0, which we may assume positive, such that $G(x)$ is increasing for $x < x_0$ and decreasing for $x > x_0$; then the above equality implies

$$\int_0^\infty \exp(-x^2) G'(x)\, dx \leq 0.$$

With some computations, the thesis can be reduced to

$$\int_0^\infty \psi(z) \exp(-z^2) G'(z) \leq 0,$$

where

$$\psi(z) = \exp(z^2) \int_z^\infty \left(\int_0^z \exp(-x^2 - y^2)(x^\alpha - y^\alpha)\, dy \right) dx,$$

and the proof is complete once we show that $\psi(z)$ is increasing for $z > 0$. But this is true because

$$\varphi'(z) = \int_z^\infty \exp(-x^2)(x^\alpha - z^\alpha)\, dx +$$
$$+ z \exp(z^2) \int_z^\infty \left(\int_0^z \exp(-x^2 - y^2)[(\alpha - 1)x^{\alpha-2} + y^\alpha x^{-2}\, dy \right) dx. \qquad \square$$

Chapter 4

Differentiability of log-concave measures

In this chapter we want to reach a recent result found by Krugova. First, we will review some introductory material, following mainly the exposition of [Bog10, Chapter 1 and 3].

When dealing with differentiability, it is more natural to consider signed measures. The space of signed measures $\mathcal{M}(E)$ is a Banach space with the norm of *total variation* $\|\mu\|_{\mathrm{TV}} = |\mu|(E)$, where $|\mu|$ is the *absolute variation*.

4.1 Directional differentiability

If $h \in E$, $\tau_h \colon E \to E$ is the translation $\tau_h(x) = x+h$ and we write $\mu_h = (\tau_{-h})_\# \mu$.

Definition 4.1. A measure $\mu \in \mathcal{M}(E)$ is said to be continuous along the vector $h \in E$ if

$$\lim_{t\to 0} \|\mu_{th} - \mu\|_{\mathrm{TV}} = 0.$$

It is said to be quasiinvariant along the vector $h \in E$ if μ and μ_{th} are equivalent for all $t \in \mathbb{R}$.

Definition 4.2 (Fomin's derivative)**.** A measure $\mu \in \mathcal{M}(E)$ is said to be differentiable along the vector $h \in E$ in Fomin's sense if, for all $A \in \mathscr{B}(E)$, there exists the finite limit

$$d_h\mu(A) \coloneqq \lim_{t\to 0} \frac{\mu(A+th) - \mu(A)}{t} = \lim_{t\to 0} \frac{\mu_{th}(A) - \mu(A)}{t}.$$

Equivalently, the function $t \mapsto \mu(A+th)$ must be differentiable at every point.

In such case, the function $d_h\mu$ is the pointwise limit of the measures $A \mapsto n\big(\mu(A+n^{-1}h) - \mu(A)\big)$, therefore it is a measure itself by Nikodym's theorem. The measure $d_h\mu$ is called Fomin's derivative and is always a signed measure because $d_h\mu(E) = 0$. This is the reason to admit signed measures from the beginning.

Definition 4.3 (Skorohod's derivative)**.** A measure $\mu \in \mathcal{M}(E)$ is said to be differentiable along the vector $h \in E$ in Skorohod's sense if, for all $f \in C_b(E)$, the function

$$t \mapsto \int_E f(x - th)\, d\mu(x) = \int_E f\, d\mu_{th}.$$

is differentiable at every point.

A measure ν on E is called Skorohod's derivative (or weak derivative) of μ along h if

$$\begin{aligned}\int_E f\, d\nu &= \lim_{t\to 0} \int_E \frac{f(x-th) - f(x)}{t}\, d\mu(x) = \\ &= \lim_{t\to 0} \int_E \frac{f \circ \tau_{th} - f}{t}\, d\mu = \\ &= \lim_{t\to 0} \int_E f\, d\Big(\frac{\mu_{th} - \mu}{t}\Big)\end{aligned}$$

for every $f \in C_b(E)$.

In the paper [Kru95], Krugova proves the following result about the differentiability of log-concave measures in the finite dimensional case.

Theorem 4.4. *Let $\mu = e^{-V}\mathscr{L}^n$ be a log-concave finite measure on $\mathbb{R}^n$, with $V\colon \mathbb{R}^n \to [-\infty, \infty]$ lower semicontinuous and convex. Then the following facts hold.*

- *The measure μ is Skorohod differentiable along any vector $h \in \mathbb{R}^n$ and*

  $$d_h\mu = -\partial_h V \mu - e^{-V}(h \cdot \hat{n})\sigma,$$

  where σ is the Hausdorff measure $\mathscr{H}^{n-1}$ restricted to the boundary ∂D of the domain $D = \mathrm{Dom}(V)$ and $\hat{n}$ is the outer normal on ∂D. (Note that $-\hat{n}\sigma$ is the distributional derivative of the $\mathrm{BV}_{\mathrm{loc}}$ function χ_D).

- *If $\lim_{y\to x} V(y) = \infty$ for $\mathscr{H}^{n-1}$-a.e. point $x \in \partial D$, then the measure μ is Fomin differentiable along any vector $h \in \mathbb{R}^n$ and*

  $$d_h\mu = -\partial_h V \mu.$$

  In particular, this is true if $\mathrm{Dom}(V) = \mathbb{R}^n$.

4.2 Gaussian measures

Here we recall some facts regarding Gaussian measures and their differentiable properties. A measure $\mu \in \mathcal{P}(E)$ is Gaussian if $f_{\#}\mu$ is a Gaussian measure on $\mathbb{R}$ for all $f \in E'$. Every Gaussian measure μ has a *barycentre* $m \in E$ characterized by

$$f(m) = \int_E f(x)\, d\mu(x) \qquad \text{for all } f \in E'.$$

μ is centred if $m = 0$, which is equivalent to saying that $f_{\#}\mu$ has zero mean for all $f \in E'$. Since every Gaussian measure is the translation of a centred one, it suffices to consider this class only.

For a centred Gaussian measure μ, X'_μ denotes the closure of X' in $L^2(\mu)$ and its element are called μ-measurable linear functionals. There exists an injective linear operator $R_\mu \colon X'_\mu \to X$ called *covariance operator* such that

$$f(R_\mu g) = \int_E f(x)g(x)\, d\mu(x) \qquad \text{for all } f \in X' \text{ and } g \in X'_\mu.$$

We use the notation $g = \hat{h}$ if $h = R_\mu g$ and $\hat{h}$ is called the μ-measurable linear functional generated by $h \in E$.

Definition 4.5. If μ is a centred Gaussian measure on E, the *Cameron-Martin space* of μ is the linear space $H(\mu) = R_\mu(X'_\mu) \subseteq E$. It is a separable Hilbert space with the norm

$$\langle h, k \rangle_{H(\mu)} = \int_E \hat{h}(x)\hat{k}(x)\, d\mu(x).$$

Indeed, $R_\mu \colon X'_\mu \to H(\mu)$ is an isometry whose inverse is $h \mapsto \hat{h}$. The corresponding norm is

$$|h|_{H(\mu)} = \|\hat{h}\|_{L^2(\mu)}.$$

The Cameron-Martin space plays a central role in the differentiability properties of a Gaussian measure.

Theorem 4.6. *Let $\mu \in \mathcal{P}(E)$ be a centred Gaussian. The Cameron-Martin space $H(\mu)$ coincides with the directions of quasiinvariance, continuity and differentiability. More precisely, for every $h \in H(\mu)$ we have that μ and μ_h are equivalent with density*

$$\frac{d\mu_h}{d\mu} = \exp\left(-\hat{h} - |h|^2_{H(\mu)}/2\right)$$

and μ admits the Fomin's derivative $d_h\mu = -\hat{h}\mu$; whereas for every $h \notin H(\mu)$ we have $\mu \perp \mu_h$.

The dichotomous character of the theorem is a consequence of a theorem by Hajek and Feldman.

Theorem 4.7. *Let $\mu, \nu \in \mathcal{P}(E)$ be Gaussian measures. Then either $\mu \sim \nu$ or $\mu \perp \nu$.*

4.3 Dichotomy for log-concave measures

A partial generalization of Theorem 4.6 is given by Krugova in [Kru97]. She proves that a log-concave measure is either differentiable in a direction or its corresponding shifts are always singular. To the present day, however, it is still not know whether any log-concave measure has at least a vector of continuity.

Theorem 4.8 (Krugova's dichotomy)**.** *Let μ be a finite log-concave measure on E and $h \in E$. Then one of two alternatives occurs:*

- *either μ is Skorohod differentiable along h and*

$$\|\mu_h - \mu\|_{\mathrm{TV}} \geq 2 - \exp\bigl(- \|d_h\mu\|_{\mathrm{TV}}/2\bigr),$$

- *or μ is mutually singular with all its shifts μ_{th} for $t \neq 0$.*

Proof. We proceed in several steps. First of all, we reduce to the finite dimensional case. Assume that the statement holds in this case. The key fact is that

$$\|\mu\|_{\mathrm{TV}} = \sup\left\{ \|\pi_\# \mu\|_{\mathrm{TV}} \;\middle|\; \pi \colon E \to \mathbb{R}^n \text{ projection} \right\}.$$

If μ is not differentiable along h, then the supremum of $\left\|d_{\pi(h)}(\pi_\#\mu)\right\|_{\mathrm{TV}}$ over the finite dimensional projections $\pi \colon E \to \mathbb{R}^n$ is infinite, therefore by the inequality we get that the supremum of $\left\|(\pi_\#\mu)_{\pi(h)} - \pi_\#\mu\right\|_{\mathrm{TV}} = 2$. This means that also $\|\mu_h - \mu\|_{\mathrm{TV}} = 2$, i.e. $\mu_h \perp \mu$.

If μ is differentiable along h, the key fact enables us to recover the inequality for the infinite dimensional case from the same dimension independent inequality applied to the finite projections.

Now we are in $\mathbb{R}^n$. By an approximation argument consisting in the convolution with Gaussian kernels, we can assume that μ has smooth positive density.

Let's start with $n = 1$. We have $\mu = e^{-V}\mathscr{L}^1$ with V smooth and strictly convex (this is another consequence of the convolution with Gaussian kernels). We already know that $d_1\mu = -V'e^{-V}\mathscr{L}^1$. □

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