Real Analysis

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Real Analysis II

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Midterm 40%

Final 40%

Homework 20%

~~presentation 20%~~

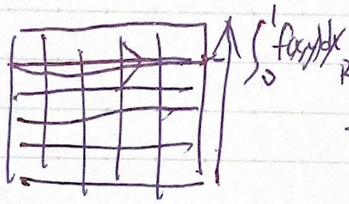
presentation 2% bonus

Fubini's theorem

Recall

$f : [0,1] \times [0,1]$

$R\int_{[0,1]\times[0,1]} f(x,y)\,dxdy$: limit of 2-d Riemann sum



$$\underline{R\int_0^1 \left(R\int_0^1 f(x,y)\,dx \right) dy}$$

Iterated integrable

Recall Thm suppose ~~$f : I_1 \times I_2 \to \mathbb{R}$ are~~

$f : [0,1] \times [0,1] \to \mathbb{R}$ is continuous

Then

$$R\int_{[0,1]\times[0,1]} f(x,y)\,dxdy = \int_0^1 \left[\int_0^1 f(x,y)dx \right] dy$$

Let $I_1 = \{ (x) = (x_1 \cdots x_n) : a_i \le x_i \le b_i \quad i = 1 \cdots n \}$ or $\mathbb{R}^n$

$I_2 = \{ y = (y_1 \cdots y_m) : c_j \le y_j \le d_j \quad j = 1 \cdots m \}$ or $\mathbb{R}^m$

$I = I_1 \times I_2 = \{ (x_1 \cdots x_n, y_1 \cdots y_m : a_i \le x_i \le b_i \quad i = 1 \cdots n, \; c_j \le y_j \le d_j \quad j = 1 \cdots m \}$

Cartition product

or $\mathbb{R}^n \times \mathbb{R}^m$ or $\mathbb{R}^n \times I_2$

or $I_1 \times \mathbb{R}^m$

Theorem (Fubini's theorem)

Let $f \in L(I)$. $I = I_1 \times I_2$. Then

(i) for almost every $x \in I_1$, $f(x,y)$ is measurable and integrable on I_2 as a function of y;

(ii) ~~As a function $\int_{I_2} f(x,y)dy$ is a measurable function~~

As a function of x, $\int_{I_2} f(x,y)dy$ is measurable and integrable on I_1 and

$$\int_I f(x,y)\,dxdy = \int_{I_1}\left[\int_{I_2} f(x,y)\,dy\right]dx$$

(i) $f : I_1 \times I_2 \to \overline{\mathbb{R}}$

Let $\tilde{f} : \mathbb{R}^n \times \mathbb{R}^m \to \overline{\mathbb{R}}$

s.t. $\tilde{f} = \begin{cases} f & \text{if } x \in I_1 \times I_2 \\ 0 & \text{if } x \notin I_1 \times I_2 \end{cases}$

$\tilde{f}$ is ~~the~~ a zero extension of f.

Then $\int_{\mathbb{R}^n \times \mathbb{R}^m} \tilde{f}\,dxdy = \int_{I_1 \times I_2} f\,dxdy$

$\int_{\mathbb{R}^m} \tilde{f}(x,y)\,dy = \int_{I_2} f(x,y)\,dy$ $\quad \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^m} \tilde{f}(x,y)\,dy\right)dx$

$= \int_{I_1}\left(\int_{I_2} f(x,y)\,dy\right)dx$

Thus, it is sufficient to prove the case $I_1 = \mathbb{R}^n$ $I_2 = \mathbb{R}^m$.

$\mathcal{F}$ $\mathcal{F}$ σ $\mathcal{F}$

~~We~~ say f ~~satisfie~~ has

σ $\mathcal{F}$

σ $\mathcal{F}$

Def. we say a function in $L(dx\,dy)$ for which Fubini's theorem is true has property $\mathcal{F}$

Lemma. A finite linear combination of functions with property $\mathcal{F}$ has property $\mathcal{F}$.

proof Suppose $f_1 \cdots f_k$ have property $\mathcal{F}$

$a_1 \cdots a_k$ are real numbers

Then Let $f = \sum_{i=1}^{k} a_i f_k$

$$\int_{I_1 \times I_2} f = \int_{I_1 \times I_2} \sum_{i=1}^{k} a_i f_i = \sum_{i=1}^{k} a_i \int_{I_1 \times I_2} f_i$$

Since $f_1 \cdots f_k$ has property $\mathcal{F}$

(i) For almost all $x \in I_1$

$f_1(x,y) \cdots f_k(x,y)$ are measurable and integrable.

(ii) As functions of x

$\int_{I_2} f_1(x,y)\,dy$ ~~is measurabe~~

$\int_{I_2} f_2(x,y)\,dy \cdots \int_{I_k} f_k(x,y)\,dy$.

are measurable and integrable.

Then and $\int_{I_1 \times I_2} f_1(x,y)\,dxdy = \int_{I_1}$

$\exists\, Z_1 \cdots Z_k \in I_1$ s.t

$|Z_i| = 0$ and

$f_k(x,y)$ is measurable and integrable

if $x \in Z_k$

Then $Z = \bigcup_{i=1}^{k} Z_i$

$$|Z| = \sum_{i=1}^{k} |Z_i| = 0$$

~~f(x,y)~~ $\sum a_i f_i(x,y)$ is measurable and integrable as a function of y if $x \notin Z$ (i holds)

Furthermore

$$\int_{I_2} f(x,y)\,dy = \int_{I_2} \sum a_i f_i(x,y)$$
$$= \sum_{i=1}^{k} a_i \int f_i(x,y)\,dy$$

is measurable and integrable as a function of x.

And

$$\int_{I_2} \int_{I_2} f(x,y)\,dy\,dx$$

$$= \int \sum_{i=1}^{k} a_i \int_{I_2} f_i(x,y)\,dy$$

$$= \sum_{i=1}^{k} a_i \int_{I_1} \int_{I_2} f_i(x,y)\,dy\,dx$$

$$= \sum_{i=1}^{k} a_i \int_{I} f_i(x,y)\,dy\,dx$$

#

Lemma Let $f_1 \cdots f_k \cdots$ have property $\mathcal{F}$. If $f_k \nearrow f$ or $f_k \searrow f$ and $f \in L(dx\,dy)$, then f has property $\mathcal{F}$.

proof If $f_k \searrow f$, then $-f_k \nearrow f$.

It suffices to prove the case $f_k \nearrow f$.

~~By the Hypers~~

$\forall i$ $\exists$ Let $Z_i \in I_1$ s.t.

$|Z_i|_n = 0$ and

$f_i(x, y)$ is a measurable and integrable function of y for $x \notin Z_i$

Let $Z = \bigcup_{k=1}^{\infty} Z_i$

For $x \notin Z$ $f_i(x,y) \geq f_1(x,y) \in L(I_2)$

$f_i(x,y)$ is measurable.

$\Rightarrow f(x,y) = \lim_{i \to \infty} f_i(x,y)$ is measurabe as a function of y

Let $h_k(x) = \int f_k(x,y)\,dy$, since $f_k(x,y) \nearrow f(x,y)$

$f_k(x,y) \geq f_1(x,y) \in L(I_2)$

By the Monotone convergence theorme.

$$h_k(x) = \int_{I_2} f_k(x,y)\,dy \nearrow \int_{I_2} f(x,y)\,dy = h(x)$$

$h_k(x)$ is measurable and integrable

on I_1 $\quad h_k(x) \geq h_1(x) \in L(I_1)$,

By the Monotone convergence theorem

$$\int_{I_1} h_k(x)\,dx \nearrow \int_{I_1} h(x)\,dx$$

On the other hand

$$f_k(x,y) \nearrow f(x,y) \quad f_k(x,y) \geq f_1(x,y) \in L(I_1 \times I_2)$$

By the M.C.T

$$\int_{I_1 \times I_2} f_k(x,y)\,dx\,dy \nearrow \int_{I_1 \times I_2} f(x,y)\,dy\,dx$$

Thus

$$\int_{I_1 \times I_2} f(x,y)\,dy\,dx = \lim_{k \to \infty} \int_{I_1 \times I_2} f_k(x,y)\,dx\,dy$$

by (iii) for f_k

$$= \lim_{k\to\infty} \int_{I_1} \left(\int_{I_2} f_k(x,y)\, \cancel{dx}\, dy \right) dx$$

$$= \lim_{k\to\infty} \int_{I_1} h_k(x)\, dx = \int_{I_1} h(x)\, dx$$

$$= \int_{I_1} \left(\int_{I_2} f(x,y)\, dy \right) dx.$$ (iii) holds for f

Since $f \in L(dx\,dy)$

$$\int_{I_1} \left(\int_{I_2} f(x,y)\, dy \right) dx = \int_{I_1 \times I_2} f(x,y)\, dy\, dx < \infty$$

~~g.Th~~, $\int_{I_2} f(x,y)\, dy < \infty$ $\forall$ a.e. $x \in I_1$ ~~a.e. in I~~

$f(x,y)$ is ~~measurable~~ integrable for ~~al~~ a.e $x \in I_1$

~~Lemma~~ If

~~straighty~~ ~~stagy~~

$\mathring{I}$ ~~open interval~~

~~$\overline{I}$~~ ~~bdd interval~~

χ_E has property ~~$\mathcal{F}$~~ ~~$\mathcal{F}$~~

~~χ_E~~ ① E is a bdd open interval

② E is a ~~partially~~ partly open interval

③ E is an open set with finite measure

④ E is an ~~set~~ of G_δ type

⑤ E is measurable

$\Rightarrow$ ⑥ ~~Fub~~ ~~meas~~ Fubini's theorem

Case 1. E is a bdd open interval in $\mathbb{R}^n \times \mathbb{R}^m$

$E = \mathring{I} = \mathring{I}_1 \times \mathring{I}_2$ where $\mathring{I}_1$ is a bdd open interval in $\mathbb{R}^n$ and $\mathring{I}_2$ is a bdd open interval in $\mathbb{R}^m$

~~χ_E~~ $\chi_E(x,y) = \begin{cases} \chi_{\mathring{I}_2} & \text{if } x \in \mathring{I}_1 \\ 0 & \text{if } x \notin \mathring{I}_1 \end{cases}$

$\chi_E(x,y)$ is measurable and integrable as a function of y for all $x \in \mathbb{R}^n$ (i) holds

If $x \in \mathring{I}_1$

$$\int \chi_E(x,y)\,dy = |I_2|_m$$

If $x \notin \mathring{I}_1$ $\int \chi_E(x,y)\,dy = 0$

$\int \chi_E(x,y)\,dy$ is measurable and integrable ~~aa~~ as a function of x

and
$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^m} \chi_E(x,y)\,dy\,dx = \int_{\mathring{I}_1} |I_2|_m\,dx + \int_{\mathbb{R}^n \setminus \mathring{I}_1} 0\,dx$$

$$= |\mathring{I}_1|_n \cdot |\mathring{I}_2|_m + 0 = |\mathring{I}_1|_n |\mathring{I}_2|_m = |\mathring{I}_1 \times \mathring{I}_2|_{n+m}$$

$$= \int_{\mathbb{R}^{n+m}} \chi_E(x,y)\,dx\,dy$$ ((ii) holds)

Case 2a If $E \in \partial I$

Then. $\chi_E(x,y) = \begin{cases} 0 & \text{if } x \notin E \subseteq \partial I \end{cases}$

~~$\chi_E(x,y)$~~ If $x \in$