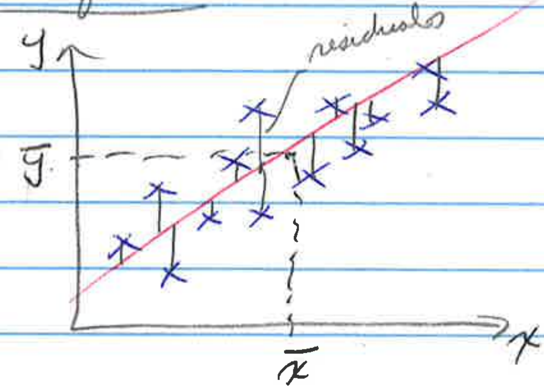
Least Squares



$y_{n\times 1}$ + $x_{n\times 1}$

$Z := \|y - \beta_1 x - \beta_0 J_n\|^2$

Find $\min_{\beta} Z$

$\Rightarrow$ Fix β_1 $\Rightarrow$ treat $y - \beta_1 x$ as a single vector

$\Rightarrow$ $\beta_0(\beta_1) = \frac{1}{n}(y - \beta_1 x)^T J_n = \bar{y} - \beta_1 \bar{x}$

$\Rightarrow$ $Z = \|y - \bar{y} J_n - \beta_1(x - \bar{x} J_n)\|^2$

$= \|\tilde{y} - \beta \tilde{x}\|^2$

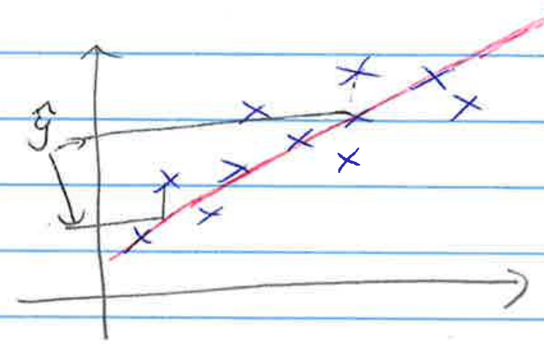
$\Rightarrow$ $\hat{\beta}_1 = \frac{\langle \tilde{y}, \tilde{x} \rangle}{\langle \tilde{x}, \tilde{x} \rangle} = \hat{\rho}_{xy} \frac{\hat{\sigma}_y}{\hat{\sigma}_x}$

$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$ Thus a linear fit line always passes through $(\bar{x}, \bar{y})$

Prediction

$Z := \|y - \beta_0 J - \beta_1 x\|^2$

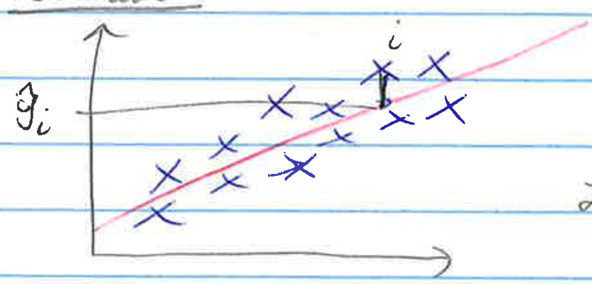
$\Rightarrow$ $\hat{y}_{pred} = \hat{\beta}_0 + \hat{\beta}_1 x_{new}$



$\hat{\beta}_0 J_n + \hat{\beta}_1 x = \hat{y}$

Least squares minimizes $\|y - \hat{y}\|^2$

Residuals

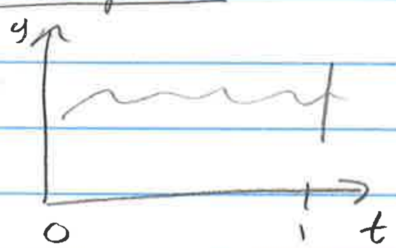


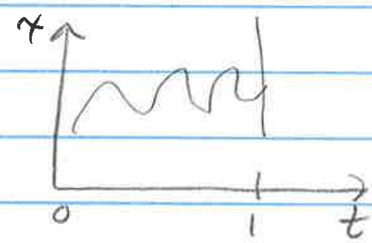
$e := y - \hat{y} = y - \hat{\beta}_0 J_n - \hat{\beta}_1 x$ (residuals)

Least squares $\Rightarrow \min \|e\|^2$

plot residuals vs. x with horizontal line at $e=0$

Generalizations





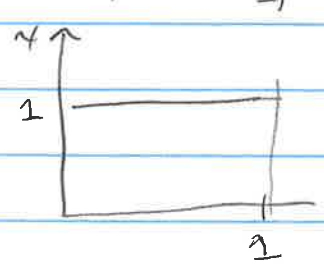
$\mathcal{L}^2[0,1]$

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt$$

$z := \|y - \beta x\|^2$, show $\hat{\beta} = \dfrac{\langle y, x\rangle}{\langle x, x\rangle}$

$$\begin{aligned} z &= \|y - \hat{\beta}x + \hat{\beta}x - \beta x\|^2 \\ &= \|y - \hat{\beta}x\|^2 + \underbrace{2\langle y - \hat{\beta}x, \hat{\beta}x - \beta x\rangle}_{0} + \underbrace{\|\hat{\beta}x - \beta x\|^2}_{\text{positive}} \\ &\geq \|y - \hat{\beta}x\|^2 \end{aligned}$$

Let $x = 1$ $\Rightarrow \hat{\beta} = \dfrac{\langle y, x\rangle}{\langle x, x\rangle} = \int_0^1 y(t)\,dt = \bar{y}$



$\tilde{y}(t) := y(t) - \bar{y}J(t)$ $\quad J(t) = 1,\ t \in [0,1]$

$\tilde{x}(t) := x(t) - \bar{x}J(t)$

$$\mathrm{Cov}(y,x) = \int_0^1 (y(t) - \bar{y}J(t))(x(t) - \bar{x}J(t))\,dt$$

$$\mathrm{var}(y) := \mathrm{cov}(y,y)$$

$$\min Z := \| y - \beta_0 J - \beta_1 x \|^2$$

$$\beta_0(\beta_1) = \bar{y} - \beta_1 \bar{x}$$

$$\Rightarrow \hat{\beta}_1 = \mathrm{Cor}(y,x)\,\frac{sd(y)}{sd(x)} \quad \text{and} \quad \hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$$

<u>Example</u>: $y(t) = t + 2t^2$, $x(t) = t$

$$\min \| y - \beta x \|^2 \Rightarrow \hat{\beta} = \frac{\langle y,x \rangle}{\langle x,x \rangle}$$

$$\hat{\beta} = \frac{\int_0^1 (t+2t^2)t\,dt}{\int_0^1 t^2\,dt} = \frac{\frac{1}{3}+\frac{1}{2}}{\frac{1}{3}} = \frac{\frac{2}{6}+\frac{3}{6}}{\frac{2}{6}} = \frac{5}{2} = 2.5$$

In R, call lm(y~x-1) to get 2.5