Stable synchronisation in Random Dynamical Systems

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1 Introduction

In this project we aim at presenting one result from the theory of synchronisation for random dynamical systems. Synchronisation has been experimentally verified many times, especially in the deterministic setting, but the theory behind it is still not completely understood. Actually, a general agreement on the definition itself of this phenomenon has not been reached: all the possible statements are usually not equivalent and depending on both the system itself and what particular aspect one wishes to consider. Our focus will be on what is commonly called noise-driven or noise-induced synchronisation. A random dynamical system comes naturally equipped with a noise space, which drives the evolution of the orbits of the various points in the state or physical space. We will look into the possibility of finding some noise realisation, under which two points will eventually mutually converge. This work closely follows the steps of Julian Newman in his paper *Necessary and sufficient conditions for stable synchronization in random dynamical systems* [6], from which most theorems and definitions have been taken.

Throughout the entirety of the project we will work under the hypotheses of memoryless and stationary noise, driving a process defined on a compact Polish metric space. Furthermore, the time of evolution of our system is always going to be one-sided. For every noise realisation we will assume that the system is right continuous in time and continuous in space.

We will develop necessary and sufficient conditions for stable synchronisation. Stable synchronisation means that almost every noise realisation brings any two points infinitesimally close one to each other, and that the orbits of every point is almost surely stable (for almost any noise realisation ω it is possible to find a neighbourhood of the points which shrinks under the action of ω). The aforementioned conditions are as follows:

1. The state space has a unique non-empty, closed, forward-invariant set K (orbits are bounded inside the set as time tends to infinity) which does not contain any proper non-empty forward-invariant subset;

2. There is a set of positive measure of noise realisations which will drive any two points in K arbitrary close together;

3. The measure of the set of noise realisations making every point asymptotically stable is positive.

Our first assumption, together with some technical tools which we will introduce in due time will allow us to extend the properties defined on K to the whole state space.

To prove this statement we will firstly discuss some general theory of random dynamical systems, only to then define minimal and accessible sets, together with some of their most essential properties. After having set up our foundations, we will introduce the concept of synchronisation in Section 3, together with a brief discussion on the specific definition we will adopt for asymptotic stability, and thus stable synchronisation. Section 3.2 will be mainly devoted to prepare all the technical tools needed in order to prove the main result of the project, which will be stated in Section 4.2. We will conclude with an example showing stable synchronisation.

2 Random Dynamical Systems

In this section we will introduce the mathematically rigorous concept of random dynamical system, together with all the needed assumptions to develop our theory. Define the discrete (resp. continuous) one-sided time $\mathbb{T}^+$ to be $\mathbb{N}_0 \coloneqq \{0\} \cup \mathbb{N}$ (resp. $\mathbb{R}_{\geq 0} \coloneqq [0, \infty)$).

Let (X, d) be a compact Polish metric space, and let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. We now wish to recall the definition of filtration.

Definition 2.1. Given a measurable space $(\Omega, \mathcal{F})$ define $\mathcal{F}_0 \coloneqq \{\emptyset, \Omega\}$, and for $i \in \mathbb{T}^+$ let $(\mathcal{F}_i)_{i\in\mathbb{T}^+}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that if $s, t \in \mathbb{T}^+$, and $s \geq t$ then $\mathcal{F}_s \subset \mathcal{F}_t$. We will call $(\mathcal{F}_i)_{i\in\mathbb{T}^+}$ a *filtration* of $\mathcal{F}$, and we define the maximal element of this ordered set with respect to set inclusion as $\mathcal{F}_\infty \coloneqq \sigma(\mathcal{F}_t \mid t \in \mathbb{T}^+)$.

Recall that for any measure $\mathbb{P}$ we define $\mathcal{N}_\mathbb{P}$ to be the family of all sets A contained in some B where $B \in \mathcal{F}$, and $\mathbb{P}(B) = 0$. A standard assumption (see Kuksin, Shirikyan [1]) is to work under the *usual hypothesis*, hence $\mathcal{F}$ and all its sub-σ-algebras $\mathcal{F}_t$ are $\mathbb{P}$-complete, which means they all contain every elements of $\mathcal{N}_\mathbb{P}$. We are now going to introduce the main object of our studies: random dynamical systems.

Definition 2.2 (Random Dynamical System)**.** Let (X, d) be a Polish metric space, $\mathbb{T}^+$ the one-sided time of evolution of the system, and $(\Omega, \mathcal{F}, (\mathcal{F}_i)_{i\in\mathbb{T}^+}, \mathbb{P})$ a filtered probability space. We call Ω the *noise space*, and X the *state space* of our system. Slightly abusing notation, we identify $\mathbb{P}|_{\mathcal{F}_\infty}$ with $\mathbb{P}$. Define the semigroup of $\mathbb{P}$-preserving Ω-transformations $(\theta^t)_{t\in\mathbb{T}^+}$ where $\theta^t : \Omega \to \Omega$ with $\theta^0 = \mathrm{id}_\Omega$ and $(\theta^t)^*\mathbb{P} = \mathbb{P}$, for all $t \in \mathbb{T}^+$. Suppose further θ^t is $(\mathcal{F}_{s+t}, \mathcal{F}_s)$-measurable for every $s, t \in \mathbb{T}^+$ (i.e. $(\theta^{-t})\mathcal{F}_s \subset \mathcal{F}_{s+t}$, where we used the identification $\theta^{-t} = (\theta^t)^{-1}$).

Define now the function

$$\varphi : \mathbb{T}^+ \times \Omega \times X \to X,$$

such that for any $\omega \in \Omega$ and $t \in \mathbb{T}^+$ the map $\varphi(t, \omega) : X \to X$, given by $\varphi(t, \omega)x = \varphi(t, \omega, x)$, is continuous and for every $t \in \mathbb{T}^+$ the map $(\omega, x) \mapsto \varphi(t, \omega, x)$ is $(\mathcal{F}_t \otimes \mathcal{B}(X), \mathcal{B}(X))$-measurable. Furthermore, assume φ satisfies the following properties

1. *Cocycle property*:

$$\varphi(t+s,\omega)x = \varphi(t,\theta^s\omega)(\varphi(s,\omega)x)$$

   for all $\omega \in \Omega$, $s, t \in \mathbb{T}^+$, and $x \in X$;

2. *Identity property*:

$$\varphi(0,\omega) = \mathrm{id}_X$$

   for all $\omega \in \Omega$.

We refer to φ as a *random dynamical system* on X.

Remark. *In the above definition we identified $\mathbb{P}$ with $\mathbb{P}|_{\mathcal{F}_\infty}$. This is due to the fact we will always require our functions to be measurable with respect to the filtration of $\mathcal{F}$, hence making the choice of using $\mathbb{P}\ |_{\mathcal{F}_\infty}$ more natural and frequent than $\mathbb{P}$.*

We can think of $\theta^t\omega$ as the *time-shift* of the noise realisation ω forward by time t. Hence $(\theta^t)_{t\in\mathbb{T}^+}$ induces a dynamical systems on the noise space Ω, which drives the random dynamical system φ. Throughout the paper we will assume two important conditions on our noise:

N1) *The noise is strictly stationary.* This is naturally given by the $\mathbb{P}$-invariance of $(\theta^t)_{t\in\mathbb{T}^+}$. Shifting in time our noise is not going to affect its probability measure.

N2) *The noise is memoryless.* For all $E \in \mathcal{F}_t$ and $F \in \mathcal{F}_\infty$ then $\mathbb{P}(E \cap \theta^{-t}(F)) = \mathbb{P}(E)\mathbb{P}(F)$

Recall that we call $(\mathbb{P}, f)$ ergodic if and only if for every $A \in \mathcal{F}$ such that $A = f^{-1}(A)$, then $\mathbb{P}(A) \in \{0,1\}$. Thanks to our second hypothesis we automatically get $(\mathbb{P}, \theta^{-t})$ is ergodic for each $t \in \mathbb{T}^+ \setminus \{0\}$. Pick $F \in \mathcal{F}_t$ such that $\theta^{-t}(F) = F$ then $\mathbb{P}(F) = \mathbb{P}(F \cap F) = \mathbb{P}(F \cap \theta^{-t}(F)) = \mathbb{P}(F)\mathbb{P}(F)$, therefore $\mathbb{P}(F) \in \{0,1\}$.

The next assumption is quite technical and its necessity will only be briefly mentioned later when defining asymptotic stability. Assume that φ is càdlàg (from French *continue à droite, limite à gauche*):

C1) *Continue à droite*: For any decreasing converging sequence $t_n \searrow t$ in $\mathbb{T}^+$, and any converging sequence $x_n \to x$ in X, we have that for any $\omega \in \Omega$

$$\lim_{n\to\infty} \varphi(t_n,\omega,x_n) = \varphi(t,\omega,x).$$

C2) *Limite à gauche*: There exists a function $\tilde{\varphi} : \mathbb{T}^+ \times \Omega \times X \to X$, such that for any strictly monotonically increasing sequence (t_n) in $\mathbb{T}^+$ converging to a limit t, and a sequence (x_n) in X converging to a limit x, then for all $\omega \in \Omega$

$$\lim_{n\to\infty} \varphi(t_n,\omega,x_n) = \tilde{\varphi}(t,\omega,x).$$

Random dynamical systems can be analysed from many different perspectives, but surely one of the most useful is studying all the possible orbits of one fixed point $x \in X$, i.e $(\varphi(t,\cdot)x)_{t\in\mathbb{T}^+}$, as an $(\mathcal{F}_t)$-adapted time homogeneous Markov process. Let us denote for $x \in X, t \in \mathbb{T}^+$ and $A \in \mathcal{B}(X)$

$$\varphi_x^t(A) = \mathbb{P}(\{\omega \in \Omega \mid \varphi(t,\omega)x \in A\})$$

the associated transition probabilities. Since we assumed that our noise was stationary then for any $s \in \mathbb{T}^+$ we have $\varphi_x^t(A) = \mathbb{P}(\{\omega \in \Omega \mid \varphi(t,\theta^s\omega)x \in A\})$. Let $\mathcal{P}(X)$ denote the space of probability measures over X, we can define for $t \in \mathbb{T}^+$ the Markov operator

$$\begin{aligned} \mathfrak{B}^t : \mathcal{P}(X) &\to \mathcal{P}(X) \\ \mu &\mapsto \mathfrak{B}^t\mu(\cdot) = \int_X \varphi_x^t(\cdot)\mu(dx) \end{aligned}$$

which gives rise to $(\mathfrak{B}^t)_{t\in\mathbb{T}^+}$, or the *Markov semigroup* under composition of functions. This operator represents the push-forward (by a time t) of a measure under the Markov process. A measure $\rho \in \mathcal{P}(X)$ is said to be stationary if it is invariant with respect to the Markov semigroup, i.e. $\mathfrak{B}^t\rho = \rho$ for any $t \in \mathbb{T}^+$. Thanks to dominated convergence theorem, the map $(t,x) \mapsto \varphi_x^t$ is continuous in x and right continuous in t (if we equip $\mathcal{P}(X)$ with its weak topology).

A third different approach to random dynamical system is from the viewpoint of skew-products. Define the map

$$\begin{aligned} \Theta : \mathbb{T}^+ \times \Omega \times X &\to \Omega \times X \\ (t,\omega,x) &\mapsto (\theta^t\omega, \varphi(t,\omega)x) \end{aligned}$$

and, in order to simplify notation we will define (as we have similarly done previously) the map $\Theta^t : (\Omega, x) \mapsto (\theta^t\omega, \varphi(t,\omega)x)$ for $t \in \mathbb{T}^+$. The set $(\Theta^t)_{t\in\mathbb{T}^+}$ is a semigroup of measurable transformations over $(\Omega \times X, \mathcal{F}_\infty \times \mathcal{B}(X))$ once equipped with the composition operator, since $\Theta^t \circ \Theta^s = \Theta^{t+s}$ thanks to the cocycle property of φ and the fact that $(\theta^t)_{t\in\mathbb{T}^+}$ is a semigroup.

Many are the connections between the skew product and the Markov process associated to a same random dynamical systems, especially when looking at the properties of the different measures we can define on these objects. Of notable importance is the result stating ρ is a stationary measure if and only if $\mathbb{P} \otimes \rho$ is an invariant measure of the semigroup $(\Theta^t)_{t\in\mathbb{T}^+}$. In addition, ρ is ergodic if and only if $\mathbb{P} \otimes \rho$ is ergodic. The proof of these two statements relies on the assumption of having memoryless noise. For a proof of these statements, see Theorem 143 in [4].

This project focuses its attention to synchronisation, which can be thought of as the phenomenon for which two distinct points in the state space eventually will share the same trajectory. In order to study such a phenomenon it is beneficial to keep track of the motions of two points at the same time. For this reason we introduce the concept of *two-point motion*. Let $\varphi \times \varphi := (\varphi \times \varphi(t,\omega))_{t\in\mathbb{T}^+,\omega\in\Omega}$ be the family of functions given by $\varphi\times\varphi(t,\omega) : X \times X \to X \times X$ where

$$\varphi \times \varphi(t,\omega)(x,y) = (\varphi(t,\omega)x, \varphi(t,\omega)y).$$

This functions will play a fundamental role in Section 4.1.

2.1 Minimal and accessible sets

We now start introducing some of the basic concepts needed to be able to state our final result. We would like to firstly introduce the concept of accessible sets from a given point.

Definition 2.3. Let $x \in X$, and let $U \subset X$ be an open subset. We will say U *is accessible from x (under φ)* if and only if

$$\mathbb{P}(\{\omega \mid \exists t \in \mathbb{T}^+ \text{ such that } \varphi(t,\omega)x \in U\}) > 0.$$

For $x \in X$, the above definition simply states that an open U is accessible from a point x only if there is a positive probability of choosing a noise realisation under which our random dynamical system will eventually map x in U. This can be easily rephrased in terms of transition probabilities: U is accessible from x if and only if there exists a time $t \in \mathbb{T}^+$ for which $\varphi_x^t(U) > 0$ (the concept of accessibility plays a crucial role in the theory of Markov processes, especially for finite dimensional state spaces). We would like to underline that the arbitrary union of open sets which are not accessible from x is still not accessible from x. If we combine this fact with the fact that our space is compact and thus second-countable, then the maximal open set non accessible from x, denoted U_x, is well-defined (in a second-countable space the arbitrary union of open sets can be covered by a finite or countable subcollection of the same open sets).

Definition 2.4. Let K be a closed subset of X. We say K is *forward-invariant* if

$$\mathbb{P}(\{\omega \mid \ \forall t \in \mathbb{T}^+, \varphi(t,\omega)K \subset K\}) = 1$$

and similarly an open set $U \subset X$ is said to be *backward-invariant* if its complement $X \setminus U$ is forward-invariant.

If one thinks about this definition in geometric terms it is quite clear what this means. A set is said to be forward-invariant whenever its forward iterates will be contained in itself for almost every noise realisation. Similarly, a set is backward-invariant when its backward iterations (understood as a preimages) will never leave the initial set for almost every noise realisation. Again, we can reformulate this statement in terms of transition probabilities: a closed set $K \subset X$ is said to be forward-invariant if and only if for all $x \in K$ and $t \in \mathbb{T}^+$ then $\varphi_x^t(K) = 1$. As before this means that if we consider any orbit of any point in K, this orbit is confined in K at all times t, $\mathbb{P}$-almost surely. Finally, the last equivalent restatement to Definition 2.4 is in terms of accessible set. A closed set K is forward-invariant if and only if for all $x \in K$ the set $X \setminus K$ is not accessible from x.

Forward-invariant sets play a fundamental role in random dynamics, in fact the support of every stationary measure ρ is forward-invariant, but even more interesting is the converse of this statement. Thanks to *Krylov-Bogolyubov theorem* (Theorem 114 in [4]) if the transition probabilities $(\varphi_x^t)_{t\in\mathbb{T}^+}$ are measurable and Feller for any $x \in X$ then every non-empty forward-invariant compact set corresponds an ergodic stationary measure ρ such that $\rho(K) = 1$. Since we have assumed our state space compact (and it is trivially forward-invariant), and claimed that our transition probabilities are continuous in x, then we must have at least one invariant measure.

Similarly as before, thanks to second-countability, the arbitrary intersection of forward-invariant sets is forward-invariant, hence we the smallest closed forward-invariant set containing $x \in X$, denoted G_x, is well-defined.

Lemma 2.1. *$U_x = X \setminus G_x$ for every $x \in X$.*

Proof. Fix $x \in X$. We clearly have $X \setminus G_x \subset U_x$: the orbit of any point in G_x is confined in G_x, which is closed, so no orbit will ever intersect $X \setminus G_x$, not even in limit, making this set not accessible from x, and hence a subset of the largest not accessible set from x, namely U_x. Thanks to continuity it is possible to show that the other direction holds as well. We now want to show U_x is backward-invariant, since then $X \setminus U_x$ would be forward-invariant and it would contain x, implying that $G_x \subset X \setminus U_x$ or $U_x \subset X \setminus G_x$. Let us proceed by contradiction. Fix $y \in X \setminus U_x$, and suppose there exists a time $t \in \mathbb{T}^+$ such that $\varphi^t_y(U_x) > 0$. Recall that the map $\xi \mapsto \varphi^t_\xi$ is continuous and that U_x is open, hence there exists an open neighbourhood V of y in $X \setminus U_x$ such that $\varphi^t_\xi(U_x) > 0$, for all $\xi \in V$. By construction V is accessible from x, and so there exists a time $s \in \mathbb{T}^+$ for which $\varphi^s_x(V) > 0$. Now by applying Chapman-Kolmogorov equality, and the fact that V is a subset of X

$$\varphi^{s+t}_x(U_x) = \int_X \varphi^t_\xi(U_x)\varphi^s_x(d\xi) \geq \int_V \varphi^t_\xi(U_x)\varphi^s_x(d\xi) > 0.$$

This is openly contradicting the fact U_x is the biggest set not accessible from x. ☐

The theory we developed so far can be easily lifted up to the two-point motion system. We will denote with $G_(x, y) \subset X \times X$ the smallest forward-invariant closed set under $\varphi \times \varphi$ containing the point $(x, y) \in X \times X$.

Remark. *For $(x, y) \in X \times X$ the projection with respect to the first (resp. second) component of $G_{(x,y)}$ is precisely G_x (resp. G_y). [Lemma 2.2.4 in [6]]*

The next definition aims to introduce one of the elements needed in order to establish the necessary and sufficient conditions for synchronisation.

Definition 2.5. A set $K \subset X$ is said to be *minimal* if it is non-empty, closed forward-invariant, and such that its does not have any proper closed forward-invariant subset.

Minimal sets relate to all the concepts we defined above. In particular the following statements are equivalent:

1. K is minimal;
2. K is the minimal element of the set of non-empty forward-invariant subsets of X with respect to set-inclusion (Note that this operation only defines a partial ordering; we can have more than one minimal set);
3. For all $x \in K$, then $G_x = K$;

We would like now to state a technical lemma which will be needed to state a useful property of minimal sets. The proof of the following can be found in (Lemma 2.2.6, [6]).

Lemma 2.2. *Let $K \subset X$ be a closed set with no non-empty closed forward-invariant subsets. For $\mathbb{P}$-almost every $\omega \in \Omega$ and for every $x \in X$ there exists a sequence of times $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{T}^+$ such that $t_n \to \infty$ as $n \to \infty$ and for which $\varphi(t_n, \omega)x \notin K$, for all $n \in \mathbb{N}$.*

From this lemma follows the next result, which can be considered a fourth equivalent definition of minimal set.

Proposition 2.3. *A compact set K is a minimal set, if and only if for any open set U intersecting K, for each point $x \in K$ and $\mathbb{P}$-almost every noise realisation $\omega \in \Omega$ there exists a time $t \in \mathbb{T}^+$ for which $\varphi(t, \omega)x \in U$. Furthermore, if K is the only minimal set in X, then this result extends to every point of the state space.*

Proof. Suppose, at first, that K is a minimal set, and U an open set intersecting K. Now $K \setminus U$ is a closed set not containing any non-empty invariant set, thanks to the assumed minimality of K. By Lemma 2.2 for $\mathbb{P}$-almost every $\omega \in \Omega$ and every $x \in K \setminus U$ we can construct a sequence of times $(t_n)_{n\in\mathbb{N}} \subset \mathbb{T}^+$ for which $\varphi(t_n, \omega)x \notin K \setminus U$, for all $n \in \mathbb{N}$. Since K is forward-invariant then we must have for all $n \in \mathbb{N}$ that $\varphi(t_n, \omega)x \in K$, implying $\varphi(t_n, \omega) \in U$.

Now suppose K is a compact forward-invariant set but not minimal, with the property that for any given point $x \in K$ and almost every noise realisation ω is possible to land in any open subset after a certain time $T \in \mathbb{T}^+$. Without loss of generality, we can assume K is not just the union of minimal sets, otherwise the contradiction is easily reached: the orbit of a point in one minimal set will never intersect an open set intersecting a different minimal set, because of the forward-invariant property of the set in which it belongs to. Denote by E the union of all the minimal sets contained in K, this is clearly forward-invariant and compact, hence $U := K \setminus E \neq \emptyset$ is open. Fix $x \in E$ and let $\omega \in \Omega$, then there exists a time $T \in \mathbb{N}$ for which $\varphi(T, \omega)x \in U$, but this induces a contradiction since E is forward-invariant by construction and hence $\varphi(\mathbb{T}^+, \Omega)x \in E$. Therefore, K is minimal. □

Now that we have a characterisation of minimal sets we are ready to start exploring the various definitions of synchronisation and stability.

3 Synchronisation

We will now delve into the main core of this project: synchronisation and stability. Although such phenomena have a very intuitive and practical meaning, their definition heavily depends on the context of the result one wishes to state. Throughout the years, many definitions have been proposed, most of which are not even equivalent one to the other, hence why now we will worry to specify all the ingredients needed to state precisely our main result.

We are interested in looking into noise-driven synchronisation which means that the property of the objects we will be analysing will depend on noise realisations or subsets of non-zero measure of the noise space Ω.

Definition 3.1. For a given $\omega \in \Omega$ a set $A \subset X$ *contracts under* ω if

$$\operatorname{diam}(\varphi(t, \omega)A) \to 0 \text{ as } t \to \infty.$$

Furthermore, for $\omega \in \Omega$ we say that $x, y \in X$ *mutually converge under* ω if and only if $\mathrm{d}(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0$ as $t \to \infty$, or equivalently $\{x, y\}$ contracts under ω.

This definition is the most general way of understanding what it means for two points to "synchronise" under a specific noise. If this property holds for any couple of points in X, and for almost every noise realisation then the system itself is said synchronising.

Definition 3.2. A random dynamical system φ is said to be *globally synchronising* if for every $x, y \in X$ then

$$\mathbb{P}(\{\omega \in \Omega \mid x \text{ and } y \text{ mutually converge under } \omega\}) = 1.$$

The following proposition is aimed at giving a relation between synchronising systems and stationary measures.

Proposition 3.1. *If φ is synchronising then it has a unique stationary measure.*

The idea of the proof is to show that given a stationary measure ρ of a synchronising random dynamical system φ, then the transition probabilities φ_x^t will converge weakly to ρ. It then will follow that ρ is the unique stationary measure of φ.

Proof. Fix a point $x \in X$, and let $g : X \to \mathbb{R}$ be a continuous function (i.e. $g \in \mathscr{C}(X)$). Since φ is synchronising, for ρ-almost every $y \in X$ and $\mathbb{P}$-almost every $\omega \in \Omega$, x and y mutually converge under ω. By continuity of g it follows

$$\lim_{t\to\infty} g(\varphi(t,\omega)x) - g(\varphi(t,\omega)y) = 0.$$

Since X is compact, g is bounded and hence we can integrate the above limit over $X \times \Omega$ and swap the sign of integration and limit thanks to the dominated convergence theorem. This means

$$\begin{aligned}
0 &= \int_{X\times\Omega} \lim_{t\to\infty} g(\varphi(t,\omega)x) - g(\varphi(t,\omega)y)(\mathbb{P}\otimes\rho)(d(\omega,y)) = \\
&= \lim_{t\to\infty} \int_{X\times\Omega} g(\varphi(t,\omega)x)(\mathbb{P}\otimes\rho)(d(\omega,y)) - \int_{X\times\Omega} g(\varphi(t,\omega)y)(\mathbb{P}\otimes\rho)(d(\omega,y)) = \\
&= \lim_{t\to\infty} \rho(X)\int_{\Omega} g(\varphi(t,\omega)x)\mathbb{P}(d\omega) - \int_{X\times\Omega} g(\pi_2(\Theta^t(\omega,y)))(\mathbb{P}\otimes\rho)(d(\omega,y)) = \\
&= \lim_{t\to\infty} \int_X g(z)\varphi_x^t(dz) - \int_{X\times\Omega} g(\pi_2(\omega,y))(\Theta^t)^*(\mathbb{P}\otimes\rho)(d(\omega,y)) = \\
&= \lim_{t\to\infty} \int_X g(z)\varphi_x^t(dz) - \int_{X\times\Omega} g(\pi_2(\omega,y))(\mathbb{P}\otimes\rho)(d(\omega,y)) = \\
&= \lim_{t\to\infty} \int_X g(z)\varphi_x^t(dz) - \mathbb{P}(\Omega)\int_X g(y)\rho(dy) = \\
&= \lim_{t\to\infty} \int_X g(z)\varphi_x^t(dz) - \int_X g(y)\rho(dy)
\end{aligned}$$

where we use the fact that for all $t \in \mathbb{T}^+$ the measure $\mathbb{P}\otimes\rho$ is Θ^t-invariant given that ρ is stationary, and the definition of the measure φ_x^t. Furthermore,

π_2 is the standard second coordinate projection. This calculation shows that for every $x \in X$ and as $t \to \infty$, then $\varphi_x^t \rightharpoonup \rho$ as we wanted. $\square$

Remark. *For this proof the condition of compactness of X can be relaxed but then this result would assert there exists at most one stationary measure for a synchronising random dynamical system. If X were not compact then we would have worked with uniformly bounded g, so $g \in \mathscr{C}_b(X)$.*

3.1 Asymptotic stability

In this section we will define the concept of asymptotic stability in a different way of what could be considered the "canonical" definition. Unfortunately, we do not have the time to delve into this problem, but a full discussion on the connections between these two definitions can be found In the appendix of [5] (note that the càdlàg assumption plays a fundamental role here).

Before we state what we mean by asymptotic stability we need to set up some notation. Throughout the rest of this project it will be common to use the notation (ω, x) to denote the trajectory or orbit of a point $(\omega, x) \in \Omega \times X$

$$\mathscr{O}((\omega, x)) \coloneqq (\varphi(t, \omega)x)_{t \in \mathbb{T}^+}.$$

This abuse of notation has been inspired by deterministic systems where orbits are identified with initial conditions. Notice that in the random case it is not enough to specify where your orbit is starting from in your state space, but it is necessary to specify what noise realisation one is considering as well.

Definition 3.3. An orbit (ω, x) is said to be *asymptotically stable* if there exists an ω-contractible neighbourhood U of x.

From now on the concept of asymptotically stable will always be in the sense of Definition 3.3.

3.2 Stable synchronisation

The main result stated and proved in this project describes necessary and sufficient conditions in order to have *stable synchronisation.* This is a specific type of synchronising systems, where in addition to synchronisation you have the possibility to always find a noise realisation for every point in the state space, under which the orbit is asymptotically stable. The focus of the next few results is on the local theory of stable trajectories.

Definition 3.4. Let U be an open non-empty subset of X, then we denote by $E_U \subset \Omega$ the set of noise realisations under which U contracts. So for all $x, y \in U$

$$\begin{aligned} E_U &\coloneqq \{\omega \in \Omega \mid \lim_{t \to \infty} d(\varphi(t, \omega)x, \varphi(t, \omega)y) = 0\} \\ &= \{\omega \in \Omega \mid \lim_{t \to \infty} \operatorname{diam}(\varphi(t, \omega)U) = 0\}. \end{aligned}$$

Now let $O \subset \Omega \times X$ be the set of all asymptotically stable points $(\omega, x) \in \Omega \times X$, and similarly for any $x \in X$ denote by

$$O_x \coloneqq \{w \in \Omega \mid (\omega, x) \in O\}$$

the *x-section of O*. Notice that for any $x \in X$ we have

$$O_x = \bigcup_{n \in \mathbb{N}} E_{B_{\frac{1}{n}}(x)}$$

where $B_r(x)$ is the usual open ball of radius r around x with respect to the usual metric d of X. It is important to notice that for $r, \delta > 0$ then $E_{r+\delta} \subseteq E_r$, hence why we need to take the union of all the possible balls as their radius tends to zero. The proof of one of the directions of the equality follows from the following consideration: if $\omega \in O_x$ then there exists an open neighbourhood U of x contracting under ω, but since (X, d) is a metric space with a countable basis, there must exists $m \in \mathbb{N}$ such that $B_{\frac{1}{m}}(x) \subset U$. Therefore, since the diameter of U tends to zero under the iterations of $\varphi(t, \omega)$, the same must happen to $B_{\frac{1}{m}}(x)$, thus $\omega \in E_{B_{\frac{1}{m}}(x)} \subset \bigcup_{n \in \mathbb{N}} E_{B_{\frac{1}{m}}(x)}$. For the other direction let $\omega \in \bigcup_{n \in \mathbb{N}} B_{\frac{1}{n}}(x)$, so there exists $N \in \mathbb{N}$ so that the ball $B_{\frac{1}{N}}(x)$ contracts under ω. It follows by definition that $\omega \in O_x$.

Let us take a look at some useful properties of the sets O and E_U.

Lemma 3.2.

1. *If $U \subset X$ is open, then E_U is $\mathcal{F}_\infty$-measurable.*
2. *The set O is $\mathcal{F}_\infty \otimes \mathcal{B}(X)$-measurable and for any $t \in \mathbb{T}^+$, $\Theta^{-t}(O) \subset O$.*

Proof. The proof of the first part can be found in [6] (Lemma 3.2.3). For the second part, consider a countable base $\mathbb{B}$ of the metric space X, and thanks to our previous discussion, it is quite easy to see $O = \bigcup_{B \in \mathbb{B}} E_B \times B$. Thus, O is $\mathcal{F} \otimes \mathcal{B}(X)$-measurable, thanks to the previous part. Finally, let $t \in \mathbb{T}^+$, then if we can find a neighbourhood of $\varphi(t, \omega)x$ which contracts under $\theta^t \omega$, then since $\varphi(t, \omega) : X \to X$ is continuous, we have a neighbourhood of x which contracts under ω. Therefore, $\Theta^{-t}(O) \subset O$. $\square$

Remark. *The second part of the previous lemma is telling us that O is backward invariant with respect to the skew product Θ. Furthermore, we would like to make notation simpler hence let for any $x \in X$ and $r > 0$:*

- $P_O(x) \coloneqq \mathbb{P}(O_x)$
- $P_r(x) \coloneqq \mathbb{P}(E_{B_r}(x))$.

Clearly since if $B_{r+1}(x)$ contracts under ω, then so does B_r, we have $P_r(x) = \mathbb{P}(B_r(x)) \geq \mathbb{P}(B_{r+1}(x)) = P_{r+1}(x)$, or equivalently $P_r(x)$ decreases as r increases. Hence $P_O(x) = \sup_{r>0} P_r(x) = \lim_{r \to 0} P_r(x)$.

Lemma 3.3. *For $x \in X$ and $t \in \mathbb{T}^+$*

$$P_O(x) \geq \int_X P_O(y) \varphi_x^t(dy).$$

Proof. Recall that we defined $\mathbb{P}$ to be θ-invariant, hence $P_O(y) = \mathbb{P}(O_y) = \mathbb{P}(\theta^{-s} O_y)$ for all $y \in X$ and $s \in \mathbb{T}^+$. For fixed $x \in X$ and $t \in \mathbb{T}^+$ the following holds:

$$
\begin{aligned}
\int_X P_O(y)\varphi_x^t(dy) &= \int_\Omega P_O(\varphi(t,\omega)x)\mathbb{P}(d\omega) \\
&= \int_\Omega \mathbb{P}(\theta^{-t}O_{\varphi(t,\omega)x})\mathbb{P}(d\omega) && \text{(by above remark)} \\
&= \int_\Omega \int_\Omega \mathbb{1}_O(\theta^t\tau, \varphi(t,\omega)x)\mathbb{P}(d\tau)\mathbb{P}(d\omega) \\
&= \int_\Omega \mathbb{1}_O(\theta^t\omega, \varphi(t,\omega)x)\mathbb{P}(d\omega) && \text{(by memoryless property)} \\
&= \int_\Omega \mathbb{1}_{\Theta^{-t}(O)}(\omega, x)\mathbb{P}(d\omega) \\
&\leq \int_\Omega \mathbb{1}_O(\omega, x)\mathbb{P}(d\omega) && \text{(since } O \text{ is } \Theta^t\text{-backward-invariant)} \\
&= P_O(x).
\end{aligned}
$$

□

Definition 3.5. We say that $x \in X$ is:

1. *almost surely stable* if $P_O(x) = 1$;
2. *potentially stable* if $P_O(x) > 0$.

Note that for x being potentially stable is equivalent to having a neighbourhood U for which $\mathbb{P}(E_U) > 0$. Moreover, if every point $x \in X$ is almost surely stable then $1 = P_O(x) = \lim_{r\to 0} P_r(x)$ everywhere on X.

The next proposition is aimed at describing the connection between asymptotically stable and potentially stable.

Proposition 3.4. *For $\mathbb{P}$-almost every $\omega \in \Omega$, for any $x \in X$ such that (ω, x) is asymptotically stable, then x is potentially stable.*

Proof. We firstly want to build the biggest set of zero measure such that all its remaining elements have the property stated in the proposition. Let $\mathbb{B}$ be a countable base of the topology of X, and define $\mathbb{B}_0 := \{B \in \mathbb{B} \mid \mathbb{P}(E_B) = 0\}$. Then $\tilde{\Omega} = \Omega \setminus \bigcup_{B\in\mathbb{B}_0} E_B$ only contains noise realisations for which there exists at least an element of the base $\mathbb{B}$ which contracts under ω. It is important to notice that, by construction, $\mathbb{P}(\bigcup_{B\in\mathbb{B}_0} E_B) = 0$, thus $\mathbb{P}(\tilde{\Omega}) = 1$. Now fix $\omega \in \tilde{\Omega}$, and pick any $x \in X$. If the pair (ω, x) is asymptotically stable, then there exists $B \in \mathbb{B}$ such that $x \in B$, $\omega \in E_B$ and $\mathbb{P}(E_B) > 0$, making x potentially stable. □

Denote by U_{ps} the subset of X of potentially stable points. This set is open since for every $x \in U_{ps}$ we have that there must exist an open neighbourhood U for which $\mathbb{P}(E_U) > 0$. Since for all points x in U, the set U itself is a contracting open neighbourhood, which contracts for a set of positive measure of noise realisations, we immediately have that $P_O(x) > 0$, implying $U \subset X_{ps}$. Furthermore, it is backward-invariant as well. Let $t \in \mathbb{T}^+$ and $x \in X \setminus U_{ps}$, hence we know that $P_O(x) = 0$. By Lemma 3.3 we have that the following holds for every $\omega \in \Omega$:

$$0 = P_O(x) \geq \int_X P_O(y)\varphi_x^t(dy) \geq P_O(\varphi(t,\omega)x).$$

Thus $P_O(\varphi(t,\omega)x) = 0$ for every $\omega \in \Omega$, or equivalently for every $t \in \mathbb{T}^+$

$$\mathbb{P}(\{\omega \in \Omega \mid \varphi(t,\omega)(X \setminus U_{ps}) \subset X \setminus U_{ps})\}) = 1,$$

which implies U_{ps} is backward-invariant.

We now want to shift our attention towards local theory, therefore we want to define sets admitting stable trajectories. Firstly, we can easily extend the definition of O_x to subsets of X. For any $A \subset X$ the set

$$O_A := \{\omega \in \Omega \mid \exists x \in A \text{ such that } (\omega, x) \text{ is asymptotically stable}\} = \bigcup_{x \in A} O_x,$$

is $\mathcal{F}_\infty$-measurable. This follows quite easily from considering the topological base $\mathbb{B}_A$ of A induced by a (countable) topological base $\mathbb{B}$ of X through the subspace topology. One can rewrite, thanks to a previous remark, $O_A = \bigcup_{B \in \mathbb{B}_A} E_B$, therefore the result follows (Lemma 3.2). Now let us have a look at the definition of a set which admits stable trajectories.

Definition 3.6. A non-empty closed forward-invariant set $K \subset X$ *admits stable trajectories* if $\mathbb{P}(O_K) > 0$.

The fact K is closed and forward-invariant assures us all its orbit will remain inside the set for all $t \in \mathbb{T}^+$, whereas the second condition allows us to say that there is at least one point which is asymptotically stable under some noise realisation belonging to a non-zero measure set. The connection with potentially stable points is immediate thanks to Proposition 3.4. Since $\mathbb{P}(O_K) > 0$, and we have that there exists at least one asymptotically stable orbit (ω, x) with $\mathbb{P}(O_x) > 0$, therefore x is potentially stable. This result can be stated as follows.

Lemma 3.5. *A non-empty closed forward-invariant subset of X admits stable trajectories if and only if at least one of its points is potentially stable.*

The next result is a technical tool which will be exploited in the proof of Theorem 3.7 and of the main result of this project.

Lemma 3.6. *Let D be a totally ordered countable set. Suppose that the the two families $(R_{n,s})_{n \in \mathbb{N}, s \in D}$ and $(S_{n,s})_{n \in \mathbb{N}, s \in D}$ of events of $\mathcal{F}$ have the following properties:*

1. *for all n and s, $S_{n,s}$ is independent of $\sigma(R_{n,t} \mid t \leq s, t \in D)$;*
2. *for all n, $\mathbb{P}(\bigcup_{s \in D} R_{n,s}) = 1$;*
3. $\lim_{n \to \infty} \inf_{s \in D} \mathbb{P}(S_{n,s}) = 1$.

Then

$$\mathbb{P}(\bigcup_{n \in \mathbb{N}} \bigcup_{s \in D} R_{n,s} \cap S_{n,s}) = 1$$

Proof. Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. By the second assumption, we can find a finite ordered subset $\{t_i\}_{i=1}^m$ of D with $t_1 < t_2 < \cdots < t_m$, and such that $\mathbb{P}(\bigcup_{i=1}^m R_{n,t_i}) > 1 - \varepsilon$. Now

$$\begin{aligned}
\mathbb{P}(\bigcup_{s\in D} R_{n,s} \cap S_{n,s}) &\geq \mathbb{P}(\bigcup_{i=1}^{m} R_{n,t_i} \cap S_{n,t_i}) \\
&\geq \mathbb{P}(\bigcup_{i=1}^{m} (R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j}) \cap S_{n,t_i}) && \text{(This is now a disjoint union)} \\
&= \sum_{i=1}^{m} \mathbb{P}((R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j}) \cap S_{n,t_i}) \\
&= \sum_{i=1}^{m} \mathbb{P}(R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j}) \mathbb{P}(S_{n,t_i}) && \text{(by assumption 1)} \\
&\geq (\sum_{i=1}^{m} \mathbb{P}(R_{n,t_i} \setminus \bigcup_{j=1}^{i-1} R_{n,t_j})) \inf_{s\in D} \mathbb{P}(S_{n,s}) \\
&= \mathbb{P}(\bigcup_{i=1}^{m} R_{n,t_i}) \inf_{s\in D} \mathbb{P}(S_{n,s}) \\
&> (1-\varepsilon) \inf_{s\in D} \mathbb{P}(S_{n,s})
\end{aligned}$$

The above deduction holds for any $\varepsilon > 0$, hence

$$\mathbb{P}(\bigcup_{s\in D} R_{n,s} \cap S_{n,s}) \geq \inf_{s\in D} \mathbb{P}(S_{n,s}).$$

If we now we take the limit as $n \to \infty$ we obtain

$$\mathbb{P}(\bigcup_{n=1}^{\infty} \bigcup_{s\in D} R_{n,s} \cap S_{n,s}) \geq \lim_{n\to\infty} \mathbb{P}(\bigcup_{s\in D} R_{n,s} \cap S_{n,s}) \geq \lim_{n\to\infty} \inf_{s\in D} \mathbb{P}(S_{n,s}) = 1$$

thanks to assumption 3. Since $\mathbb{P}$ is a probability measure, the result follows. □

The next theorem we will introduce is the foundation of the main result of this project. Lemma 3.5 told us that admitting stable trajectories is equivalent to have at least one potentially stable point for a non-empty closed forward-invariant set. Under the stronger assumption of a minimal set, we are going to show that every point will be almost surely stable. This is mainly due to the already mentioned fact that we can equip such sets with ergodic measures. The previous lemma will help us finishing the proof.

Theorem 3.7. *Suppose that K is a minimal set which admits stable trajectories, $\mathbb{P}(O_K) > 0$, then every point is K is almost surely stable. Furthermore, if K is the only minimal set in X, then every point $x \in X$ will be almost surely stable and will have the property that*

$$\mathbb{P}(\{\omega \mid \mathrm{d}(\varphi(t,\omega)x, K) \to 0 \text{ as } n \to \infty\}) = 1$$

Proof. Let K be a minimal set admitting stable trajectories. By minimality of K we know there exists an ergodic invariant measure ρ whose support is precisely K. We have (by Lemma 3.5) at least one point which is potentially stable, but U_{ps} is open and backward-invariant as we have seen previously: since K is minimal, then $U_{ps} = K$. So every point in K is potentially stable. The next step is to show that that these points are actually almost surely stable. Recall we denoted by O the set of all asymptotically stable points which was $\mathcal{F}_\infty \otimes \mathcal{B}(X)$-measurable and Θ-backward-invariant. Let us focus our attention to the behaviour of φ restricted to K. The measure $\mathbb{P}\otimes\rho$ is ergodic with respect to Θ and $\Theta^{-t}(O) = O$ for all $t \in \mathbb{T}^+$. By definition, $\mathbb{P}\otimes\rho(O) \in \{0,1\}$, but since $K = U_{ps}$ then $\mathbb{P}\otimes\rho(O) > 0$, which immediately implies that ρ-almost every point in K is almost surely stable. We want now lift this result from ρ-almost every point to every point in K. In order to do that we will need Lemma 3.6.

For p any almost surely stable point of K and $x \in X$, we can define for $n \in \mathbb{N}$, and $s \in \mathbb{T}^+$ the sets

$$R_{n,s} \coloneqq \{\omega \in \Omega \mid \varphi(s,\omega)x \in B_{\frac{1}{n}}(p)\}$$
$$S_{n,s} \coloneqq \theta^{-s} E_{B_{\frac{1}{n}}(p)}.$$

The family of sets $\{R_{n,s}\}_{n\in\mathbb{N},s\in\mathbb{T}^+}$ is telling us what noise realisations will allow the point x to be arbitrarily close to p, whereas the family $\{S_{n,s}\}$ is collecting all the different ω for which p is asymptotically stable. Since K is minimal and $B_{\frac{1}{n}}(p)$ is an open set intersecting K non-trivially for any fixed $n \in \mathbb{N}$ we can apply Proposition 2.3 in order to obtain a time l for which $\mathbb{P}$-almost every ω realises $\varphi(l,\omega)x \in B_{\frac{1}{n}}(p)$. This means that $\lim_{s\to\infty}\mathbb{P}(R_{n,s}) = \mathbb{P}(\bigcup_{s\in\mathbb{T}^+} R_{n,s}) = 1$ for every n. Furthermore, $\mathbb{P}(S_{n,s}) = P_{\frac{1}{n}}(p)$ independently from s since $\mathbb{P}$ is independent of θ. But this clearly implies that $\mathbb{P}(S_{n,s}) = 1$ as $n \to \infty$, uniformly in s (since it is independent). The points p and x where picked arbitrarily hence it follows that for any $n \in \mathbb{N}$, and $s \in \mathbb{T}^+$ the event $S_{n,s}$ is independent of $\sigma(R_{n,t} \mid t \leq s, t \in \mathbb{T}^+)$. By Lemma 3.6 we can conclude that $\mathbb{P}(\bigcup_{n\in\mathbb{N}}\bigcup_{s\in\mathbb{T}^+} R_{n,s} \cap S_{n,s}) = \mathbb{P}(O_x) = 1$ which means that x is almost surely stable, implying that every point in K is almost surely stable.

The last part of this proof concerns extending the results just proven to the case where our state space X has only one minimal set K. The second part of Proposition 2.3 allow us, for any $x \in X$, to find a time $l \in \mathbb{T}^+$ granting us that the orbit of x will intersect $B_{\frac{1}{n}}(p)$ for $\mathbb{P}$-almost every ω, whenever $n \in \mathbb{N}$ is fixed. Thus, notice that for any fixed $n \in \mathbb{N}$, $s \in \mathbb{T}^+$, and $\omega \in R_{n,s} \cap S_{n,s}$ we have that $\lim_{t\to\infty} \mathrm{d}(\varphi(t,\omega)x, \varphi(t-s,\theta^s\omega)p) = 0$. For $\mathbb{P}$-almost every $\omega \in \Omega$ then $\varphi(t-s,\theta^s\omega)p \in K$ for any $s \in \mathbb{T}^+$ and $t \geq s$. Therefore we can conclude thanks to Lemma 3.6 that $\mathrm{d}(\varphi(t,\omega)x, K) \to 0$ as $t \to \infty$ for $\mathbb{P}$-almost every ω.

□

Remark. *If we now suppose that the transition probabilities induced by the random dynamical system φ have a unique stationary measure ρ with the property that $\mathbb{P}\otimes\rho(O) = 1$, then we know that every point in the state space is almost surely stable.*

We are now finally ready to define the precise type of synchronisation our main result will be concerned with.

Definition 3.7. A random dynamical system φ is said to be *(globally) stably synchronising* if φ is synchronising and every point in X is almost surely stable.

Thanks to the previous remark for a synchronising system it is "enough" to have a unique stationary measure ρ under which O has full $\mathbb{P} \otimes \rho$-measure.

4 Necessary and sufficient conditions for synchronisation

Before stating the main result we will like to discuss the concept of two-point motion, and specifically two-point contractibility.

4.1 Two-point motion

So far we have always been concerned about the behaviour of orbits of single points. One of the most fascinating sides of dynamical systems hides behind its ability to tell us informations about the asymptotic behaviour of different points at the same time. For this reason we wish to further develop the theory of two-point motions which was introduced earlier in this project.

Given any set $A \subset X$, we denote $\Delta_A := \{(x,x) \mid x \in A\} \subset X \times X$: this can be thought of as the "diagonal" of the set $A \times A$. It is fundamental to notice that Δ_A is always forward-invariant under the $\varphi \times \varphi$ iterations. Many of the concepts we introduced before can be redefined under the perspective of two-point motions. Given $x, y \in X$ we say that (x, y) is *contractible under φ* if and only if every neighbourhood of Δ_X is accessible from (x, y) under $\varphi \times \varphi$, i.e. $\mathbb{P}(\{\omega \mid \exists t \in \mathbb{T}^+ \text{ s.t } \varphi \times \varphi(t,\omega)(x,y) \in A\}) > 0$ for A any neighbourhood of Δ_X. Notice that if for some time t and noise realisation ω one has $\varphi \times \varphi(t,\omega)(x,y) \in \Delta_X$, then $\varphi(t,\omega)x = \varphi(t,\omega)y$, which means that the two points x and y are synchronised, in the sense that their orbits coincide after the time t. The definition of contractible under φ relies on the fact that every neighbourhood of Δ_X is accessible from (x, y) meaning that their orbit under $\varphi \times \varphi$ will eventually tend to Δ_X. From this perspective this definition agrees with the one of mutual convergence of (x, y) under φ (See Definition 3.1). Actually, since $X \times X$ is a compact metric space, one can check the easier condition that for all $\varepsilon > 0$

$$\mathbb{P}(\{\omega \mid \exists t \in \mathbb{T}^+ \text{ s.t. } \mathrm{d}(\varphi(t,\omega)x, \varphi(t,\omega)y) < \varepsilon\}) > 0.$$

Again we are more interested in theory involving sets instead of singular points, hence for a given non-empty closed invariant set $K \subset X$, *φ is two-point contractible on K* if for any $x, y \in K$, (x, y) is contractible under φ. Equivalently, we can say that φ is two-point contractible over K if and only if there are no non-empty forward invariant subsets of $K \times K \setminus \Delta_K$. If there were such a forward-invariant set F then we would be able to find an ε-neighbourhood V_ε of Δ_K, which does not intersect F, but any point $(x, y) \in F$ would never land in V_ε for any time t or noise realisation ω.

Otherwise, φ is two-point contractible over K if a contraction between $x, y \in K$ (where $x \neq y$) always happens with non-zero probability, or in symbols:

$$\mathbb{P}(\{\omega \mid \exists t \in \mathbb{T}^+ \text{ s.t. } \mathrm{d}(\varphi(t,\omega)x, \varphi(t,\omega)y) < \mathrm{d}(x,y)\}) > 0.$$

If we have that φ is two-point contractible on X then we must have a unique minimal set. Suppose we had two (disjoint) minimal sets $K_1, K_2 \subset X$ then we must have that $K_1 \times K_1$ and $K_2 \times K_2$ both intersect Δ_X non-trivially, since $\Delta_{K_1}, \Delta_{K_2} \subset \Delta_X$. But this means that for any point $(x, y) \in X \times X$ the obit of this point will have to intersect every possible ε-neighbourhood of Δ_{K_1}, and Δ_{K_2}. Without loss of generality suppose that $(x, y) \in K_1 \times K_1$ for example, then $\varphi \times \varphi(t, \omega)(x, y) \in K_1 \times K_1$ at all time and for $\mathbb{P}$-almost every noise realisation, and hence the probability of intersecting any neighbourhood of Δ_{K_2} is zero, contradicting the fact the minimal subset of a set X which is two-point contractible is itself two-point contractible. This last statement directly follows from the fact that minimal sets are forward-invariant.

We now want to state a condition which will allow us to determine if a minimal set is two-point contractible. This proposition essentially proves that if we can find a full-measure subset of a minimal set (with respect to the stationary measure induced by the minimal set) which is two-point contractible, then the property can be lifted to the whole invariant set.

Proposition 4.1. *Let $K \subset X$ be a minimal set, support of a stationary measure ρ. Suppose $A \subset K$ has the following properties*

1. *$\rho(A) = 1$;*
2. *the interior of A is non empty with respect to the subspace topology induced on K;*
3. *every pair of points in A is contractible.*

Then φ is two-point contractible on K.

Proof. Fix $x, y \in K$. Recall that we denote by $G_{(x,y)}$ the smallest forward invariant set containing (x, y), then, thanks to the third assumption, it is enough to show that $G_{(x,y)} \cap A \times A \neq \emptyset$. If that were the case then we have that (x, y) would be contractible under φ. Define

$$B := \{x \in K \mid \varphi_x^t(A) = 1, \forall t \in \mathbb{T}^+\}.$$

Since ρ is stationary and $\rho(A) = 1$ then $\rho(B) = 1$, which means that B is non-empty. By a previous remark the projection of $G(x, y)$ in the first and second coordinates is precisely G_x and G_y respectively, so they are both equal to K (by minimality).

Fix $(u, v) \in G_{(x,y)}$ where $u \in B$. Now let $U \subset X$ open, such that $U \cap K$ is a non-empty subset of A; then U is accessible from v (thanks to Proposition 2.3), with full probability $\mathbb{P}$. Recall that $t \mapsto \varphi_x^t(U)$ is right continuous, and hence we can find a time $s \in \mathbb{T}^+$ such that $\varphi_v^s(U) > 0$. Now, thanks to the fact that K is forward-invariant, we infer $\varphi_v^s(A) \geq \varphi_v^s(U) > 0$, since $U \subset A$. Therefore, by construction, we have a $\mathbb{P}$-positive measure set of noise realisations for which both $\varphi(s, \omega)x$ and $\varphi(s, \omega)y$ are both in A. So $G_{(u,v)}$, and automatically $G_{(x,y)}$, has non-trivial intersection with $A \times A$. □

4.2 Main result

We are now fully able to state and prove necessary and sufficient conditions in order to have stable synchronisation.

Theorem 4.2. *A random dynamical system φ is stably synchronising if and only if the following conditions hold:*

1. *there exists a unique minimal set K of the state space X;*
2. *φ is two-point contractible on K;*
3. *K admits stable trajectories.*

Remark. *As we have seen before the condition on minimal sets can be easily substituted by the existence of a unique stationary measure ρ for the Markov operator $\mathfrak{B}$. In that case $K = supp(\rho)$.*

Proof. Suppose that φ is synchronising, then it follows immediately that φ is two-point contractible over X, and by the above discussion there exists a unique minimal set K. Hence conditions 1 and 2 hold. If φ is stably synchronising then, by definition, K admits stable trajectories. Hence we have proved one direction.

Now we wish to prove the converse. Thanks to Theorem 3.7, since we assumed that there exists a unique minimal set whose points are all almost surely stable, then every point in X is almost surely stable. We are left with showing that φ is synchronising. In order to do this we will use Lemma 3.6. Let C be a closed non-empty forward-invariant set of $X \times X$. By the second part of Theorem 3.7 all points in X tend to K with probability one, thus since both C and $K \times K$ are closed, they must intersect, non-trivially. We clearly have that Δ_K is minimal under $\varphi \times \varphi$. By our third assumption, we have that $C \cap \Delta_K \neq \emptyset$, but more precisely $\Delta_K \subset C$. Hence Δ_K is the unique minimal set under $\varphi \times \varphi$.

Fix $x, y \in X$, and $p \in K$. For any $n \in \mathbb{N}$ and $s \in \mathbb{T}^+$ define

$$\begin{aligned} R_{n,s} &:= \{\omega \mid \varphi \times \varphi(s,\omega)(x,y) \in B_{\frac{1}{n}}(p) \times B_{\frac{1}{n}}(p)\} \\ S_{n,s} &:= \theta^{-s}(E_{B_{\frac{1}{n}}(p)}). \end{aligned}$$

By mimicking the proof of Theorem 3.7, Proposition 2.3 tells us that $\mathbb{P}(\bigcup_{s\in\mathbb{T}^+} R_{n,s}) = 1$ for all n. Furthermore, since p is almost surely stable we have again that $\mathbb{P}(S_{n,s}) \to \infty$ as $n \to \infty$, uniformly in s. For any n, s, if $\omega \in R_{n,s} \cap S_{n,s}$ then $\mathrm{d}(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0$ as $t \to \infty$, by construction. Therefore, by Lemma 3.6 we have that $\mathrm{d}(\varphi(t,\omega)x, \varphi(t,\omega)y) \to 0$ as $t \to \infty$ for $\mathbb{P}$-almost every ω, proving that φ is synchronising. $\square$

5 Example

We would now like to present an example where our main result can be exploited to show stable synchronisation.

Let our state space be the unit circle $\mathbb{S}^1$, and let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth map. Identify the unit circle with $\mathbb{S}^1 \simeq \mathbb{R}/\mathbb{Z}$, so that every point is represented by an equivalence class $[x] = \{x + n \mid n \in \mathbb{Z}\}$, where $x \in \mathbb{R}$. This induces a natural metric on $\mathbb{R}/\mathbb{Z}$ given by

$$\mathrm{d}([x],[y]) = \min_{x'\in[x],y'\in[y]} |x' - y'|$$

for all $[x], [y] \in \mathbb{R}/\mathbb{Z}$. Under this metric, $\mathrm{diam}(\mathbb{S}^1)=\frac{1}{2}$. Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$, and define by $F : \mathbb{R} \to \mathbb{R}$ the lift of f to $\mathbb{R}$. The following diagram:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{F} & \mathbb{R} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle q} \\ \mathbb{R}/\mathbb{Z} & \xrightarrow[f]{} & \mathbb{R}/\mathbb{Z} \end{array}$$

commutes, which means that $f([x]) = [F(x)]$. Note that the function $F(x) - x$ has to be 1-periodic, since F is the lift of a function defined on the circle. So for all $n \in \mathbb{Z}$ and all $x \in \mathbb{R}$ we have $F(x+n)-(x+n) = F(x)-x$ or equivalently

$$F(x+n) = F(x) + n. \tag{1}$$

Let $([0,1)^{\mathbb{N}}, \mathcal{B}([0,1))^{\otimes\mathbb{N}}, \nu^{\otimes\mathbb{N}})$ be a probability space, where ν is the Lebesgue measure over $[0,1)$. Define, for $n \in \mathbb{N}$ the projection of the first n-elements of our noise realisation $c_n : [0,1)^{\mathbb{N}} \to [0,1)^n$ where $c_n((\alpha_i)_{i\in\mathbb{N}}) = (\alpha_i)_{i=1}^n$. We can use this function to define a filtration $\mathcal{F}_n \coloneqq \sigma(c_n)$ of $\mathcal{B}([0,1))^{\otimes\mathbb{N}}$. Finally, for all $n \in \mathbb{N}$ define the shift operator $\theta^n : [0,1)^{\otimes\mathbb{N}} \to [0,1)^{\otimes\mathbb{N}}$ as $\theta^n((\omega_i)_{i\in\mathbb{N}}) = (\omega_{i+n})_{i\in\mathbb{N}}$. Therefore, $([0,1)^{\mathbb{N}}, \mathcal{B}([0,1))^{\otimes\mathbb{N}}, \nu^{\otimes\mathbb{N}}, (\mathcal{F}_n), (\theta^n))$ is a memoryless noise space.

We can now define our random dynamical system in the following way. For any $\alpha \in (0,1)$ let us define the α-perturbation of the map f

$$f_\alpha([x]) = [F(x+\alpha) - \alpha],$$

which goes from $\mathbb{R}/\mathbb{Z}$ to $\mathbb{R}/\mathbb{Z}$. If we want to look at these maps from the perspective of $\mathbb{S}^1$, these are nothing more than α-rotations of f. For all $\alpha \in [0,1)$, let $R_\alpha : \mathbb{S}^1 \to \mathbb{S}^1$ represent the rotation by $2\pi\alpha$ radians of the circle, then $f_\alpha(x) = (R_{-\alpha} \circ f \circ R_\alpha)(x)$, for all $x \in \mathbb{S}^1$. We will use both these two definitions from now on.

Definition 5.1. We say that $\alpha \in (0,1)$ is a subperiod of f if and only if $f_\alpha(x) = f(x)$ for all x in the state space.

If f is a rotation to start with then we have that every α is a subperiod of f, since rotations permute. This is easily seen with the second definition we have given, but nevertheless it holds true even if we use the first definition. Let $f([x]) = e^{2\pi i(x+\xi)}$, for $\xi \in (0,1)$, then its lift F is simply given by $F(x) = x + \xi$. Therefore, for all $\alpha \in (0,1)$, we have that

$$f_\alpha([x]) = [F(x+\alpha) - \alpha] = [x + \xi + \alpha - \alpha] = [F(x)] = f([x])$$

as we wanted. It follows quite easily that, given any $\alpha_1, \alpha_2 \in (0,1)$, α_1 is a subperiod of f if and only if α_1 is a subperiod of f_{α_2}. Let us make this statement more precise. Suppose that α_1 is a subperiod of f, then we clearly see that for any $z \in \mathbb{S}^1$

$$\begin{aligned} (f_{\alpha_2})_{\alpha_1}(z) &= (R_{-\alpha_1} \circ R_{-\alpha_2} \circ f \circ R_{\alpha_2} \circ R_{\alpha_1})(z) \\ &= (R_{-\alpha_2} \circ R_{-\alpha_1} \circ f \circ R_{\alpha_1} \circ R_{\alpha_2})(z) \\ &= (R_{-\alpha_2} \circ f \circ R_{\alpha_2})(z) && \text{(by assumption)} \\ &= f_{\alpha_2}(z). \end{aligned}$$

A similar calculation gives us the other implication. If we suppose that for all $z \in \mathbb{S}^1$ we have $(f_{\alpha_2})_{\alpha_1} = f_{\alpha_2}$ then the following holds:

$$\begin{aligned} f(z) &= (R_{\alpha_2} \circ f_{\alpha_2} \circ R_{-\alpha_2})(z) \\ &= (R_{\alpha_2} \circ (f_{\alpha_2})_{\alpha_1} \circ R_{-\alpha_2})(z) && \text{(by assumption)} \\ &= (R_{\alpha_2} \circ R_{-\alpha_1} \circ R_{-\alpha_2} \circ f \circ R_{\alpha_2} \circ R_{\alpha_1} \circ R_{-\alpha_2})(z) \\ &= (R_{\alpha_2} \circ R_{-\alpha_2} \circ R_{-\alpha_1} \circ f \circ R_{\alpha_1} \circ R_{\alpha_2} \circ R_{-\alpha_2})(z) \\ &= (R_{-\alpha_1} \circ f \circ R_{\alpha_1})(x) \\ &= f_{\alpha_1}(z). \end{aligned}$$

Finally, we can define our random dynamical systems as

$$\varphi(n, (\alpha_i)) = f_{\alpha_n} \circ \cdots \circ f_{\alpha_1},$$

and we clam that this is stably synchronising. We will be proving this result by checking the conditions of the main theorem we have stated in Section 4.2. But firstly, we need to discuss a couple of properties of φ. Recall that a dynamical system is said to be *minimal* if and only if every orbit is dense in the state space. In the specific case of random dynamics we say that φ is minimal over X if every non-empty open subset of X is accessible from any point in X.

Proposition 5.1. *φ is minimal over $\mathbb{S}^1$ if and only if f is not a rational rotation.*

Proof. Suppose f is a rotation, then $\varphi(n, (\alpha_i)) = f^n$. If f is a rational rotation, we know by standard results from the theory of deterministic dynamical systems that any orbit of f is periodic, proving that in this situation φ is not dense in $\mathbb{S}^1$. We can exclude this case.

Now if we assume that f is not a rational rotation, there must exist a point $p \in \mathbb{R}$ such that

$$F(p) - p = \omega \tag{2}$$

where ω is an irrational number. Let us define a function $\alpha : \mathbb{S}^1 \to [0, 1)$, such that for $y \in \mathbb{S}^1$ we have $[p - \alpha(y)] = [y]$, or equivalently $[y + \alpha(y)] = [p]$. The scope of this function is to measure the angular distance (normalised in $[0, 1)$) between p and any point y in the circle. This is going to be useful to define a noise realisation which will make φ intersect any arbitrary open set in the state space.

The next part of the proof is based on the idea that irrational circle rotations are minimal. Let U be a non-empty open subset of $\mathbb{S}^1$, and fix $x \in \mathbb{S}^1$. Denote by x' one of the preimages of x under the quotient map q. Let $n \in \mathbb{N}$ be such that $[x' + n\omega] \in U$. As we have discussed before $\tilde{F}(x) = x + \omega$ is the lift of an irrational rotation $\tilde{f}$, hence by minimality, the orbit of x under $\tilde{f}$ will intersect the open set U at the n-th iteration. Recursively define $\alpha_1 \coloneqq \alpha(x)$, and $\alpha_s = \alpha(f_{\alpha_{s-1}} \circ \cdots \circ f_{\alpha_1}(x))$, where $2 \leq s \leq n$. Then we can check

$$\begin{aligned} f_{\alpha_1}([x]) &= [F(x + \alpha_1) - \alpha_1] \\ &= [F(x + \alpha(x)) - \alpha_1] \\ &= [F(p + m) - \alpha_1] && \text{(for } m \in \mathbb{Z}) \\ &= [F(p) + m - \alpha_1] && \text{(by Eq. (1))} \\ &= [p + \omega - \alpha(x)] && \text{(by Eq. (2))} \\ &= [x + \omega] \end{aligned}$$

and similarly

$$(f_{\alpha_2} \circ f_{\alpha_1})([x]) = [p + \omega - \alpha_2] = [p + \alpha(f_{\alpha_1}(x)) + \omega] = [x + \omega + \omega] = [x + 2\omega].$$

By induction it follows that for any $\alpha \in [0,1)^{\mathbb{N}}$, whose first n entries are precisely the previously defined $(\alpha_1, \dots \alpha_n)$ we have

$$\varphi(n, \alpha)[x] = f_{\alpha_n} \circ \cdots \circ f_{\alpha_1}([x]) = [x + n\omega] \in U.$$

Now, for a sufficiently small neighbourhood V of $(\alpha_1, \dots \alpha_n)$ in $[0,1)^n$, $\varphi(n,\alpha)[x] \in U$, where $\alpha \in c_n^{-1}(V)$. Recall that the measure ν is just the Lebesgue measure, hence it has full support on $[0,1)$, therefore giving us that $\nu^{\otimes \mathbb{N}}(c_n^{-1}(V)) > 0$, implying that U is accessible from x. Since x and U were arbitrary, φ is minimal. □

Next, we want to establish some conditions in order to have two-point contractions.

Proposition 5.2. *φ is two-point contractible if and only if f admits no subperiods.*

Proof. Firstly assume that f has a subperiod $\alpha \in (0,1)$. We have seen before how this implies that α is then a subperiod for all f_{α_1}, with $\alpha_1 \in [0,1)$. Without loss of generality, we can assume $\alpha \in (0, \frac{1}{2}]$, given that our choice of metric implies diam$(\mathbb{S}^1)$=$\frac{1}{2}$. Then for all $[x],[y] \in \mathbb{S}^1$ such that $d([x],[y]) = \alpha$ it follows that for all $n \in \mathbb{N}$, and $\omega \in [0,1)^{\mathbb{N}}$ the distance is preserved under the iterations, i.e. $\mathrm{d}(\varphi(n,\omega)[x], \varphi(n,\omega)[y]) = \alpha$.

In order to see this assume (without loss of generality) $x, y \in [0,1)$ with $x = y + \alpha$ (hence $\mathrm{d}([x],[y]) = \alpha$) and consider f_{α_i} for any $\alpha_i \in [0,1)$. Then

$$\begin{aligned}
\mathrm{d}(f_{\alpha_i}([x]), f_{\alpha_i}([y])) &= \min \mid F(x + \alpha_i) - \alpha_i - F(y + \alpha_i) + \alpha_i \mid \\
&= \min \mid F(x + \alpha_i) - F(x + \alpha + \alpha_i) \mid \\
&= \min \mid F(x + \alpha_i) - F(x + \alpha_i) - \alpha \mid \qquad \text{(since } \alpha \text{ is a subperiod)} \\
&= \alpha.
\end{aligned}$$

It then follows that $\mathrm{d}(\varphi(n,\omega)[x], \varphi(n,\omega)[y]) = \alpha$, and thus φ is not two-points contractible.

In order to prove the converse, assume that f has no subperiods. Fix any $\alpha \in [0, \frac{1}{2})$ (again we can focus to only this case thanks to our choice of metric). Define a function

$$\begin{aligned}
g : \mathbb{R} &\to \mathbb{R} \\
z &\mapsto F(z + \alpha) - F(z) - \alpha
\end{aligned}$$

and since we have assumed that f has no subperiods, then there must exists at least one $q \in \mathbb{R}$ such that $g(q) \neq 0$. Assume, for now, $g(q) > 0$. Fix $\xi \in (0, g(q))$, then by continuity of F we have that for $\frac{\xi}{2} > 0$ there exists $0 < \varepsilon < \min\{\frac{\xi}{2}, 1\}$ such that for all $|x - q| < \varepsilon$ then $|F(x) - F(q)| < \frac{\xi}{2}$. This means

$$F(x) - F(q) - (x - q) < \frac{\xi}{2} + \varepsilon < \frac{\xi}{2} + \frac{\xi}{2} < g(q)$$

for all $x \in [q, q+\varepsilon)$. Recall that any number $x \in \mathbb{R}$ can be decomposed as $x = \lfloor x \rfloor + \langle x \rangle$, where $\lfloor \cdot \rfloor$ is the floor function, and $\langle \cdot \rangle$ is the fractional part function. Pick $r \in \mathbb{N}_{\geq 2}$ so that $\langle r\alpha \rangle \in [0, \varepsilon)$, so

$$
\begin{aligned}
\sum_{i=0}^{r-1} g(q+i\alpha) =& g(q) + g(q+\alpha) + \cdots + g(q+(r-1)\alpha) \\
=& \cancel{F(q-\alpha)} - F(q) - \alpha + \cancel{F(q+2\alpha)} - \cancel{F(q-\alpha)} - \alpha + \cdots + \\
& + F(q+r\alpha) - \cancel{F(q+(r-1)\alpha)} - \alpha \\
=& F(q+r\alpha) - F(q) - r\alpha \\
=& F(q + \lfloor r\alpha \rfloor + \langle r\alpha \rangle) - F(q) - \lfloor r\alpha \rfloor - \langle r\alpha \rangle \\
=& F(q + \langle r\alpha \rangle) - F(q) - \langle r\alpha \rangle \qquad \text{(by Property 1 of } F) \\
<& g(q)
\end{aligned}
$$

given that $q + \langle r\alpha \rangle \in [q, q+\varepsilon)$. By comparing the first and last line of the previous derivation we immediately notice that there must exists at least one $i \in \{1, \ldots, r-1\}$ such that $g(q+ir) < 0$. A symmetric argument can be used under the hypothesis $g(q) < 0$, in order to conclude that there must exists at least one $j \in \mathbb{N}$ such that $g(q+j\alpha) > 0$. By the intermediate value theorem, g must assume the value zero at at least one point. Therefore, by continuity of g we can fix $p \in \mathbb{R}$ such that $g(p) \in [-\alpha, 0)$.

Fix now $[x], [y] \in \mathbb{S}^1$ such that $d([x],[y]) = \alpha$. Without loss of generality, denote by x', and y' the two lifts of $[x]$, $[y]$, respectively, such that $|x' - y'| = \alpha$. If we define $\beta := \langle p - x' \rangle$ then

$$
\begin{aligned}
\mathrm{d}(f_\beta([x]), f_\beta([y])) &= \mathrm{d}([F(x'+\beta)], [F(y'+\beta)]) \\
&= \mathrm{d}([F(p)], [F(p+\alpha)]) \\
&= \min_{m,n\in\mathbb{Z}} |F(p) + n - F(p+\alpha) - m| \\
&= \min_{m,n\in\mathbb{Z}} |g(p) + \alpha + m - n| \\
&= |g(p) + \alpha| < \alpha.
\end{aligned}
$$

By continuity, we can finally find a set $U \subset [0,1)$ such that

$$\mathrm{d}(f_{\tilde{\beta}}([x]), f_{\tilde{\beta}}([y])) < \alpha$$

for all $\tilde{\beta} \in U$. Lastly, $\nu^{\otimes\mathbb{N}}(c_1^{-1}(U)) = \nu(U) > 0$, which means that φ is two-point contractible. $\square$

Lastly, we need to briefly introduce the concept of *Lyapunov exponents*. In the next few paragraphs we are going to state some specific results useful to study the example we are developing. For proofs or a more detailed discussion of all the upcoming claims I recommend reading Example 4 from [5].

Assume that if f is not a rational rotation then the Lebesgue measure l on $\mathbb{S}^1$ is the only stationary distribution for φ (to be more precise l is stationary with respect to the Markov transition probabilities induced by φ). Since l is stationary it follows that $\nu^{\otimes\mathbb{N}} \otimes l$ is the only invariant measure over $[0,1)^{\mathbb{N}} \times \mathbb{S}^1$. By the structure theorem of invariant probability measure, $\nu^{\otimes\mathbb{N}} \otimes l$ is ergodic, and so must be l.

In general, Lyapunov exponents describe the expansion rate of an orbit. When working on a space equipped with an ergodic measure it is possible to define it independently from the orbit one wants to analyse, thanks to the Birkhoff's Ergodic Theorem. In our case

$$\mathrm{Lya}^n(l) = \int_{[0,1)^{\mathbb{N}}\times\mathbb{S}^1} \log\|d\varphi(n,\omega)|_x\|(\nu^{\otimes\mathbb{N}}\otimes l)(d\omega, dx)$$

where $\|\cdot\|$ is the standard Riemannian metric on $\mathbb{S}^1$. We will assume the following sufficient test for stable trajectories holds in our specific example (its proof relies on Pesin theory and the paper [2] by Le Jan, way beyond the scope of this project).

Remark (Sufficient condition for almost sure stability)**.** *Suppose the spacial derivatives of φ are sufficiently well-controlled in (t, x, ω) over bounded ranges of t (see [2]); if there exists an $n \in \mathbb{N}$ such that $\mathrm{Lya}^n(l) < 0$ then l-almost every point in $\mathbb{S}^1$ is almost surely stable.*

Thanks to these last assumptions we are ready to prove that our example is indeed stably synchronising.

Theorem 5.3. *Suppose that f has no subperiods, and that there exists $n \in \mathbb{N}$ such that $\mathrm{Lya}^n < 0$. Then φ is stably synchronising.*

Proof. Since f has no subperiods, we can immediately discard the possibility that f is a rational rotation. By Proposition 5.1 we automatically have that $\mathbb{S}^1$ is the only minimal set in our state space. By Proposition 5.2 we have that φ is two-point contractible. Finally, thank to our remark, $\mathbb{S}^1$ admits stable trajectories, and thus by our main theorem 4.2 we can conclude φ is stably synchronising. □

6 Conclusions

The main aim of this project was to explore the sufficient and necessary conditions in order to achieve stable synchronisation. It appears evident from the example in the previous section that these conditions are in reality quite hard to prove for a general random dynamical system, especially the almost sure stability property. The condition on Lyapunov exponents does not solve the problem since computing such objects is still an open problem in many cases. It would be interesting to see whether it would be possible to weak some of the conditions in Theorem 4.2 and still obtain some sort of control over stable synchronisation.

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