M4R Research Project

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Morse Decomposition of Random and Set-valued Dynamical Systems

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Declaration of Originality

I hereby declare this is my own work except where otherwise stated.

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Abstract

We introduce the concept of Morse decomposition for topological dynamical systems, and we proceed to adapt such ideas to the classes of random and set-valued dynamical systems, while building up all the necessary technical knowledge. The reason why we look at these specific classes is given by the natural way of constructing a set-valued system from a RDS, under mild assumptions. We focus on the correspondence between attractors and Morse decompositions of random dynamical systems induced by strictly monotonically increasing random diffeomorphisms acting upon an interval and their associated set-valued counterpart. After establishing a preliminary relation between minimal invariant sets and random fixed points, we proceed to determine sufficient conditions to estimate the cardinality of the set-valued finest Morse decomposition of this class, and make some considerations on the corresponding Morse decomposition for the random system.

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Chapter 1

Introduction

Dynamical systems is the study of the time evolution of given initial conditions under a formal deterministic rule. Such evolutions are called orbits or trajectories of a point. Of particular interest is the asymptotic behaviour of such orbits; for example, this can lead to a better understanding of differential equations which cannot be solved analytically. In this project we look at the concepts of *attractors* and *repellers*: subsets of the state space which pull orbits forward, or backward in time, respectively. We work under the assumption of a compact metric state space, in order to ensure the existence of such objects.

Both of these types of sets induce an "area of influence" called the *basin of attraction* or the *basin of repulsion* for attractors and repellers, respectively. To every attractor it is possible to uniquely associate a repeller by exploiting the basin of attraction, such a construction is called an *attractor-repeller pair*. Therefore, an increasing finite chain of attractors induces a decreasing finite chain of repellers and by looking at specific intersections between these structures one can define *Morse sets*, which yield a *Morse decomposition* of the system, as in [CK14].

For every point not contained in a given Morse decomposition the asymptotic behaviour is completely described by two different Morse sets. In some cases it is possible to obtain the *finest* Morse decomposition, which gives the most complete description of the asymptotic dynamics of a system.

In this project we will be focusing on *random dynamical systems* (RDS); these are the combination of two dynamical systems where the first one, which acts upon a noise space, determines which member of a family of functions is applied to the state space at each time step. The order in which functions are selected from the family is called a noise realisation. Examples of random dynamical systems arise as solutions of stochastic differential equations, or Markov processes.

We will use the approach introduced by Ochs in 1999 ([Och99]), and later continued by Crauel, Duc and Siegmund in 2004 ([CDS04]) to adapt the definition of Morse decomposition to RDS's. In order to do so, we will formalise the concept of *weak attractors*; these rely upon convergence in probability and will lead to the definition of *(weak) Morse decomposition.*

By considering all possible noise realisations at every time step of a given RDS one induces a set-valued map, which in turn yields a *set-valued dynamical system*; this removes the random component. We will adapt the definitions of attractors and repellers in order to describe the concept of Morse decompositions for this new system. This was initially done by Li in [Li07], we will use ideas from Lamb, Rasmussen, and Rodrigues ([LRR15]) to make the connection to the single-valued counterpart clearer.

Finally, we will look at the relation between the Morse decompositions of RDS's and set-valued systems acting upon intervals. We will start from some of the ideas presented in [Ras18] on strictly monotonic increasing diffeomorphisms prescribed on a compact interval, to then independently look at the RDS and set-valued dynamics they produce. Of fundamental importance in our work will be the idea that there exists a correspondence between (minimal) attractors for the set-valued case and attractive random fixed points. Under the additional assumption of additive bounded noise for the random dynamical system, this result will be used to motivate correspondences between Morse decompositions. We will finish by showing it is possible to establish the existence of a finest Morse decomposition for the set-valued dynamical system, and how this strongly suggests the existence of a finest Morse decomposition for the associated RDS.

Let us briefly summarise how the material has been organised. The second chapter of this project aims at illustrating Morse decompositions in the case of deterministic single-valued dynamical systems. This will give motivation to why we are interested by such structures. In the third chapter we will provide all the necessary background in probability theory and random dynamical systems needed to define weak attractors, repellers, and Morse decomposition. Furthermore, we will discuss finest Morse decompositions. We will then proceed in Chapter 4 to introduce set-valued dynamical systems and to adapt the definitions given in Chapter 2 to this new case. Finally in Chapter 5 we will focus on the interval case. We will carefully set up both the random dynamical systems and their associated set-valued systems representing such a class, whilst looking at the correspondences between their attractors, and therefore their Morse decompositions. In the last section we will sharpen our focus by adding the restriction of bounded additive noise, which will turn out to be enough to determine sufficient conditions for the existence of finest Morse decomposition for set-valued systems. We will conclude the project with a brief discussion on the existence of a finest Morse decomposition for random dynamical systems.

Chapter 2

Topological Dynamical Systems

This chapter will act as an introduction and a motivation to Morse decomposition theory. We will work in the deterministic setting of topological dynamical systems, as in [CDS04] and [CK14]. Firstly, we will introduce the concept of topological dynamical systems, to then proceed to define attractors, repellers and Morse decomposition, together with some of its properties.

Definition 2.1 (Topological dynamical system)**.** Let (X, τ) be a topological space. A *topological dynamical system* is a continuous map

$$\begin{aligned} \varphi : \mathbb{R} \times X &\to X \\ (t, x) &\mapsto \varphi(t, x) \end{aligned}$$

such that the family of maps $(\varphi(t) := \varphi(t, \cdot))_{t \in \mathbb{R}}$ satisfies the flow properties:

- *Identity property*: $\varphi(0) = \mathrm{Id}_X$
- *Cocycle property*: $\varphi(t + s) = \varphi(t) \circ \varphi(s)$

for all $t, s \in \mathbb{R}$.

Remark. Clearly $(\varphi(t))_{t \in \mathbb{R}}$ is a family of homeomorphisms of X, where $\varphi^{-1}(t) = \varphi(-t)$ for all $t \in \mathbb{R}$. ⧫

From now on we will assume (X, d) is a compact metric space. Loosely speaking, we would like to define an attractor to be a set which pulls close sets towards itself. In order to quantify such a behaviour we will introduce the concept of *Hausdorff distance* and *Hausdorff semi-distance.*

1. Let x be a point and B a subset of X. Their distance is given by

$$\tilde{d}(x, B) := \inf_{b \in B} d(x, b);$$

2. This induces a semi-distance (known as the *Hausdorff semi-distance*) between any two subsets A, B of X:

$$\mathrm{dist}(A, B) := \sup_{a \in A} \tilde{d}(a, B) = \sup_{a \in A} \inf_{b \in B} d(a, b).$$

   This operator might not be symmetric and $\mathrm{dist}(A, B) = 0$ does not imply $A = B$, hence why the name semi-distance;

3. In order to turn the Hausdorff semi-distance into a distance it is enough to define for $A, B \subset X$
$$d_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\}.$$

The compactness of X ensures the Hausdorff distance between two non-empty sets always takes finite values. This now allows to state the definition of attractor and repeller.

Definition 2.2. Given a topological dynamical system φ over a compact metric space (X, d) we define:

1. *Attractor*: Let $A \subseteq X$ be a non-empty, compact, and invariant under φ set, i.e. $\varphi(t)A = A$ for all $t \in \mathbb{R}$. A is said to be an attractor if and only if there exists an open neighbourhood U of A such that
$$\lim_{t\to\infty} \text{dist}(\varphi(t)U, A) = 0.$$
U is said to be the *attractor neighbourhood* of A.

2. *Repeller*: Let $R \subseteq X$ be a non-empty, compact, and invariant under φ set. R is said to be a repeller if and only if there exists an open neighbourhood V of R such that
$$\lim_{t\to-\infty} \text{dist}(\varphi(t)V, R) = 0.$$
V is said to be the *repeller neighbourhood* of R.

The above definition of attractor and repeller can be restated in terms of ω-limit and α-limit sets, and hence avoiding the use of the Hausdorff distance. Recall that for a set $Y \subset X$, we define its ω-limit set:

$$\omega(Y) = \{y \in X \mid \exists (y_n)_{n\in\mathbb{N}} \text{ in } Y \text{ and } t_n \to \infty \text{ in } \mathbb{R} \text{ s.t. } \lim_{n\to\infty} \varphi(t_n, y_n) = y\}$$

and, similarly, its α-limit set

$$\alpha(Y) = \{y \in X \mid \exists (y_n)_{n\in\mathbb{N}} \text{ in } Y \text{ and } t_n \to \infty \text{ in } \mathbb{R} \text{ s.t. } \lim_{n\to-\infty} \varphi(t_n, y_n) = y\}.$$

Remark. Restricting our attention to compact metric spaces ensures us that none of the above defined sets is ever going to be empty. Furthermore, notice that

$$\bigcup_{y\in Y} \omega(\{y\}) \subset \omega(Y)$$

and an identical inclusion holds for α-limit sets. This follows from the fact that the sequence of points in Y is allowed to vary (see Example 2.1.2). ♦

Definition 2.2 is the one that will be use as an inspiration in the later chapters to adapt the concepts of attractors and repeller to different classes of dynamical systems. The concept of ω-limit set and α-limit set can be used to equivalently redefine such concepts.

Definition 2.3. A compact, invariant set A is an attractor if it admits a neighbourhood $U \subset X$ such that $A = \omega(U)$. Similarly, a compact, invariant set R is a repeller if it admits a neighbourhood $V \subset X$ such that $V = \alpha(V)$.

Notice that for the time-reversed system $\hat{\varphi}(t, x) = \varphi(-t, x)$ attractors correspond to repellers and vice versa. This second definition is usually preferred when working with topological dynamical systems since it avoids the use of Hausdorff distances and it allows us to efficiently define the concept of attractor-repeller pair.

Proposition 2.4. *Let φ be a topological dynamical system over a compact metric space (X, d), and let A be an attractor. Then*

$$R := \{x \in X \mid \omega(x) \cap A = \emptyset\}$$

is a repeller. (A, R) is called an attractor-repeller pair.

Proof. In order to prove this proposition we first need to establish the following result.
Claim. For every attractor neighbourhood N there exists $t^* > 0$ such that $\overline{\varphi([t^*, \infty), N)} \subset \mathring{N}$.

Without loss of generality we can assume N is a closed attractor neighbourhood. Since N is a neighbourhood of A, we can find $\varepsilon > 0$ for which $B_\varepsilon(A) := \bigcup_{a \in A} B_\varepsilon(a) \subset \mathring{N}$, where $B_\varepsilon(a)$ are the usual open balls, with respect to the metric d, around a point a with radius ε. Now we would like to show there exists $t^* > 0$ such that $\varphi([t^*, \infty), N) \subset B_{\frac{\varepsilon}{2}}(A)$. Suppose for a contradiction this was not true, then we could find a sequence (t_n, x_n) with $x_n \in N$, and $t_n \to \infty$ such that $\varphi(t_n, x_n) \notin B_{\frac{\varepsilon}{2}}(A)$. This contradicts the fact that $A = \omega(N)$, hence there must exist $t^* > 0$ for which

$$\varphi([t^*, \infty), N) \subset B_{\frac{\varepsilon}{2}}(A) \subset \mathring{N}.$$

From this we can infer $\overline{\varphi([t^*, \infty), N)} \subset \overline{B_{\frac{\varepsilon}{2}}(A)}$, but this is exactly what we wanted since:

$$\overline{\varphi([t^*, \infty), N)} \subset \overline{B_{\frac{\varepsilon}{2}}(A)} \subset B_\varepsilon(A) \subset \mathring{N}.$$

We can now proceed proving the main proposition.
Claim. $R = \{x \in X \mid \omega(x) \cap A = \emptyset\}$ is a repeller.

Let N be a compact attractor neighbourhood of A, and choose $t^* > 0$ such that $\varphi([t^*, \infty), N) \subset N$. Define an open set

$$V := X \backslash \overline{\varphi([t^*, \infty), N)}$$

and we claim $\alpha(V) = R$. Notice that $X = N \cup V$, and $\varphi((-\infty, -t^*], V) \subset X \backslash N$. The latter can be proven by contradiction. Suppose there exists a $p \in N$ for which we can find a time $t \geq t^*$, and $x \in V$ so that $p = \varphi(-t, x)$. Thanks to the cocycle property $x = \varphi(t, \varphi(-t, x)) = \varphi(t, p)$, which means that $x \in \varphi([t^*, \infty), N)$, contradicting the fact $x \in V := X \backslash \overline{\varphi([t^*, \infty), N)}$.

So we have that $\alpha(V) \subset X \backslash N \subset V$. By construction $\alpha(V)$ is a repeller with repelling neighbourhood V. For every $x \in \alpha(V)$ then we immediately get that $\omega(x) \subset \alpha(V)$, by the invariance of limit sets. Since $\alpha(V) \cap A = \emptyset$, then $\alpha(V) \subset R$.

We are left with the other direction of the inclusion to prove. If $x \in R$ then $x \notin N$, thus $x \in V$ and $\omega(x) \cap A = \emptyset$. This is equivalent to have that for all $t \in \mathbb{R}$ then $\omega(\varphi(t, x)) \cap A = \emptyset$, thanks to the invariance of limit sets. We can hence infer $\varphi(t, x) \in R \subset V$, for all times t. By using the identity and cocycle properties of topological dynamical systems we can build a sequence of points in R and times $t_n \to \infty$ so that we have $x \in \alpha(V)$, as we wanted. Therefore $R = \alpha(V)$, or equivalently R is a repeller.

□

2.1 Morse Decomposition

We have just established that attractors and repellers always come in pairs, and this holds even for some interesting limit cases. We will assume that $\emptyset$ is always an attractor, therefore if X is the underlying space of our dynamical system, then X is its associated repeller, and vice versa. This will allow us, given a chain of attractors $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X$, to use their associated repellers to create the bare bones of the asymptotic behaviour of our dynamical system.

Definition 2.5. Let φ be a topological dynamical system over a compact metric space (X, d), and for $i = 1, \dots, n$, let the tuples (A_i, R_i) be attractor-repeller pairs such that:

$$\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = X \qquad \text{and} \qquad X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset$$

Then the family $(\mathcal{M}_i)_{i=1,\dots,n}$ of subsets of X defined by

$$\mathcal{M}_i = A_i \cap R_{i-1}$$

for $1 \leq i \leq n$, is called a *Morse decomposition of* X, and each $\mathcal{M}_i$ is called a *Morse set.*

This is the most intuitive way of defining a Morse decomposition, and it will be mimic in later chapters when adapting this notion to more complicated systems. Nevertheless, this is not the only possible way of defining such a structure. Before we look at an equivalent statement let us introduce the concept of isolated invariant sets:

Definition 2.6. A set $K \subset X$ is said to be *isolated invariant* if $\varphi(t, K) = K$ for all $t \in \mathbb{R}$, and there exists a neighbourhood N of K, containing K in its interior, such that if $\varphi(t, x) \in N$ for all $t \in \mathbb{R}$ then $x \in K$.

Example 2.1.1. Consider the following ODE over the interval $[0, 1] \subset \mathbb{R}$ since

$$\dot{x} = \begin{cases} x(\cos(\frac{\pi}{x}) - 1) & \text{for } x \in (0, 1] \\ 0 & \text{for } x = 0 \end{cases}$$

The equilibria of this system are $x_0 = 0$ and $x_n = \frac{1}{2n}$, for $n \in \mathbb{N}_0$ since $\cos(\frac{\pi}{x_n}) - 1 = \cos(2\pi n) - 1 = 0$. For every $n \in \mathbb{N}_0$, $\{x_n\}$ is an invariant isolated set, whereas $\{x_0\}$ is not isolated. Indeed, for every neighbourhood of 0 it is possible to find an n big enough, such that $x_n \neq 0$ belongs to the neighbourhood.

Thanks to this new notion we are ready to give an equivalent definition of Morse decomposition.

Definition 2.7. Given a topological dynamical system φ over a compact metric space (X, d) we call a finite collection of non empty, pairwise disjoint, compact isolated invariant sets $(\mathcal{M}_i)_{i=1,\dots,n}$ a *Morse decomposition of* X if and only if

1. for all $x \in X$ then the limit sets $\omega(x)$, $\alpha(x) \subset \bigcup_{i=1}^{l} \mathcal{M}_i$;

2. suppose there are $\mathcal{M}_{j_0}, \mathcal{M}_{j_1}, \dots, \mathcal{M}_{j_n}$ and $x_1, \dots, x_n \in X \setminus \bigcup_{i=1}^{l} \mathcal{M}_i$ with $\alpha(x_i) \subset \mathcal{M}_{j_{i-1}}$ and $\omega(x_i) \subset \mathcal{M}_{j_i}$ for $i = 1, \dots, n$, then $\mathcal{M}_{j_0} \neq \mathcal{M}_{j_n}$.

We will call $\mathcal{M}_i$ *Morse sets.*

A proof of the equivalence between Definition 2.5 and Definition 2.7 can be found in chapter 8 of [CK14]. This second definition contains all the most important properties of Morse decompositions, and it allows us to introduce the concept of ordering of Morse sets.

Proposition 2.8. *Let $\{\mathcal{M}_1, \dots, \mathcal{M}_n\}$ be a Morse decomposition, then we can define an order by the relation $\mathcal{M}_i \preceq \mathcal{M}_j$ if there exist indices $j_0, \dots, j_n$ such that $j_0 = i$, $j_n = j$ and*

$$\alpha(x_{j_i}) \subset \mathcal{M}_{j_{i-1}} \text{ and } \omega(x_{j_i}) \subset \mathcal{M}_{j_i} \qquad \text{for } i = 1, \dots, n.$$

Proof. We have to check this relation is an equivalence relation, but reflexivity and transitivity are immediate. By the second property of Definition 2.7, we have that if $\mathcal{M}_i \preceq \mathcal{M}_j$ and $\mathcal{M}_j \preceq \mathcal{M}_i$ then $\mathcal{M}_i = \mathcal{M}_j$ □

This last results, essentially, tells us that the flow of a dynamical systems goes from a smaller to a bigger set (with respect to the order induced by $\preceq$). Given two decompositions $\{\mathcal{M}_1, \dots, \mathcal{M}_l\}$ and $\{\mathcal{M}'_1, \dots, \mathcal{M}'_{l'}\}$, we say that the first one is *finer* than the second one if for any $j \in \{1, \dots, l'\}$ there exists $i \in \{1, \dots, l\}$ such that $\mathcal{M}_i \subset \mathcal{M}'_j$. A *finest Morse decomposition* $(\tilde{\mathcal{M}}_i)$, if it exists, it is characterised by the fact that any finer decomposition is equal to $(\tilde{\mathcal{M}}_i)$, and therefore it is unique. Furthermore, it is possible, given two decompositions, or finitely many of them, to generate a new Morse decomposition by considering all the possible intersections between all the various Morse sets (even when empty). In [CK14] is possible to find examples of why the intersection of infinitely many decompositions not always define a Morse decomposition.

Now would like now to present a fully worked example on how to compute ω and α-limit sets, and Morse decompositions. This combines Example 8.1.7 and Example 8.2.12 in [CK14].

Example 2.1.2. Consider the following ODE

$$\dot{x} = x(x-1)(x-2)^2(x-3)$$

over the compact interval $X = [0, 3]$, equipped with the Euclidean metric (notice that this example can be easily extended to any finite number of terms). Now let $\phi(t, x)$ be the solution to the differential equation with initial condition $\phi(0, x) = x$ for $x \in X$. These solutions are unique and exist at all time $t \in \mathbb{R}$, by Picard-Lindelöf theorem. Define the dynamical system:

$$\begin{aligned} \varphi : \mathbb{R} \times X &\to X \\ (t, x) &\mapsto \varphi(t, x) = \phi(t, x). \end{aligned}$$

For simplicity of notation let $\omega(x) \coloneqq \omega(\{x\})$. Hence

$$\omega(x) = \begin{cases} \{0\} & \text{for } x = 0 \\ \{1\} & \text{for } x \in (0, 2) \\ \{2\} & \text{for } x \in [2, 3) \\ \{3\} & \text{for } x = 3 \end{cases} \quad \text{and} \quad \alpha(x) = \begin{cases} \{0\} & \text{for } x \in [0, 1) \\ \{1\} & \text{for } x = 1 \\ \{2\} & \text{for } x \in (1, 2] \\ \{3\} & \text{for } x \in (2, 3] \end{cases}$$

where these can be calculated by considering when $\dot{x}$ is positive over X, and by noticing that $0, 1, 2, 3$ are all fixed points of φ. What happens when we consider the ω-limit set of a set? For example we claim that for $Y = [a, b]$ where $a \in (0, 1]$ and $b \in [2, 3)$, we have $\omega(Y) = [1, 2]$. Firstly, $[1, 2] \in \omega(Y)$, since by the above discussion we immediately have $\{1,2\} \in \omega(Y)$ and for every $x \in (1,2)$ we know $\lim_{t\to-\infty} \varphi(t,x) = 2$. Thanks to this latter consideration, we can define a sequence for every $n \in \mathbb{N}$ with $t_n = n$ and $x_n = \phi(-n,x) \in (1,2) \subset Y$ so that

$$\varphi(t_n, x_n) = \varphi(n, \varphi(-n,x)) = \varphi(0,x) = x$$

for every $n \in \mathbb{N}$, thanks to the cocycle and identity properties. Since this procedure holds for any $x \in (1,2)$ we can conclude $[1,2] \subset \omega(Y)$. For the reversed direction of inclusion, note that $\lim_{t\to\infty} \varphi(t,a) = 1$, and for all $y \in (a,1]$ and $t \geq 0$ the following holds

$$\mathrm{d}(\varphi(t,y),1) \leq \mathrm{d}(\varphi(t,a),1)).$$

Consider a sequence of points $(y_n)_{n\in\mathbb{N}}$ in $[a,1)$, and $t_n \to \infty$, then we can conclude

$$\lim_{n\to\infty} \mathrm{d}(\varphi(t_n,y_n),1) \leq \lim_{n\to\infty} \mathrm{d}(\varphi(t_n,a),1) = 0.$$

We can therefore infer that no point $x \in (0,1)$ can be in $\omega(Y)$, and by a symmetric argument, no point in $(2,3)$ can be in $\omega(Y)$ either. Obviously, $0,3 \notin \omega(Y)$, and therefore we can conclude $\omega(Y) = [1,2]$. If we let $Y = [1,2]$ then

$$\bigcup_{x\in[1,2]} \omega(x) = \{1,2\} \subset [1,2] = \omega([1,2])$$

We can compute few Morse decompositions for this example using Definition 2.5. We can use our previous computation to construct the following chains of attractors and associated repellers):

$$\begin{aligned} A_0 = \emptyset \subsetneq A_1 = [1,2] \subsetneq A_2 = X \\ R_0 = X \supsetneq R_1 = \{0\} \cup \{3\} \supsetneq R_2 = \emptyset. \end{aligned}$$

These generate the following Morse sets (recall $\mathcal{M}_i = A_i \cap R_{i-1}$):

$$\begin{aligned} \mathcal{M}_1 &= [1,2] (= A_1) \\ \mathcal{M}_2 &= \{0\} \cup \{3\} (= R_1). \end{aligned}$$

We can establish $\mathcal{M}_2 \preceq \mathcal{M}_1$. A different choice of attractors might induce a different Morse decomposition. Indeed if we consider the following two chains:

$$\begin{aligned} A'_0 = \emptyset \subsetneq A'_1 = \{1\} \subsetneq A'_2 = [1,2] \subsetneq A'_3 = [1,3] \subsetneq A'_4 = X \\ R'_0 = X \supsetneq R'_1 = \{0\} \cup [2,3] \supsetneq R'_2 = \{0,3\} \supsetneq R'_3 = \{0\} \supsetneq R'_4 = \emptyset \end{aligned}$$

these generate

$$\begin{aligned} \mathcal{M'}_1 &= \{1\} \cap [0,3] = \{1\} (= A'_1) \\ \mathcal{M'}_2 &= [1,2] \cap (\{0\} \cup [2,3]) = \{2\} \\ \mathcal{M'}_3 &= [1,3] \cap \{0,3\} = \{3\} \\ \mathcal{M'}_4 &= [0,3] \cap \{0\} = \{0\} (= R'_3). \end{aligned}$$

This is actually the finest Morse decomposition, and $\mathcal{M'}_4 \preceq \mathcal{M'}_1$, and $\mathcal{M'}_3 \preceq \mathcal{M'}_2 \preceq \mathcal{M'}_1$. Note that for any Morse decomposition of n elements, we will always have

$$\mathcal{M}_1 = A_1 \cap R_0 = A_1 \cap X = A_1$$

and

$$\mathcal{M}_n = A_n \cap R_{n-1} = X \cap R_{n-1} = R_{n-1},$$

hence our first element is always an attractor, whereas the last one is always a repeller.

Chapter 3

Random Dynamical Systems

We will now shift our attention towards random dynamical systems (RDS's). In contrast to topological dynamical systems, these objects allow the qualitative study of processes perturbed by random noise. RDS's naturally arise as solutions of a large class of stochastic differential equations, and they are strictly connected with Markov processes as we will soon see.

In this chapter, after a brief discussion about some of the elementary properties of random dynamical systems, we will look at stationary and invariant measures, and establish a connection between the two. In the third section we will develop all the needed theory on random sets in order to be able to define weak attractors and weak repellers. Finally we look into (weak) Morse decompostions and what we mean by finest decomposition. Most of the notions defined in the following sections are been adapted from [KS12], [CDS04] and [Cal14].

3.1 Preliminaries

We will denote by $\mathbb{T}$ the set which keeps track of the time evolution of our RDS. It can be equal to $\mathbb{R}$ or $\mathbb{Z}$ if we are talking about *continuous* or *discrete*, respectively, *two-sided time*, or $\mathbb{R}_0^+ \coloneqq \{x \in \mathbb{R} \mid x \geq 0\}$ or $\mathbb{N}_0 \coloneqq \mathbb{N} \cup \{0\}$ if we are talking about *continuous* or *discrete*, respectively, *one-sided time*. We will almost immediately assume to be working with two sided times since it is the most natural environment for defining attractors and repellers, since they describe the forward and backward asymptotic behaviour of the system.

Definition 3.1 (Random dynamical systems)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and define the function

$$\theta : \mathbb{T} \times \Omega \to \Omega$$

to be measurable with respect to $\mathcal{F}$. Furthermore, assume that for all $t, s \in \mathbb{T}$ we have $\theta_{s+t} = \theta_s \circ \theta_t$ where

$$\begin{aligned} \theta_t : \Omega &\to \Omega \\ \omega &\mapsto \theta_t(\omega) = \theta(t, \omega). \end{aligned}$$

and θ_t preserves $\mathbb{P}$ for every $t > 0$, i.e. $(\theta_t)^*\mathbb{P} \coloneqq \mathbb{P} \circ \theta_t^{-1} = \mathbb{P}$, and $(\theta_t)_{t \in \mathbb{T}}$ is ergodic.

Given the tuple $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{T}})$ and a metric space (X, d), let

$$\varphi : \mathbb{T} \times \Omega \times X \to X$$

be a measurable map for which the *cocycle property*

$$\varphi(t+s, \omega) = \varphi(t, \theta_s\omega) \circ \varphi(s, \omega)$$

holds for every $t, s \in \mathbb{T}$, and where $\varphi(t, \omega) : X \to X$ is such that $\varphi(t, \omega)x = \varphi(t, \omega, x)$. This latter function is assumed to be continuous for all $t \in \mathbb{T}$ and $\omega \in \Omega$. Finally, assume that the *identity property* holds for any $\omega \in \Omega$

$$\varphi(0, \omega) = \mathrm{Id}_X.$$

φ will be said *random dynamical system.*

From this definition it is clear that a random dynamical system is the combination of two different systems: θ acting on Ω and influencing the iterations of $\varphi(t, \omega)$ over X. The prospective of *skew-products* will allow us to keep track of the time evolution of the two systems at the same time. That is done by defining a mapping

$$\begin{aligned} \Theta : \mathbb{T} \times \Omega \times X &\to \Omega \times X \\ (t, \omega, x) &\mapsto (\theta_t\omega, \varphi(t, \omega)x) \end{aligned}$$

which is $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B}(X), \mathcal{F} \otimes \mathcal{B}(X))$-measurable. It will be convenient to think of this object as a semigroup. For any $t \in \mathbb{T}$ one can define $\Theta^t : \Omega \times X \to \Omega \times X$ as $\Theta^t(\omega, x) = \Theta(t, \omega, x) = (\theta^t\omega, \varphi(t, \omega)x)$, and therefore define a semigroup $(\Theta^t)_{t\in\mathbb{T}}$ thanks to the cocycle property of φ, and the fact that $(\theta^t)_{t\in\mathbb{T}}$ is a semigroup as well. This is steadily checked for any $t, s \in \mathbb{T}$, $\omega \in \Omega$ and $x \in X$:

$$\begin{aligned} \Theta^s(\Theta^t(\omega, x)) &= \Theta^s(\theta_t\omega, \varphi(t, \omega)x) \\ &= (\theta_s\theta_t\omega, \varphi(s, \theta_t\omega)(\varphi(t, \omega)x)) \\ &= (\theta_{s+t}\omega, \varphi(s+t, \omega)x) = \Theta^{s+t}(\omega, x). \end{aligned}$$

From now onward we are always going to assume the following:

- The state space X is compact and a Polish metric space (complete separable metric space) equipped with its Borel σ-algebra $\mathcal{B}(X)$.
- The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete. A probability space is said to be *complete* if for all $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ then for all $A \subset B$ it follows that $A \in \mathcal{F}$.
- For each $\omega \in \Omega$ the function $\varphi(\cdot, \omega, \cdot) : \mathbb{R} \times X \to X$ is continuous.
- $\mathbb{T}$ is double sided, and hence $\varphi(t, \omega)^{-1} = \varphi(-t, \omega)$. Hence, $\varphi(t, \omega)$ is an homeomorphism of X for all $t \in \mathbb{T}$, and $\omega \in \Omega$.

Recall that a *filtration* is an ascending chain of sub-σ-algebras with respect to set inclusion. In symbols, if I is a fully ordered index set, then $(\mathcal{F}_i)_{i\in I}$ is a filtration of the σ-algebra $\mathcal{F}$ if and only if every $\mathcal{F}_i$ is a sub-σ-algebra of $\mathcal{F}$ and for $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t$.

Any given random dynamical system naturally induces a filtration on its σ-algebra $\mathcal{F}$. Given any $p, q \in \mathbb{T}$ with $p < q$, let $\mathcal{F}_{[p,q]}$ the sub-σ-algebra generated by the random variables $\omega \mapsto \varphi(t, \theta_s\omega)u : \Omega \to X$ together with all the sets of measure zero of $\mathcal{F}$, where $p \leq s < q$, and $0 < t \leq q - s$ and $u : \Omega \to X$ is a random variable. This definition immediately tells us that for any $p, q \in \mathbb{T}$ then the random variable $\omega \mapsto \varphi(q, \theta_p\omega)u$ is $\mathcal{F}_{[p,p+q]}$-measurable. We can define the following important σ-algebras:

$$\begin{aligned}
\mathcal{F}_{[-\infty,q]} &= \sigma(\mathcal{F}_{[p,q]} \mid p \in \mathbb{T}, p < q)\\
\mathcal{F}_{[q,\infty]} &= \sigma(\mathcal{F}_{[q,p]} \mid p \in \mathbb{T}, p > q)\\
\mathcal{F}_{[-\infty,\infty]} &= \sigma(\mathcal{F}_{[p,q]} \mid p, q \in \mathbb{T}, p < q).
\end{aligned}$$

These allow us to formally define the *filtration induced by* φ as $(\mathcal{F}_t)_{t\in\mathbb{T}}$, where $\mathcal{F}_t := \mathcal{F}_{[-\infty,t]}$. Filtrations represent our knowledge of the behaviour of a certain process up to a specific point in time; they allow us to make sense of concept such as past and future. With this in mind we call $\mathcal{F}_- := \mathcal{F}_{[-\infty,0]}$ the *past of* φ and $\mathcal{F}_+ := \mathcal{F}_{[0,\infty]}$ the *future of* φ.

Definition 3.2. A random dynamical system φ is said to be *Markov* if the past and future σ-algebra are independent.

We now shift our attention to the study of the trajectories of points. This can be done in few different ways, but one of the most useful is to approach the problem from a probabilistic viewpoint by exploiting the Markov chain naturally induced by every random dynamical system. Indeed, the function φ gives us a natural way of defining the transition probabilities of such a process. For $x \in X$, for all $t \geq 0$, and $A \in \mathcal{B}(X)$ define

$$T_x^t(A) := \mathbb{P}(\{\omega \in \Omega \mid \varphi(t, w)x \in A\}).$$

Denote by $T := (T_x^t)_{x\in X, t\geq 0}$ the set of all transition probabilities. Thanks to this we can now introduce the concept of stationary measure.

Definition 3.3. A probability measure $\rho : \mathcal{B}(X) \mapsto [0, 1]$ is said to be a *stationary measure* if for all $A \in \mathcal{B}(X)$ and all $t \geq 0$

$$\rho(A) = \int_X T_x^t(A)\, \rho(dx).$$

Furthermore, such a measure is said to be ergodic if and only if for all invariant sets $A \in \mathcal{B}(X)$, i.e. $T_x^t(A) = 1$ for all $t \geq 0$ and for ρ-almost every $x \in A$, then $\rho(A)$ has either zero or full measure.

Definition 3.3 is implicitly using the concept of *Markov operator*. Let $\mathcal{P}(X)$ denote the space of probability measures over X then we can define the semigroup of Markov operators for any $t \geq 0$

$$\begin{aligned}
\mathfrak{B}^t &: \mathcal{P}(X) \to \mathcal{P}(X)\\
&\mu \mapsto \mathfrak{B}^t\mu(\cdot) = \int_X T_x^t(\cdot)\, \mu(dx)
\end{aligned}$$

and it represent the push-forward of a measure induced by the Markov chain. With this new notation, stationary measures ρ of a random dynamical system correspond to invariant measures of the Markov operator $\mathfrak{B}^t\rho = \rho$, for all $t \geq 0$.

Denote by $\mathcal{B}_b(X)$ the space of all bounded Borel measurable functions mapping X to $\mathbb{R}$, then it is possible to define the dual operator to $\mathfrak{B}^t$ as follows:

$$\begin{aligned}
\mathfrak{B}_*^t &: \mathcal{B}_b(X) \to \mathcal{B}_b(X)\\
&f \mapsto (\mathfrak{B}_*^t f)(x) := \int_X f(y) T_x^t(dy)
\end{aligned}$$

As a last definition, which will be useful in the later sections, we introduce the concept of Feller operator. Let $\mathcal{C}_b(X)$ be the space of continuous and bounded functions from X to $\mathbb{R}$.

Definition 3.4. The semigroup $(\mathfrak{B}^t_*)_{t\geq 0}$ is said to be *Feller* if for any $f \in \mathcal{C}_b(X)$ then $\mathfrak{B}^t_* f \in \mathcal{C}_b(X)$, for all $t \geq 0$.

Since the semigroup $(\mathfrak{B}^t_*)_{t\geq 0}$ is completely determined by the transition probabilities T, then one can simply say that T is Feller.

In this specific case, where our Markov process is being induced by a random dynamical system, we automatically get that its transition probabilities are Feller, since $\varphi(t,\omega)$ is continuous over X for all $(t,\omega) \in \mathbb{T} \times \Omega$ (see [KS12]).

3.2 Stationary and skew-product invariant measures

We briefly mentioned how one can induce a Markov process from a random dynamical system. This section aims at exploring a useful connection between the two viewpoints from a measure-theoretical perspective. Let us now recall some useful results and definitions from probability theory. Just for the next result assume X is only a topological space.

Definition 3.5 (Tightness)**.** Let M be a collection of probability measures defined on a measurable space $(X, \mathcal{B}(X))$. Then M is said to be *tight* if for all $\varepsilon > 0$ there exists a compact set $K(\varepsilon) \subset X$ such that for all $\mu \in M$ then

$$\mu(K_\varepsilon) \geq 1 - \varepsilon.$$

Tightness can be seen as a way of saying that the mass of every measure of the collection is mostly concentrated on a compact set. Suppose now that X is a Polish space, and that we equip $\mathcal{P}(X)$, the space of probability measures defined over $(X, \mathcal{B}(X))$, with its weak*-topology. If we consider a sequence of measures $(\mu_n)_{n\in\mathbb{N}} \subset \mathcal{P}(X)$ then it follows (by Prohorov's theorem [Li18]) that there exists a measure $\mu \in \mathcal{P}(X)$ and a subsequence $(\mu_{n_j})_{j\in\mathbb{N}}$ such that $\mu_{n_j} \rightharpoonup \mu$ weakly. Thus, tightness can be understood in the sense of sequentially compactness.

Tightness is a surprisingly useful property and can be used to show existence of stationary measures for a Markov process.

Theorem 3.6 (Krylov-Bogolyubov theorem)**.** *Let T be a Feller transition probability set of a Markov process defined on a Polish metric space X. Suppose there exists a point $z \in X$ such that $\{T^t_z(\cdot)_{t\geq 0}\}$ is tight, then the Markov process admits at least one stationary measure.*

A proof for this well known result can be found in [Li18]. In the special case of X compact then we have that Krylov-Bogolyubov theorem always holds.

Corollary 3.7. *Suppose that T is a Feller transition probabilities set over a compact Polish metric space X. Then the Markov process admits at least one stationary measure.*

Proof. Since X is compact and all the measures in T are probability measures whose support is in X then for any $z \in X$ and $t \geq 0$, we have $T^t_z(X) = 1$. This means that T is tight, and Feller by assumption. By Krylov-Bogolyubov theorem this corollary follows. □

Thanks to the initial assumptions we stated in the previous section it is always possible to find at least one stationary measure. We are now interested in investigating possible relations between stationary measures and invariant measures for the skew-product Θ.

Let us denote by $\mathcal{P}(\Omega \times X, \mathbb{P})$ the space of probability measures defined on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X))$, whose projection on Ω is precisely $\mathbb{P}$.

Definition 3.8. A measure $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ said to be Θ-*invariant* if and only if for all $A \in \mathcal{F} \otimes \mathcal{B}(X)$, and $t \geq 0$ then $(\Theta^t)^*\eta(A) = \eta((\Theta^t)^{-1}(A)) = \eta(A)$. Furthermore, we say that η is Θ-*ergodic* if for any set $A \in \mathcal{F} \otimes \mathcal{B}(X)$ such that $\eta((\Theta^t)^{-1}(A) \setminus A) = 0$, then $\eta(A) \in \{0, 1\}$.

Remark. We have assumed to be working under the assumption of $\mathbb{T}$ being two-sided. This would technically allow us to use the fact

$$(\Theta^t)^*\eta(A) = \eta((\Theta^t)^{-1}(A)) = \eta(\Theta^{-t}(A)),$$

thus allowing us to extend the previous definition to every $t \in \mathbb{T}$. The decision to restrict our attention to only positive times allows us to state a more general definition, which can be used in different contexts and it is consistent with the literature. ♦

Since invariant measures for Θ are defined over a product space it is natural to wonder if there is a way to split such measures into the relative components on each space. Firstly

Definition 3.9. A family of measures $\{\mu_\omega\}_{\omega\in\Omega} \subset \mathcal{P}(X)$ is said to be a *random probability measure* if for all $A \in \mathcal{B}(X)$ the function $\omega \to \mu_\omega(A)$ going from Ω to $\mathbb{R}$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable.

So far we have been considering the space $\mathcal{P}(\Omega \times X, \mathbb{P})$ where the Ω-projection of every measure is $\mathbb{P}$, but what can we say about the projection on the space X? In order to answer this question we must introduce the concept of disintegration of measures.

Proposition 3.10. *Every measure $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ admits a disintegration*

$$\eta(d\omega, dx) = \mu_\omega(dx)\mathbb{P}(d\omega)$$

where $\{\mu_\omega\}_{\omega\in\Omega}$ is a random probability measure on X. This decomposition is unique up to a set of measure zero, which means that if $\{\mu'_\omega\}_{\omega\in\Omega}$ is a second disintegration of η then $\mu_\omega = \mu'_\omega$ for almost every $\omega \in \Omega$.

A proof of this proposition can be found in [Dud02]. This is particularly useful when computing integrals. Let $f : \Omega \times X \to \mathbb{R}$ be a bounded measurable function then

$$\int_{\Omega\times X} f(\omega, x)\eta(d\omega, dx) = \int_\Omega \int_X f(\omega, x)\mu_\omega(dx)\mathbb{P}(d\omega)$$

Now we are ready to answer our initial question: what is the projection of our measure η on X? This is simply given by the expectation of its disintegration. If π_X represents the projection on X then

$$(\pi_X)^*\eta = \mathbb{E}(\mu_\omega) = \int_\Omega \mu_\omega \mathbb{P}(d\omega).$$

Remark. Disintegrations give us a condition to check Θ-invariance of a measure. Let $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$, and denote by $\{\mu_\omega\}$ its disintegration. Then η is Θ-invariant, if and only if for any $t \geq 0$, and for almost every $\omega \in \Omega$

$$(\varphi(t,\omega))^* \mu_\omega = \mu_{\theta_t \omega}$$

For a proof see [KS12]. ♦

Recall that $(\mathcal{F}_t)_{t\in\mathbb{T}}$ denotes the filtration induced by a RDS φ, and that we called $\mathcal{F}_-$ or $\mathcal{F}_0$ the past of φ. In what follows we want to consider Θ-invariant measures on $\Omega \times X$, with the specific property that their disintegration is measurable with respect to the past.

Definition 3.11. Let $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ be a Θ-invariant measure. η is said to be *Markov* if its disintegration $\{\mu_\omega\}$ is $\mathcal{F}_0$ measurable. This is equivalent to saying that for any $A \in \mathcal{B}(X)$, the mapping $\omega \mapsto \mu_\omega(A)$ is $(\mathcal{F}_0, \mathcal{B}(\mathbb{R}))$-measurable.

Actually in order to ensure the Markov property for a measure it is necessary and sufficient to show that its disintegration is $\mathcal{F}_t$ measurable, for some $t \in \mathbb{T}$ (Proposition 4.2.8 in [KS12]).

We are now ready to state the main results of this section. This theorem is going to give us a way of relating (Markov) Θ-invariant measures and stationary measures in Markov random dynamical systems.

Theorem 3.12 (Ledrappier-Le Jan-Crauel theorem)**.** *Let φ be a Markov random dynamical system, then the following hold:*

1. *Let $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ be a Markov Θ-invariant measure, and let $\{\mu_\omega\}$ denote its disintegration. Then $\mathbb{E}(\mu_\omega)$ is a stationary measure for φ. If $\eta' \in \mathcal{P}(\Omega \times X, \mathbb{P})$ is a second Markov Θ-invariant measure with disintegration $\{\mu'_\omega\}$, and such that $\mathbb{E}(\mu_\omega) = \mathbb{E}(\mu'_\omega)$, then $\eta = \eta'$.*

2. *Suppose that μ is a stationary measure for φ, then for every sequence $t_k \to +\infty$, there exists a set $\tilde{\Omega}$, with $\mathbb{P}(\tilde{\Omega}) = 1$, such that for every $\omega \in \Omega$ the following weak limit exists:*

$$\mu_\omega = \lim_{k\to\infty} (\varphi(t_k, \theta_{t_k}\omega))^* \mu.$$

Furthermore, given any other sequence $t'_k \to +\infty$ then the family of measures $\{\mu'_\omega\}$ it defines agree with $\{\mu_\omega\}$ almost surely. Finally, the measure $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ defined by the disintegration $\{\mu_\omega\}$ is a Markov Θ-invariant measure, and its projection on X is precisely $(\pi_X)^*\eta = \mathbb{E}(\mu_\omega) = \mu$.

Thus, this theorem gives a one-to-one correspondence between Markov Θ-invariant measures and stationary measures for φ.

Proof. See Theorem 4.2.9 in [KS12]. □

Remark. In the second part of the theorem we inferred that it is possible to construct a measure over $\Omega \times X$ from given a disintegration. This is true, but since the family of measured constructed by limit is only defined for almost every ω, we just need to impose for every $\omega \in \Omega \setminus \tilde{\Omega}$ that $\mu_\omega = \gamma$, where $\gamma \in \mathcal{P}(X)$ is arbitrary. ♦

This result can be further extended if one considers ergodic measures to start with. The next result is inspired by Proposition 3.49 in [New16].

Corollary 3.13. *If $\eta \in \mathcal{P}(\Omega \times X, \mathbb{P})$ is assumed to be Markov, Θ-invariant and Θ-ergodic, then the corresponding measure $\mathbb{E}(\mu_\omega) \in \mathcal{P}(X)$, in the sense of Theorem 3.12, is stationary and ergodic for the Markov process induced by φ.*

Conversely, if $\mu \in \mathcal{P}(X)$ is a stationary ergodic measure for the Markov process induced by φ, then its corresponding Markov Θ-invariant measure is also Θ-ergodic.

3.3 Random attractors and repellers

We would now like to investigate what it means to be an attractor and repeller in the setting of random dynamics. In order to simplify notation we will improperly use d instead of dist when considering Hausdorff semi-distances. The material presented in this section mostly follows the work of Crauel in [CDS04], but where possible we have adjusted some of the proofs and the definitions in order to present the material in a neater way.

Before even approaching concepts such as invariant sets, we need to introduce the idea of random sets. This necessity comes from the fact that we are interested to later on define properties in the "almost surely" sense, given the probabilistic nature of random dynamical systems.

Definition 3.14. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and (X, d) a Polish metric space. Denote by 2^X the power set of X. We define a *random set* to be a set-valued map

$$\begin{aligned} V : \Omega &\to 2^X \\ \omega &\mapsto V(\omega) \subset X \end{aligned}$$

where the mapping $\omega \mapsto \inf_{v \in V(\omega)} \mathrm{d}(x, v)$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every $x \in X$. We call $V(\omega)$ the ω-fibre of V.

If O is a random set and $O(\omega)$ is an open set in X for almost every $\omega \in \Omega$, then we say that O is a *open random set*. Similarly, F is said to be a *closed random set* if the mapping $\omega \mapsto F^{\complement}(\omega) := (F(\omega))^{\complement}$ is an open random set. Notice this is equivalent to saying that almost every fibre $F(\omega)$ is closed in X. In a similar fashion, if K is a random set, and $K(\omega)$ is compact in X for almost every ω then we say that K is a *compact random set*.

We will now look at a particular type of compact random set which will play a fundamental role in the next chapters.

Definition 3.15. A measurable map $a : \Omega \to X$ is said to be a *random fixed point* if for all $\omega \in \Omega$, $x \in X$, and $t \in \mathbb{T}$ then

$$\varphi(t, \omega) a(\omega) = a(\theta_t \omega).$$

We can now easily extend the definition of neighbourhood of a set. U is a *random neighbourhood* of a random set V if it is a random open set and for almost every $\omega \in \Omega$, $V(\omega) \subset U(\omega)$.

Finally, we are ready to adapt some of the definitions we explored in Chapter 2. Firstly, let us introduce the definition of invariant and isolated sets in the random context.

Definition 3.16. Let V be a random set, and φ an RDS over a two-sided time.

1. V is said *forward invariant* if for almost every $\omega \in \Omega$ and all $t \geq 0$

$$\varphi(t, \omega)V(\omega) \subset V(\theta_t \omega).$$

2. V is said *invariant* if for almost every $\omega \in \Omega$ and all $t \geq 0$

$$\varphi(t, \omega)V(\omega) = V(\theta_t \omega).$$

3. V is said to be *isolated* if it is invariant and there exists a random neighbourhood U of V where for any random variable x whose orbit is always in U, $\varphi(t, \omega)x(\omega) \in U(\theta_t \omega)$ for every $t \in \mathbb{T}$, then $x(\omega) \in V(\omega)$ almost surely.

The definition of attractors and repellers now follows immediately.

Definition 3.17. A random compact set A, invariant under φ, is called *(weak local) attractor of* φ if there exists an open forward-invariant neighbourhood U of A, i.e. $A(\omega) \subset U(\omega)$ almost surely, and such that for any random closed set V in U, and any $\varepsilon > 0$:

$$\lim_{t \to \infty} \mathbb{P}(\{\mathrm{d}(\varphi(t, \omega)V(\omega), A(\theta_t \omega)) > \varepsilon\}) = 0. \tag{1}$$

U is called the *attracting neighbourhood* of A.

A random compact set R, invariant under φ, is called *(weak local) repeller of* φ if there exists a random open neighbourhood U of R such that for any random closed set V in U, and any $\varepsilon > 0$:

$$\lim_{t \to -\infty} \mathbb{P}(\{\mathrm{d}(\varphi(t, \omega)V(\omega), R(\theta_t \omega)) > \varepsilon\}) = 0. \tag{2}$$

U called the *repeller neighbourhood* of R.

Remark. The above definition turns out to be the most natural way of extending the concept of attractors and repellers from a deterministic to random setting. One might be tempted to define an attractor such that the limit in the above statement is not taken in probability, but in the classical sense:

$$\lim_{t \to \infty} \mathrm{d}(\varphi(t, \omega)V(\omega), A(\theta_t \omega)) = 0,$$

but even if this approach feels more natural, and it is valid, it generates various problems. This more "physical" or "naïve" approach leads to a poor asymptotic behaviour. The behaviour is studied on a "moving fibre", which means that the target fibre in the limit is not fixed, and therefore the limit rarely converges ω-wise. A better "mathematical" approach is to study such limits from $-t$ to 0, as $t \to \infty$, considering that in this way we fix our attention on the fibre at $t = 0$, on which ω-wise convergence has to be expected. Therefore, we are going to use *pullback limits*:

$$\lim_{t \to \infty} \mathrm{d}(\varphi(t, \theta_{-t}\omega)V(\theta_{-t}\omega), A(\omega)) = 0.$$

Random sets attracting with such limits are called *pullback attractors*. Clearly these two approaches generate different attractors, but if we consider convergence in probability we have that

$$\mathbb{P}(\{\mathrm{d}(\varphi(t,\omega)V(\omega), A(\theta_t\omega))\}) = \mathbb{P}(\{\mathrm{d}(\varphi(t,\theta_{-t}\omega)V(\theta_{-t}\omega), A(\omega)\})$$

by the invariance of $\mathbb{P}$ under θ_t. This whole discussion directly extends to repellers as well. ♦

Given this set up, we would like to state and prove some useful properties of attractors and repellers. Before doing this we need a small technical lemma.

Lemma 3.18. *Let U be a forward invariant random set for φ, then for every $\omega \in \Omega$, $t \geq 0$, and $s \leq t$ the following holds:*

$$\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) \subset \varphi(s,\theta_{-s})U(\theta_{-s}\omega)U(\theta_{-s}\omega).$$

Proof. Note that $t-s \geq 0$, so

$$\begin{aligned}\varphi(t,\theta_{-t}\omega)U(\theta_{-t}\omega) &= \varphi(s+t-s,\theta_{-t}\omega)\\ &= \varphi(s,\theta_{-s}\omega)(\varphi(t-s,\theta_{-t}\omega)U(\theta_{-t}\omega))\\ &\subset \varphi(s,\theta_{-s}\omega)U(\theta_{t-s}(\theta_{-t}\omega))\\ &= \varphi(s,\theta_{-s}\omega)U(\theta_{-s}\omega).\end{aligned}$$

Notice the second and third equalities are given by the cocycle property of φ, and the inclusion follows by A being forward invariant. □

We will now give the definition of basin of attraction and repulsion of an attractor and repeller in the random sense, together with their main properties.

Proposition 3.19. *1. Let A be an attractor with attracting neighbourhood U. Define the random set $\mathcal{A}(A)$ fibre-wise as*

$$\mathcal{A}(A)(\omega) := \{x \in X \mid \varphi(t,\omega)x \in U(\theta_t\omega),\ \textit{for some } t \geq 0\}.$$

We will call $\mathcal{A}(A)$ the basin of attraction of A, and it has following properties:

(a) $\mathcal{A}(A)$ is invariant;

(b) $\mathcal{A}(A)(\omega) = \bigcup_{t\geq 0}\varphi(-t,\theta_t\omega)U(\theta_t\omega) = \lim_{T\to\infty}\varphi(-T,\theta_T\omega)U(\theta_T\omega)$;

(c) Any closed random set $V \subset \mathcal{A}(A)$ is attracted to A in the sense of Equation (1).

2. Let R be a repeller with repeller neighbourhood U. Define the random set $\mathcal{R}(R)$ fibre-wise as

$$\mathcal{R}(R)(\omega) := \{x \in X \mid \varphi(t,\omega)x \in U(\theta_t\omega),\ \textit{for some } t \leq 0\}.$$

We will call $\mathcal{R}(R)$ the basin of repulsion of R, and it has following properties:

(a) $\mathcal{R}(R)$ is invariant;

(b) $\mathcal{R}(R)(\omega) = \bigcup_{t\leq 0}\varphi(-t,\theta_t\omega)U(\theta_t\omega) = \lim_{T\to\infty}\varphi(T,\theta_{-T}\omega)U(\theta_{-T}\omega)$;

(c) Any closed random set $V \subset \mathcal{R}(R)$ is repelled from R in the sense of Equation (2).

Proof. We will only prove the first statement, the second one follows symmetrically.

Claim. $\mathcal{A}(A)(\omega) = \bigcup_{t\geq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega) = \lim_{T\to\infty} \varphi(-T, \theta_T\omega)U(\theta_T\omega)$

If $x \in \mathcal{A}(A)(\omega)$, for $\omega \in \Omega$, then there exists $t \geq 0$ such that $\varphi(t, \omega)x \in U(\theta_t\omega)$, and since we are working under the hypothesis of a two-sided time, this is equivalent to $x \in \varphi(-t, \theta_t\omega)U(\theta_t\omega)$. So we can conclude that $x \in \bigcup_{t\geq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega)$. Similarly if $x \in \bigcup_{t\geq 0} \varphi(-t, \theta_t\omega)U(\theta_t\omega)$, then there exists at least one time $\tilde{t} \geq 0$ for which $x \in \varphi(-\tilde{t}, \theta_{\tilde{t}}\omega)U(\theta_{\tilde{t}}\omega)$. By our assumption on $\mathbb{T}$ we can conclude $x \in \mathcal{A}(A)(\omega)$.

Now, if we apply Lemma 3.18 to $-T \leq -t \leq 0$

$$\bigcup_{0\leq t\leq T} \varphi(-t, \theta_t\omega)U(\theta_t\omega) = \varphi(-T, \theta_T\omega)U(\theta_T\omega). \tag{3}$$

Therefore, for every ω we have that $\varphi(-T, \theta_T\omega)$ is increasing in T, and hence $\mathcal{A}(A)(\omega) = \lim_{T\to\infty} \varphi(-T, \theta_T\omega)$. Thanks to this result we have that $\mathcal{A}(A)$ is an open random set.

Claim. $\mathcal{A}(A)$ is invariant.

Note that for any $s \geq 0$, thanks to the cocycle property, we have

$$\varphi(s, \omega)(\varphi(-T, \theta_T\omega)U(\theta_T\omega)) = \varphi(s - T, \theta_{T-s}\theta_s\omega)U(\theta_{T-s}\theta_s\omega).$$

Now, as $T \to \infty$, left hand side increases to $\varphi(s, \omega)\mathcal{A}(A)(\omega)$, whereas the right hand side increases to $\mathcal{A}(A)(\theta_s\omega)$, yielding $\varphi(s, \omega)\mathcal{A}(A)(\omega) = \mathcal{A}(A)(\theta_s\omega)$ for all $s \geq 0$.

Claim. A attracts any closed random set in $\mathcal{A}(A)$.

Let V be a closed random set in $\mathcal{A}(A)$, so $V(\omega) \subset \mathcal{A}(A)(\omega) = \bigcup_{t\geq 0} \varphi(-t, \theta_{-t}\omega)U(\theta_t\omega)$. Thanks to the fact that V is fibre-wise closed and Equation (3), then there exists $T = T(\omega) \geq 0$ such that $V(\omega) \subset \varphi(-T, \theta_T\omega)U(\theta_T\omega)$. Since we assumed $\mathbb{T}$ to be two-sided, and forward invariance of U we have that for any $t \geq T$

$$\varphi(t, \omega)V(\omega) \subset \varphi(t - T, \theta_T\omega)U(\theta_T\omega) \subset U(\theta_t\omega).$$

Define a random variable n by

$$n(\omega) = \inf\{n \in \mathbb{N} \mid \varphi(n, \omega)V(\omega) \subset U(\theta_n\omega)\},$$

and for $m \geq 0$, the set

$$\Omega_m = \{\omega \mid n(\omega) \leq m\}$$

Notice that $\Omega = \bigcup_{m\geq 0} \Omega_m$, and $\Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \dots$ Therefore, for any $\varepsilon > 0$, we can find $N_0 \in \mathbb{N}$ such that $\mathbb{P}(\Omega_{N_0}) < 1 - \frac{\varepsilon}{2}$. Define

$$Z(\omega) = \begin{cases} V(\omega) & \text{for } \omega \in \Omega_{N_0} \\ A(\omega) & \text{for } \omega \notin \Omega_{N_0}. \end{cases}$$

By construction, Z is a closed random set such that $\varphi(t, \omega)Z(\omega) \subset U(\theta_t\omega)$ for any ω and $t \geq 0$, which implies $\mathrm{d}(\varphi(t, \omega)Z(\omega), A(\theta_t\omega)) \xrightarrow{t\to\infty} 0$ in probability. We can find $T(\varepsilon) > 0$ such that $\mathbb{P}(\mathrm{d}(\varphi(t, \omega)Z(\omega), A(\theta_t\omega)) > \varepsilon) \leq \frac{\varepsilon}{2}$ for all $t \geq T(\varepsilon)$, which allows us to estimate

$$\begin{aligned}\mathbb{P}(\{\omega \mid \mathrm{d}(\varphi(t, \omega)V(\omega), A(\theta_t\omega)) > \varepsilon\}) &\leq \mathbb{P}(\{\omega \mid \mathrm{d}(\varphi(t, \omega)V(\omega), A(\theta_t\omega)) > \varepsilon\} \cap \Omega_{N_0}) + \mathbb{P}(\Omega_{N_0}^{\complement}) \\ &\leq \mathbb{P}(\{\omega \mid \mathrm{d}(\varphi(t, \omega)Z(\omega), A(\theta_t\omega)) > \varepsilon\}) + \mathbb{P}(\Omega_{N_0}^{\complement}) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

Therefore, A attracts V in probability. □

It is important to notice that the basin of attraction is always well-defined, which means it is independent from the attractor neighbourhood. Obviously the same holds for basins of repulsion (see Corollary 5.1 in [CDS04]). The next proposition is key in developing Morse theory as we have already seen in the previous chapter.

Proposition 3.20. *Let A be an attractor for φ over X with basin of attraction $\mathcal{A}(A)$. The random compact set R fibre-wise defined*

$$R(\omega) \coloneqq X \setminus \mathcal{A}(A)(\omega)$$

is called the repeller corresponding to A, and the couple (A, R), an attractor-repeller pair.

Proof. See Proposition 5.1 in [CDS04]. □

A fundamental property of attractor-repeller pairs is the following.

Theorem 3.21 ([CDS04])**.** *Suppose that the two attractors A_1 and A_2 are nested: $A_1 \subsetneq A_2$. Then their associated repellers R_1, and R_2, respectively, are nested in the opposite direction: $R_1 \supsetneq R_2$.*

Now we can easily see how to extend the definition of Morse decomposition to this setting.

3.4 Morse decomposition for random dynamical systems

This section will predominantly focus on adapting the definition of Morse decomposition for RDS's. Since we have decided to work with convergence in probability we will see in Chapter 5 that the next definition will allow us to discard some pretty serious pathological behaviours.

Definition 3.22. Let A, B be two random sets. We will say that A *is almost surely strictly contained in* B if $A(\omega) \subsetneq B(\omega)$ for almost every $\omega \in \Omega$. Similarly, A *is almost surely contained in* B if $A(\omega) \subseteq B(\omega)$ for almost every $\omega \in \Omega$. Finally, A is *almost sure equal* to B if A is almost surely contained in B and B is almost surely contained in A.

With this new definition we can proceed to introduce the Morse decomposition for random dynamical system.

Definition 3.23. Let φ be a random dynamical system over a compact state space X, which admits a family of attractor-repeller pairs $(A_i, R_i)_{i=0}^n$ with the properties:

$$\emptyset =: A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n \coloneqq X \quad \text{and} \quad X =: R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n \coloneqq \emptyset.$$

Furthermore, suppose that the strict set inclusions in one of the two chains is in the almost sure sense as in Definition 3.22. The *(weak) Morse decomposition* of X is given by the family of subset $(\mathcal{M}_i)_{i=1}^n$ defined

$$\mathcal{M}_i \coloneqq A_i \cap R_{i-1} \qquad \text{for } i = 1, 2, \dots n$$

where each of these subset is called *Morse set*.

Remark. The definition we just stated is stronger to the one given by Crauel, Duc and Siegmund in [CDS04]. We decided to add an extra specification on the set inclusion used for defining the nested sequences of attractors and repellers. We will now briefly explain why we need it.

For what follows we will call *strong set inclusion* the one given by simply looking at random sets are subsets of $\Omega \times X$. Let A_1, and A_2 be two attractors, such that $A_1 \subsetneq A_2$ in the strong sense, but where $A_1(\omega) = A_2(\omega)$ for almost every $\omega \in \Omega$, and $A_1(\omega) \subsetneq A_2(\omega)$ for every other noise realisation ω. Clearly A_1 is almost everywhere equal to A_2. Since we are working with weak attractors, and hence with limits in probability, this means that A_1 and A_2 admit the same basin of attraction, leading then to the same associated repeller. This obviously needs to be avoided. Whenever we will build a Morse decomposition we will always start the construction from the chain for which the set inclusion is given in the almost everywhere sense, otherwise stated. ♦

One can notice the strong resemblance between the definition of Morse sets in the deterministic and random setting. Actually, they share quite a few topological properties.

Proposition 3.24. *Morse sets are non-empty, invariant and pairwise disjoint.*

Proof. Let $(\mathcal{M}_i)_{i=1}^n$ be a Morse decomposition of X. In the case where the systems admits no attractors, hence $A_0 = \emptyset, A_1 = X$, and $R_0 = X, R_1 = \emptyset$, the only Morse set is $\mathcal{M}_1 = \emptyset$. Hence if assume that our system admits at least one attractor, then thanks to Theorem 3.21 we automatically have that Morse sets are non-empty.

By definition, attractors and repellers are invariant, and since the intersection of invariant sets is invariant, Morse sets are invariant sets. Showing that Morse sets are pairwise disjoint follows from their definition. Let $\mathcal{M}_i, \mathcal{M}_j$ be two different Morse sets, and $1 \leq i < j \geq n$. Then we know $A_i \subsetneq \cdots \subsetneq A_j$ and $R_{i-1} \supsetneq \cdots \supsetneq R_{j-1}$. Therefore,

$$\mathcal{M}_i \cap \mathcal{M}_j = (A_i \cap R_{i-1}) \cap (A_j \cap R_{j-1}) = A_i \cap R_{j-1} \subset Aj - 1 \cap R_{j-1} = \emptyset.$$

Finally we wish to prove that any $\mathcal{M}_i$ is isolated. In order to do so define the random variables mapping from Ω to $\mathbb{R}$

$$\delta_A(\omega) := \min_{m \in \mathcal{M}_i(\omega)} d(m, A_{i-1}(\omega)) = \min_{m \in \mathcal{M}_i(\omega)} \min_{a \in A_{i-1}(\omega)} d(m, a)$$

and

$$\delta_R(\omega) := \min_{m \in \mathcal{M}_i(\omega)} d(m, R_i(\omega)) = \min_{m \in \mathcal{M}_i(\omega)} \min_{r \in R_i(\omega)} d(m, r).$$

Similarly we can define the random variable $\delta(\omega) := \min(\delta_A, \delta_R)$, this function measures the minimal distance between the i-th Morse set, and the $(i-1)$-th attractor (containing $\mathcal{M}_{\rangle-\infty}$) or i-th attractor (containing $\mathcal{M}_{i+1}$). Hence this random variable tells us, in a rough way, how far three consecutive Morse sets are. For almost every $\omega \in \Omega$ define the random neighbourhood $U(\omega) := B_{\delta(\omega)}(\mathcal{M}_i(\omega)) = \bigcup_{m \in \mathcal{M}_i(\omega)} \{x \in X \mid d(x, m) < \delta(\omega)\}$. Assume there is a random variable x satisfying $\varphi(t, \omega)x(\omega) \in U(\omega_t(\omega))$ for all $t \in \mathbb{T}$, and $\mathbb{P}(\{x(\omega) \notin \mathcal{M}_i(\omega))\}) > 0$. In the second hypothesis we assumed that there exist noise realisations for which $\mathcal{M}$ is not an isolated set in order to get a contradiction. Notice, by the definition of attractor-repeller pair, $A_i^{\complement}$ is the basin of repulsion of R_i, and symmetrically, $R_{i-1}^{\complement}$ is the basin of attraction of A_{i-1}. Hence there exists a time $T > 0$ such that $\mathbb{P}(\{\mathrm{d}(\varphi(T, \omega)x(\omega), A_{i-1}(\omega)) < \delta(\omega)\} > 0$ and $\mathbb{P}(\{\mathrm{d}(\varphi(-T, \omega)x(\omega), R_i(\omega)) < \delta(\omega)\}) > 0$, contradicting our contradiction hypothesis. □

What kind of information can we determine from the the Morse decomposition of a compact space X? Firstly, note that if we a random dynamical system φ over a compact state space X with Morse decomposition $(\mathcal{M}_i)_{i=1}^n$, and attractor sequence $(A_i)_{i=0}^n$, then we can restrict our dynamical system to A_i, for any i, by invariance of attractors. In this context $(A_{i-1}, \mathcal{M}_i)$ is an attractor repeller pair in the compact metric space A_i. Thus, Morse decomposition gives an insight on the dynamic of our system inside attractors.

It is important to notice that, given a Morse decomposition, it is possible to recover the initial chain of attractors and repellers. Firstly, note that $\mathcal{M}_1 = A_1$ and $\mathcal{M}_n = R_{n-1}$, since both chains always start with either the empty set or the state space. The attractor repeller pair (A_{n-1}, R_{n-1}) is an attractor-repeller pair in $A_n = X$. So, we can recover A_{n-1} by taking the complement of the basin of repulsion of $R_{n-1} = \mathcal{M}_n$. We can repeat this procedure inductively, together with our previous discussion, to recover $(A_i)_{i=0}^n$,and $(R_i)_{i=0}^n$. Suppose we know A_i, then A_{i-1} will be the complement of the basin of repulsion of $\mathcal{M}_i$. Then one can compute the chain of attractors from the repellers. Therefore, the Morse decomposition fully determines the asymptotic behaviour in terms of attractor and repellers of a dynamical systems.

Theorem 3.25. *Let $(\mathcal{M}_i)_{i=1}^n$ be a Morse decomposition of the compact metric space X given by the attractor-repeller pairs $((A_i, R_i))_{i=0}^n$ of the random dynamical system φ. The compact random set*

$$\mathfrak{M}(\omega) = \bigcup_{i=1}^{n} \mathcal{M}_i(\omega)$$

determines the asymptotic behaviours of φ. More precisely:

1. *For any $x \in X$ then $\mathbb{P}(\{\omega \in \Omega \mid \lim_{t\to\pm\infty} \mathrm{d}(\varphi(t,\omega)x, \mathfrak{M}(\theta_t\omega)) = 0\}) = 1$;*

2. *Let x be a X-valued random variable such that*

$$\lim_{t\to\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), \mathcal{M}_i(\theta_t\omega)) = 0 \quad \text{and} \quad \lim_{t\to-\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), \mathcal{M}_j(\theta_t\omega)) = 0$$

   hold in probability, then $i \geq j$, with equality holding if and only if $x \in \mathcal{M}_i$ almost surely;

3. *Let $x_1, \ldots, x_p \in X$ be X-valued random variables and assume that for any x_i, there exists an index $j_i \in \{1, 2, \ldots, p\}$ such that x_i is attracted by $\mathcal{M}_{j_i}$ and repelled by $\mathcal{M}_{j_{i-1}}$, then $j_0 \leq j_p$. In addition, $j_0 < j_p$ if and only if there is at least one x_i such that $\mathbb{P}(\{x(\omega \notin \mathfrak{M})\}) > 0$, otherwise $j_0 = \cdots = j_p$.*

Proof. First of all we have showed that every $\mathcal{M}_i$ is both compact and invariant, and since $\mathfrak{M}$ is defined as a finite union of Morse sets, it is automatically compact and invariant.

In order to prove the first statement recall that $X = R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n = \emptyset$, and by definition of Morse sets, $M_i \subset R_i^{\complement} \cap R_{i-1}$ for any i. By exploiting the fact that $(R_i^{\complement} \cap R_{i-1})_{i=0}^n$ partitions X, we can define a partition on Ω.

Let us define for $i = 1, \dots, n$ the sets

$$\Omega_i := \{\omega \in \Omega \mid x(\omega) \in R_i^{\complement}(\omega) \cap R_{i-1}(\omega)\}$$

and notice that all these sets are not only pairwise disjoint but $\bigcup_{i=1}^n \Omega_i = \Omega$. Now for $r > 0$

$$\begin{aligned}\mathbb{P}(\{\mathrm{d}(\varphi(t,\omega)x(\omega), \mathfrak{M}(\theta_t\omega)) > r\}) &= \sum_{i=1}^n \mathbb{P}(\{\mathrm{d}(\varphi(t,\omega)x(\omega), \mathfrak{M}(\theta_t\omega)) > r\} \cap \Omega_i) \\ &= \sum_{i=1}^n \mathbb{P}(\{\mathrm{d}(\varphi(t,\omega)x(\omega), \mathcal{M}_i(\theta_t\omega)) > r\} \cap \Omega_i) \\ &= \sum_{i=1}^n \mathbb{P}(\{\mathrm{d}(\varphi(t,\omega)x(\omega), A_i \cap R_{i-1}(\theta_t\omega)) > r\} \cap \Omega_i)\end{aligned}$$

Since for every term of the sum we have that $\varphi(t,\omega)x(\omega) \in R_i^{\complement}(\omega) \cap R_{i-1}(\theta_t\omega)$, then the random variable is attracted by $A_i(\theta_t\omega) \subset R_i^{\complement}(\theta_t\omega)$. This means that we can make every term in the sum as small as we wish, just by considering a further in time iteration of our RDS, meaning that x is attracted to $\mathfrak{M}$, forward in time.

By considering the partition given by $\tilde{\Omega}_i := \{\omega \in \Omega \mid x(\omega) \in A_{i-1}^{\complement}(\omega) \cap A_i(\omega)\}$ for $1 \le i \le n$, it is possible to show, by running a similar argument as before, that x is attracted to $\mathfrak{M}$ backward in time.

In order to prove our second assertion, consider the Morse set $\mathcal{M}_i$. Since x is repelled by it, this means that

$$\begin{aligned}1 &= \mathbb{P}(\{\omega \in \Omega \mid \lim_{t\to-\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), A_j(\theta_t\omega) \cap R(\theta_t\omega)_{j-1}) = 0\}) \\ &= \mathbb{P}(\{\omega \in \Omega \mid \lim_{t\to-\infty} \mathrm{d}(\varphi(t,\theta_{-t}\omega)x(\theta_{-t}\omega), A_j(\omega) \cap R(\omega)_{j-1}) = 0\}) \\ &= \mathbb{P}(\{\omega \in \Omega \mid \lim_{t\to\infty} \mathrm{d}(\varphi(-t,\theta_t\omega)x(\theta_t\omega), A_j(\omega) \cap R(\omega)_{j-1}) = 0\}).\end{aligned}$$

Hence for any $\varepsilon > 0$, and $\delta > 0$, we can find a time $T \in \mathbb{T}$ such that for all $s \ge T$ then

$$\mathbb{P}(\varphi(-s,\theta_s\omega)x(\theta_s\omega) \in \bar{B}_j(A_j(\omega)) \ge 1 - \varepsilon.$$

By taking δ small enough we have that $\bar{B}_j(A_j(\omega))$ is in the basin of attraction of A_j with positive probability. Hence, x will be attracted to A_j with positive probability, but x is attracted by $\mathcal{M}_i$ as well. We now have that x is attracted by A_j and R_{i-1}, and by definition, our attractors are nested, and so we must have $\emptyset \ne A_j \cap R_{i-1} \subset A_j \cap A_{i-1}^{\complement}$. If we assume that $j < i$, we immediately know that $A_j \subset A_{i-1}$, but then $A_j \cap A_{i-1}^{\complement} = \emptyset$, which is a contradiction.

Suppose now $i = j$, and in order to reach a contradiction that $x \notin \mathcal{M}_i (= \mathcal{M}_j)$ almost surely, or more concretely that either $\mathbb{P}(\{x \notin A_i\}) > 0$ or $\mathbb{P}(\{x \notin R_{i-1}\}) > 0$. Let us analyse both cases separately. In the former instance we have that

$$\mathbb{P}(\{\lim_{t\to-\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), R_i(\theta_t\omega)) = 0\}) = \mathbb{P}(\{x \notin A_i\})$$

since if $x(\omega)$ is not in $A_i(\omega)$ then it is either in $R_i(\omega)$ or in its basin of repulsion. Thus, x is repelled by R_i with non-zero probability, but this is in open contradiction with x being repelled by $\mathcal{M}_i$. Recall repellers are assumed nested so R_i and $\mathcal{M}_i = A_i \cap R_{i-1}$ would have to intersect but

$$R_i \cap \mathcal{M}_i = R_i \cap A_i \cap R_{i-1} \subset R_i \cap A_i = \emptyset.$$

Assume now the latter case, and deduce similarly

$$\mathbb{P}(\{\lim_{t\to\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), R_{i-1}(\theta_t\omega)) = 0\}) = \mathbb{P}(\{x \notin R_{i-1}\}) > 0.$$

This induces a contradiction, since x is being attracted with positive probability by both A_{i-1} and $\mathcal{M}_i$, which need to have non-empty intersection since attractors are nested, but

$$A_{i-1} \cap \mathcal{M}_i = A_{i-1} \cap R_{i-1} \cap A_i \subset A_{i-1} \cap R_{i-1} = \emptyset.$$

We can therefore conclude that attraction and repulsion of x by the same Morse set implies $x \in \mathcal{M}_i$ almost surely. The other direction is obvious, if we assume without loss of generality that $x \in \mathcal{M}_i$ almost surely and $\mathbb{P}(\{\lim_{t\to-\infty} \mathrm{d}(\varphi(t,\omega)x(\omega), \mathcal{M}_j) = 0\}) = 1$ then by invariance of Morse sets we have $\mathcal{M}_j \subset \mathcal{M}_i$ almost surely. Since Morse sets are pairwise disjoint this means $\mathcal{M}_i = \mathcal{M}_j$.

Finally, the last claim of the theorem follows from what we have just proved. We have just showed that if x_k is repelled by $\mathcal{M}_{j_{k-1}}$ and attracted by $\mathcal{M}_{j_k}$ then $j_{k-1} \leq j_k$, for any $1 \leq k \leq p$. Equality does not hold if and only if $\mathbb{P}(\{x_k \notin \mathfrak{M}\}) > 0$ for some k. □

In the final part of this chapter we would like to briefly discuss the idea of finest Morse decomposition for a random system. Before doing so we would like to look into an example for two reasons: to convince ourselves of why we needed to modify the definition of Morse decomposition and to take inspiration for defining what we mean by finest Morse decomposition.

Example 3.4.1. For the next example we will work with the naïve (or strong as we previously defined) set-inclusion between attractors and repellers. Let us denote by $(\mathcal{M}_i)_{i=1}^n$ a Morse decomposition for φ, and $\emptyset = A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n = [a,b]$ be the associated chain of attractors. Let $\Xi \subset \Omega$ be a non-empty zero measure set. We can define the random set

$$A_1'(\omega) = \begin{cases} A_1(\omega) & \text{for } \omega \in \Omega \setminus \Xi \\ \emptyset & \text{for } \omega \in \Xi \end{cases}$$

Since A_1 is a weak attractor and $A_1(\omega) = A_1'(\omega)$ up to a set of measure 0, we have that A_1' is a weak attractor as well. This is given by the fact that convergence in probability, compactness as a random set, being φ-invariant are almost everywhere properties. Denote by R_1' the associated repeller to A_1'. Notice that

$$\emptyset \subsetneq \tilde{A}_1' \subsetneq \tilde{A}_1$$

in what we earlier defined as "strong sense". We have created a longer chain of attractors, which yields a bigger (cardinality-wise) decomposition than $(\tilde{\mathcal{M}}_i)_{i=1}^n$. Furthermore, this newly defined decomposition, which will be denoted by $(\tilde{\mathcal{N}}_j)_{j=1}^{n+1}$ is finer since

$$\begin{aligned}
&\tilde{\mathcal{N}}_1 = \tilde{A}_1' \subsetneq \tilde{A}_1 = \tilde{\mathcal{M}}_1;\\
&\tilde{\mathcal{N}}_2 = \tilde{A}_1 \cap \tilde{R}_1' \subsetneq \tilde{A}_1 = \tilde{\mathcal{M}}_1;\\
&\tilde{\mathcal{N}}_3 = \tilde{\mathcal{M}}_2\\
&\vdots\\
&\tilde{\mathcal{N}}_{n+1} = \tilde{\mathcal{M}}_n.
\end{aligned}$$

If working on a probability Hausdorff space equipped with an absolutely continuous measure for example, this process can be repeated *ad libitum* by taking a longer and longer sequence of countable points, thus always generating a bigger (cardinality-wise) and finer Morse decomposition.

The main takeaway from the previous example is that without some restrictions on the set inclusion operation it will be impossible to define the concept of finest Morse decomposition, since it will never exist.

Definition 3.26. Let $(\mathcal{M}_i)_{i=1}^n$ and $(\mathcal{N}_j)_{j=1}^m$ be two Morse decompostitions of the same RDS φ. We say that $(\mathcal{N}_j)_{j=1}^m$ is *finer* than $(\mathcal{M}_i)_{i=1}^n$ if for every $i \in \{1, \ldots, n\}$ there exists $j \in \{1, \ldots, m\}$ such that $\mathcal{N}_j \subseteq \mathcal{M}_i$, where the set inclusion has to be considered in the almost everywhere sure sense. A decomposition whose finer Morse decomposition is only itself is said to be the *finest Morse decomposition*.

As a final remark, finest Morse decompositions, if they exist, are unique.

Chapter 4

Set-valued dynamical systems

In the previous chapter we introduced and worked with the concept of random dynamical systems. Now we would like to delve into the theory of set-valued dynamical systems, and in particular we would like to define what we mean by Morse decomposition in this case. One might wonder why we wish to do so. The answer is quite simple: it is possible to associate to every random dynamical system φ a set-valued system by considering the map $\varphi(\cdot, \Omega, \cdot)$, where Ω represents the noise space. Before looking at this construction we need to build some foundations for what it has to come. Here we return to deterministic theory and thus similar to the one presented in the section on topological dynamical systems, for this reason many of the proofs will not be here developed. In order to make the distinction between set-valued dynamical systems and topological dynamical systems clearer, we will refer to the latter as single-valued systems. In this chapter we mainly refer back to the work of Lamb, Rasmussen and Rodrigues in [LRR15], Li in [Li07], and McGehee in [McG92].

By set-valued dynamical systems we refer to multi-valued maps, which satisfy the usual cocycle and identity property. Let us first introduce the state space for the dynamic. Let (X, d) denote a compact metric space equipped with metric $\mathrm{d}(\cdot,\cdot)$. We recall that when considering the distance between sets, this will have to be understood in the sense of Hausdorff semi-distance $\mathrm{dist}(\cdot,\cdot)$ which we will denote again simply by $\mathrm{d}(\cdot,\cdot)$. The Hausdorff distance, on the other hand, will be denoted by $d_H(\cdot,\cdot)$.

Given a set $A \subset X$ and $\varepsilon > 0$ we denote the ε-neighbourhood of A as

$$B_\varepsilon(A) = \bigcup_{x \in A} B_\varepsilon(x).$$

Finally denote by $\mathcal{K}(X)$ the set of all compact subsets of X, and notice that since X is assumed compact it is enough for a set to be closed in order to be an element of $\mathcal{K}(X)$. If we equip $\mathcal{K}(X)$ with the Hausdorff distance d_H, then it becomes a metric space. Furthermore, recall that if X is compact (resp. complete), then $\mathcal{K}(X)$ is compact (resp. complete).

In this section we will work under the assumption of one-sided time, hence $\mathbb{T} := \{0\} \cup \mathbb{N}$ in the discrete case, or $\mathbb{T} = 0 \cup \mathbb{R}^+$ in the continuous case.

Definition 4.1 (Set-valued dynamical systems)**.** A *set-valued dynamical system* is a map

$$\Phi : \mathbb{T} \times X \to \mathcal{K}(X)$$

such that $\Phi(1, X) = X$ if we assume $\mathbb{T}$ discrete, and $\Phi(t, X) = X$ for all $t \in \mathbb{T}$ in the continuous case. Φ must satisfy the following properties:

1. Φ is upper semicontinuous, i.e. $\forall (t,x) \in \mathbb{T} \times X$
$$\limsup_{(s,y)\to(t,x)} \Phi(s,y) \subset \Phi(t,x);$$
2. For any $x \in X$ the map $t \mapsto \Phi(t,x)$ is continuous with respect to the Hausdorff distance
3. *Identity property*: For all $x \in X$
$$\Phi(0,\cdot) = \{x\};$$
4. *Cocycle property*: For all $s,t \in \mathbb{T}$ and $x \in X$
$$\Phi(s+t,x) = \Phi(s,\Phi(t,x)).$$

As we anticipated at the beginning of this section, we would like now to be able to understand Morse sets and Morse decomposition in this new setting. In order to that however, we firstly need to introduce the concept of attractors and repellers. We will see that the definition of attractor can be easily extended to set-valued dynamics, but what about repellers? We have just assumed to be working with a one-sided time, thus we will now make sense of the concept of "backwards in time".

Definition 4.2 (Dual of a set-valued dynamical system)**.** For a set-valued dynamical system $\Phi : X \to \mathcal{K}(X)$ let the *dual of* Φ to be a map $\Phi^* : \mathbb{T} \times X \to \mathcal{K}(X)$, where
$$\Phi^*(t,y) := \{x \in X \mid y \in \Phi(t,x)\}$$

Remark. In the case where Φ is a single-valued map defined over a two-sided time, i.e. an invertible topological dynamical system, then Φ^* coincides with the system under time reversal. ♦

We can then interpret the dual of Φ as a formal way of defining backward iterations, but unfortunately this is not exactly a set-valued system. Let us now list some of its most useful properties:

Proposition 4.3 ([LRR15])**.** *The dual of a set-valued dynamical system enjoys the identity and cocyle properties, and it is upper semicontinuous.*

Proof. The identity and cocycle properties follow directly from definitions. Let $x \in X$ and $t, T \in \mathbb{T}$ then:

- Identity property:
$$\Phi^*(0,x) = \{y \in X \mid x \in \Phi(0,y)\} = \{y \in X \mid x \in \{y\}\} = \{x\};$$
- Cocycle property:
$$\begin{aligned}
\Phi^*(T+t,x) &= \{y \in Y \mid x \in \Phi(T+t,y)\} \\
&= \{y \in X \mid x \in \Phi(T,\Phi(t,y))\} \\
&= \{y \in X \mid \exists z \in \Phi(t,y) \text{ s.t. } x \in \Phi(T,z)\} \\
&= \{y \in X \mid \exists z \in \Phi(t,y) \text{ s.t. } z \in \Phi^*(T,x)\} \\
&= \{y \in X \mid \Phi(t,y) \cap \Phi^*(T,x)\} \\
&= \{y \in X \mid \exists z \in \Phi^*(T,x) \text{ s.t. } z \in \Phi(t,y)\} \\
&= \{y \in X \mid \exists z \in \Phi^*(T,x) \text{ s.t. } y \in \Phi^*(t,z)\} \\
&= \{y \in X \mid y \in \Phi^*(T,\Phi^*(t,x))\} \\
&= \Phi^*(T,\Phi^*(t,x))
\end{aligned}$$

Now we are left proving upper semicontinuity. Once again, fix $(t, x) \in \mathbb{T} \times X$, and let $(t_n, x_n)_{n\in\mathbb{N}}$ be a sequence tending to (t, x). We want to show that $\limsup_{n\to\infty} \Phi^*(t_n, x_n) = \limsup_{n\to\infty}\{y \in X \mid x_n \in \Phi(t_n, y)\} \subset \Phi^*(t, x)$. In order to prove this inclusion, let $z \in \limsup_{n\to\infty}\{y \in X \mid x_n \in \Phi(t_n, y)\}$. We have a subsequence $(x_{n_j})_{j\in\mathbb{N}}$ of $(x_n)_{n\in\mathbb{N}}$, and $z_j \to z$ as $j \to \infty$, with the property that for every j then $x_{n_j} \in \Phi(t_{n_j}, z_j)$. Now by upper semicontinuity of Φ we conclude that $x \in \Phi(t, z)$, or equivalently $z \in \Phi^*(t, x)$. Since z was arbitrary, this precisely means $\limsup_{n\to\infty}\{y \in X \mid x_n \in \Phi(t_n, y)\} \subset \Phi^*(t, x)$. □

In order to simplify notation let for $t \in \mathbb{T}$ and $x \in X$

$$\Phi^t(x) := \Phi(t, x), \qquad \text{and} \qquad \Phi^t(A) = \bigcup_{x\in A} \Phi^t(x).$$

We adopt analogous notation for the dual Φ^*. Proceeding in a similar fashion as we did for single valued maps we now will define for all $t \geq 0$, and $A \subset X$, a non empty compact subset of X:

- *Forward-invariant* if $\Phi^t(A) \subseteq A$;
- *Invariant* if $A = \Phi^t(A)$.
- *Minimal invariant* if A is invariant and it contains no proper forward-invariant subsets.

It is always possible to construct an invariant set for any given compact metric space. Let us (re)introduce the concept of *ω-limit set*: for any subset A of X define

$$\omega(A) = \bigcap_{t\geq 0} \overline{\bigcup_{k\geq t} \Phi^k(A)} = \limsup_{t\to\infty} \Phi^t(A)$$

and by Theorem 5.9 in [McG92] this set is compact and invariant. Studying these types of objects allows us to understand the asymptotic behaviour of our system.

Remark. In the paper *Attractors for closed relations on compact Hausdorff spaces* by McGehee ([McG92]), the author proves these properties for ω-limit sets, whereas the definition we adopted is known as *weak ω-limit set*. The difference arises when considering closed relations instead of functions. For the entirety of this project we will always focus on functions, and therefore ω-limit sets agree with weak ω-limit sets. ♦

Instead of looking at the entire orbit of a point, which is a collection of sets, we might be interested in a specific selection of points in these sets.

Definition 4.4. A *trajectory* is a mapping γ from an interval $I \subset \mathbb{R}$ to X such that if $t_1 < t_2$ then $\gamma(t_2) \in \Phi^{t_2-t_1}(\gamma(t_1))$.

If I agrees with $\mathbb{R}$, we say that γ is a *complete trajectory*, and a *complete trajectory through* $x \in X$ is a complete trajectory with $\gamma(0) = x$.

This new concept allows to look at the asymptotics of Φ in a new way. In fact, we can introduce the *ω-limit set of a trajectory* γ defined over an interval of the form $[a, \infty)$ as

$$\omega(\gamma) := \{x \in X \mid \exists t_n \to \infty \text{ s.t } \gamma(t_n) \to x\}$$

and similarly if γ is defined on an interval of the form $(\infty, a]$ then we can define the *α-limit set of a trajectory* as

$$\alpha(\gamma) := \{x \in X \mid \exists t_n \to -\infty \text{ s.t. } \gamma(t_n) \to x\}.$$

These objects will help us to make more explicit the similarities between set-valued and single-valued dynamics. Now let us define the concept of attractors in this new setting in the sense of [LRR15].

Definition 4.5. A compact Φ-invariant subset S of X is said to be an *attractor* if there exists $\xi > 0$ such that

$$\lim_{t\to\infty} \mathrm{d}(\Phi^t(B_\xi(S), S) = 0.$$

We define its *domain of attraction* as

$$\mathcal{A}(S) \coloneqq \{x \in X \mid \lim_{t\to\infty} \mathrm{d}(\Phi^t(x), S) = 0\}$$

The set $\mathcal{A}(S)$ is open and forward-invariant. As we have previously seen, we now need to define the concept of repeller. In order to do that let us prove a lemma first. This result relies on the equivalent definition of domain of attraction

$$\mathcal{A}(S) = \{x \in X \mid \forall T \geq 0, \exists V \text{ neighbourhood of } \Phi^T(x) \text{ s.t. } \lim_{t\to\infty} \mathrm{d}(\Phi^t(V), S) = 0\}$$

where S is an attractor (see Lemma 4.2 in [LRR15]). We decided against using this as our main definition since it is less intuitive, although more useful in proofs.

Lemma 4.6. *Let S be an attractor in X then the set $S^* = X \setminus \mathcal{A}(S)$ is Φ^*-invariant.*

Proof. We wish to show that for any $t \geq 0$, $(\Phi^*)^t(S^*) = S^*$. As per usual, let us show both directions of set inclusion.

Let us start by showing $(\Phi^*)^t(S^*) \subseteq S^*$. Suppose, for a contradiction, that there exists $t \geq 0$ such that $(\Phi^*)^t(S^*) \setminus S^*$ is non-empty. Let $x \in (\Phi^*)^t(S^*) \setminus S^* = (\Phi^*)^t(S^*) \cap (S^*)^\complement = (\Phi^*)^t(S^*) \cap \mathcal{A}(S)$. Therefore, on one hand we have $x \in \mathcal{A}(S)$, but on the other by definition of dual system, $\Phi^t(x) \cap S^* \neq \emptyset$. This is clearly a contradiction with our initial assumption since $\mathcal{A}(S)$ is forward-invariant and so $\Phi^t(\mathcal{A}(S)) \subset \mathcal{A}(S)$, for all $t \geq 0$.

For the converse, similarly assume there exists $t \geq 0$ such that $S^* \setminus (\Phi^*)^t(S^*)$ is non-empty. Therefore, let $x \in S^* \setminus (\Phi^*)^t(S^*)$, then $\Phi^t(x) \cap S^* = \emptyset$, which means $\Phi^t(x) \subset \mathcal{A}(S)$. Next we wish to show that this would imply $x \in \mathcal{A}(S)$, which would induce a contradiction since x is an element of S^*. In order to show the claim fix $T \geq 0$ and analyse the two different cases separately.

Firstly assume $T \leq t$. The set $\mathcal{A}(S)$ is open and $\Phi^t(x)$ is compact. This means we can find a $\gamma > 0$ for which

$$\overline{B_\gamma(\Phi^t(x))} \subset \mathcal{S}.$$

Thanks to the cocycle property we have that $\Phi^{t-T}(\Phi^T(x)) = \Phi^t(x)$, and together with the fact that Φ is upper semi-continuous we can infer there exists $\delta > 0$ such that

$$\Phi^{t-T}(\overline{B_\delta(\Phi^T(x))}) \subset \overline{B_\gamma(\Phi^t(x))} \subset \mathcal{A}(S).$$

Since S is assumed to be an attractor, all compact subsets of $\mathcal{A}(M)$ are attracted in semi-distance to S. This indeed implies that $x \in \mathcal{A}(S)$.

If now we assume that $T > t$, $\mathcal{A}(S)$ is invariant and $\Phi^t(x) \subset \mathcal{A}(S)$, then $\Phi^T(x)$ is a compact subset of $\mathcal{A}(S)$. As before, $\mathcal{A}(S)$ is open and so it is possible to find a compact neighbourhood of $\Phi^T(x)$ which is attracted to S. $\square$

Definition 4.7. Let S be an attractor. The complementary set

$$S^* := X \setminus \mathcal{A}(S)$$

is called *repeller dual to* S. Similarly, define the *repelling neighbourhood* of S^* as

$$\mathcal{R}(S^*) = X \setminus S.$$

The previous Lemma precisely tells us that repellers are Φ^*-invariant. Beware, repellers might not be Φ-invariant (hence why we introduced the concept of dual systems). We have now recovered the concept of *attractor-repeller pair* in the set-valued setting. If we now suppose that we have two nested attractors $S_1 \subset S_2$, then $\mathcal{A}(S_1) \subset \mathcal{A}(S_2)$. This can be seen via the second definition of domain of attraction. If $x \in \mathcal{A}(S_1)$, then for all $T \geq 0$, we have a neighbourhood V of $\Phi^T(x)$ with the property that $\lim_{t\to\infty} \mathrm{d}(\Phi^t(V), S_1) = 0$. But

$$\mathrm{d}(\Phi^t(V), S_1) = \sup_{x\in\Phi^t(V)} \inf_{y\in S_1} d(x,y) \geq \sup_{x\in\Phi^t(V)} \inf_{y\in S_2} \mathrm{d}(x,y) = \mathrm{d}(\Phi^t(V), S_2)$$

since $S_1 \subset S_2$, therefore by taking the limit as $t \to \infty$, it follows that $\Phi^t(V)$ tends to S_2 in semi-distance. Given that T was arbitrary chosen, we can conclude that $x \in \mathcal{A}(S_2)$. If $S_1 \subset S_2$ are both attractors then not only $\mathcal{A}(S_1) \subset \mathcal{A}(S_2)$, but more importantly $S_1^* = X \setminus \mathcal{A}(S_1) \supset X \setminus \mathcal{A}(S_2) = S_2^*$. In order to keep the notation consistent with the one used in the previous chapters we will denote attractors by the letter A and their associated repellers by R.

Intuitively, the asymptotic behaviour of trajectories will be affected by the presence of attractors and repellers. Let (A, R) be an attractor-repeller pair, and γ a complete trajectory through $x \in X$, then if follows that

- If $\omega(\gamma) \cap R \neq \emptyset$ then $\gamma(\mathbb{R}) \subset R$;
- If $\alpha(\gamma) \cap A \neq \emptyset$ then $\gamma(\mathbb{R}) \subset R$;
- If $x \neq A$ then $\alpha(\gamma) \subset R$;
- If $x \neq R$ then $\omega(\gamma) \subset A$.

The proof of these facts follows from the classic theory of single-valued dynamical systems (see [Li07]).

4.1 Morse decomposition for set-valued dynamical systems

We can finally define what we mean by Morse decomposition in this new setting.

Definition 4.8. Let X be a compact metric space, admitting a family of attractor-repeller pairs $(A_i, R_i)_{i=0}^n$ for a set-valued dynamical system Φ, satisfying

$$\emptyset := A_0 \subsetneq A_1 \subsetneq \cdots \subsetneq A_n := X \quad \text{and} \quad X := R_0 \supsetneq R_1 \supsetneq \cdots \supsetneq R_n := \emptyset.$$

We define the *Morse decomposition* of X to be an ordered collection of sets $(\mathcal{M}_i)_{i=1}^n$ where every *Morse set* is defined as

$$\mathcal{M}_i := A_i \cap R_{i-1}$$

for $1 \leq i \leq n$.

As we have previously done we now wish to state some of the most useful properties of Morse sets, and in particular how to recover the original chain of attractors given a Morse decomposition.

Proposition 4.9. *Let $(\mathcal{M}_i)_{i=1}^n$ be a Morse decomposition of X, and $\emptyset = A_0 \subsetneq \cdots \subsetneq A_n = X$ be the corresponding chain of attractors. The following statements hold:*

1. *For any $1 \leq i \leq n$, the tuple $(A_{i-1}, \mathcal{M}_i)$ is an attractor-repeller pair in A_i;*
2. *$\mathcal{M}_i$ are all compact and pairwise disjoint sets;*
3. *Let γ be a complete trajectory which is not contained in a Morse set, then there exists $i < j$ such that $\alpha(\gamma) \subset \mathcal{M}_i$, and $\omega(\gamma) \subset \mathcal{M}_j$;*
4. *Morse sets completely determine their associated attractor chain. If we let $\mathcal{W}^u(\mathcal{M}_i) \coloneqq \{x \in X \mid \exists \gamma \text{ through } x, \text{ s.t. } \alpha(\gamma) \subset \mathcal{M}_i\}$ then for any $1 \leq k \leq n$*
$$A_k = \bigcup_{1 \leq i \leq k} \mathcal{W}^u(\mathcal{M}_i)$$
5. *If X is isolated, then so are all A_k.*

Proof. The proof of these facts follows from the single-valued case. See Theorem 3.7 in [Li07]. □

As we previously did, we will end this chapter with the definition of finest Morse decomposition, which is essentially the same as for topological dynamical systems.

Definition 4.10. Let $(\mathcal{M}_i)_{i=1}^n$ and $(\mathcal{N}_j)_{j=1}^m$ be two Morse decompositions of Φ. We say that $(\mathcal{N}_j)_{j=1}^m$ is *finer* than $(\mathcal{M}_i)_{i=1}^n$ if and only if for every $i \in \{1, \ldots . n\}$ there exists $j \in \{1, \ldots, m\}$ such that $\mathcal{N}_j \subset \mathcal{M}_i$. A Morse decomposition whose finer Morse decomposition is only itself it said to be the *finest Morse decomposition.*

Once again finest Morse decompositions not always exist but if they do they are unique.

Chapter 5

The interval case

The aim of this chapter is to study and compare the Morse decompositions of a random dynamical systems and their associated set-valued counterpart iterating over a compact interval. First of all we need to discuss the set-up. For a given compact, connected set $\Delta \subset \mathbb{R}^d$, which is not just a single point, define the probability space $(\Delta, \mathcal{B}(\Delta), \nu)$, where ν is any absolutely continuous measure with respect to the $\mathbb{R}^d$-Lebesgue measure, restricted to Δ.

Definition 5.1. Let X be a compact metric Polish space, and Δ a closed domain of $\mathbb{R}^n$. A *random diffeomorphism* is a smooth map $f : \Delta \times X \to X$, such that $\omega \mapsto f(\omega, x)$ is a non-constant map depending from a parameter ω sampled from Δ by a measure with a smooth density function, and such that $x \mapsto f(\omega, x)$ is a $\mathcal{C}^\infty$-diffeomorphism for every $\omega \in \Delta$.

Now the closed and bounded interval $[a, b] \subset \mathbb{R}$, once equipped with the metric induced by the euclidean metric of $\mathbb{R}$, is a compact Polish metric space. Consider a random diffeomorphism $f : \Delta \times [a, b] \to [a, b]$., and let us denote $f(\alpha, \cdot) : [a, b] \to [a, b]$ by f_α, for any $\alpha \in \Delta$. Assume the following:

H1) $f_\alpha : [a, b] \to [a, b]$ is strictly monotonically increasing for all $\alpha \in \Delta$;

H2) $\nu(\{\alpha \in \Delta \mid f_\alpha(x) = x\}) < 1$ for all $x \in [a, b]$;

H3) $\nu(\{\alpha \in \Delta \mid f_\alpha(a) = a\}) = \nu(\{\alpha \in \Delta \mid f_\alpha(b) = b\}) = 0$;

H4) $\bigcup_{\alpha \in \Delta} f_\alpha([a, b]) = [a, b]$.

as in [Ras18]. Therefore, f induces a random dynamical system. Let $(\Delta^{\mathbb{Z}}, \mathcal{B}(\Delta)^{\otimes \mathbb{Z}}, \nu^{\otimes \mathbb{Z}})$ be the probability space associated with the bi-infinite sequences whose components are sampled from Δ. This probability space is well-defined, thanks to Kolmogorov extension theorem, where the consistency conditions on the measures $\nu^{\otimes n}$, $n \in \mathbb{N}$, are satisfied by our hypothesis on ν (See [Li18]). Our previous assumption that Δ was not just a single point was made in order to ensure that the noise space $\Delta^{\mathbb{Z}}$ had sensible meaning. We now define the shift operator as $\theta_t : \Delta^{\mathbb{Z}} \to \Delta^{\mathbb{Z}}$, where

$$(\theta_0(\alpha_n)_{n \in \mathbb{Z}})_k = \alpha_k \quad \text{and} \quad (\theta_t(\alpha_n)_{n \in \mathbb{Z}})_k = \alpha_{t+k}$$

for all $k \in \mathbb{Z}$, and $t \in \mathbb{Z} \setminus \{0\}$. Notice that in this case, where we have a discrete two-sided time, $(\theta_t)_{t \in \mathbb{Z}}$ is a group under function composition, since $\theta_t^{-1} = \theta_{-t}$, and θ_0 is the identity element.

We define our random dynamical system $\varphi : \mathbb{Z} \times \Delta^{\mathbb{Z}} \times [a, b] \to [a, b]$ as

$$\varphi(n, (\alpha_n)_{n\in\mathbb{Z}}, x) := \begin{cases} (f_{\alpha_{n-1}} \circ f_{\alpha_{n-2}} \circ \cdots \circ f_{\alpha_1} \circ f_{\alpha_0})(x) & n \in \mathbb{Z}^+ \\ x & n = 0 \\ (f^{-1}_{\alpha_n} \circ \cdots \circ f^{-1}_{\alpha_{-2}} \circ f^{-1}_{\alpha_{-1}})(x) & n \in \mathbb{Z}^- \end{cases}$$

Notice this ensures the identity property. Now, in order to simplify notation let us write

$$\tilde{\alpha} = (\ldots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \ldots)$$

The cocycle property of φ follows quite easily by the definition. If we assume $t, s \geq 0$ then

$$\begin{aligned} \varphi(t + s, \alpha)x &= (f_{\alpha_{t+s-1}} \circ f_{\alpha_{t+s-2}} \circ \cdots \circ f_{\alpha_s} \circ f_{\alpha_s - 1} \circ \cdots \circ f_{\alpha_1})(x) \\ &= (f_{\alpha_{t+s-1}} \circ f_{\alpha_{t+s-2}} \circ \cdots \circ f_{\alpha_s})(\varphi(s, \alpha)x) \\ &= (f_{(\theta_s\alpha)_{t-1}} \circ f_{(\theta_s\alpha)_{t-2}} \circ \cdots \circ f_{(\theta_s\alpha)_0})(\varphi(s, \alpha)x) \\ &= \varphi(t, \theta_s\alpha)(\varphi(s, \alpha)x). \end{aligned}$$

All the other cases are quite similar.

Next we wish to prove a couple of properties of $(\nu^{\otimes\mathbb{Z}}, (\theta_t))$. Intuitively, our bi-infinite sequences are given by i.i.d. samplings from $[-1, 1]$, and for this reason we expect our noise to be stationary, i.e. $(\theta_t)^*\nu^{\otimes\mathbb{Z}} = \nu^{\otimes\mathbb{Z}}$ for all $t \in \mathbb{T}$. Furthermore, we need to check that $(\nu^{\otimes\mathbb{Z}}, (\theta_t)_{t\in\mathbb{Z}})$ is ergodic. In order to prove these properties we will use the standard fact that our product sigma algebra is generated by the sigma algebras of cylinder sets. Firstly, for any $m, n \in \mathbb{T}$, we can define the following projection maps:

$$\begin{aligned} c_{m,n} : \Delta^{\mathbb{Z}} &\to \Delta^{m+n+1} \\ (\ldots, \alpha_0, \alpha_1, \ldots) &\mapsto (\alpha_m, \ldots, \alpha_n) \end{aligned}$$

and these define a filtration, which coincides with the filtration of induced by φ as a RDS (see [KS12])

$$\mathcal{F}_{[p,q]} = \sigma(\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid c_{p,q}(\alpha) \in A, \text{ where } A \in \bigotimes_{i=0}^{q-p} \mathcal{B}(\Delta)) = \sigma(c_{p,q})$$

for any $p, q \in \mathbb{T}$, and $p \leq q$. Recall

$$\bigotimes_{i=0}^{q-p} \mathcal{B}(\Delta) = \sigma(E_0 \times \cdots \times E_{q-p} \mid E_i \text{ is open in } \Delta,\ 0 \leq i \leq q - p),$$

thus we can define the notion of (bi-infinite) *cylinder sets* as

$$I^{p,q}_{(E_i)_{i=0}^{q-p}} = c^{-1}_{p,q}(E_0 \times \cdots \times E_{q-p})$$

where $(E_i)_{i=0}^{q-p}$ is a collection of open subsets of Δ. Furthermore, notice that

$$\nu^{\otimes\mathbb{Z}}(I^{p,q}_{(E_i)_{i=0}^{q-p}}) = \prod_{i=0}^{q-p} \nu(E_i)$$

by construction. One can then redefine the filtration sets $\mathcal{F}_{[p,q]}$ as the σ-algebras given by all the possible $I^{p,q}$, but more importantly, cylinder sets completely generate our infinite product σ-algebra $\mathcal{B}(\Delta)^{\otimes\mathbb{Z}}$. It will be then enough to show that the properties we want to prove for $\nu^{\otimes\mathbb{Z}}$ hold for any cylinder set to then be able to extend them for arbitrary sets in $\Delta^{\mathbb{Z}}$.

Let us start by proving our noise is stationary. Fix any $t, p, q \in \mathbb{T}$, let $I^{p,q}_{(E_i)_{i=0}^{q-p}}$ be a cylinder set where $(E_i)_{i=0}^{q-p}$ is a collection of open subsets of Δ. In order to simplify notation let us relabel $I^{p,q}_{(E_i)_{i=0}^{q-p}} = I^{p,q}$. Then

$$\begin{aligned}(\theta_t)^*\nu^{\otimes\mathbb{Z}}(I^{p,q}) &= \nu^{\otimes\mathbb{Z}}(\theta_t^{-1}(I^{p,q})) = \nu^{\otimes\mathbb{Z}}(\theta_{-t}(I^{p,q})) \\ &= \nu^{\otimes\mathbb{Z}}(I^{p-t,q-t}) = \prod_{i=0}^{(q-t)-(p-t)} \nu(E_i) \\ &= \prod_{i=0}^{q-p} \nu(E_i) = \nu^{\otimes\mathbb{Z}}(I^{p,q}),\end{aligned}$$

and so we have that for all $t \in \mathbb{T}$, the measure $\nu^{\otimes\mathbb{Z}}$ is θ_t-invariant. Next, we would like to show $\nu^{\otimes\mathbb{Z}}$ is mixing. What follows is just the sketch of the proof.

Definition 5.2. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and let $f : \Omega \to \Omega$ be measurable. The measure μ is said to be *mixing* if and only if for all $A, B \in \mathcal{F}$ then

$$\lim_{t\to\infty} \mu(A \cap f^{-t}(B)) = \mu(A)\mu(B).$$

So now let $p, q, r, s \in \mathbb{T}$, with $p \leq q$ and $r \leq s$. Let $(E_i)_{i=0}^{q-p}$ and $(G_j)_{j=0}^{s-r}$ be two collections of open subsets of Δ. Consider the two cylinder sets $I^{p,q}_{(E_i)_{i=0}^{q-p}}$, which we will denote by $I^{p,q}$, and $I^{r,s}_{(G_j)_{j=0}^{s-r}}$, which we will denote by $\tilde{I}^{r,s}$. We can find a $T \in \mathbb{N}$ such that for all $t \geq T$ we have $p - T \leq q - T < r \leq s$. This implies that for all $t \geq T$, then $\theta_t^{-1}(I^{p,q}) = I^{p-t,q-t}$ and $\tilde{I}^{r,s}$ are two disjoint cylinder sets, in the sense that the σ-algebras to which they belong, namely $\mathcal{F}_{[p-t,q-t]}$ and $\mathcal{F}_{[r,s]}$ are independent. Once again, this follows from the fact that we are sampling process for constructing bi-infinite sequences is given by independent, identically distributed random variables. This is equivalent to saying that $(\nu^{\otimes\mathbb{Z}}, (\theta_t))$ is mixing.

This is actually telling us something more. The system φ is Markov as well because of the independence between past and future σ-algebras, once again given by our identically and independent sampling from Δ.

Corollary 5.3. *Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, and $f : \Omega \to \Omega$ be a measurable function. If μ is mixing, then it is ergodic.*

Proof. Let $A \in \mathcal{F}$ be an invariant set, i.e. $f^{-1}(A) = A$. Then

$$\mu(A) = \mu(A \cap A) = \mu(A \cap f^{-1}(A)) = \lim_{t\to\infty} \mu(A \cap f^{-t}(A)) = \mu(A)\mu(A)$$

which, since μ is a probability measure, means $\mu(A) \in \{0, 1\}$. □

We can therefore conclude that $(\nu^{\otimes\mathbb{Z}}, (\theta_t))$ is ergodic.

5.1 Associated set-valued system and minimal invariant sets

It is possible to associate to such random dynamical systems a set-valued counterpart. Let us denote by $\mathcal{K}([a,b])$ the set of all compact sets in $[a,b]$, then

$$\begin{aligned} F : [a,b] &\to \mathcal{K}([a,b]) \\ x &\mapsto F(x) := \bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(1,\tilde{\alpha})(x)\} = \bigcup_{\alpha\in\Delta} \{f_\alpha(x)\}. \end{aligned}$$

The fact that $\bigcup_{\alpha\in\Delta}\{f_\alpha(x)\} \in \mathcal{K}([a,b])$ follows from the fact that Δ is compact. The iterations of such a map defines a set-valued dynamical system, under the assumption $F^0(x) = x$. In this case we clearly have $\mathbb{T} = \mathbb{N} \cup \{0\}$. We need to check that the iterations of F are well-defined. In order to do so we introduce the "extended" map

$$\begin{aligned} \hat{F} : \mathcal{K}([a,b]) &\to \mathcal{K}([a,b]) \\ A &\mapsto \hat{F}(A) = \bigcup_{x\in A} F(x) = f(\Delta, A) \end{aligned}$$

and so for any $n \geq 1$ we can equivalently rewrite $F^n(x)$ as $\hat{F}^n(\{x\})$. This follows from noting that $F(x) = \hat{F}(\{x\})$, and similarly for any $A \in \mathcal{F}([a,b])$, $F(A) = \bigcup_{x\in A} F(x) = \hat{F}(A)$. We need to make sure that $\hat{F}$ is actually well-defined itself. Let us recall that the space $\mathcal{K}([a,b])$ has been equipped with the Hausdorff distance $d_H(\cdot,\cdot)$. The map $\hat{F}$ is well-defined since f is, and its co-domain is given by $\mathcal{K}([a,b])$ since for all $A \in \mathcal{K}([a,b])$, then the set $\Delta \times A$ is compact, and thus $\hat{F}(A) = f(\Delta, A)$ is compact since it is the continuous image of a compact set (recall that f was assumed smooth).

Having established that the iterates of F are well-defined for any $n \geq 2$ we can write $F^n = \bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}}\{\varphi(n,\alpha)(x)\}$. We equip $\mathbb{N}\cup\{0\}$ with the discrete topology, hence if $(n_j)_{j\in\mathbb{N}}$ is a converging sequence $n_j \to n$ for $j \to +\infty$ then we must have that the sequence becomes eventually constant, since every point in $\mathbb{N}\cup\{0\}$ is isolated.

It will then be enough to show that the extended map is continuous to then immediately infer that F is not only upper semicontiuous, but actually continuous. Since we equipped the space $\mathcal{K}([a,b])$ with its metric topology, then it will be enough to check that the pre-image of any open ball in $\mathcal{K}([a,b])$, is open in $\mathcal{K}([a,b])$. For any $A \in \mathcal{K}([a,b])$, define for any $\varepsilon > 0$, the open ball around A of radius ε as

$$B_\varepsilon(A) = \{B \in \mathcal{K}([a,b] \mid d_H(A,B) < \varepsilon)\}.$$

But now notice $\hat{F}^{-1}(B_\varepsilon(A)) = \pi_2(f^{-1}(B_\varepsilon(A))$, where π_2 is the projection map of the second coordinate. This is manifestly open since f is continuous and projection maps are open maps. Therefore, $\hat{F}$ is continuous. Now if we take any sequence $(t_n, x_n) \to (t,x)$ in $(\mathbb{N}\cup\{0\}) \times X$, we know that for n big enough, our sequence will be of the form (t, x_n), and since the composition of continuous functions is continuous will have that

$$\lim_{n\to\infty} F^{t_n}(x_n) = \lim_{n\to\infty} F^t(x_n) = \lim_{n\to\infty} \hat{F}^t(x_n) = \hat{F}^t(x) = F^t(x).$$

The cocycle property is steadily proved for any natural numbers n, m and any $x \in [a, b]$:

$$\begin{aligned} F^n(F^m(x)) &= F^n(\bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(m,\tilde{\alpha})(x)\}) = \\ &= \bigcup_{\tilde{\beta}\in\Delta^{\mathbb{Z}}} \{\varphi(n,\tilde{\beta})\{(\bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(n,\tilde{\alpha})(x)\})\} \\ &= \bigcup_{\tilde{\beta}\in\Delta^{\mathbb{Z}}} \bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(n,\tilde{\beta})(\varphi(m,\tilde{\alpha})(x))\} \\ &= \bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(n,\theta_m\tilde{\alpha})(\varphi(m,\tilde{\alpha})(x))\} \\ &= \bigcup_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}} \{\varphi(n+m,\tilde{\alpha})(x))\} \\ &= F^{n+m}(x) \end{aligned}$$

where the fourth equality follows from the fact that in this specific case our noise does not have any dependence from the past.

Finally, we need to show that $t \to F^t(x)$ is continuous in with respect to the Hausdorff distance. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{Z}$ such that it converges to $t \in \mathbb{Z}$. Then we wish to show for any $x \in [a, b]$ that $\lim_{n\to\infty} d_H(F^{t_n}(x), F^t(x)) = 0$, but as we have seen before the sequence (t_n) must become constantly equal to t for n big enough, and therefore this follows by properties of metrics. We can conclude that F is effectively a set-valued dynamical system.

Remark. We have previously assumed that Δ is compact, connected and not a point, and that $\alpha \to f(x, \alpha)$ is non-constant for all $\alpha \in \Delta$. This means that Δ has non-empty interior, and hence there exists a $\varepsilon > 0$ and a $T \in \mathbb{N}$ such that $F^T(x)$ contains a ball of radius ε, for all $x \in [a, b]$. This will be fundamental in proving Proposition 5.5. ♦

Now (H4) can be restated in terms of set-valued systems as

H4) F([a,b])=[a,b].

The focus of the next few results will be on minimal invariant sets for the system we have just described. We will now show that in our case there always exists at least one minimal invariant set. The main ideas are from [LRR15].

Proposition 5.4. *There exists at least one F-minimal invariant set in $[a, b]$*

Proof. Consider the collection of forward-invariant sets $\mathcal{C} := \{A \in \mathcal{K}([a, b]) \mid F(A) \subset A\}$. This is not empty thanks to assumption (H4). The set inclusion operation gives a partial order on $\mathcal{C}$. Let $\mathcal{C}'$ be a chain of totally ordered elements of $\mathcal{C}$, i.e. $\mathcal{C}' = \{A_i\}_{i\in I}$, and $k < j$ implies $A_j \subset A_k$, for $k, j \in I$, and I an index set. Now denote by $\tilde{A} = \bigcap_{i\in I} A_i$. This is non-empty and we have that $\tilde{A} \in \mathcal{K}([a, b])$ since the intersection of closed sets is closed and $\tilde{A} \subset A_i$ for any $i \in I$. This means that every totally ordered chain has a minimal element. By Zorn's Lemma there exists at least one minimal element in $\mathcal{C}$. This minimal element is a minimal invariant set for F, since it does not contain any forward-invariant set. □

Now that we have established existence of F-minimal invariant sets let us focus on some of their most useful properties. This has been adapted from [LRR15].

Proposition 5.5. *The F-minimal invariant sets of a set-valued system F are finitely many and pairwise disjoint.*

Proof. Firstly, let us prove that they are pairwise disjoint. Let E_i, E_j be two minimal invariant sets, and assume that their intersection is non-trivial. If we pick $z \in E_i \cap E_j$ then $z \in E_i$ hence $F(z) \in F(E_i) = E_i$, and similarly, $z \in E_j$ implies $F(z) \in F(E_j) = E_j$. Therefore, $E_i \cap E_j$ is forward-invariant, $F(E_i \cap E_j) \subset E_i \cap E_j$. Because of this we have that the ω-limit set of $E_i \cap E_j$ is again in $E_i \cap E_j$ by construction. This induces a contradiction since $\omega(E_i \cap E_j)$ is forward-invariant and lies in both E_i, and E_j which were assumed invariant. We can conclude that minimal invariant sets have to be pairwise disjoint.

Now we can prove that a systems admits finitely many minimal invariant sets. Suppose we have infinitely many of these sets, and denote by E one of them. By previous remark there exists an $\varepsilon > 0$ such that for a certain $T \in \mathbb{N}$ the set $F^T(x)$ contains a ball of radius ε. By the fact that K is minimal then this ball must lie in K. This holds for all the minimal invariant sets. Consider the open covering $\{B_\varepsilon(x)\}_{x\in[a,b]}$, where $B_\varepsilon(x)$ denotes the open ball of ε-radius centred at x. By compactness of $[a, b]$, we must have a finite set $S \subset [a, b]$ such that $\{B_\varepsilon(x)\}_{x\in S}$ is still an open covering of the state space. We have just proved that the minimal invariant sets are all disjoint, and they all contain at least one ball of radius ε. Having infinitely many minimal invariant sets implies that there cannot exists a finite subcovering made of ε-balls of $[a, b]$, contradicting the definition of compactness. This is a contradiction, and therefore, we have finitely many minimal invariant sets. □

Remark. These last two proposition hold in much greater generality under appropriate assumptions on the state space X, and structure of F (see [LRR15]). ⧫

Since we assumed that f_α was increasing for any $\alpha \in \Delta$ we can conclude that if a minimal invariant set K is the union of different disjoint components, we must have that all the components map precisely into themselves. If this were not the case, by the invariance property $F(K) = K$ the components would permute, but then at least one of them would move in a direction different from the one imposed by f_α being increasing for every noise realisation, hence contradicting one of our hypotheses. Similarly as before, we can only have finitely many of these components since they all must contain a ball of radius $\varepsilon > 0$, thanks to the above discussion. For this same reason, all these components cannot have empty interior, so we can conclude that, in this case, all minimal invariant sets are actually compact intervals of the form $K = [k_1, k_2]$.

The next result is of fundamental importance in order to understand the core of this project. It clearly states a connection between random dynamical systems and their correspondent set-valued systems, in particular between their Morse decompositions. It is inspired by [Ras18].

Proposition 5.6. *Let K be a minimal F-invariant set, then the ergodic invariant Markov measure μ induced by K is supported on a random fixed point $a : \Delta^{\mathbb{Z}} \to [a, b]$. This means that its disintegration is of the form $\mu_{\tilde{\alpha}} = \delta_{a(\tilde{\alpha})}$, for almost every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$.*

Before proving this result let us state a useful proposition, adapted from Theorem 1.3 in [ZH07], which will give us an ulterior way of relating random dynamical systems to their associated set-valued counterparts.

Proposition 5.7. *Under our current hypotheses, φ possesses a finite number of ergodic stationary measures $\mu_1, \ldots, \mu_n$ whose supports are the minimal F-invariant sets in $[a, b]$. All stationary measures of φ are linear combinations of $\mu_1, \ldots, \mu_n$.*

The idea behind the proof of Proposition 5.6 is to use the fact that every minimal F-invariant set K, thanks to Proposition 5.7, supports an ergodic stationary measure μ. This measure corresponds to an ergodic invariant Markov measure whose disintegration is given by $\mu_{\tilde{\alpha}} = \lim_{n\to\infty} \varphi(n, \theta_{-n}\tilde{\alpha})^{*}\mu$, for almost every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$, by Ledrappier-Le Jan-Crauel theorem. This disintegration will have to then agree almost surely with the one constructed through the random family of Dirac delta measures supported on a random fixed contained in K.

Proof of Proposition 5.6. Let $K = [k_-, k_+]$ be a minimal invariant set of F, and pick any $\tilde{\alpha} = (\alpha_n)_{n=1}^{\infty} \in \Delta^{\mathbb{Z}}$. Define the sequences

$$a_n^-(\tilde{\alpha}) \coloneqq \varphi(n, \theta_{-n}\tilde{\alpha})k_- \quad \text{and} \quad a_n^+(\tilde{\alpha}) \coloneqq \varphi(n, \theta_{-n}\tilde{\alpha})k_+$$

for $n \in \mathbb{N}$. Firstly note that

$$\begin{aligned} a_1^-(\tilde{\alpha}) &= \varphi(1, \theta_{-1}\tilde{\alpha})k_- = f_{\alpha_{-1}}(k_-) \geq k_-, \\ a_1^+(\tilde{\alpha}) &= \varphi(1, \theta_{-1}\tilde{\alpha})k_+ = f_{\alpha_{-1}}(k_+) \geq k_+, \end{aligned}$$

since K is forward-invariant for F. We will now prove that the sequence $a_n^-(\tilde{\alpha})$ is increasing; one can show that the sequence $a_n^+(\tilde{\alpha})$ is decreasing by similar arguments to the ones presented here. Assume, as an inductive hypothesis, that $a_n^-(\tilde{\alpha}) \geq a_{n-1}(\tilde{\alpha})$. Since the set K is forward-invariant for F, then for every $\alpha \in \Delta$ we must have that $f_\alpha(k_-) \geq k_-$, hence in particular

$$f_{-n-1}(k_-) \geq k_-. \tag{4}$$

Recall that the composition of strictly monotonically increasing functions is once again strictly monotonic increasing, thus the function

$$f_{\alpha_{-1}} \circ \cdots \circ f_{\alpha_{-n}} : [a, b] \to [a, b]$$

is increasing. If we apply the previous function to Inequality (4) we have

$$\begin{aligned} & (f_{\alpha_{-1}} \circ \cdots \circ f_{\alpha_{-n}})(f_{\alpha_{-n-1}}(k_-)) \geq f_{\alpha_{-1}} \circ \cdots \circ f_{\alpha_{-n}}(k_-) \\ \iff & \varphi(n+1, \theta_{-n-1}\tilde{\alpha})k_- \geq \varphi(n, \theta_{-n}\tilde{\alpha})k_- \\ \iff & a_{n+1}^-(\tilde{\alpha}) \geq a_n^-(\tilde{\alpha}), \end{aligned}$$

as we claimed. Therefore the sequences $a_n^-(\tilde{\alpha})$, and symmetrically $a_n^+(\tilde{\alpha})$ are respectively increasing and decreasing for every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$.

The two sequences converge since they are monotonic and bounded. Let $a_\infty^-(\tilde{\alpha})$ and $a_\infty^+(\tilde{\alpha})$ be the limits of $(a_n^-(\tilde{\alpha}))_n$ and $(a_n^+(\tilde{\alpha}))_n$, respectively. The functions $a_\infty^\pm(\tilde{\alpha})$ are random fixed points: they are measurable since they are pointwise limits of measurable functions, and for every $n \in \mathbb{Z}$ we have $\varphi(n, \tilde{\alpha})a_\infty^\pm(\tilde{\alpha}) = a_\infty^\pm(\theta_n\tilde{\alpha})$ by continuity of φ. These induce two families of measures $\{\delta_{a_\infty^-(\tilde{\alpha})}\}_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}}$ and $\{\delta_{a_\infty^+(\tilde{\alpha})}\}_{\tilde{\alpha}\in\Delta^{\mathbb{Z}}}$. For any $A \in \mathcal{B}([a, b])$, the map

$$q^\pm : \alpha \mapsto \delta_{a_\infty^\pm(\tilde{\alpha})}(A) = \begin{cases} 1 & a_\infty^\pm(\tilde{\alpha}) \in A \\ 0 & a_\infty^\pm(\tilde{\alpha}) \notin A \end{cases}$$

is measurable since

$$(q^\pm)^{-1}(\{1\}) = (a_\infty^\pm)^{-1}(A) \in \mathcal{B}(\Delta)^{\otimes\mathbb{Z}} \quad \text{and} \quad (q^\pm)^{-1}(\{0\}) = (a_\infty^\pm)^{-1}(A^\complement) \in \mathcal{B}(\Delta)^{\otimes\mathbb{Z}}.$$

The set memberships follow from the measurability of $a^{\pm}_{\infty}$ and $A, A^{\complement} \in \mathcal{B}(X)$. Thus, our families of measures define two random probabilities measures. It follows quite immediately that $q^{\pm}$ is $\mathcal{F}_{-}$-measurable. The limit which defines $a^{\pm}_{\infty}(\tilde{\alpha})$ is taken in the pull-back sense, and as such it only depends on the past of $\tilde{\alpha}$.

Finally, since $a^{\pm}_{\infty}$ are two random fixed points, and since our two families of measures only contain Dirac measures, we have

$$\varphi(t,\tilde{\alpha})^{*}\delta_{a^{\pm}_{\infty}(\tilde{\alpha})} = \delta_{\varphi(t,\tilde{\alpha})a^{\pm}_{\infty}(\tilde{\alpha})} = \delta_{a^{\pm}_{\infty}(\theta_t\tilde{\alpha})}.$$

yielding that these random probability measures are two disintegrations of Markov Θ-invariant probability measures. By Ledrappier-Le Jan-Crauel theorem (Theorem 3.12) to each family of measures there corresponds a stationary measure

$$\mu^{+} = \mathbb{E}(\delta_{a^{+}_{\infty}(\tilde{\alpha})}) \quad \text{and} \quad \mu^{-} = \mathbb{E}(\delta_{a^{-}_{\infty}(\tilde{\alpha})}),$$

both supported on K. By Proposition 5.7 we know there exists a unique ergodic stationary measure ξ supported on K. Since all stationary measures are linear combinations of ergodic stationary measures, in this case follows, by uniqueness of ξ, that $\mu^{+} = \xi = \mu^{-}$. By Ledrappier-Le Jan-Crauel theorem we can conclude that $\delta_{a^{-}_{\infty}(\tilde{\alpha})} = \delta_{a^{+}_{\infty}(\tilde{\alpha})}$ almost surely. Therefore, for almost every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$

$$a^{+}_{\infty}(\tilde{\alpha}) = a^{-}_{\infty}(\tilde{\alpha}) =: a(\tilde{\alpha}),$$

and the ergodic stationary measure ξ supported on K induces the disintegration

$$\xi_{\tilde{\alpha}} = \delta_{a(\tilde{\alpha})}$$

for almost every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$. □

Remark. We would like to highlight, once again, the important correspondence exploited in the second part of this proof. To every minimal F-invariant set there corresponds a unique ergodic stationary measure for φ, which induces a unique disintegration of a Markov ergodic Θ-invariant measure (up to a set of measure zero), and vice versa. Thus, any Markov random dynamical systems, their associated set-valued counterpart, and their induced Markov process are connected from measure-theoretical point of view. ♦

Our assumptions on monotonicity will now play a key role in what follows. The random fixed point acting fibre-wise as the support of the disintegration of the ergodic invariant measure μ supported on the minimal invariant set we have just found is actually an attractor.

Proposition 5.8. *Let K be an F-invariant minimal set, then it contains precisely one local attractor for φ.*

Proof. As we have seen before we can consider $K = [k_{-}, k_{+}]$ to be our F-minimal invariant set and define, for almost every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$, the functions $a(\tilde{\alpha}) = \lim_{n\to\infty} \varphi(n, \theta_{-n}\tilde{\alpha})k_{-} = \lim_{n\to\infty} \varphi(n, \theta_{-n}\tilde{\alpha})k_{+}$. Define the compact random set $A(\tilde{\alpha}) := \{a(\tilde{\alpha})\}$. This is well-defined and φ-invariant since $a : \Delta^{\mathbb{Z}} \to K$ is a random fixed point. Now thanks to the fact that f_{α} is strictly monotonic increasing for every $\alpha \in \Delta$, we can show that the forward-invariant open random set U, fibre-wise defined as $U(\tilde{\alpha}) := (k_{-}, k_{+})$ is an attracting random neighbourhood of A. Before proving this, we would like to be sure that the random fixed does not accumulate on the boundary of K with positive probability.

Thanks to the assumption $(H3)$ we know that $\nu(\{\alpha \in \Delta \mid f_\alpha(k_-) = k_-\}) < 1$, hence, by construction of a,

$$\nu^{\otimes\mathbb{Z}}(\{\tilde{\alpha} \in \Delta^{\otimes\mathbb{Z}} \mid a(\tilde{\alpha}) = k_-\}) = 0.$$

An identical argument shows that $a(\tilde{\alpha}) = k_+$ with zero probability. Thus, $a(\tilde{\alpha}) \in (k_-, k_+)$ almost surely. Since all the functions f_α, for $\alpha \in \Delta$ are strictly monotonically increasing, we always have $f_\alpha((k_-, k_+)) = (f_\alpha(k_-), f_\alpha(k_+))$. This and the fact that both boundary points k_- and k_+ converge with probability one to the random fixed point, they imply

$$\begin{aligned}
&\nu^{\otimes\mathbb{Z}}(\{\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid \lim_{n\to\infty} \mathrm{d}(\varphi(n, \tilde{\alpha})U(\tilde{\alpha}), \{a(\theta_n\tilde{\alpha})\}) = 0\}) \\
=&\nu^{\otimes\mathbb{Z}}(\{\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid \lim_{n\to\infty} \mathrm{d}(\varphi(n, \theta_{-n}\tilde{\alpha})(k_-, k_+), \{a(\tilde{\alpha})\}) = 0\}) \\
=&\nu^{\otimes\mathbb{Z}}(\{\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid \lim_{n\to\infty} \mathrm{d}((a_n^-(\tilde{\alpha}), a_n^+(\tilde{\alpha})), \{a(\tilde{\alpha})\}) = 0\}) = 1.
\end{aligned}$$

Therefore, A is a local attractor of φ. The uniqueness of the attractor follows from the fact that the attracting neighbourhood is the interior of the F-minimal invariant set. □

The following corollary is immediate.

Corollary 5.9. *If the state space $[a, b]$ contains only one F-minimal invariant set K, then there exists only one attractor for φ.*

Proof. From the previous proposition we know that the minimal invariant set contains a local attractor for φ. As we did in the proof of Proposition 5.6, we can define, for every $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$, two sequences

$$\begin{aligned}
a_-^n(\tilde{\alpha}) &:= \varphi(n, \theta_{-n}\tilde{\alpha})a \\
a_+^n(\tilde{\alpha}) &:= \varphi(n, \theta_{-n}\tilde{\alpha})b
\end{aligned}$$

which are, respectively, increasing and decreasing since $[a, b]$ is F-invariant (assumption (H4)). These are bounded, hence both converge to some limit in the state space. The two limits have to be almost everywhere equal to the fibre-wise description of the random attractor since the system admits only one F-minimal invariant set. Thus, the forward-invariant closed random set $A(\tilde{\alpha}) = [a, b]$ is an attracting neighbourhood of the attractor in K. Assumptions (H3) and (H4) grants us that almost every orbit starting on the boundary of the state space maps in the interior, and thus in our attracting neighbourhood. □

5.2 Equivalence between Morse decompositions

In the next section, we would like to focus on comparing the similarities and differences of the Morse decompositions induced by the set-valued and random systems. From the previous corollary we can conclude that we need different F-minimal invariant sets in order to generate multiple local attractors for φ.

Of interest will be the situation where we will have a countable sequence of F-invariant nested sets: we will only have one random local attractor for φ, whilst possibly generating multiple nested attractor for F. In this case the set-valued finest Morse decomposition will be composed of more sets than the finest Morse decomposition of the random dynamical system. Firstly, let us look into an example when this equivalence goes wrong.

Example 5.2.1. One of the easiest examples comes from the class of random dynamical systems with additive bounded noise, which we will soon explore in much greater detail.

Consider a strictly monotonically increasing $\mathcal{C}^\infty$-diffeomorphism $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 1$. Assume that $\frac{df}{dx}(0) = 1$, and $\frac{d^2 f}{dx^2}(x) > 0$, which means that, in a neighbourhood of 0, f intersects $y = x + 1$, and it locally lies above such a line. Suppose there exists a values $a < 0$, and $c > 0$ such that $f(a) = a + 1$ and $f(c) = c - 1$.

Let us now construct a random dynamical system with additive bounded noise from f. Let $\Delta = [-1, 1]$, and define $f_\alpha(x) = f(x) + \alpha$, for $\alpha \in \Delta$. Let $\nu = \frac{1}{2}\lambda$ be the measure defined over Δ, for λ the Lebesgue measure on $\mathbb{R}$. Now, just by following the construction we previously described, we obtain a random dynamical system over the noise space $[-1, 1]^{\mathbb{Z}}$ with associated measure $\nu^{\otimes\mathbb{Z}}$.

Remark. One of the most useful properties of one-dimensional RDS with additive bounded noise is that there exists two functions $f_{max}(x)$, and $f_{min}(x)$ which point-wise bound, from above and below respectively, $f_\alpha(x)$ for any $\alpha \in \Delta$. Such functions can be exploited to efficiently compute the invariant sets. If there are two points $m < n$ in the domain of f such that $f_{min}(m) = m$, and $f_{max}(n) = n$ then the set $[m, n]$ is invariant for the set-valued function $F(x) = \bigcup_{\alpha\in\Delta}\{f_\alpha(x)\}$. If we add the restriction that m, n are consecutive then $[m, n]$ is a minimal F-invariant set. This last remark is the key point used in this example to generate two finest Morse decompositions which will not agree. From now on minimal invariant sets will always be in the sense of F-invariant, unless otherwise stated. ♦

In our example we have that $f_{min}(x) = f(x) - 1$, which we will denote $f_{-1}(x)$, and similarly $f_{max}(x) = f(x)+1$, namely $f_1(x)$. By our previous hypothesis we know $f_1(c) = c$, $f_{-1}(0) = 0$, and $f_{-1}(a) = a$. Given the convexity of f around 0 and the fact that $f_1(b) = b$, one can see there is at least one point $b \in (0, c)$ such that $f_{-1}(b) = b$.

In order to simplify our next discussion we will assume that the only fixed points in $[a, c]$ of f_1 or f_{-1} are $a < 0 < b < c$. From now on we will set $X := [a, c]$ as the state space of our random dynamical system φ induced by f. Before proceeding any further we have to check all the hypotheses we made at the beginning of this chapter. $f_\alpha(x) = f(x) + \alpha$ is strictly monotonically increasing for all $\alpha \in \Delta$, since f is assumed so. For any fixed $x \in X$, $f_\alpha(x) = f(x) + \alpha = x$ holds for at most one α, call it α_0. But $\nu(\{\alpha_0\}) = 0$, so we can conclude $\nu(\{\alpha \in \Delta \mid f_\alpha(x) = x\}) = 0$ for all $x \in X$. Similarly, combining the fact that $f_\alpha([a, b]) \subset [a, b]$, and the fact that the set of noise realisations giving rise to fixed points has measure zero, we have

$$\nu(\{\alpha \in \Delta \mid f_\alpha(a) > a\}) = \nu(\{\alpha \mid f_\alpha(c) < c\}) = 1.$$

Finally, $F(X) = \bigcup_{\alpha\in\Delta}\{f_\alpha(X)\} = X$, by our previous discussion on the properties of F-invariant sets for random dynamical systems with additive bounded noise. Therefore, hypotheses (H1) to (H4) all hold. Moreover, by our specific choice of measure over Δ, and the regularity assumption on f, we have that not only

$$\begin{aligned} \tilde{f} : \Delta \times X &\to X \\ (\alpha, x) &\mapsto \tilde{f}(\alpha, x) = f_\alpha(x) \end{aligned}$$

smooth (it is a composition of smooth functions), but it is a random diffeomorphism.

Firstly focus on the Morse decomposition of the set-valued system F over X. Consider now the only three invariant sets:

$$X = [a, c], \qquad K_1 = [0, c], \qquad K_2 = [b, c]$$

where K_2 is the only minimal one.

Lemma 5.10. *K_1 is an attractor for F.*

Proof. Since we are working under the assumption of an additive bounded noise random dynamical system induced by a strictly monotonic increasing function, it will be sufficient to show that for any $p \in (a,0)$ then $\lim_{t\to\infty}|f_{-1}^t(p)| = 0$, to then deduce that $\lim_{t\to\infty} \mathrm{d}(F^t((p,c]),[0,c]) = 0$, in semi-distance.

Claim. $f(x) \geq x+1$

The Taylor's expansion of f around 0 yields

$$\begin{aligned} f(x) &= f(0) + \frac{df}{dx}(0)x + \frac{1}{2}\frac{d^2 f}{dx^2}(0)x^2 + h_2(x)x^2 \\ &= 1 + x + \left(\frac{1}{2}\frac{d^2 f}{dx^2}(0) + h_2(x)\right)x^2 \end{aligned}$$

where $h_2(x)$ is the Peano form of the remainder, i.e. $\lim_{x\to 0} h_2(x) = 0$. Thus, by fixing $\varepsilon > 0$ small enough so that for $x \in (-\varepsilon, \varepsilon)$, together with the assumption that $\frac{d^2 f}{dx^2} > 0$, we have

$$\frac{1}{2}\frac{d^2 f}{dx^2}(0) + h_2(x) > 0,$$

and hence $f(x) > x+1$ for $x \in (-\varepsilon,0)\cup(0,\varepsilon)$. Fix $r \in (-\varepsilon, 0)$ such that $f(r) > r+1$. Now assume that, without loss of generality, there exists $q \in (a,r)$ so that $f(q) < q+1$, then we have that by the intermediate value theorem, a point $m \in (q,r)$ such that $f(m) = m+1$. This is a contradiction since it implies $f_{-1}(m) = f(m) - 1 = m - 1 + 1 = m$, but we assumed that the only fixed points of f_{-1} are $\{a, 0, b\}$. We can conclude that $f(x) > x-1$ on $(a,0)$, and by similar reasons we can extend this result over $(a,0)\cup(0,b)$.

From the claim it follows that $f_{-1}(x) > x$ for $x \in (a,0)\cup(0,b)$. By the theory of deterministic dynamical systems, we have that the point a is repelling, whereas the point 0 is attractive for f_{-1}. Now since f_{-1} is a lower bound for the iterations of F, and it tends to zero for any point in $(a,0)$, then we have $\lim_{t\to\infty} d(F^t((a,c]),[0,c]) = 0$ where $[0,c]$, F-invariant and compact, is our attractor. Notice that this tells us that $\mathcal{A}([0,c]) = (a,c]$. ☐

Remark. The technique used in the previous proof can be used to show that any F-invariant set is an attractor for F, under the condition that the set does not share a boundary point with an F^*-invariant set, where F^* is the dual system to F. This follows from the fact that if we consider the boundary of any F-invariant set, A we will surely have an f_{min} fixed point, call it p, as the leftmost, or smallest, boundary point, and an f_{max} fixed point, call it q as the rightmost, or biggest, boundary point. Since f_α is always assumed increasing, in the specific case of additive bounded noise, one sees immediately that under the assumption of disjoint F and F^*-invariant sets, f_{min} has to be above the identity line locally at the left of p, and f_{max} has to be below the identity line locally at the right of q. By the same strategy adopted in the previous proof this implies that A is an attractor.

Symmetrically, this shows that any F^*-invariant set is a repeller, under condition that it does not share a boundary point with an F-invariant set. What happens in the limit case has been analysed by Lamb, Rasmussen, Rodrigues in [LRR15]. ♦

Corollary 5.11. *K_2 is an attractor for F.*

This corollary follows immediately from Lemma 5.10, which also tells us that $\mathcal{A}([b,c]) = (0,c]$. So our chain of attractors is given by

$$\emptyset =: A_0 \subset [b,c] =: A_1 \subset [0,c] =: A_2 \subset X =: A_3$$

and the corresponding chain of repellers (built by considering the complement of the basins of attraction):

$$X =: R_0 \supset [a,0] =: R_1 \supset \{a\} =: R_2 \supset \emptyset =: R_3$$

These two chains generate the following Morse decomposition:

$$\begin{aligned}
\mathcal{M}_1 &= [b,c](= A_1);\\
\mathcal{M}_2 &= \{0\};\\
\mathcal{M}_3 &= \{a\}(= R_2).
\end{aligned}$$

Remark that this decomposition is described by two singletons $\{a\}$ and $\{0\}$, and a minimal F-invariant set $[b,c]$, which by definition it does not contain any other invariant sets. Considering we have stated that Morse sets are invariant, this immediately tells us that the above decomposition is the finest.

What about the Morse decomposition generated by the RDS φ? We know that the only F-minimal invariant set is $[b,c]$, and thus by Corollary 5.9 we know there exists a unique attractor $\tilde{A}_1$, given by the random fixed point $s(\tilde{\alpha})$ for $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$ as defined in the proof of Proposition 5.8. So by definition we will only have one associated repeller. In order to compute such an object let us first define the sets

$$\tilde{\Omega} := \{\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid \varphi(n,\tilde{\alpha})a = a, \forall n \geq 0\}$$

and

$$\tilde{\Xi} := \{\tilde{\alpha} \in \Delta^{\mathbb{Z}} \mid \exists T_0 > 0 \text{ s.t. } \forall t \geq T_0,\ \varphi(t,\tilde{\alpha})x = 0 \text{ where } x \in [a,0]\}$$

which are both clearly in $\mathcal{B}(\Delta)^{\otimes \mathbb{Z}}$ and have zero measure, thanks to (H3) and (H2), respectively. Following our footstep in the proof of Proposition 5.8, we have that the basin of attraction of $\tilde{A}_1$ for any $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$ is

$$\mathcal{A}(\tilde{A}_1)(\tilde{\alpha}) = \begin{cases} (a,c] & \text{for } \tilde{\alpha} \in \tilde{\Omega} \\ (0,c] & \text{for } \tilde{\alpha} \in \tilde{\Xi} \\ [a,c] & \text{otherwise.} \end{cases}$$

This is an open random set since $[a,c]$ has been equipped with the subspace topology, and therefore $(a,c]$, $(0,c]$, and $[a,c]$ are all open.

We are now able to fibre-wise construct our attractor

$$\tilde{R}_1(\tilde{\alpha}) = \begin{cases} \{a\} & \text{for } \tilde{\alpha} \in \tilde{\Omega} \\ [a,0] & \text{for } \tilde{\alpha} \in \tilde{\Xi} \\ \emptyset & \text{otherwise.} \end{cases}$$

The Morse decomposition of φ is defined for $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$ as follows

$$\tilde{\mathcal{M}}_1(\tilde{\alpha}) = \{s(\tilde{\alpha})\}; \qquad \text{and} \qquad \tilde{\mathcal{M}}_2(\tilde{\alpha}) = \begin{cases} \{a\} & \text{for } \tilde{\alpha} \in \tilde{\Omega} \\ [a,0] & \text{for } \tilde{\alpha} \in \tilde{\Xi} \\ \emptyset & \text{otherwise.} \end{cases}$$

This is again a finest Morse decomposition, given the nature of our Morse sets: $\tilde{\mathcal{M}}_1$ is always a point, whereas $\tilde{\mathcal{M}}_2$ is almost surely empty. Given Definition 3.23 the result follows. Clearly, the cardinality of these two decompositions is different, but do they describe different asymptotic behaviours? If one compares

$$\mathfrak{M} = \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3 = \{a\} \cup \{0\} \cup [b, c]$$

and

$$\tilde{\mathfrak{M}} = \bigcup_{\tilde{\alpha} \in \Delta^{\mathbb{Z}}} \tilde{\mathcal{M}}_1(\tilde{\alpha}) \cup \tilde{\mathcal{M}}_2(\tilde{\alpha}) \subset [a, 0] \cup [b, c]$$

it is evident that one decomposition is finer than the other. But there is more to it. Actually, $\tilde{\mathfrak{M}}$ takes into account all possible noise realisations. The interval $[a, 0]$ is a repeller with zero probability since $\nu^{\otimes\mathbb{Z}}(\tilde{\Omega}) = \nu^{\otimes\mathbb{Z}}(\tilde{\Xi}) = 0$, which means that is does not play an important role in describing the asymptotic behaviour of φ since we have always been interested in weak repellers, and therefore in convergence in probability. On the other hand, in the set-valued case we have two distinct attractors which both influence the asymptotic behaviour of F.

Remark. As we have just seen, when working with convergence in probability, events happening with probability zero are irrelevant for the global dynamics, but they can induce some bizarre Morse decompositions. Recall that in Definition 3.23 we had the condition that at least one of the two chains must be defined with respect to the almost surely strict set inclusion. In this example the chain of attractors satisfies such a condition, whereas for the chain of repellers one can see that R_1 is almost surely equal to the empty set. ♦

Let us now go back to the general case, and recall that our state space was given by $[a, b]$ It is natural to ask if there is a way to determine a priori if the Morse decompositions of a random dynamical system, and its associated set-valued counterpart will have the same cardinality and comparable behaviours?

We will only be able to provide an overview of such correspondences for the specific class of random dynamical systems with additive bounded noise.

Definition 5.12 (Additive bounded noise)**.** The random diffeomorphisms f is said to induce a *random dynamical system with additive bounded noise* if and only if there exists functions

$$g : [a, b] \to [a, b] \quad \text{and} \quad \xi : \Delta \to \mathbb{R}$$

such that f can be decomposed as follows:

$$f(\alpha, x) = g(x) + \xi(\alpha)$$

where $\alpha \in \Delta$ and $x \in [a, b]$. Clearly, g is a $\mathcal{C}^\infty$-diffeomorphism, and ξ is a smooth non-constant random variable with smooth probability density distribution. The noise is bounded since Δ is compact, and therefore ξ is bounded.

As we have seen in Example 5.2.1, there are cases where the Morse decomposition for the set-valued dynamical system contains more sets than the one induced by the RDS. This is given by the fact that $f_{min}(x)$ at $x = 0$ was "tangent" to the identity line $y = x$. We wish to generalise the idea behind such points.

Definition 5.13 (f_α-tangency points)**.** Given $f_\alpha \in \Delta$, we say that $x \in (a, b)$ is an f_α-*tangency point* if $f_\alpha(x) = x$, and $f'_\alpha(x) = 1$.

Remark. An f_α-tangency point $\tilde{x}$ is precisely a fixed point of f_α, and a critical point of the function $f_\alpha(x) - x$, such that $f_\alpha(\tilde{x}) = 0$. ♦

For instance, in Example 5.2.1 we had that $x = 0$ was a f_{-1}-tangency point. We have clearly seen that if we allow f_α-tangency points then it is possible to find two Morse decompositions which do not agree. For what follows we will say that a set A is a *minimal F-attractor* if A is minimal F-invariant set, and an attractor for F. These minimal attractors for F will each of them contain an attractive random fixed point.

Theorem 5.14. *Let $f : \Delta \times [a, b] \to [a, b]$ be a random diffeomorphism inducing a random dynamical system φ with additive bounded noise, and satisfying hypotheses (H1)-(H4). Let F denote its associated set-valued dynamical system, and F^* the dual of such system. Suppose that there exists no f_α-tangency points, for any $\alpha \in \Delta$, and that the system admits "n" minimal F-invariant sets. Then both F and φ admit a Morse decomposition of cardinality either $2n$ or $2n - 1$.*

Proof. The idea of this proof is to show that since there are no f_α-tangency points, all F-invariant minimal sets and F^*-invariant sets are disjoint and that all the intersection between the graph of $F(x)$ and the identity line $y = x$ is a finite union of compact intervals whose boundary points are of a specific form. This will allow us to deduce that the number of minimal attractors for F is the same as the one of attractive random fixed points for φ. By considering the maximal chain that it is possible to obtain by exploiting the minimal F-attractors the result will follow.

Recall that since f induces a random dynamical system with additive bounded noise, thus thanks to Definition 5.12 it can be decomposed as

$$f(\alpha, x) = g(x) + \xi(\alpha).$$

We have assumed that ξ is smooth, and since Δ is compact, connected and not a single point, we conclude that $\xi(\Delta)$ is a connected, compact subset of $\mathbb{R}$, and since we know that ξ is non-constant, it follows that $\xi(\Delta)$ is a closed interval. Let us denote it by $\xi(\Delta) := [v_-, v_+]$. As we did in our previous example, let us denote f_{v_-} by f_{min}, and f_{v_+} by f_{max}: these two functions bound the behaviour of F.

We want to now look at the intersections of the graph of F, that we will denote by $\Gamma := \{(x, y) \mid x \in [a, b], y \in F(x)\}$, and the identity line $y = x$. In order to do that let us consider the diagonal set $\Delta_{[a,b]} := \{(x, x) \mid x \in [a, b]\}$, and so we will be looking at

$$I := \pi_1(\Gamma \cap \Delta_{[a,b]}),$$

where π_1 is the projection of the first coordinate.

Firstly we need to show that I is non-empty, but this follows immediately from the assumption (H4). Since $[a, b]$ is F-invariant, as we had discussed, this means that a is a fixed point of f_{min} and b is a fixed point of f_{max}. As we have seen F is continuous then we have that I is surely non-empty.

Let us focus for a second on the possible behaviours of f_{min} at a. We know a is a fixed point for f_{min}, and by smoothness it follows that if f_{min} is above the identity line locally at the right of a, then a will be a point repeller, whereas if f_{min} is locally below the line then we will have a connected component of I having a as a boundary point. Finally, f_{min} cannot locally agree with the identity line otherwise we would be generating infinitely many f_{min} tangency points.

A similar reasoning can be applied to b. The point b is a fixed point of f_{max}, there are only two possibilities: either f_{max} is above the identity line locally at the left of b, or it is below. In the former case, we must have that there exists a connected component of I whose boundary point is b. In the latter, b is a point repeller, since the entire graph of F lies below the identity line locally at the left of b. As before, f_{max} cannot agree with the identity line.

Next we want to show that I can be written as the disjoint union of closed intervals and at most the singletons $\{a\}$ or $\{b\}$. We can immediately rule out the possibility that I will be generate by the disjoint union of singletons z other than $\{a\}$ or $\{b\}$, since that would imply that either z is an f_{min}-tangency point or an f_{max}-tangency point, but this is impossible. Furthermore, notice that both Γ and $\Delta_{[a,b]}$ are compact, and thus I itself is compact. So if we look at the connected components of $I \setminus \{a, b\}$ we now know that they are all compact intervals. The boundary points of such intervals can only be fixed points of f_{min} or f_{max}, since as we previously said these two functions bound the behaviour of F.

Claim. The boundary points of the connected components of I are precisely one fixed point of f_{min} and one fixed point of f_{max}.

A point $z \in [a, b]$ cannot be at the same time a fixed point of f_{min}, and f_{max} since it would imply that $v_- = v_+$, or equivalently that $\xi(\Delta)$ is precisely a point, which is impossible. Let us now suppose, without loss of generality, that the boundary points of one of the intervals which are a connected component are two fixed points of f_{min}. Denote this interval by $[p, q]$. By mean value theorem there exists a point $k \in (p, q)$ such that

$$f'_{min}(k) = \frac{f_{min}(q) - f_{min}(p)}{q - p} = \frac{q - p}{q - p} = 1$$

Define the function

$$h(x) = f_{min}(x) + k - f_{min}(k),$$

and we claim that $\gamma \in \Delta$ such that $h(x) = f_\gamma(x)$. This follows immediately by rewriting $h(x) = g(x) + k - g(k)$, according to Definition 5.12, and noticing that $g(k) + v_- \leq k \leq g(k) + v_+$ by the fact that $k \in [p, q]$, and $[p, q]$ is a subset of I. This implies that $k - g(k) \in [v_-, v_+] = \xi(\Delta)$, thus there exists $\gamma \in \Delta$ such that $f_\gamma(x) = h(x)$. This is a contradiction because k is then a f_γ-tangency point. A symmetric argument holds in the case that the two boundary points of a connected component of I are both fixed points of f_{max}. Therefore, this proves the claim.

Notice that all connected components of $I \setminus \{a, b\}$ do not admit the possibility of containing any f_{min} or f_{max} fixed point in their interior, since such points would have to be tangency points. Thus, the previous claim tells us that all the connected components of I are either minimal F-invariant or minimal F^*-invariant sets, and these are all disjoint.

Denote by I' the set containing all the connected components I such that their left boundary point is a fixed point of f_{min}, and their right boundary point is a fixed point of f_{max}. This set precisely contains all minimal F-invariant sets in $[a, b]$, and thus every element in I' contains an attractive random fixed point by Proposition 5.6 and Proposition 5.8. By the remark after Lemma 5.10, we know that every element in I' is a minimal attractor for F. Remark that $\{a\}$ or $\{b\}$ can only be repellers because of the local behaviour of f_{min} and f_{max} around a and b respectively that we previously discussed. So I contains all minimal F-attractors in $[a, b]$.

By using the complete ordering of $\mathbb{R}$, we will number the elements of I' starting with A_1 for the leftmost interval, and finishing with A_n for the rightmost interval. Before we further proceed in the proof, recall that the *convex hull* of a set A is defined by

$$\operatorname{hull}(A) \coloneqq \bigcap_{\substack{A \subseteq C \\ C \text{ convex}}} C.$$

Since all A_i are disjoint, the union of two A_i is still an attractor for F, and so is the convex hull of the union.

Claim. We can write a chain of attractors of length $2n-1$ whose last element is given by $\operatorname{hull}(\bigcup_{i=1}^{n} A_i)$.

If we only have one set A_1 then this trivially gives an attractor chain of length 1. If we have two sets A_1, A_2 then the maximal chain is given by

$$\begin{aligned} C_1 &= A_1; \\ C_2 &= A_1 \cup A_2; \\ C_3 &= \operatorname{hull}(A_1 \cup A_2). \end{aligned}$$

thus, its length is 3. Now suppose that for I' having cardinality n, the chain induced by $(A_i)_{i=1}^{n}$ has length $2n-1$. Now assume that the cardinality of I' is given by $n+1$. We know that the longest chain for n minimal F attractors is given by $2n-1$ elements, and $C_{2n-1} = \operatorname{hull}(\bigcup_{i=1}^{n} A_i)$. Consider now the element A_n. This will induce two more sets:

$$\begin{aligned} C_{2n} &= C_{2n-1} \cup A_{n+1}; \\ C_{2n+1} &= \operatorname{hull}(C_{2n-1} \cup A_{n+1}) = \operatorname{hull}(\bigcup_{i=1}^{n+1} A_i). \end{aligned}$$

inducing thus a chain of length $2n+1 = 2(n+1)-1$, as we expected. Now we need to show that such a chain is maximal. Given $n+1$ minimal F-invariant sets, we can generate at most $n+1$ chain elements by taking the union of all the different minimal F-attractors, and n other elements by taking the convex hull of the sets. This clearly implies that we can get a chain with at most $2n+1$ elements, as we wanted.

Let $C_0 \coloneqq \emptyset$. If we have that $a \in A_1$ and $b \in A_n$, then $\operatorname{hull}(\bigcup_{i=1}^{n} A_i) = [a,b]$, and thus our maximal chain of attractors induces a Morse decomposition for F of $2n-1$ elements. Otherwise, we need to set $C_{2n} = [a,b]$, and the attractor chain $C_0 \subsetneq C_1 \subsetneq \cdots \subsetneq C_{2n}$ induces a Morse decomposition of $2n$ elements.

The argument we just developed holds for the attractors given by the random fixed points of φ, and since we have established a relation between minimal F-attractors and these objects we can conclude the theorem holds. $\square$

Remark. The definition of f_α-tangency points can be strengthen by adding the condition that $f''_\alpha \neq 0$ at the tangency point. This would allows us to weaken the assumptions of Theorem 5.14, and the proof would still hold with just few minor adjustments, mostly in the part where we invoked the mean value theorem.

If one wishes to do so, it will become necessary to develop a theory for which we allow f_α to be parallel to the identity line. Some extra restrictions might then be needed. ♦

As we did in Example 5.2.1 we want to look at the set $\mathfrak{M}$ of the decomposition of F described in Theorem 5.14. Let us consider once again the set I, but this time let us look at the connected components having as left boundary point a fixed point of f_{max} and as right boundary point a fixed point of f_{min}, together with any possibly the singletons $\{a\}$ or $\{b\}$. Call the set containing all such components or points J'. All the elements of J' show repulsive behaviour and by construction those are minimal F^*-invariant sets. So, similarly as we did for the elements of I', we call the elements of J' minimal F-repellers. Clearly, the union over all elements of I' and the union over all elements of J' partition I, and they both contain the minimal structures describing the asymptotic behaviour of F. Therefore, it is natural to suppose that

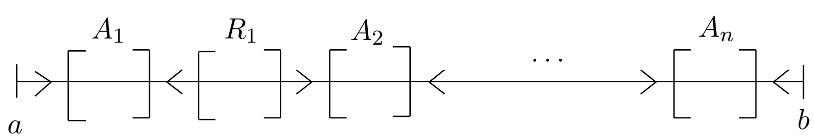
$$\mathfrak{M} = I = \bigcup_{A \in I'} A \cup \bigcup_{R \in J'} R. \tag{5}$$

Before proving such a statement we need to state a small lemma.

Lemma 5.15. *Under the assumptions of Theorem 5.14, between every two minimal F-attractors there is always a minimal F-repeller, and vice versa.*

Proof. The proof of this statement follows from the fact that every f_α is strictly monotonically increasing, in conjunction with the absence of f_α-tangency points. It is here omitted in favour of concision of exposition. □

We will prove Equation (5) only in one specific case, the others will similarly follow. Suppose that both $\{a\}$ and $\{b\}$ are elements of J', and that the cardinality of I is $n > 0$. Let us use the letter A to denote attractors in I', and the letter R to denote the repellers in J'. Give those sets an increasing oreder as we did in the proof of Theorem 5.14. By doing so, A_1 is the leftmost element of I', and the numbering increases up to A_n, the rightmost minimal F-attractor in $[a, b]$. The intervals in J', which are precisely $n - 1$ thanks to Lemma 5.15. Considering $\{a\}$ and $\{b\}$ we have a total of $n + 1$ repellers. The following figure summarises the situation we have just described.



By applying the algorithm described in the proof of Theorem 5.14 we proceed to inductively build our sequence of attractors, denoted by $\tilde{A}$, and repellers, denoted by $\tilde{R}$.

You can find here few iterations which aim at clarifying the algorithm:

$$\begin{aligned}
\tilde{A}_0 &= \emptyset & \tilde{R}_0 &= [a,b] \\
\tilde{A}_1 &= A_1 & \tilde{R}_1 &= \{a\} \cup \mathrm{hull}(R_1 \cup b) \\
\tilde{A}_2 &= A_1 \cup A_2 & \tilde{R}_2 &= \{a\} \cup R_1 \cup \mathrm{hull}(R_2 \cup b) \\
\tilde{A}_3 &= \mathrm{hull}(A_1 \cup A_2) & \tilde{R}_3 &= \{a\} \cup \mathrm{hull}(R_2 \cup b) \\
\tilde{A}_4 &= \mathrm{hull}(A_1 \cup A_2) \cup A_3 & \tilde{R}_4 &= \{a\} \cup R_2 \cup \mathrm{hull}(R_3 \cup b) \\
\tilde{A}_5 &= \mathrm{hull}(A_1 \cup A_2 \cup A_3) & \tilde{R}_5 &= \{a\} \cup \mathrm{hull}(R_3 \cup b) \\
&\vdots & &\vdots \\
\tilde{A}_{2n-2} &= \mathrm{hull}(\bigcup_{i=1}^{n-1} A_i) \cup A_n & \tilde{R}_{2n-2} &= \{a\} \cup R_{n-1} \cup \{b\} \\
\tilde{A}_{2n-1} &= \mathrm{hull}(\bigcup_{i=1}^{n} A_i) & \tilde{R}_{2n-2} &= \{a\} \cup \{b\} \\
\tilde{A}_{2n} &= [a,b] & \tilde{R}_{2n} &= \emptyset.
\end{aligned}$$

From this it is immediate to compute the finest set-valued Morse decomposition of such a system:

$$\begin{array}{ccc}
\mathcal{M}_1 = A_1, & \mathcal{M}_2 = A_2, & \mathcal{M}_3 = R_1, \\
\mathcal{M}_4 = A_3, & \mathcal{M}_5 = R_2, & \mathcal{M}_6 = A_4, \\
 & \vdots & \\
\mathcal{M}_{2n-2} = A_n, & \mathcal{M}_{2n-1} = R_{n-1}, & \mathcal{M}_{2n} = \{a\} \cup \{b\}.
\end{array}$$

Finally we can have

$$\mathfrak{M} = \bigcup_{i=1}^{2n} \mathcal{M}_i = A_1 \cup A_2 \cup R_1 \cup \cdots \cup R_{n-1} \cup (\{a\} \cup \{b\}) = I$$

As we said before it is possible to analyse all the other possible different behaviours of a or b using the exact same reasoning we just discussed, and thus showing that Equation (5) always holds.

This discussion leads us to one of our final results. Recall that a Morse decomposition $(\mathcal{M}_i)_{i=1}^n$ is said to be finer than a second Morse decomposition $(\mathcal{M}'_i)_{j=1}^m$ if for any $j \in \{1, \ldots, m\}$ then there exists an $i \in \{1, \ldots, n\}$ such that $\mathcal{M}_i \subset \mathcal{M}'_j$. A finest Morse decomposition is then a decomposition such that any finer decomposition is equal to the original decomposition.

Corollary 5.16. *The Morse decomposition described by Theorem 5.14 is the finest Morse decomposition for F.*

Proof. It will be enough to show that any F-attractor in $[a,b]$ can be written in terms of unions and convex hulls of union of minimal F-attractors, to then deduce that the attractor chain we generated in the proof of Theorem 5.14 gives us the finest Morse decomposition of F.

Let A be an F-attractor in $[a, b]$. Without loss of generality we can assume it to be connected. We exclude the possibility of A being a single attractive point thanks to the absence of f_α-tangency points. We claim that the connected component is actually a closed interval. The connected components of an attractor is an attractor as well. As such it is an F-invariant compact set, and therefore a closed interval, as we have seen in the proof of Theorem 5.14.

Recall that I' is the set of all connected components of I which have an f_{min} fixed point as left boundary point and an f_{max} fixed point as a right boundary point.

Claim. There exists $A_i, A_j \in I'$ such that $a_- \in A_i$ and $a_+ \in A_j$.

By definition, A is an F-invariant set and therefore

$$f_{min}(a_-) = a_- \quad \text{and} \quad f_{max}(a_+) = a_+.$$

Supposing that $A \notin I'$, otherwise this claim in trivial, implies that f_{max} intersects the identity line $y = x$ at least once in (a_-, a_+). Call d the first time f_{max} intersects the identity line. By assumption, d is not a tangency point, which immediately tells us that $[a_-, d] \in I'$, and thus $A_i = a_-$. If we now focus on a_+ and f_{min} we have to first notice that the intersections of f_{min} and the identity line are finitely many: every intersection generates a connected component of I and these are finitely many. Thus, we can let e denote the biggest f_{min} fixed point in (a_-, a_+) and hence $A_j = [e, a_+]$ will give us the minimal F-attractor we were seeking.

Since A was assumed connected, we have that $A = \text{hull}(A_i \cup A_j)$. A was arbitrary, which means that every F-attractor can be written in terms of unions and convex hulls of elements of I', as we wanted. By what we have just showed it immediately follows that $\text{hull}(\bigcup_{A \in I'} A)$ is the biggest attractor for F. The attractor chain described in the proof of Theorem 5.14 was the maximal chain of attractors which could have been generated with elements of I' and the use of $\cup$ and hull.

We have now deduced that it is not possible to generate a longer chain of attractors, or repellers. From the discussion we had before stating this corollary we know what every minimal attractor of I' is a Morse set, and so is every element of J'. Denote by $(\mathcal{M}_i)_{i=1}^n$ the Morse decomposition induced by Theorem 5.14, and suppose $(\mathcal{N}_j)_{j=1}^m$ was a finer Morse decomposition. From what we just showed $n = m$. Furthermore, if $(\mathcal{N}_j)_{j=1}^m \neq (\mathcal{M}_i)_{i=1}^n$, then we would have that at least one $\mathcal{N}_j$ would be strictly contained in one of the $\mathcal{M}_i$. This is clearly impossible since the Morse sets $\mathcal{M}_i$ are precisely given by minimal attractors or repellers, or boundary points of the state space. □

The last consideration we would to make is about the relation between I and random fixed points in $[a, b]$. As we had discussed before there is a relation between elements of I' and attractive random fixed points, made explicit by Proposition 5.6. It is clearly possible to obtain such a relation even for the elements of J' which are not singletons. Notice that the proof of Proposition 5.6 perfectly works for the dual system F^*, and this allows us to relate every interval in J', which are F^*-invariant sets, to attractive random fixed points for the time reversed system φ^*, where $\varphi^*(t, \alpha, x) = \varphi(-t, \alpha, x)$ for any $t \in \mathbb{Z}$, $\tilde{\alpha} \in \Delta^{\mathbb{Z}}$ and $x \in [a, b]$. These attractive fixed points for the time reversed system φ^*, generate nothing more than repellers for the system φ, thus they will be called *repulsive random fixed points*. We then have a relation between minimal F^*-attractors, or minimal F-repellers and repulsive random fixed points.

The only points which we need to carefully consider are the endpoints of our state space. In the case they are not one of the boundary points of some minimal F-attractor, then we know they exhibit repulsive behaviour. If at least one of the a or b shows such a behaviour, then we know that the convex hull of all the minimal F-attractors is strictly contained in $[a, b]$. This consideration basically reduces our case to what we previously analysed in Example 5.2.1. The hull of the minimal attractors gives an attractor similar to K_1, and we can mimic the construction of $\tilde{R}_1$ again. Beware this set will likely going to almost surely empty.

This strongly suggests the possibility that the Morse decomposition described by Theorem 5.14 for φ could be the finest. This seems to be the case since our weak attractors are described by random fixed points, in a certain sense, the "minimal" objects in the random world.

Chapter 6

Conclusion

The initial correspondence between random and set-valued dynamical systems which inspired this project, indeed revealed some useful connections between their Morse decompositions, especially for the class of systems considered in the last chapter.

After having briefly sketched Morse decompositions for topological dynamical systems, we have adapted those definitions to random and set-valued dynamical systems. In the last chapter we studied the specific class of random dynamical systems φ with additive bounded noise, induced by strictly monotonically increasing random diffeomorphism, and their associated set-valued dynamical systems F. By combining Ledrappier-Le Jan-Crauel theorem with Theorem 1.3 from [ZH07] we deduced a first correspondence between F-minimal invariant sets and random fixed points. The theme of the rest of the chapter revolved around exploiting this core idea to develop a relation between Morse decompositions. We proceeded to showing, through an example, that the finest Morse decompositions of φ and F might lead to different asymptotic behaviours. In order to exclude such possibilities, we introduced the concept of f_α-tangency points, which in turn was used to establish a sufficient condition to estimate the cardinality of the finest Morse decomposition of F. We conjectured the F finest decomposition induces a finest Morse decomposition for φ.

This conjecture is supported by the idea that random fixed points are the equivalent of minimal F sets for RDS's. This should be enough to show that any finer Morse decomposition, in the sense of almost surely strict inclusion (Definition 3.22), can only be given by empty sets, which is impossible. One could further expand this example by finding a different, and possibly weaker condition, for the existence of these finest Morse decomposition, as we have proposed in the Remark after the proof of Theorem 5.14.

We believe that an expansion of this project can be achieved by modifying the state space from an interval to the circle $\mathbb{S}^1$. In this case, strictly monotonically increasing diffeomorphisms would be replaced by orientation preserving diffeomorphisms. If the random system φ the diffeomorphisms induce is globally synchronising, which means the distance between the iterations of any two distinct points under the same noise realisation tends to zero, this will lead to a unique stationary measure over $\mathbb{S}^1$. This would therefore imply that there exist one attractive random fixed point distributed over the entirety of $\mathbb{S}^1$, and similarly a repelling one. We believe this could be exploited to create some interesting Morse decompositions.

Instead of working with weak attractors, as we did, one might wish to use strong pullback attractors to define Morse decomposition for RDS. This could potentially lead to new sufficient conditions for the existence of finest Morse decompositions, and more in general to new relations between F and φ.

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