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KALMAN FILTERING FOR SPACECRAFT ATTITUDE ESTIMATION*

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Abstract

Several schemes in current use for sequential estimation of spacecraft attitude using Kalman filters are examined. These differ according to their treatment of the attitude error, namely: using the complete four-component quaternion; using a truncated quaternion in which one of the components has been eliminated; or using a quaternion referred to approximate body-fixed axes. These schemes are examined for the case of a spacecraft carrying line-of-sight attitude sensors and three-axis gyros whose measurements are corrupted by noise on both the drift rate and the drift-rate ramp. The analysis of the covariance is carried out in detail. The historical development of Kalman filtering of attitude is reviewed.

1. Introduction

The present report reviews the methods of Kalman filtering in attitude estimation and their development over the last two decades. This review is not intended to be complete but is limited to algorithms suitable for spacecraft equipped with three-axis gyros and line-of-sight attitude sensors. These are the systems to which we feel that Kalman filtering is most applicable.

Throughout this report the attitude is represented by the quaternion. The development of the Kalman filter for the quaternion representation was motivated by the requirement of real-time autonomous attitude determination for attitude control and the annotation of science data. The quaternion parameterization was chosen for several practical reasons: (1) the prediction equations are treated linearly, (2) the representation is free from singularities (thus, the gimbal-lock situation is avoided), and (3) the attitude matrix is algebraic in the quaternion components (thus, eliminating the need for transcendental functions).

The dynamic equations for the spacecraft attitude pose many difficulties in the filter modeling. In particular, the external torques and the distribution of momentum due to the use of rotating or rastering instruments lead to significant uncertainties in the modeling. For autonomous spacecraft the use of inertial reference units as a model replacement permits the circumvention of these problems. In this representation the angular velocity of the spacecraft is obtained from the gyro data. The kinematic equations are used to obtain the attitude state and this is augmented by means of additional state-vector components for the gyro biases. Thus, gyro data are not treated as observations and the gyro noise appears as state noise rather than as observation noise.

The use of the quaternion as the attitude state presents some difficulty in the application of the filter equations. This difficulty is due to the lack of independence of the four quaternion components, which are related by the constraint that the quaternion have unit norm. This constraint results in the singularity of the covariance matrix of the quaternion state. The various ways to treat or circumvent this difficulty make up the major part of this report. Each of the filters discussed predicts the quaternion state in the same manner as if each component were independent. Different approaches to the update equations and to the covariance representation and its propagation give rise to the different methods presented.

Section 2 of this report reviews the relevant literature on Kalman filtering as applied to attitude estimation and on related topics. Since the attitude problem is nonlinear the vehicle for optimal estimation is the extended Kalman filter, which is reviewed in Section 3 without derivation. Attitude kinematics are reviewed in Section 4, with emphasis on the quaternion representation. A discussion of gyros used in the model replacement mode and of line-of-sight attitude sensors is presented in Section 5.

The state equation and the equation for attitude prediction are derived in Section 6. The various approaches to the filtering equations are discussed in the succeeding five sections. Section 12 reviews the advantages of the different methods. Several results of interest have been gathered in two appendices.

The intent of the present work is to provide a complete account of what now seem to be the relevant filtering algorithms rather than to present a new approach. An attempt has been made to present the current methods in a common framework in order to produce a useful reference for future applications.

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2. Historical Survey

Early Applications of Kalman Filtering

The Kalman filter [1, 2], which was originally developed as a tool in linear estimation theory, was soon applied to nonlinear orbital guidance and navigation problems in the Apollo program by Schmidt and his collaborators [3-5]. Almost simultaneously with Kalman's work, Swerling [6, 7] developed a recursive theory for satellite navigation which differed from Kalman's in the treatment of the process noise and was not carried out as completely. Standard treatments of Kalman filtering can be found in the review of Sorenson [8] and the textbooks of Jazwinski [9] and Gelb [10].

An accurate and complete historical survey of Kalman filtering for attitude estimation is not possible since the earliest applications were directed towards national defense and hence could not be published in the open literature. The earliest published reference is by Farrell [11, 12], who studied the extent to which Kalman filtering of crude attitude measurements from sun sensors and magnetometers could provide attitude accuracy equivalent to that obtained without smoothing from more elaborate instrumentation. Farrell represented the attitude by Euler angles and assumed torque-free motion in the attitude prediction. Cherry and O'Connor [13], in their design of the lunar excursion module autopilot, considered sequential estimation of the disturbance torques induced by the ascent or descent propulsion system. Potter and Vander Velde [14] used Kalman filtering theory to determine the optimum mixing of gyroscope and star tracker data in an attitude determination system. Generally, as remarked by Sabroff [15], the application of Kalman filtering to attitude estimation had not shown impressive results up to 1967. Aside from insufficient study, the lack of real success in applying optimal estimation was caused by the inability to model the system dynamics accurately.

Continued effort in this area was evidenced at a symposium on spacecraft attitude determination in 1969, at which six papers were presented on the application of Kalman filtering to this problem. Foudriat [16] and Arneson and Nelson [17] considered spin-stabilized satellites, while Ribarich [18] and Lesinski [19] were interested in the dual-spin case. Pauling, Jackson, and Brown [20] and Toda, Heiss, and Schlee [21] studied the Space Precision Attitude Reference System (SPARS), which used gyro measurements in a model replacement mode with periodic star-sensor updates. Pauling et al. [20] used the attitude matrix for state prediction and Euler angles for updates, while Toda et al. [21] used the quaternion for prediction and incremental error angles in the update. Jackson [22] also contributed a paper to the symposium on the application of nonlinear estimation theory to the attitude determination problem, an application that was also studied by Kau, Kumar, and Granley [23].

In a 1971 review of strapdown navigation by Edwards [24], Kalman filtering is not treated but the use of the quaternion and error angles for state prediction is discussed. A review article by Schmidtbauer, Samuelsson, and Carlsson [25] extends the history to 1973 and contains a classification and discussion of extended Kalman filter methods for attitude determination, with specific emphasis on algorithms suitable for onboard computation.

Attitude Representations

The nonlinear aspects of rotational kinematics are discussed in many texts [26-30]. Approaches which treat the spacecraft axes as uncoupled (thus ignoring commutation errors) have been studied by Potter and Vander Velde [14], Ribarich [18], Schmidtbauer et al. [25], and Farrenkopf [31]. This approximation leads to independent linear estimation problems for the three spacecraft axes, and closed-form expressions for the steady-state estimation errors can be obtained in some cases [14, 25, 31]. Uncoupled-axis Kalman filters employing both gyro data and dynamics modeling have been used in onboard attitude control systems, in the NASA International Ultraviolet Explorer and Solar Maximum Mission [32] spacecraft, for example.

Stuelpnagel [33] and Markley [34] discuss different parameterizations of the attitude that have been used when the decoupled-axis approximation is not applicable. Early strapdown navigation systems used the direction-cosine matrix as the representation of the attitude [20, 24]. Due to round-off, quantization, and truncation errors in the attitude propagation, this procedure results in an attitude matrix that is not orthogonal [35, 36]. Various orthogonalization schemes for the attitude matrix were developed; but Giardina, Bronson, and Wallen [36] proved that an optimal orthogonalization (one that minimizes the sum of squares of the differences between the elements of the propagated matrix and those of the orthogonalized matrix) requires a computationally expensive matrix square root. Because of this problem and the redundancy of the nine-parameter direction cosine representation, it has not been widely used recently.

The three-parameter Euler-angle representation [26-30, 33, 34] was used in several early applications of Kalman filtering to attitude estimation [11, 16, 17, 19, 20, 23]. However the kinematic equations for Euler-angles involve nonlinear and computationally expensive trigonometric functions, and the angles become undefined for some rotations (the gimbal-lock situation), which cause problems in Kalman filtering applications. Despite these difficulties, the Euler angle representation continues to be used for the attitude estimation of spinning spacecraft. Stuelpnagel [33] discusses two other three-parameter representations: the exponential of a 3x3 skew-symmetric matrix (rotation vector representation) and the Cayley parameterization (not to be confused with the Cayley-Klein parameters). The latter is equivalent to the Gibbs vector parameterization [34]. Neither of these representations has found a Kalman filtering application, to our knowledge. Stuelpnagel proves that no three-parameter representation can be both global and nonsingular.

The global nonsingular four-parameter representation of the attitude in terms of Euler's symmetric parameters, or equivalently the four components of a quaternion, is discussed by many authors [26-30, 33-37]. Quaternions were invented by Hamilton [38] in 1843; their use in attitude dynamics simulations was promoted by Robinson [39] and by Mitchell and Rogers [40]. The attitude matrix computed from a quaternion (as a homogeneous quadratic function) is orthogonal if the sum of squares of the quaternion components is unity. If propagation errors result in a violation of this constraint, the quaternion can be renormalized by dividing its components by the (scalar) square root of the sum of their squares; and Giardina et al. [36] showed that the attitude matrix computed from the renormalized quaternion is identical to the one given by their optimal orthogonalization. The application of quaternions to strapdown guidance, with error analyses, was discussed by Wilcox [35] and Mortenson [41]. Quaternion kinematics has been the subject of several recent studies [42-48]. Of particular concern has been the best method for extracting a quaternion from an attitude matrix [49-52]. A recent review of quaternion relations has been given by Friedland [53].

The advantages of the quaternion parameterization have led to its frequent use in attitude determination systems. One example is an attempt by Lefferts and Markley [54] to model the attitude dynamics of the NIMBUS-6 spacecraft, which indicated that dynamics modeling with elaborate torque models could still not give acceptable attitude determination accuracy. For this reason, most applications of quaternion attitude estimation have used gyros in the dynamic model replacement mode. These include the work by Toda et al. [21] mentioned above, by the group at TRW for the Precision Attitude Determination System (PADS) [55], which was incorporated in the attitude determination system for the High Energy Astronomy Observatory (HEAO) mission [56-58], by Yong and Headley [59] for highly maneuverable spacecraft, by Murrell [60] in a design for the NASA Multimission Modular Spacecraft, by Sorenson, Schmidt, and Goka [61] in a square-root filtering application, by Shuster and coworkers [62-64] for a microprocessor-based onboard attitude determination system, and by Markley [65] for an autonomous navigation study.

Gyro Noise Models

The first papers describing statistical models of gyro drifts were by Newton [66] and Hammon [67] in 1960. Newton considered additive white noise in the gyro drifts, while Hammon assumed that the gyro drift rates were an exponentially correlated random process. Dushman [68] considered a drift rate model obtained by adding a random walk component to Hammon's autocorrelation function. Early implementations of gyro noise models in Kalman filters were generally incomplete. Thus, Potter and Vander Velde [14] include only the random walk term in the drift rate, while both Pauling et al. [20] and Toda et al. [21] have only the white noise term. Farrenkopf [31, 69] considered a gyro model including all the terms discussed above. This model was used in the subsequent HEAO system development [56, 57].

3. The Kalman Filter

We review in this section the principal equations for the extended Kalman filter [9, 10] in order to introduce the necessary notation for the sections which follow.

The state equation may be written as

$$\frac{d}{dt}\,\underline{x}(t) = \underline{f}(\underline{x}(t),t) + g(\underline{x}(t),t)\,\underline{w}(t) \tag{1}$$

where $\underline{x}(t)$ is the state vector and $\underline{w}(t)$, the process noise, is a Gaussian white-noise process whose mean and covariance function are given by

$$E\{\underline{w}(t)\} = \underline{0} \tag{2}$$

$$E\{\underline{w}(t)\underline{w}^T(t)\} = Q(t)\delta(t-t') \tag{3}$$

"E" denotes the expectation and "T" the matrix transpose. The initial mean and covariance of the state vector are given by

$$E\{\underline{x}(t_o)\} \equiv \hat{\underline{x}}(t_o) = \underline{x}_o \tag{4}$$

$$E\{(\underline{x}(t_o) - \underline{x}_o)(\underline{x}(t_o) - \underline{x}_o)^T\} \equiv P(t_o) = P_o \tag{5}$$

Prediction

Given the initial conditions on the state vector and the state covariance matrix, the minimum-variance estimate of the state vector at a future time t is given in the absence of measurements by the conditional expectation

$$\hat{\underline{x}}(t) = E\{\underline{x}(t)\,|\,\hat{\underline{x}}(t_o) = \underline{x}_o\} \tag{6}$$

This predicted estimate satisfies the differential equation

$$\frac{d}{dt}\,\hat{\underline{x}}(t) = E\{\underline{f}(\underline{x}(t),t)\} \equiv \hat{\underline{f}}(\underline{x}(t),t) \tag{7}$$

which we write approximately as

$$\frac{d}{dt}\,\hat{\underline{x}}(t) = \underline{f}(\hat{\underline{x}}(t),t) \tag{8}$$

Equation (8) may be integrated formally to give

$$\hat{\underline{x}}(t) = \underline{\phi}(t,\hat{\underline{x}}(t_o),t_o) \tag{9}$$

The state-error vector and covariance matrix are defined by

$$\Delta\underline{x}(t) = \underline{x}(t) - \hat{\underline{x}}(t) \tag{10}$$

$$P(t) = E\{\Delta\underline{x}(t)\,\Delta\underline{x}^T(t)\} \tag{11}$$

Neglecting terms which are higher than first order in the state-error vector and the process noise, the state-error vector satisfies the differential equation

$$\frac{d}{dt}\,\Delta\underline{x}(t) = F(t)\,\Delta\underline{x}(t) + G(t)\,\underline{w}(t) \tag{12}$$

where

$$F(t) \equiv \frac{\partial}{\partial \underline{x}} \underline{f}(\underline{x},t)\Big|_{\hat{\underline{x}}(t)} \qquad (13)$$

$$G(t) \equiv g(\hat{\underline{x}}(t),t) \qquad (14)$$

Equation (12) may be integrated formally to give

$$\Delta\underline{x}(t) = \Phi(t,t_o)\Delta\underline{x}(t_o) + \int_{t_o}^{t} \Phi(t,t')G(t')\underline{w}(t')dt' \quad (15)$$

where $\Phi(t,t_o)$ is the transition matrix, which satisfies

$$\frac{\partial}{\partial t}\Phi(t,t_o) = F(t)\,\Phi(t,t_o) \qquad (16)$$

$$\Phi(t_o,t_o) = I \qquad (17)$$

Note that for non-linear systems $\Phi(t,t_o)$ also depends implicitly on $\hat{\underline{x}}(t_o)$, which for notational convenience will be suppressed.

The predicted covariance matrix satisfies the Riccati equation

$$\frac{d}{dt}P(t) = F(t)P(t) + P(t)F^T(t) + G(t)Q(t)G^T(t) \quad (18)$$

which may be integrated to give

$$P(t) = \Phi(t,t_o)P(t_o)\Phi^T(t,t_o) + \int_{t_o}^{t} \Phi(t,t')G(t')Q(t')G^T(t')\Phi^T(t,t')dt' \quad (19)$$

In more compact notation, $\hat{\underline{x}}_k(-)$ and $P_k(-)$ denote the predicted values of the state vector and state covariance matrix at time t_k, and $\hat{\underline{x}}_k(+)$ and $P_k(+)$ denote the same quantities immediately following a measurement at time t_k. Thus, in obvious notation

$$\hat{\underline{x}}_{k+1}(-) = \underline{\phi}(t_{k+1},\, \hat{\underline{x}}_k(+),\, t_k) \qquad (20)$$

$$P_{k+1}(-) = \Phi_k P_k(+)\Phi_k^T + N_k \qquad (21)$$

Filtering

The measurement vector at time t_k is related to the state vector by

$$\underline{z}_k = \underline{h}(\underline{x}_k) + \underline{v}_k \qquad (22)$$

where $\underline{v}_k$, the measurement noise, is a discrete Gaussian white-noise process

$$E\{\underline{v}_k\} = \underline{0} \qquad (23)$$

$$E\{\underline{v}_k\underline{v}_{k'}^T\} = R_k\,\delta_{kk'} \qquad (24)$$

The minimum-variance estimate of $\underline{x}_k$ immediately following the measurement is given by

$$\hat{\underline{x}}_k(+) = \hat{\underline{x}}_k(-) + K_k[\underline{z}_k - \underline{h}(\hat{\underline{x}}_k(-))] \qquad (25)$$

where the Kalman gain matrix is given by

$$K_k = P_k(-)H_k^T[H_kP_k(-)H_k^T + R_k]^{-1} \qquad (26)$$

and the measurement sensitivity matrix is given by

$$H_k = \frac{\partial \underline{h}(\underline{x})}{\partial \underline{x}}\Big|_{\hat{\underline{x}}_k(-)} \qquad (27)$$

The covariance matrix immediately following the measurement is given by

$$P_k(+) = (I-K_kH_k)P_k(-) \qquad (28)$$

$$= (I-K_kH_k)P_k(-)(I-K_kH_k)^T + K_kR_kK_k^T \qquad (29)$$

Equation (29) is the starting point for many efficient and numerically stable factored forms of the Kalman filter. [70]

In the sections which follow, the implementation of the above equations is examined for three different representations of the attitude estimation problem. In each case, explicit expressions are developed for the transition matrix Φ and the sensitivity matrix H.

4. Attitude Kinematics

In the systems investigated in the present study the attitude is represented by the quaternion defined as

$$\bar{q} = \begin{bmatrix} \vec{q} \\ q_4 \end{bmatrix} \qquad (30)$$

where

$$\vec{q} = \hat{n}\sin(\theta/2) \qquad q_4 = \cos(\theta/2) \qquad (31)$$

The unit vector $\hat{n}$ is the axis of rotation and θ is the angle of rotation. Quaternions will always be denoted by an overbar. The quaternion possesses three degrees of freedom and satisfies the constraint

$$\bar{q}^T\bar{q} = 1 \qquad (32)$$

The attitude matrix is obtained from the quaternion according to the relation

$$A(\bar{q}) = (|q_4|^2 - |\vec{q}|^2)\, I_{3\times3} + 2\vec{q}\vec{q}^T + 2q_4[\underset{\sim}{\vec{q}}] \quad (33)$$

where

$$[\underset{\sim}{\vec{q}}] = \begin{bmatrix} 0 & q_3 & -q_2 \\ -q_3 & 0 & q_1 \\ q_2 & -q_1 & 0 \end{bmatrix} \quad (34)$$

This notation will be used generally for any 3×3 skew-symmetric matrix generated from a three-vector. The convention will be followed that A is the matrix which tranforms representations of vectors in the reference (usually geocentric inertial) coordinate system to representations in the body fixed coordinate system.

In contrast to the usual convention for quaternion composition established by Hamilton [38], the product of two quaternions will be written in the same order as the corresponding rotation matrices. Thus,

$$A(\bar{q}')\, A(\bar{q}) = A(\bar{q}' \bullet \bar{q}) \quad (35)$$

The composition of quaternions is bilinear, i.e.,

$$\bar{q}' \bullet \bar{q} = [\bar{q}']\, \bar{q} \quad (36)$$

with

$$[\bar{q}'] = \begin{bmatrix} q_4' & q_3' & -q_2' & q_1' \\ -q_3' & q_4' & q_1' & q_2' \\ q_2' & -q_1' & q_4' & q_3' \\ -q_1' & -q_2' & -q_3' & q_4' \end{bmatrix} \quad (37)$$

or

$$\bar{q}' \bullet \bar{q} = \{\bar{q}\}\, \bar{q}' \quad (38)$$

with

$$\{\bar{q}\} = \begin{bmatrix} q_4 & -q_3 & q_2 & q_1 \\ q_3 & q_4 & -q_1 & q_2 \\ -q_2 & q_1 & q_4 & q_3 \\ -q_1 & -q_2 & -q_3 & q_4 \end{bmatrix} \quad (39)$$

The rate of change of the attitude matrix with time defines the angular velocity vector, $\vec{\omega}$

$$\frac{d}{dt} A(t) = [\underset{\sim}{\vec{\omega}}(t)]\, A(t) \quad (40)$$

The corresponding rate of change of the quaternion is given by

$$\frac{d}{dt} \bar{q}(t) = \frac{1}{2}\, \Omega(\vec{\omega}(t))\, \bar{q}(t) \quad (41)$$

with

$$\Omega(\vec{\omega}) \equiv \begin{bmatrix} 0 & \omega_3 & -\omega_2 & \omega_1 \\ -\omega_3 & 0 & \omega_1 & \omega_2 \\ \omega_2 & -\omega_1 & 0 & \omega_3 \\ -\omega_1 & -\omega_2 & -\omega_3 & 0 \end{bmatrix} \quad (42)$$

It will be convenient in later sections to define the four-component quantity

$$\bar{\omega} = \begin{bmatrix} \vec{\omega} \\ 0 \end{bmatrix} \quad (43)$$

whence

$$\frac{d}{dt} \bar{q}(t) = \frac{1}{2}\, \bar{\omega}(t) \bullet \bar{q}(t) \quad (44)$$

If the direction of $\vec{\omega}$ is constant over the time interval of interest or if the "rotation vector" defined by

$$\Delta\vec{\theta} = \int_t^{t+\Delta t} \vec{\omega}(t')\, dt' \quad (45)$$

is small, then the solution of Eq. (41) is

$$\bar{q}(t+\Delta t) = M(\Delta\vec{\theta})\, \bar{q}(t) \quad (46)$$

where

$$M(\Delta\vec{\theta}) = \cos(|\Delta\vec{\theta}|/2)\, I_{4\times4} + \frac{\sin(|\Delta\vec{\theta}|/2)}{|\Delta\vec{\theta}|}\, \Omega(\Delta\vec{\theta}) \quad (47)$$

5. Sensor Models

The spacecraft being studied is assumed to be equipped with three-axis gyros and line-of-sight attitude sensors.

Gyro Models

We use a simple but realistic model for gyro operation developed by Farrenkopf [31] and applied to the HEAO mission by Hoffman and McElroy [57]. In this model the spacecraft angular velocity is related to the gyro output vector $\vec{u}$ according to

$$\vec{\omega} = \vec{u} - \vec{b} - \vec{n}_1 \quad (48)$$

The vector $\vec{b}$ is the drift-rate bias and $\vec{n}_1$ is the drift-rate noise. $\vec{n}_1$ is assumed to be a Gaussian white-noise process

$$E\,\{\vec{n}_1(t)\} = \vec{0} \quad (49)$$

$$E\,\{\vec{n}_1(t)\vec{n}_1^T(t')\} = Q_1(t)\delta(t-t') \quad (50)$$

The drift-rate bias is itself not a static quantity but is driven by a second Gaussian white-noise process, the gyro drift-rate ramp noise

$$\frac{d}{dt} \vec{b} = \vec{n}_2 \quad (51)$$

with

$$E\,\{\vec{n}_2(t)\} = \vec{0} \quad (52)$$

$$E\,\{\vec{n}_2(t)\vec{n}_2^T(t')\} = Q_2(t)\delta(t-t') \quad (53)$$

The two noise processes are assumed to be uncorrelated

$$E\,\{\vec{n}_1(t)\vec{n}_2^T(t')\} = 0 \quad (54)$$

In general, $\vec{u}$, $\vec{b}$, $\vec{n}_1$, and $\vec{n}_2$ will be linear combinations of the outputs of three or more gyros, which need not be aligned along the spacecraft axes.

An alternative equation for the drift-rate bias

$$\frac{d}{dt} \vec{b} = -\vec{b}/\tau + \vec{n}_2 \tag{55}$$

gives rise to an exponentially correlated noise term in the model considered by Hammon [67]. In order to reproduce realistic gyro data, a superposition of several drift terms, with different values of the time constant may be needed [68]. Since relatively persistent drifts are observed in actual gyro operation, at least one time constant must be very large. Letting τ be infinite, which leads to Eq. (51), is adequate for most applications.

In the propagation equations to be used in the Kalman filter, Eq. (48) is assumed to be integrated continuously. The model thus assumes that the gyros are used in a rate mode. In practice, however, rate-integrating gyros are used, which sense the spacecraft angular rates continuously but are sampled at discrete intervals. The spacecraft attitude is also propagated at discrete intervals, equal to either the gyro sampling interval or some multiple thereof. If the attitude update interval is much shorter than the Kalman filter update interval, as it always is in practice, the approximation of continuous gyro updates will be good.

Line-of-Sight Attitude Sensors

The line-of-sight attitude sensor considered here is any sensor for which the measured quantity depends solely on the direction of some object in the sensor coordinate system. Typical of such sensors are vector Sun sensors and fixed-head star trackers.

The direction of a body in the sensor coordinate system $\vec{p}_S$ is related to the direction in the reference coordinate system $\vec{p}_R$ according to

$$\vec{p}_S = TA(\bar{q})\vec{p}_R \tag{56}$$

where $A(\bar{q})$ is the spacecraft attitude and T is the sensor alignment matrix. Note that the measurement depends explicitly on the attitude but not on parameters such as gyro biases.

It will be assumed throughout this report that the sensor measurements are scalar and uncorrelated. In general the correlation between the measurements is small. When this is not the case a set of uncorrelated measurements can always be achieved by choosing a representation in which the measurement covariance matrix is diagonal.

6. The State Equation

The spacecraft attitude state is given by the attitude quaternion and the gyro drift-rate bias vector

$$\underline{x}(t) = \begin{bmatrix} \bar{q}(t) \\ \vec{b}(t) \end{bmatrix} \tag{57}$$

and thus has dimension seven. The quaternion and the bias vector have been shown to satisfy the coupled differential equations

$$\frac{d}{dt} \bar{q}(t) = \frac{1}{2} \Omega(\vec{u}(t)-\vec{b}(t)-\vec{n}_1(t))\, \bar{q}(t) \tag{58}$$

$$\frac{d}{dt} \vec{b}(t) = \vec{n}_2(t) \tag{59}$$

Noting that the matrix function Ω is linear and homogeneous in its argument and defining the 4×3 matrix function $\Xi(\bar{q})$ by

$$\Omega(\vec{b})\bar{q} = \Xi(\bar{q})\vec{b} \tag{60}$$

Eq. (58) may be rewritten as

$$\frac{d}{dt} \bar{q}(t) = \frac{1}{2} \Omega(\vec{u}(t)-\vec{b}(t))\bar{q}(t) - \frac{1}{2} \Xi(\bar{q}(t))\vec{n}_1(t) \tag{61}$$

Equations (59) and (61) are now in the same form as Eq. (1) above.

The matrix $\Xi(\bar{q})$ has the explicit form

$$\Xi(\bar{q}) = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{62}$$

The properties of the matrix $\Xi(\bar{q})$ are discussed in Appendix A.

Prediction

The predicted state vector is defined as in Sec. 2 Taking the expectation of Eqs. (59) and (61) leads, within the approximation of Eq. (8), to

$$\frac{d}{dt} \hat{\bar{q}} = \frac{1}{2} \Omega(\hat{\vec{\omega}})\, \hat{\bar{q}} \tag{63}$$

$$\frac{d}{dt} \hat{\vec{b}} = \vec{0} \tag{64}$$

where

$$\hat{\vec{\omega}} = \vec{u} - \hat{\vec{b}} \tag{65}$$

is the estimated angular velocity.

From Eq. (64) we see immediately that $\hat{\vec{b}}$ is constant over the prediction interval. Thus, $\hat{\vec{\omega}}$ depends only on $\vec{u}(t)$ and the initial value of the state vector. Therefore, Eq. (63) can be integrated directly to give

$$\hat{\bar{q}}(t) = \Theta(t,t_k)\, \hat{\bar{q}}(t_k) \tag{66}$$

with

$$\frac{\partial}{\partial t} \Theta(t,t_k) = \frac{1}{2} \Omega(\hat{\vec{\omega}}(t))\, \Theta(t,t_k) \tag{67}$$

$$\Theta(t_k,t_k) = I_{4x4} \tag{68}$$

When the direction of $\hat{\bar{\omega}}(t)$ is constant throughout the time interval or the angular displacement of the axes is small, then $\theta(t,t_k)$ can be approximated by Eq. (47) above.

7. The Seven-Dimensional Covariance Representation

The simplest approach to express the state-error vector and the covariance matrix is to define these in terms of the complete state vector as in Eqs. (10-11) above.

Prediction

The seven-dimensional state-error vector satisfies the differential equation

$$\frac{d}{dt}\,\Delta\underline{x}(t) = F(t)\,\Delta\underline{x}(t) + G(t)\,\underline{w}(t) \tag{69}$$

with

$$F(t) = \left[\begin{array}{c|c} \frac{1}{2}\,\Omega(\hat{\bar{\omega}}) & -\frac{1}{2}\,\Xi(\hat{\bar{q}}) \\ \hline 0_{3x4} & 0_{3x3} \end{array}\right] \tag{70}$$

$$G(t) = \left[\begin{array}{c|c} -\frac{1}{2}\,\Xi(\hat{\bar{q}}) & 0_{4x3} \\ \hline 0_{3x4} & I_{3x3} \end{array}\right], \tag{71}$$

$$\underline{w}(t) = \begin{bmatrix} \vec{n}_1(t) \\ \vec{n}_2(t) \end{bmatrix} \tag{72}$$

and therefore

$$Q(t) = \left[\begin{array}{c|c} Q_1(t) & 0_{3x3} \\ \hline 0_{3x3} & Q_2(t) \end{array}\right] \tag{73}$$

$F(t)$ and $G(t)$ are determined straightforwardly from Eqs. (13), (14), (59), and (61) above.

The transition matrix has the general form

$$\Phi(t,t\ ) = \left[\begin{array}{c|c} \Theta(t,t_o) & \Psi(t,t_o) \\ \hline 0_{3x4} & I_{3x3} \end{array}\right] \tag{74}$$

where $\Theta(t,t_o)$ is given by Eqs. (67-68) above and

$$\frac{\partial}{\partial t}\,\Psi(t,t_o) = \frac{1}{2}\,\Omega(\hat{\bar{\omega}}(t,t_o))\Psi(t,t_o) - \frac{1}{2}\,\Xi(\hat{\bar{q}}(t)) \tag{75}$$

subject to

$$\Psi(t_o,t_o) = 0_{4x3} \tag{76}$$

Equation (75) may be integrated directly to yield

$$\Psi(t,t_o) = -\frac{1}{2}\int_{t_o}^{t} \Theta(t,t')\ \Xi(\hat{\bar{q}}(t'))dt' \tag{77}$$

Filtering

Since the measurements are assumed to be scalar and to depend only on the attitude and not the bias vector, as discussed in Sec. 5, it follows that the sensitivity matrix is a seven-dimensional row vector of the form

$$H = [\ell,\ \vec{0}^T] \tag{78}$$

where

$$\ell = \left(\frac{\partial h}{\partial \vec{p}_S}\ \frac{\partial \vec{p}_S}{\partial \bar{q}}\right)\Bigg|_{\hat{\bar{q}}(-)} \tag{79}$$

Define the vector $\vec{r}$ by

$$\vec{r} \equiv \left[\frac{\partial h}{\partial \vec{p}_S}\ T\right]^T \tag{80}$$

then it can be shown that

$$\ell = [\frac{\partial}{\partial \bar{q}}\ \ \vec{r}\cdot A(\bar{q})\vec{p}_R]|_{\hat{\bar{q}}(-)}$$

$$= 2(\vec{r} \times \hat{\vec{p}}_B)^T\ \Xi^T(\hat{\bar{q}}(-)) \tag{81}$$

where

$$\hat{\vec{p}}_B = A(\hat{\bar{q}}(-))\vec{p}_R \tag{82}$$

8. Singularity of the Covariance Matrix

The covariance matrix for the seven-dimensional state-vector is singular. This follows immediately from the constraint on the quaternion norm so that

$$\Delta\bar{q}^T\ \hat{\bar{q}} = 0 \tag{83}$$

and hence

$$\begin{bmatrix} \hat{\bar{q}}(t) \\ \vec{0} \end{bmatrix}$$

is a null vector of $P(t)$.

This singularity is difficult to maintain numerically due to the accumulation of round-off error. If fact, $P(t)$ may even develop a negative eigenvalue. The simplest way to maintain the singularity is to represent $P(t)$ by a matrix of smaller dimension. There are three possible approaches to accomplishing this, two of which lead to the same result. The examination of these three approaches forms the subject matter for the remainder of this report.

9. Reduced Representation of the Covariance Matrix

A 6x6 representation of the covariance matrix is induced by the form of the transition matrix prescribed above.

Prediction

It can be shown (see Appendix A) that

$$\Theta(t,t') \equiv \Xi(\hat{\bar{q}}(t,t')) = \Xi(\hat{\bar{q}}(t))\, A(t,t') \tag{84}$$

where

$$A(t,t') = A(\hat{\bar{q}}(t))A^T(\hat{\bar{q}}(t')) \tag{85}$$

is the 3 x 3 rotation matrix which transforms the estimated attitude matrix from time t' to time t. Substituting this into Eq. (77) leads to

$$\Psi(t,t_o) = \Xi(\hat{\bar{q}}(t))K(t,t_o) \tag{86}$$

with

$$K(t,t_o) = -\frac{1}{2}\int_{t_o}^{t} A(t,t')\, dt' \tag{87}$$

It is also shown in Appendix A that

$$\Theta(t,t_o) = \Xi(\hat{\bar{q}}(t))A(t,t_o)\Xi^T(\hat{\bar{q}}(t_o)) + \hat{\bar{q}}(t)\hat{\bar{q}}^T(t_o) \tag{88}$$

Substituting these expressions into the expression for the transition matrix gives

$$\Phi(t,t_o) = S(\hat{\bar{q}}(t))\bar{\Phi}(t,t_o)S^T(\hat{\bar{q}}(t_o)) + \left[\begin{array}{c|c} \hat{\bar{q}}(t)\hat{\bar{q}}^T(t_o) & 0_{4x3} \\ \hline 0_{3x4} & 0_{3x3} \end{array}\right] \tag{89}$$

where

$$S(\hat{\bar{q}}(t)) = \left[\begin{array}{c|c} \Xi(\hat{\bar{q}}(t)) & 0_{4x3} \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{90}$$

$$\bar{\Phi}(t,t_o) = \left[\begin{array}{c|c} A(t,t_o) & K(t,t_o) \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{91}$$

The second term of the right member of Eq. (89) annihilates $P(t_o)$ on the left and $P(t)$ on the right. Thus, if the 6x6 covariance matrix $\tilde{P}(t)$ is defined by

$$\tilde{P}(t) = S^T(\hat{\bar{q}}(t))P(t)S(\hat{\bar{q}}(t)) \tag{92}$$

this satisfies the integrated Riccati equation of the form

$$\tilde{P}(t) = \bar{\Phi}(t,t_o)\, \tilde{P}(t_o)\, \bar{\Phi}^T(t,t_o) + \int_{t_o}^{t} \bar{\Phi}(t,t')\tilde{G}(t')Q(t')\tilde{G}^T(t')\bar{\Phi}^T(t,t')dt' \tag{93}$$

where

$$\tilde{G}(t) \equiv S^T(\hat{\bar{q}}(t))G(t) = \left[\begin{array}{c|c} -\frac{1}{2} I_{3x3} & 0_{3x3} \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{94}$$

In obtaining Eq. (93) repeated use has been made of the equation

$$\Xi^T(\hat{\bar{q}})\, \hat{\bar{q}} = \vec{0} \tag{95}$$

At any time the 7x7 covariance matrix P(t) can be reconstructed via

$$P(t) = S(\hat{\bar{q}}(t))\, \tilde{P}(t)\, S^T(\hat{\bar{q}}(t)) \tag{96}$$

Filtering

The Kalman filter may also be mechanized using $\tilde{P}(t)$. Define the 1x6 matrix $\tilde{H}_k$ and the 6x1 matrix $\tilde{K}_k$ according to

$$\tilde{H}_k = H_k\, S(\hat{\bar{q}}_k(-)) \tag{97}$$

$$\tilde{K}_k = \tilde{P}_k(-)\tilde{H}_k^T[\tilde{H}_k\tilde{P}_k(-)\tilde{H}_k^T + R_k]^{-1} \tag{98}$$

where H_k is as in Sec. 8. Then it is easy to show that

$$\tilde{P}_k(+) = (I_{6x6} - \tilde{K}_k\, \tilde{H}_k)\, \tilde{P}_k(-) \tag{99}$$

$$K_k = S(\hat{\bar{q}}_k(-))\, \tilde{K}_k \tag{100}$$

In implementing this method the 7x7 covariance matrix need never be computed, the 6x6 covariance is propagated and updated using Eqs. (93) and (99), respectively. The expression for the 1x6 sensitivity matrix $\tilde{H}_k$ can be simplified; but this expression will be derived in Sec. 11, since it falls out more directly in that approach. That section contains an independent derivation of the prediction and filtering algorithms which turn out to be identical to those presented here.

10. The Truncated Covariance Representation

The most obvious approach to reducing the dimension of the covariance matrix is to delete one of the quaternion components from the state-error vector. We may let this be the fourth component, although in principal any component could be deleted.

The truncated state-error vector is then

$$\Delta\underline{x} = \begin{bmatrix} \Delta\vec{q} \\ \Delta\vec{b} \end{bmatrix} \tag{101}$$

This state-error vector cannot be treated as simply as the 7-dimensional state-error vector of Sec. 8. In Sec. 8, the four components of the quaternion were treated as independent variables with the normalization constraint being maintained by the form of the equation of motion. In the present case the fourth component of the quaternion error must be determined from the constraint according to

$$\Delta q_4 = -\frac{1}{\hat{q}_4}\,\hat{q}\cdot\Delta\vec{q} \tag{102}$$

Thus, partial differentiation must be treated differently in the two cases.

Prediction

The truncated covariance matrix is defined as

$$P_y(t) = E\,\{\Delta\underline{y}(t)\,\Delta\underline{y}^T(t)\} \tag{103}$$

which satisfies

$$P_y(t) = \Phi_y(t,t_o)\,P_y(t_o)\,\Phi_y^T(t,t_o) + \int_{t_o}^{t}\Phi_y(t,t')G_y(t')Q(t')G_y^T(t')\Phi_y^T(t,t')\,dt' \tag{104}$$

where

$$\Phi_y(t,t_o) = \left[\begin{array}{c|c} \Theta_y(t,t_o) & \Psi_y(t,t_o) \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{105}$$

is the 6x6 transition matrix. $G_y(t)$ is defined below.

The submatrices of $\Phi_y(t;t_o)$ are easily determined from the corresponding submatrices of $\Phi(t,t_o)$. This is simpler than attempting to determine $\Phi_y(t,t_o)$ by direct integration of a 6-dimensional state error equation.

Note that

$$[\Theta_y(t,t_o)]_{ij} \equiv \left(\frac{\partial\Delta q_i(t)}{\partial\Delta q_j(t_o)}\right)_y \tag{106}$$

$$[\Psi_y(t,t_o)]_{ij} \equiv \left(\frac{\partial\Delta q_i(t)}{\partial\Delta b_j(t_o)}\right)_y \tag{107}$$

where the subscript "y" indicates that the partial derivatives are to be computed subject to the constraint. No similar constraint exists in the corresponding expression for $\Theta(t,t_o)$ and $\Psi(t,t_o)$, the constraint being satisfied by the state equation. Noting this fact leads immediately to

$$[\Theta_y(t,t_o)]_{ij} = [\Theta(t,t_o)]_{ij} - \frac{\hat{q}_j(t_o)}{\hat{q}_4(t_o)}\,[\Theta(t,t_o)]_{i4} \tag{108}$$

$$[\Psi_y(t,t_o)]_{ij} = [\Psi(t,t_o)]_{ij} \tag{109}$$

and also

$$[G_y(t)]_{ij} = [G(t)]_{ij} \tag{110}$$

The 7x7 covariance matrix can be recovered from $P_y(t)$ by constructing the missing elements according to

$$E\{\Delta q_4(t)\Delta y_\ell(t)\} = -\frac{1}{\hat{q}_4(t)}\sum_{i=1}^{3}\hat{q}_i(t)\,[P_y(t)]_{i\ell} \tag{111}$$

$$E\{\Delta q_4(t)\Delta q_4(t)\} = \frac{1}{|\hat{q}_4(t)|^2}\sum_{i=1}^{3}\sum_{j=1}^{3}\hat{q}_i(t)[P_y(t)]_{ij}\,\hat{q}_j(t) \tag{112}$$

Filtering

Analogously to Eqs. (78) and (79) in Sec. 7 the sensitivity matrix is given by

$$H_y = [\ell_y,\ \vec{\delta}^T] \tag{113}$$

where the 3-dimensional row vector ℓ_y is given by

$$\ell_y = \left[\frac{\partial h}{\partial\hat{\vec{p}}_S}\left(\frac{\partial\hat{\vec{p}}_S}{\partial\vec{q}}\right)_y\right]\Bigg|_{\hat{\bar{q}}(-)} \tag{114}$$

which reduces to

$$(\ell_y)_i = \ell_i - (\hat{q}_i/\hat{q}_4)\ell_4 \tag{115}$$

and ℓ is given by Eq. (81) above.

In implementing the Kalman filter in this case also the 7x7 covariance matrix need never be computed. The state-vector is updated by computing

$$\Delta\hat{\underline{y}}(+) = K_y(z - h(\hat{\bar{q}}(-))) \tag{116}$$

where K_y is computed from $P_y(-)$ and H_y using Eq. (26). $\Delta\hat{q}_4(+)$ is obtained from Eq. (102) and the updated state-vector is given by

$$\hat{\underline{x}}(+) = \hat{\underline{x}}(-) + \Delta\hat{\underline{x}}(+) \tag{117}$$

Note from Eq. (102) that truncation can lead to large errors when $\hat{q}_4$ is small. This can be avoided by always deleting the quaternion component which is largest in magnitude.

11. The Body-Fixed Covariance Representation

In this section we develop an approximate body-referenced representation of the state-vector and covariance matrix. The quaternion error in this representation is expressed not as the arithmetic difference between the true and the estimated quaternion but as the quaternion which must be composed with the estimated quaternion in order to obtain the true quaternion. Since this incremental quaternion corresponds almost certainly to a small rotation, the fourth component will be close to unity (to second order in the vector components) and hence all the attitude information of interest is contained in the three vector components. Therefore, the six-component object defined by the vector components of the incremental quaternion and the drift bias vector will provide a non-redundant representation of the state error. This representation will turn out to be identical to that developed in Sec. 9.

If the infinitesimal attitude error angles are defined as twice the vector components of the incremental quaternion, then this treatment becomes very similar to that employed by several other authors. [21, 59-65].

Define the error quaternion as

$$\delta\bar{q} = \bar{q} \bullet \hat{\bar{q}}^{-1} \tag{118}$$

and the six-dimension body-referenced state vector as

$$\bar{\underline{x}} = \begin{bmatrix} \delta\vec{q} \\ \vec{b} \end{bmatrix} \tag{119}$$

From Eqs. (30) and (62) we have that

$$\bar{q} = \{\hat{\bar{q}}\}\,\delta\bar{q} = [\Xi(\hat{\bar{q}}) \,\vdots\, \hat{\bar{q}}]\,\delta\bar{q} \tag{120}$$

whence

$$\delta\vec{q} = \Xi^T(\hat{\bar{q}})\bar{q} \tag{121}$$

$$\delta q_4 = \hat{\bar{q}}^T\bar{q} \tag{122}$$

and

$$\bar{\underline{x}} = \left[\begin{array}{c|c} \Xi^T(\hat{\bar{q}}) & 0_{3x3} \\ \hline 0_{3x4} & I_{3x3} \end{array}\right] \begin{bmatrix} \bar{q} \\ \vec{b} \end{bmatrix} \tag{123}$$

$$= S^T(\hat{\bar{q}})\underline{x} \tag{124}$$

From Eq. (95) it then follows that

$$\hat{\bar{\underline{x}}} = S^T(\hat{\bar{q}})\hat{\underline{x}} = \begin{bmatrix} \vec{0} \\ \hat{\vec{b}} \end{bmatrix} \tag{125}$$

$$\Delta\bar{\underline{x}} = S^T(\hat{\bar{q}})\Delta\underline{x} = \begin{bmatrix} \delta\vec{q} \\ \Delta\vec{b} \end{bmatrix} \tag{126}$$

Note, however, that although

$$\Delta\underline{x} = S(\hat{\bar{q}})\,\Delta\bar{\underline{x}} \tag{127}$$

it happens that

$$\hat{\underline{x}} \neq S(\hat{\bar{q}})\hat{\bar{\underline{x}}} \tag{128}$$

The results of Sec. 9 may now be derived in short order.

<u>Prediction</u>

From

$$\frac{d}{dt}\bar{q} = \frac{1}{2}\bar{\omega} \bullet \bar{q} \tag{129}$$

$$\frac{d}{dt}\hat{\bar{q}} = \frac{1}{2}\hat{\bar{\omega}} \bullet \hat{\bar{q}} \tag{130}$$

it follows immediately that

$$\frac{d}{dt}\delta\bar{q} = \frac{1}{2}[\bar{\omega} \bullet \delta\bar{q} - \delta\bar{q} \bullet \hat{\bar{\omega}}]$$

$$= \frac{1}{2}[\hat{\bar{\omega}} \bullet \delta\bar{q} - \delta\bar{q} \bullet \hat{\bar{\omega}}] + \frac{1}{2}\delta\bar{\omega} \bullet \delta\bar{q} \tag{131}$$

where

$$\delta\bar{\omega} = \begin{bmatrix} \delta\vec{\omega} \\ 0 \end{bmatrix} = \begin{bmatrix} \vec{\omega} - \hat{\vec{\omega}} \\ 0 \end{bmatrix} = \begin{bmatrix} -\Delta\vec{b} - \vec{n}_1 \\ 0 \end{bmatrix} \tag{132}$$

Now

$$\frac{1}{2}[\hat{\bar{\omega}} \bullet \delta\bar{q} - \delta\bar{q} \bullet \hat{\bar{\omega}}] = \begin{bmatrix} -\hat{\vec{\omega}} \times \delta\vec{q} \\ 0 \end{bmatrix} \tag{133}$$

$$\delta\bar{\omega} \bullet \delta\bar{q} = \delta\bar{\omega} + O(|\delta\vec{\omega}||\delta\vec{q}|) \tag{134}$$

Thus, neglecting second-order terms

$$\frac{d}{dt}\delta\vec{q} = -\hat{\vec{\omega}} \times \delta\vec{q} - \frac{1}{2}\Delta\vec{b} - \vec{n}_1 \tag{135}$$

$$\frac{d}{dt}\delta q_4 = 0 \tag{136}$$

from which it follows that the state-error equation may be written

$$\frac{d}{dt}\Delta\bar{\underline{x}}(t) = \tilde{F}(t)\Delta\bar{\underline{x}} + \tilde{G}(t)\,\underline{w}(t) \tag{137}$$

where

$$\tilde{F}(t) = \left[\begin{array}{c|c} [\hat{\underline{\omega}}(t)] & -\frac{1}{2}I_{3x3} \\ \hline 0_{3x3} & 0_{3x3} \end{array}\right] \tag{138}$$

$$\tilde{G}(t) = \left[\begin{array}{c|c} -\frac{1}{2}I_{3x3} & 0_{3x3} \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{139}$$

$\tilde{G}$ is identical to Eq. (94) of Sec. 9.

The transition matrix may be written as

$$\tilde{\Phi} = \left[\begin{array}{c|c} \Theta & \Psi \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{140}$$

with

$$\frac{\partial}{\partial t}\Theta(t,t_o) = [\hat{\underline{\omega}}(t)]\,\Theta(t,t_o) \tag{141}$$

$$\frac{\partial}{\partial t}\tilde{\Psi}(t,t_o) = [\underline{\hat{\omega}}(t)]\tilde{\Psi}(t,t_o) - \frac{1}{2}I_{3x3} \tag{142}$$

subject to

$$\tilde{\theta}(t_o,t_o) = I_{3x3}, \quad \tilde{\Psi}(t_o,t_o) = 0_{3x3} \tag{143}$$

It now follows immediately that

$$\tilde{\theta}(t,t_o) = A(t,t_o) \tag{144}$$

$$\tilde{\Psi}(t,t_o) = -\frac{1}{2}\int_{t_o}^{t} A(t,t')\,dt' = K(t,t_o) \tag{145}$$

so that

$$\tilde{\Phi}(t,t_o) = \left[\begin{array}{c|c} A(t,t_o) & K(t,t_o) \\ \hline 0_{3x3} & I_{3x3} \end{array}\right] \tag{146}$$

as in Eq. (91) of Sec. 9.

The covariance matrix is defined as

$$\tilde{P}(t) = E\{\Delta\tilde{\underline{x}}(t)\,\Delta\tilde{\underline{x}}^T(t)\} \tag{147}$$

and from Eq. (127) it follows that

$$\tilde{P}(t) = S^T(\hat{\bar{q}}(t))\,P(t)\,S(\hat{\bar{q}}(t)) \tag{148}$$

$$P(t) = S(\hat{\bar{q}}(t))\,\tilde{P}(t)\,S^T(\hat{\bar{q}}(t)) \tag{149}$$

as in Sec. 9, and $\tilde{P}(t)$ satisfies the Riccati equation

$$\frac{d}{dt}\tilde{P}(t) = \tilde{F}(t)\tilde{P}(t)+\tilde{P}(t)\tilde{F}^T(t)+\tilde{G}(t)Q(t)\tilde{G}^T(t) \tag{150}$$

whose integral form is simply Eq. (93).

__Filtering__

In analogy with the previous sections

$$\tilde{H} = [\tilde{\ell},\ \tilde{0}^T] \tag{151}$$

where the three-dimensional row vector $\tilde{\ell}$ is given by

$$\tilde{\ell} = \left(\frac{\partial h}{\partial \vec{p}_S}\ \frac{\partial \vec{p}_S}{\partial(\delta\vec{q})}\right)\Bigg|_{\hat{\bar{q}}(-)} \tag{152}$$

From

$$A(\bar{q}) = A(\delta\bar{q})\,A(\hat{\bar{q}}(-)) \tag{153}$$

it follows that

$$\tilde{\ell} = \left(\vec{r}\cdot\frac{\partial}{\partial(\delta\vec{q})}A(\delta\bar{q})\hat{\vec{p}}_B\right)\Bigg|_{\hat{\bar{q}}(-)} \tag{154}$$

where

$$\hat{\vec{p}}_B \equiv A(\hat{\bar{q}}(-))\vec{p}_R \tag{155}$$

Noting that

$$A(\delta\bar{q}) = I_{3x3} + 2\,[\delta\vec{q}] \tag{156}$$

the differentiation may be carried out immediately to give

$$\tilde{\ell} = 2\,(\vec{r}\times\hat{\vec{p}}_B)^T \tag{157}$$

The implementation of these algorithms is identical to those of Sec. 9. An apparent but not actual difference is given by the state vector update where

$$\Delta\hat{\tilde{\underline{x}}}(+) = \begin{bmatrix} \delta\hat{\vec{q}}(+) \\ \Delta\hat{\vec{b}}(+) \end{bmatrix} = \tilde{K}\,(z - h(\hat{\bar{q}}(-))) \tag{158}$$

from which we have

$$\hat{\bar{q}}(+) = \delta\bar{q}(+) \otimes \hat{\bar{q}}\,(-) \tag{159}$$

$$\hat{\vec{b}}(+) = \hat{\vec{b}}(-) + \Delta\hat{\vec{b}}(+) \tag{160}$$

with

$$\delta\hat{\bar{q}}(+) = \begin{bmatrix} \delta\hat{\vec{q}}(+) \\ 1 \end{bmatrix} \tag{161}$$

Noting from Eqs. (37) and (60) that

$$\delta\bar{q} \otimes \hat{\bar{q}} = \hat{\bar{q}} + \Omega(\delta\hat{\vec{q}})\hat{\bar{q}} \tag{162}$$

$$= \hat{\bar{q}} + \Xi(\hat{\bar{q}})\delta\hat{\vec{q}} \tag{163}$$

we see that this algorithm is identical to implementing Eq. (100) above.

If instead of the incremental quaternion the incremental error angles given by

$$\delta\vec{q} = \frac{1}{2}\,\delta\vec{\theta} \tag{164}$$

are used, the expressions simplify somewhat in that the factors of 1/2, 1/4, and 2 in the expressions for $\tilde{F}(t)$, $\tilde{G}(t)$, $K(t,t_o)$ and $\tilde{\ell}$ disappear. The implementation is otherwise unchanged.

12. Conclusions

In all the implementations of the Kalman filter presented above the propagation of the state vector is performed identically using Eqs. (63) and (64). In the propagation the four components of the quaternion are treated as independent variables. Within numerical errors the constraint on the quaternion normalisation is maintained by the structure of Eq. (63).

This normalisation can be destroyed by the accumulation of round-off error and the linearisation approximation inherent in the update equation. This need not cause concern in the propagation of the quaternion since this operation is linear. Care must be taken, however, when the attitude matrix is to be calculated or when an update is to be performed. The renormalization of the quaternion can then be carried out using an algorithm such as the one in Appendix B.

It is in the propagation of the state covariance matrix and the update of the state vector and covariance matrix that differences in method appear. The three approaches presented above: the use of the full 7x7 covariance matrix; the truncated 6x6 representation; and the 6x6 body representation, will now be examined.

The 7x7 representation presented in Sec. 7 is the most direct but also the most burdensome computationally, since the number of elements in the covariance matrix is largest. A more significant problem, however, is the need to maintain the singularity of the covariance matrix, which is of rank 6. As pointed out in Sec. 8, round-off errors can lead to terms which raise the rank of the matrix and even lead to negative eigenvalues.

The truncated 6x6 representation of the covariance matrix maintains the proper rank but does not result in a computational saving. This is true because the full 7x7 transition matrix must be computed at each time and the related 6x6 transition matrix Φ_y computed only at the time of an update. the burden of reducing the dimension of the transition matrix cancels the savings resulting from the implementation of the update equations in a space of smaller dimension.

The 6x6 body representation of the covariance matrix presented in Secs. 9 and 11 preserves the proper rank of the covariance matrix while simplifying the computation considerably. The transition matrix and the covariance matrix are computed as 6x6 matrices throughout. Of particular value in this representation is the fact that the elements of the covariance matrix have a simple interpretation in terms of gyro bias errors and angular errors in the body frame. In addition, the matrices $\tilde{G}$ and $\tilde{H}$ have a particularly simple form.

In any Kalman filter implementation the largest computational burden is imposed in general by the computation of the transition matrix and the contribution of the process noise to the state covariance matrix. These quantities are not needed at the same level of accuracy as the state vector and hence may be computed at much larger intervals than is required by the state-vector propagation. For this same reason, approximate forms of these quantities can also be implemented. These are given in Appendix B.

Appendix A - The Matrix $\Xi(\bar{q})$

The matrix $\Xi(\bar{q})$ defined in Eq. (62) has some very useful properties which greatly simplify the computation of the covariance matrix and the transition matrices. In this appendix the following results will be proved.

$$\frac{d}{dt}\Xi(\bar{q}(t)) = \frac{1}{2}\Omega(\vec{\omega})\Xi(\bar{q}(t)) - \Xi(\bar{q}(t))[\underline{\vec{\omega}}] \qquad (A-1)$$

$$\Theta(t,t_o)\Xi(\bar{q}(t_o)) = \Xi(\bar{q}(t))\Lambda(t,t_o) \qquad (A-2)$$

$$\Lambda(t,t_o) = \Xi^T(\bar{q}(t))\Theta(t,t_o)\Xi(\bar{q}(t_o)) \qquad (A-3)$$

$$\Theta(t,t_o) = \Xi(\bar{q}(t))\Lambda(t,t_o)\Xi^T(\bar{q}(t_o)) + \bar{q}(t)\bar{q}^T(t_o) \qquad (A-4)$$

In the above equations the quaternion is assumed to be deterministic, that is

$$\frac{d}{dt}\bar{q}(t) = \frac{1}{2}\Omega(\vec{\omega})\bar{q}(t) \qquad (A-5)$$

$$\bar{q}(t) = \Theta(t,t_o)\bar{q}(t_o) \qquad (A-6)$$

and, hence, truly corresponds to $\hat{\bar{q}}(t)$ elsewhere in this report. However, to simplify the notation carets will not be written over $\bar{q}$ and $\vec{\omega}$. Also, although this is not written explicitly, $\vec{\omega}$ may depend on the time.

To prove Eq. (A-1) note that by definition for any three-vector $\vec{c}$

$$\Xi(\bar{q})\vec{c} = \Omega(\vec{c})\bar{q} \qquad (A-7)$$

Differentiating Eq. (A-7) leads to

$$\dot{\Xi}(\bar{q})\vec{c} + \Xi(\bar{q})\dot{\vec{c}} = \Omega(\dot{\vec{c}})\bar{q} + \Omega(\vec{c})\dot{\bar{q}} \qquad (A-8)$$

Let the value of $\vec{c}$ at time t be arbitrary and let the time development of $\vec{c}$ be given by

$$\dot{\vec{c}} = [\underline{\vec{\omega}}]\,\vec{c} = -\vec{\omega}\times\vec{c} \qquad (A-9)$$

Substituting Eqs. (A-5) and (A-9) into Eq. (A-8) leads to

$$\dot{\Xi}(\bar{q})\vec{c} + \Xi(\bar{q})[\underline{\vec{\omega}}]\vec{c}$$
$$= -\Omega(\vec{\omega}\times\vec{c})\bar{q} + \frac{1}{2}\Omega(\vec{c})\Omega(\vec{\omega})\bar{q} \qquad (A-10)$$

It is a simple matter to prove that

$$\Omega(\vec{\omega}\times\vec{c}) = -\frac{1}{2}[\Omega(\vec{\omega})\Omega(\vec{c})-\Omega(\vec{c})\Omega(\vec{\omega})] \qquad (A-11)$$

Substituting this expression into Eq. (A-10) above, recalling Eq. (A-7), and noting that $\vec{c}(t)$ is arbitrary leads directly to Eq. (A-1).

To prove Eq. (A-2) examine the quantity

$$C(t,t_o) \equiv \Theta(t,t_o)\Xi(\bar{q}(t_o)) - \Xi(\bar{q}(t))\Lambda(t,t_o) \qquad (A-12)$$

Differentiating this expression and using Eq. (A-1) leads to

$$\frac{\partial}{\partial t}C(t,t_o) = \frac{1}{2}\Omega(\vec{\omega})C(t,t_o) \qquad (A-13)$$

which may be integrated directly to yield

$$C(t,t_o) = \Theta(t,t_o)C(t_o,t_o) \qquad (A-14)$$

But from Eq. (A-12)

$$C(t_o,t_o) = 0_{4x3} \tag{A-15}$$

and Eq. (A-2) follows.

Equation (A-3) follows directly by applying the relation

$$\Xi^T(\bar{q})\Xi(\bar{q}) = I_{3x3} \tag{A-16}$$

to Eq. (A-2)

To obtain Eq. (A-4) note that the converse relation

$$\Xi(\bar{q})\Xi^T(\bar{q}) = I_{4x4} - \bar{q}\bar{q}^T \tag{A-17}$$

applied to Eq. (A-2) leads to

$$\theta(t,t_o)(I_{4x4} - \bar{q}(t_o)\,\bar{q}^T(t_o))$$

$$= \Xi(\bar{q}(t))A(t,t_o)\Xi^T(\bar{q}(t_o)) \tag{A-18}$$

Noting

$$\theta(t,t_o)\bar{q}(t_o)\bar{q}^T(t_o) = \bar{q}(t)\bar{q}^T(t_o) \tag{A-19}$$

Equation (A-4) follows immediately.

Appendix B - Numerical Considerations

We will only consider quantities in the body-fixed representation since computations are easiest in this representation. If desired, these results can be transformed into the other representations.

Transition Matrix

The rotation matrix is given by Eq. (85)

$$A(t,t_o) = A(\hat{\bar{q}}(t))A^T(\hat{\bar{q}}(t_o)) \tag{B-1}$$

which requires only a matrix multiplication, since the attitude matrix is required at the update times for observation processing. The second matrix factor on the right is the transpose of $A(\hat{\bar{q}}_o(+))$, while the matrix that is computed is $A(\hat{\bar{q}}_o(-))$. These are related by

$$A(\hat{\bar{q}}_o(+)) = A(\delta\hat{\bar{q}}_o)A(\hat{\bar{q}}_o(-)) \tag{B-2}$$

where $\delta\hat{\bar{q}}_o$ is the incremental quaternion at time t_o, and Eq. (156) can be used for $A(\delta\hat{\bar{q}}_o)$.

The K matrix is given by Eq. (87)

$$K(t,t_o) = -\frac{1}{2}\int_{t_o}^{t} A(t,t')\,dt' \tag{B-3}$$

This integral cannot be computed in real time since the rotation matrix must be known for t>t'. This objection can be removed by writing

$$A(t,t')=A(t,t_o)A(t_o,t') = A(t,t_o)A^T(t',t_o) \tag{B-4}$$

which gives

$$K(t,t_o) = -\frac{1}{2}A(t,t_o)\int_{t_o}^{t} A^T(t',t_o)dt' \tag{B-5}$$

Real-time computation and integration of the rotation matrix is quite expensive and can be avoided if an approximate form of K is adequate. this is derived by assuming that $\hat{\underline{\omega}}$ is constant over the interval and that $|\hat{\underline{\omega}}|(t-t_o)$ is small. Then, ignoring cubic and higher terms

$$A(t,t') = I_{3x3}+[\hat{\underline{\omega}}](t-t') + \frac{1}{2}[\hat{\underline{\omega}}]^2(t-t')^2 \tag{B-6}$$

Substituting into Eq. (B-3) and integrating gives

$$K(t,t_o)=-\frac{1}{2}(t-t_o)\{I_{3x3} + \frac{1}{2}[\hat{\underline{\omega}}](t-t_o) + \frac{1}{6}[\hat{\underline{\omega}}]^2(t-t_o)^2\} \tag{B-7}$$

Comparing this with the symmetric and skew-symmetric parts of Eq. (B-6) gives

$$K(t,t_o)=-\frac{1}{24}(t-t_o)[8I_{3x3}+5A(t,t_o)-A^T(t,t_o)] \tag{B-8}$$

This is convenient for computation since the matrix $A(t,t_o)$ has already been computed.

Process Noise

The contribution of the process noise to the state covariance matrix is given by the second term of Eq. (93)

$$\tilde{N}(t,t_o)=\int_{t_o}^{t}\tilde{\Phi}(t,t')\tilde{G}(t')Q(t')\tilde{G}^T(t')\tilde{\Phi}^T(t,t')dt'$$

$$\equiv \left[\begin{array}{c|c} \tilde{N}_{11}(t,t_o) & \tilde{N}_{12}(t,t_o) \\ \hline \tilde{N}^T_{12}(t,t_o) & \tilde{N}_{22}(t,t_o)\end{array}\right] \tag{B-9}$$

where

$$\tilde{N}_{11}(t,t_o) = \int_{t_o}^{t}[\tfrac{1}{4}A(t,t')Q_1(t')A\;(t,t') + K(t,t')Q_2(t')K^T(t,t')]dt' \tag{B-10}$$

$$\tilde{N}_{12}(t,t_o) = \int_{t_o}^{t} K(t,t')Q_2(t')dt' \tag{B-11}$$

$$\tilde{N}_{22}(t,t_o) = \int_{t_o}^{t} Q_2(t')dt' \tag{B-12}$$

Equation (B-12) need not be simplified. Substituting Eqs. (B-3) and (B-4) into Eqs. (B-10) and (B-11) and changing orders of integration give

$$\tilde{N}_{12}(t,t_o) = -\frac{1}{2}\Lambda(t,t_o)\int_{t_o}^{t}\Lambda^T(t',t_o)\tilde{N}_{22}(t',t_o)dt' \quad \text{(B-13)}$$

$$\tilde{N}_{11}(t,t_o) = \frac{1}{4}\Lambda(t,t_o)\int_{t_o}^{t}\Lambda^T(t',t_o)[Q_1(t')-2\tilde{N}_{12}(t',t_o)$$

$$-2\tilde{N}_{12}^T(t',t_o)]\Lambda(t',t_o)dt' \ \Lambda(t,t_o) \quad \text{(B-14)}$$

When, in addition to the approximations of Eqs. (B-6) through (B-8) it is further assumed that $Q_1(t)$ and $Q_2(t)$ are time independent, the above expressions simplify to

$$\tilde{N}_{11}(t,t_o) = \frac{1}{48}(t-t_o)\{[4I_{3x3}+\Lambda(t,t_o)-\Lambda^T(t,t_o)]Q_1$$

$$+2\Lambda(t,t_o)Q_1\Lambda^T(t,t_o)\}$$

$$+\frac{1}{960}(t-t_o)^3\{[30I_{3x3}+11\Lambda(t,t_o)-7\Lambda^T(t,t_o)]Q_2$$

$$+6\Lambda(t,t_o)Q_2\Lambda^T(t,t_o)\} + \text{transpose} \quad \text{(B-15)}$$

$$\tilde{N}_{12}(t,t_o) = -\frac{1}{48}(t-t_o)^2[10I_{3x3}+3\Lambda(t,t_o)-\Lambda^T(t,t_o)]Q_2 \quad \text{(B-16)}$$

$$\tilde{N}_{22}(t,t_o) = (t-t_o)\ Q_2 \quad \text{(B-17)}$$

Normalization of the Quaternion

In general, the unit normalization of the quaternion will not be maintained due to the accumulation of round-off error and approximations within the filter.

The last can be eliminated most easily in Sec. 11 by replacing Eq. (161) by

$$\delta\hat{\bar{q}}(+) = \begin{bmatrix} \delta\hat{\vec{q}}(+) \\ \delta\hat{q}_4(+) \end{bmatrix} \quad \text{(B-18)}$$

with

$$\delta\hat{q}_4(+) = 1 - \frac{1}{2}|\delta\hat{\vec{q}}(+)|^2 \quad \text{(B-19)}$$

which has been used by other authors. The normalization is now preserved to order $|\delta\hat{\vec{q}}(+)|^4$.

The simplest method to eliminate round-off error (or otherwise restore the norm) is to multiply the quaternion periodically by the factor

$$(3 + \hat{\bar{q}}^T\hat{\bar{q}})/(1 + 3\hat{\bar{q}}^T\hat{\bar{q}}) \quad \text{(B-20)}$$

If the error in the norm before multiplication is ϵ, it will be reduced after multiplication to order $\epsilon^3/32$.

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