Lec 13: Basics of Quantum error correction

Classical code recap

Quantum error correction basics

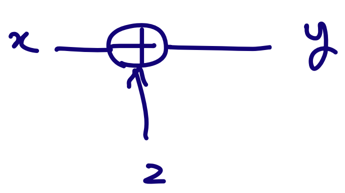
Bit flip & phase flip example

Shor code

Thm

$X \quad Z \longrightarrow$ all errors

Classical codes



$x, y, z \in \{0,1\}^n$

$z \in \mathcal{E} \subseteq \{0,1\}^n$

error set

$x \in \mathcal{C}$ codebook

For fixed $\mathcal{E}$ want to choose $\mathcal{C}$ such that we can find x from y for all $z \in \mathcal{E}$

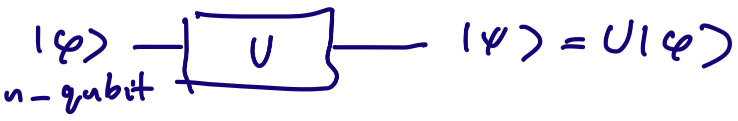
$\Longleftrightarrow \quad x + \mathcal{E} \cap x' + \mathcal{E} = \emptyset \quad \forall\, x, x' \in \mathcal{C} \quad x \neq x'$

$\mathrm{Dec} : \{0,1\}^n \longrightarrow \mathcal{C}$

When $\mathcal{E} = \{ z : |z| \leqslant w \}$, min dist $\mathcal{C} \geqslant 2w+1$

– Linear codes – Examples

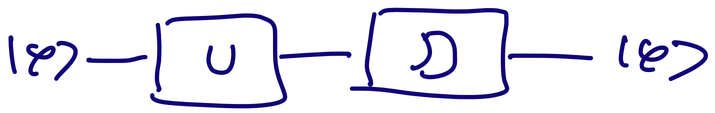
Quantum code



$U \in \mathcal{E} \subseteq$ all unitaries on n-qubits

$|\varphi\rangle \in \mathcal{C}$ subspace of $\mathbb{C}^{2^n}$

$\forall\, U \in \mathcal{E}$, we want to recover $|\varphi\rangle$ from $U|\varphi\rangle$



$\forall \ U \in \mathcal{E}$

$U\mathcal{C}U^{\dagger} \quad V\mathcal{C}V^{\dagger}$

orthogonal

Example $X^a = \begin{cases} 1 & a=0 \\ X & a=1 \end{cases}$

$\{ X^{a_1} \otimes \cdots \otimes X^{a_n} : |a| \leqslant w \}$ applying at most w bit flip

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$C \subseteq \{0,1\}^n$ with min distance $2w+1$ Dec

$\mathcal{C} = \operatorname{span}_{x \in C} |x\rangle$ subspace $(\mathbb{C}^2)^{\otimes n}$

Π_z proj onto $\operatorname{span}_{x \in C} |x+z\rangle$ $\quad \Pi_z \cdot \Pi_{z'} = 0$ if z, z'

$\{\Pi_z\}_{|z| \leqslant w}$ povm onto $\mathbb{C}^2$

Decoder measure $\{\Pi_z\}$ apply $X^{z_1} \otimes \cdots \otimes X^{z_n}$

Rep code $C = \{0^n, 1^n\}$ $\quad |\varphi\rangle = \frac{1}{\sqrt{2}}(|0^n\rangle + |1^n\rangle)$

Apply Z on one qubit $\quad \frac{1}{\sqrt{2}}(|0^n\rangle - |1^n\rangle)$

$\mathcal{E} = \{ Z^{a_1} \otimes \cdots \otimes Z^{a_n} : |a| \leqslant w \}$

$C \subseteq \{0,1\}^n$ min dist $2w+1$

$\mathcal{C} = \operatorname{span}_{x \in C} H^{\otimes n} |x\rangle$

Shor code 1 qubit to 9 qubits

dim 2 subspace of $\mathbb{C}^{2^9}$

$$|0\rangle \longmapsto \frac{1}{2\sqrt{2}}\left(|000\rangle + |111\rangle\right)^{\otimes 3}$$

$$|1\rangle \longmapsto \frac{1}{2\sqrt{2}}\left(|000\rangle - |111\rangle\right)^{\otimes 3}$$

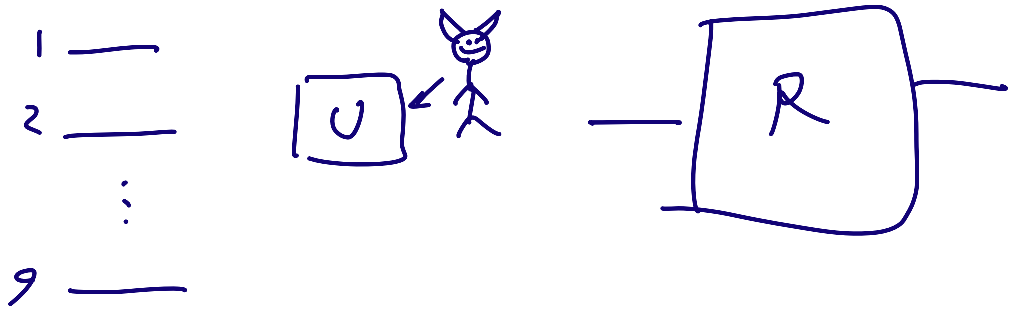
$|0\rangle \longmapsto |+\rangle^{\otimes 3}$ $\xrightarrow{\text{use repetition code}}$

$|1\rangle \longmapsto |-\rangle^{\otimes 3}$

$$\mathcal{E} = \{ X^{a_1} \otimes \cdots \otimes X^{a_9} \text{ or } Z^{a_1} \otimes \cdots \otimes Z^{a_9} \quad |a| \leq 1 \}$$

$$(XZ)^{a_1} \otimes \cdots \otimes (XZ)^{a_9}$$

Thm: Shor code can correct any unitary acting on single qubit



Step 1 $\mathcal{C}$ is a code $\mathcal{E} = \{ U_1, \ldots, U_k \}$

$U_i \,\&\, U_i^\dagger$ are orthogonal

We can correct $\mathcal{E} = \{ V = \sum_i \alpha_i U_i \}$

Define Π_i proj onto $U_i \,\&\, U_i^\dagger$

Apply POVM $\{\Pi_i\}_i$

Apply $U_i^\dagger$ when output of meas. is i

$|\varphi\rangle \in \mathcal{C}$ $\quad V = \sum \alpha_i U_i \longrightarrow \underbrace{\sum \alpha_i U_i |\varphi\rangle}_{\text{orthogonal}}$

$\xrightarrow[\text{meas.}]{\text{after}} \quad \propto \Pi_i \sum_j \alpha_j U_i |\varphi\rangle$

when the output is i

$$\propto U_i |\varphi\rangle$$

Step 2:

Pauli op $\quad 1 \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & \\ 0 & -1 \end{pmatrix}, Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\mathbb{C}^{2\times 2}$ lin. ind.

$U \quad 2\times 2 \quad = \alpha_0 1 + \alpha_1 X + \alpha_2 Y + \alpha_3 Z$

Thm: if code $\mathcal{C}$ can correct $\{ P_1 \otimes \cdots \otimes P_n :$

$P_i \in \{1, X, Y, Z\}$ & $\# i : P_i \neq 1 \leq w \}$

$\Longrightarrow$ it can correct the noise on any w qubits