Axioms and Prices
A trade off: mathematical generality versus computational applicability

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1 Introduction

Modern mathematics using set theory and classical logic is very general and very powerful. But this comes at a price, which is at least twofold. Concrete constructions are often not made as set existence axioms make them superfluous. And potential decideability is often neglected, if it comes with a loss of generality. Often the axiom of choice is just assumed at the beginning and then it is not questioned later. So we suggest to add these points to the modern discussions and also history on the subject. In our considerations we look at reverse mathematics, ultrafinitism, Hilbert's finitism and program and Harvey Friedman's grand conjecture. First we start with more details on these programs and then bring examples for this way of looking at mathematics. In a way it has two faces: a dialectic and an algorithmic one. In our opinion the algorithmic face should nowadays be emphasised more strongly.

2 The neglected success of calculism

[Barendregt(2013)] put it well on p.26: "After the notion of general computability had been captured, it was proved by Church [1936] and Turing [1936] that validity or provability of many mathematical questions was not decidable. Certain important theories are decidable, though. Tarski [1951] showed that the theory of real closed fields (and hence elementary geometry) is decidable. An essential improvement was given by Collins [1975]. In Buchberger [1965] a method to decide membership of finitely generated ideals in certain polynomial rings was developed...See also Börger, Grädel and Gurevich [2001] for other examples of decidable theories. It is the success of calculism, even if partial, that has been neglected in mathematics based on set theory or logic alone."

We suggest to bring relevant results directly after the general theorems. These algorithmic frames should have a clear syntactic origin, be algorithmic and decidable. So a computer program should exist which can run like in ultra-finitism. Often they will be less general than the axiomatic results. Also often they came historically first and were later generalised. Sometimes stronger set existence axioms are needed sometimes not. This should be discussed much more as in reverse mathematics.

The following quote by Barendregt from private communication from 2022 puts it well: "More than in books Calculism appears as software. It is like asking: where can I find literature about motorization in the early twentieth century? It is a 'happening'!

There are major systems like Mathematica, Maple, Cayley, Sage, Lie, that all are in the spirit of Calculemus.

In this respect Intuitionistic Logic is relevant. The statement: "For all x there exists a y such that A(x,y) and this y can be computed from x" is very hard to formulate in say classical set theory. But not in intuitionistic theories."

3 Reverse Mathematics and RCA0 as foundation

[Stillwell] gives a good introduction into this topic and introduces the axiom system RCA0 of recursive computable analysis. This seems to be a borderline case. Whenever more is needed this should be critically discussed and frames should be investigated, which are less general, but within the boundaries of RCA0. This can be seen as a version of Harvey Friedman's grand conjecture or a realisation of Hilbert's program to use only finitistic means. One advantage of RCA0 is that it is provably consistent in full first-order Peano arithmetic.

Examples that go beyond are Nash's existence proof of the Nash equilibrium as it uses Brouwer's fixpoint theorem, which goes beyond RCA0, the general Hahn Banach proof as it uses transfinite induction (it should be noted that Helly's original proof for the special case C(a,b) only used "normal" induction and does not go beyond RCA0) and Gödel's proof of his incompleteness theorem as it uses a non standard interpretation. We suggest to find algorithmic frames for special cases if possible. If this is not possible then the axiomatic result has no "algorithmic image", which is a certain kind of sign of course.

One good example is the existence of a Nash equilibrium for certain games like rock/scissor/paper or no limit heads up poker. Nash's general proof checks the conditions of Brouwer's fix point theorem and concludes that at least one such equilibrium must exist. But this has two problems:

1) It uses Brouwer's fix point theorem, which is outside of RCA0 and not constructive. In fact historically it was very difficult to fix this constructive gap.

2) If a construction is used then the problem is that it is a rational converging

sequence. But of course all the limits of the Nash weights can not be put into a running computer program. Only approximations. This leads to an error epsilon > 0. Halfjokingly this might be called "the price of using the classical logic". For poker it is for the Cepheus bot 0.000986 big blinds per game on expectation.

So the Nash proof does not prove that the Nash strategy can be played in poker. It only proofs that an error bound epsilon > 0 exist within which the computer program can play a strategy near the Nash strategy. And the really interesting games can not be solved exactly. Most interesting is then, how low the error bound epsilon can be made with the modern computer precision.

For easier games like rock/scissors/paper (or Hofstadter's Undercut game) on the other hand all 3 Nash weighs are rational and equal to 1/3 and a computer program can play Nash exactly. Here the pure rational Simplex algorithm can fill the constructive gap in the Nash proof.

So "Nash exists" has 2 different meanings.

4 Constructive mathematics as lingua franca of the mathematical universe

Advantages of algorithmic frames are similar to those of constructive mathematics and of course all axiomatic results should also be viewed at from this point of view. For example Bauer,p.492:

"We initially set out to understand the difference between the classical and the constructive world of mathematics, only to have discovered that there are not two but many worlds, some of which simply cannot be discounted as logicians' contrivances. Excluded middle as the dividing line between the worlds is immaterial in comparison with having Cantor's paradise shattered into an unbearable plurality of mathematical universes.

5. Acceptance

As exciting as a multiverse of mathematics may be, the working mathematician has no time to spend their career wandering from one world to another. Nevertheless, they surely are curious about the newly discovered richness of mathematics, they welcome ideas coming from unfamiliar worlds, and they strive to make their own work widely applicable. They will find it easier to accomplish these goals if they speak the lingua franca of the mathematical multiverse—constructive mathematics. Some aspects of constructive mathematics are just logical hygiene: avoid indirect proofs in favor of explicit constructions, detect and eliminate needless uses of the axiom of choice, know the difference between a proof of negation and a proof by contradiction. Of course, constructivism goes deeper than that. The stringent working conditions of constructive worlds require an economy of thought which is disheartening at first but eventually pays off with vistas of new mathematical landscapes that are proscribed by orthodox mathematics. We may at best offer a few glimpses."

We suggest at least to add algorithmic frames to the axiomatic discussions.

The other lingua franca of the mathematical universe by the way is naive set theory, not axiomatic set theory, for example Tignol, page 8:

"Although considering sets of numbers and properties of such sets was clearly alien lo the patterns of thinking until the nineteenth century, it would be futile to ignore the fact that (naive) set theory has now pervaded all levels of mathematical education. Therefore, free use will be made of the definitions of some basic algebraic structures such as field and group, at the expense of lessening some of the most original discoveries of Gauss, Abel and Galois."

And many modern books just gives Galois' impossibility proof that not all roots of rational polynomials can be written as radicals. They do not give the construction of the radicals, if they exist. But this construction is also an important part of mathematics.

5 Algorithmic frames as one aim

One aim of algorithmic mathematics should be algorithmic frames. The problem has a clear syntactic origin within the mathematical universe and algorithmic calculation and in the end the question can always be decided. Often this will be less general than the axiomatic result as the axiom of choice opens more options. This is the price of algorithmic mathematics. Examples are

1) Geometric ruler and compass constructions. This algorithmic construction frame is very large. The parallel axiom is independent of the other 4 Euclidean axioms. Here this means that other frames should be taken for different geometries and different algorithmic frames. The construction proofs can usually be directly taken as they are like the doubling of the cube.

2) Divine Proportions

Divine Proportions: Rational Trigonometry to Universal Geometry is a 2005 book by mathematician Norman J. Wildberger on a proposed alternative approach to Euclidean geometry and trigonometry, called rational trigonometry. The book suggests to replace the usual basic quantities of trigonometry, Euclidean distance and angle measure, by squared distance and the square of the sine of the angle, respectively. This is logically equivalent to the standard development. But constructivly there are differences. For example for finite fields Wildberger's constructions can be done as well. It is typical that not all what is equivalent in the axiomatic world translates to the same algorithmic frames.

3) Regular continued fractions for quadratic irrationals

In this frame it can also algorithmically decided, whether such a number is rational or not. With the general definition that the equivalance class of Cauchy sequences it looks more like metaphysics. It would be good, if such frames are always given directly after the general axiomatic definitions and theorems. In this case the theorems on the rationality of numbers. Such a frame gives the question real meaning and decideability and in this case also a quickly converging rational sequence, which all best rational approximations, if the given quadratic number is irrational.

The frame is Quadratic equation with rational coefficients, the algorithm produces a sequence of rational approximants and the algorithm ends, if the number is rational and gets periodic if not. It can also be decided with a priori bounds.

4) Nash equilibrium

For certain games all rational Nash weights can be calculated with the Simplex algorithm. This proof in an algorithmic frame is constructive and completely different from Nash's proof, where the conditions of Brouwer's fix point theorem must be checked. But this goes beyond RCA0, it is not constructive and the Nash weighs might not be rational. So the generality here definitely has a clear price.

And it is not easy to pay the price as the history (quoted from the English Wikipedia on Brouwer's fix-point theorem): "Brouwer's celebrity is not exclusively due to his topological work. The proofs of his great topological theorems are not constructive,[46] and Brouwer's dissatisfaction with this is partly what led him to articulate the idea of constructivity. He became the originator and zealous defender of a way of formalising mathematics that is known as intuitionism, which at the time made a stand against set theory.[47] Brouwer disavowed his original proof of the fixed-point theorem. The first algorithm to approximate a fixed point was proposed by Herbert Scarf.[48] A subtle aspect of Scarf's algorithm is that it finds a point that is almost fixed by a function f, but in general cannot find a point that is close to an actual fixed point."

And if Brouwer's fix point theorem is replaced by his approximate fix point theorem than an error bound epsilon > 0 comes into play. Then only an approximate Nash strategy can be played, if the computer precision allows it. Here really deep and fascinating research questions come into play:

1) Are there purely rational Nash weighs for a game like poker, where the Simplex can not directly be used? Or can it be proved that the Nash weighs are not rational?

2) If an approximation must be used, how easy is it to get a low epsilon?

3) How good is an approximate Nash strategy compared to the best humans and other computer programs in the game in question like e.g. poker?

Using Nash's existence proof these deep and really fascinating questions can not even be seen. A clear price of that non constructive proof strategy.

In general it seems that these problems arise from uncontrolled infinities. In the Nash cash they come from a limit, but only approximations can be used and so an error bound epsilon > 0 must come into play. The axiom of choice makes it possible to deal with uncontrolled infinities as the existence of the limit is given by the axiom. But when a computer program shall exist, then the infinity must be controlled and an error bound epsilon > 0 introduced. This is the price that must be paid.

So one could add to the classical proofs:

1) Is the law of excluded middle used (Intuitionism). Is this real or does it make the proof only shorter? For which subspaces can it be done without?

2) Is the proof within RCA0 (Reverse Mathematics). If not which other axioms are needed? For which subspaces is it within RCA0?

3) Is the proof constructive or does it use the axiom of choice)? For example "every vector space has a basis" proved with Zorn's Lemma uses the axiom of choice. The proof does absolutely nothing to help constructing a basis.

4) Construct as large algorithmic frames as possible (Calculism)

Especially Calculism seems to be underrated by modern mathematicians and seen as Informatics.

All problems with syntatic Origin within Mathematics -> Algorithm -> Decision Syntax

are then solved by a computer program and part of the inner truth of mathematics. When the proofs can not be connected to algorithmic frames they are in the air and Calculism brings them back to earth.

Adding this to the modern books would make them a bit bigger but would help understanding the nature of the proofs much better.

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