Sharif University of Technology

Aerospace Engineering Department

Analysis of an Incompressible Flow Inside a Back-Ward Facing Step Using Artificial Compressibility Method

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1. Description of the Problem

The flow past a backward-facing step is a good test case to examine the accuracy and performance of a numerical method implemented. Figure 1 shows the geometry of this test case and the defined important flow parameters. Although the geometry is simple, it contains complex flow features associated with separation and reattachment. The main difficulty in applying numerical schemes to this test case is the high dependence of the recirculation regions with respect to grid size and grid resolution.

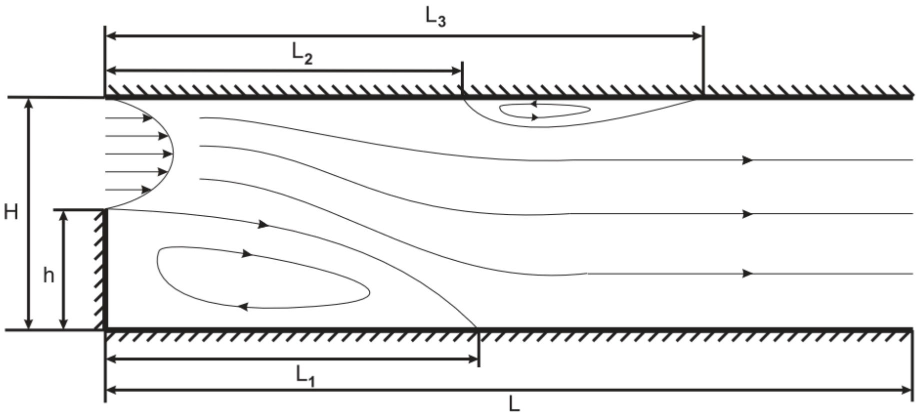


Figure 1. Geometry of backward facing step including important flow parameters

2. Governing Equations

Two-dimensional, Cartesian coordinate, Dimensional, nonconservative form of the Incompressible Navier-Stokes equations:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial x}=\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+\frac{1}{\rho}\frac{\partial p}{\partial y}=\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

2.1. Non-dimensionalization

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0$$

$$\frac{\partial u^*}{\partial t^*}+u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}+\frac{\partial p^*}{\partial x^*}=\frac{1}{Re}\left(\frac{\partial^2 u^*}{\partial x^{*2}}+\frac{\partial^2 u^*}{\partial y^{*2}}\right)$$

$$\frac{\partial v^*}{\partial t^*}+u^*\frac{\partial v^*}{\partial x^*}+v^*\frac{\partial v^*}{\partial y^*}+\frac{\partial p^*}{\partial y^*}=\frac{1}{Re}\left(\frac{\partial^2 v^*}{\partial x^{*2}}+\frac{\partial^2 v^*}{\partial y^{*2}}\right)$$

The variables in the equations above are nondimensionalized as follows:

$$t^*=\frac{tu_\infty}{L}\qquad x^*=\frac{x}{L}\qquad y^*=\frac{y}{L}$$

$$u^*=\frac{u}{u_\infty}\qquad v^*=\frac{v}{u_\infty}\qquad p^*=\frac{p}{\rho_\infty u_\infty^2}$$

From now on, we shall drop the * from non-dimensional parameters for the sake of simplicity.

2.2. Artificial Compressibility

The continuity equation is modified by inclusion of a time-dependent term and is given by:

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Where β can be interpreted as the "artificial compressibility" of the fluid. Following the equation of state, the compressibility can be related to a pseudo-speed of sound and to an artificial density by the following relations:

$$p=\beta\tilde{\rho}=\tilde{a}^2\tilde{\rho}\qquad \begin{cases}\beta=\text{artificial compressibility,}\\ \tilde{a}=\text{artificial sound speed}=\beta^{\frac{1}{2}}\end{cases}$$

Where $0.1<\beta<10$.

Thus, the steady incompressible Navier-Stokes equations (two-dimensional Cartesian coordinates) are expressed in a pseudo-transient form as:

$$\frac{1}{\beta}\frac{\partial p}{\partial \tau}+\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{\partial p}{\partial y}+\frac{1}{\mathrm{Re}}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

Above equations are written in a flux vector form as:

$$\frac{\partial Q}{\partial t}+\frac{\partial F}{\partial x}+\frac{\partial G}{\partial y}=\frac{1}{\mathrm{Re}}N\nabla^2 Q$$

$$Q=\begin{pmatrix}p\\u\\v\end{pmatrix},\ F=\begin{pmatrix}\beta u\\u^2+p\\uv\end{pmatrix},\ G=\begin{pmatrix}\beta v\\uv\\v^2+p\end{pmatrix},\ N=\begin{pmatrix}0&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

3. Implicit Discretization

$$\Delta Q\quad+\Delta t\left[\frac{\partial}{\partial x}(A\Delta Q)+\frac{\partial}{\partial y}(B\Delta Q)-\frac{N}{\mathrm{Re}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Delta Q\right]$$
$$=\Delta t\left[-\frac{\partial E^n}{\partial x}-\frac{\partial F^n}{\partial y}+\frac{N}{\mathrm{Re}}\left(\frac{\partial^2 Q}{\partial x^2}+\frac{\partial^2 Q}{\partial y^2}\right)\right]$$

Or

$$\left\{I+\Delta t\left[\frac{\partial A}{\partial x}+\frac{\partial B}{\partial y}-\frac{N}{\mathrm{Re}}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\right]\right\}\Delta Q=RHS$$

For the two-dimensional problems under consideration, the implicit formulation will result in a block penta-diagonal system of equations. As we are aware, the solution of such a system is expensive. To overcome this problem, approximate factorization will be implemented. Second, to overcome any possible instability in the solution, artificial viscosity (damping terms) will be added to the equation. Thus, with the stated considerations, the FDE is formulated as:

$$\left[I+\Delta t\left(\frac{\partial A}{\partial x}-\frac{N}{\mathrm{Re}}\frac{\partial^2}{\partial x^2}\right)+\varepsilon_i(\Delta x)^2\frac{\partial^2}{\partial x^2}\right]\Delta Q^* \\ = RHS-\varepsilon_e\left[(\Delta x)^4\frac{\partial^4}{\partial x^4}+(\Delta y)^4\frac{\partial^4}{\partial y^4}\right]Q$$

A second-order central difference approximation is applied spatially to the above to provide:

$$\begin{aligned}
&\Delta Q^*_{i,j}+\frac{\Delta t}{2\Delta x}(A_{i+1,j}\Delta Q^*_{i+1,j}-A_{i-1,j}\Delta Q^*_{i-1,j})\\
&-\frac{N}{Re}\frac{\Delta t}{(\Delta x)^2}(\Delta Q^*_{i+1,j}-2\Delta Q^*_{i,j}+\Delta Q^*_{i-1,j})+\epsilon_i(\Delta Q^*_{i+1,j}-2\Delta Q^*_{i,j}+\Delta Q^*_{i-1,j})\\
&=\Delta t\Big\{-\frac{E_{i+1,j}-E_{i-1,j}}{2\Delta x}-\frac{F_{i,j+1}-F_{i,j-1}}{2\Delta y}+\frac{N}{Re}\frac{Q_{i+1,j}-2Q_{i,j}+Q_{i-1,j}}{(\Delta x)^2}\\
&+\frac{N}{Re}\frac{Q_{i,j+1}-2Q_{i,j}+Q_{i,j-1}}{(\Delta y)^2}\Big\}-\epsilon_e\Big\{(Q_{i-2,j}-4Q_{i-1,j}+6Q_{i,j}-4Q_{i+1,j}+Q_{i+2,j})\\
&+(Q_{i,j-2}-4Q_{i,j-1}+6Q_{i,j}-4Q_{i,j+1}+Q_{i,j+2})\Big\}
\end{aligned}$$

Equation above is now rearranged to provide a block tridiagonal system as follows:

$$\left[-\left(\frac{\Delta t}{2\Delta x}\right)A_{i-1,j}-\left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right)+\varepsilon_i\right]\Delta Q^*_{i-1,j}+\left[I+\left(\frac{2N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right)-2\varepsilon_i\right]\Delta Q^*_{i,j} \\ +\left[\left(\frac{\Delta t}{2\Delta x}\right)A_{i+1,j}-\left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right)+\varepsilon_i\right]\Delta Q^*_{i+1,j}=RHS_{i,j}$$

and

$$\left[-\left(\frac{\Delta t}{2\Delta y}\right)B_{i,j-1}-\left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right)+\varepsilon_i\right]\Delta Q_{i,j-1}+\left[I+\left(\frac{2N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right)-2\varepsilon_i\right]\Delta Q_{i,j} \\ +\left[\left(\frac{\Delta t}{2\Delta y}\right)B_{i,j+1}-\left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right)+\varepsilon_i\right]\Delta Q_{i,j+1}=\Delta Q^*_{i,j}$$

These equations are written in a compact form as:

$$\overline{\mathbf{A}}_{i,j}\Delta Q^*_{i-1,j}+\overline{\mathbf{B}}_{i,j}\Delta Q^*_{i,j}+\overline{\mathbf{C}}_{2,j}\Delta Q^*_{i+1,j}=RHS_{i,j}$$

$$\overline{\overline{\mathbf{A}}}_{i,j}\Delta Q_{i-1,j}+\overline{\overline{\mathbf{B}}}_{i,j}\Delta Q_{i,j}+\overline{\overline{\mathbf{C}}}_{2,j}\Delta Q_{i+1,j}=\Delta Q^*_{i,j}$$

Where

$$\overline{\mathbf{A}}_{i,j} = -\left(\frac{\Delta t}{2\Delta x}\right)A_{i-1,j} - \left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right) + \varepsilon_i$$

$$\overline{\mathbf{B}}_{i,j} = I + \left(\frac{2N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right) - 2\varepsilon_i$$

$$\overline{\mathbf{C}}_{i,j} = \left(\frac{\Delta t}{2\Delta x}\right)A_{i+1,j} - \left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta x)^2}\right) + \varepsilon_i$$

$$\overline{\overline{\mathbf{A}}}_{i,j} = -\left(\frac{\Delta t}{2\Delta y}\right)B_{i-1,j} - \left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right) + \varepsilon_i$$

$$\overline{\overline{\mathbf{B}}}_{i,j} = I + \left(\frac{2N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right) - 2\varepsilon_i$$

$$\overline{\overline{\mathbf{C}}}_{i,j} = \left(\frac{\Delta t}{2\Delta y}\right)B_{i+1,j} - \left(\frac{N}{\mathrm{Re}}\frac{\Delta t}{(\Delta y)^2}\right) + \varepsilon_i$$

4. Boundary Conditions:

For i = 2, the compact form of the equations becomes:

$$\overline{\mathbf{A}}_{2,j}\Delta Q^*_{1,j} + \overline{\mathbf{B}}_{2,j}\Delta Q^*_{2,j} + \overline{\mathbf{C}}_{2,j}\Delta Q^*_{3,j} = RHS$$

The left boundary condition controls the equation for $\Delta Q^*_{1,j}$. Since the velocity components are invariant at the left boundary:

$$\Delta u^*_{1,j} = \Delta v^*_{1,j} = 0$$

The pressure, however, is not specific in various conditions. Therefore, setting $\left.\frac{\partial p^*}{\partial n}\right)_{1,j} = 0$ using 2nd-order forward FDE yields:

$$\frac{-3p^*_{1,j} + 4p^*_{2,j} - p^*_{3,j}}{2\Delta x} = 0 \rightarrow p^*_{1,j} = \frac{1}{3}\left(4p^*_{2,j} - p^*_{3,j}\right)$$

In Delta form:

$$\Delta p^*_{1,j} = \frac{4}{3}\Delta p^*_{2,j} - \frac{1}{3}\Delta p^*_{3,j}$$

$$\Delta Q_{1,j}^* = \begin{pmatrix} \Delta p_{1,j}^* \\ \Delta u_{1,j}^* \\ \Delta v_{1,j}^* \end{pmatrix} = \begin{pmatrix} \frac{4}{3}\Delta p_{2,j}^* - \frac{1}{3}\Delta p_{3,j}^* \\ 0 \\ 0 \end{pmatrix}$$

Thus, the first term in the compact equations becomes:

$$\overline{\mathbf{A}}_{2,j}\Delta Q_{1,j}^* = \begin{pmatrix} \bar{A}_{2,j}^{11} & \bar{A}_{2,j}^{12} & \bar{A}_{2,j}^{13} \\ \bar{A}_{2,j}^{21} & \bar{A}_{2,j}^{22} & \bar{A}_{2,j}^{23} \\ \bar{A}_{2,j}^{31} & \bar{A}_{2,j}^{32} & \bar{A}_{2,j}^{33} \end{pmatrix} \begin{pmatrix} \frac{4}{3}\Delta p_{2,j}^* - \frac{1}{3}\Delta p_{3,j}^* \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}\bar{A}_{2,j}^{11} & 0 & 0 \\ \frac{4}{3}\bar{A}_{2,j}^{21} & 0 & 0 \\ \frac{4}{3}\bar{A}_{2,j}^{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta p_{2,j}^* \\ \Delta u_{2,j}^* \\ \Delta v_{2,j}^* \end{pmatrix} + \begin{pmatrix} -\frac{1}{3}\bar{A}_{2,j}^{11} & 0 & 0 \\ -\frac{1}{3}\bar{A}_{2,j}^{21} & 0 & 0 \\ -\frac{1}{3}\bar{A}_{2,j}^{31} & 0 & 0 \end{pmatrix} \begin{pmatrix} \Delta p_{3,j}^* \\ \Delta u_{3,j}^* \\ \Delta v_{3,j}^* \end{pmatrix}$$

$$\overline{\mathbf{A}}_{2,j}\Delta Q_{1,j}^* = \begin{pmatrix} \frac{4}{3}\bar{A}_{2,j}^{11} & 0 & 0 \\ \frac{4}{3}\bar{A}_{2,j}^{21} & 0 & 0 \\ \frac{4}{3}\bar{A}_{2,j}^{31} & 0 & 0 \end{pmatrix} \Delta Q_{2,j}^* + \begin{pmatrix} -\frac{1}{3}\bar{A}_{2,j}^{11} & 0 & 0 \\ -\frac{1}{3}\bar{A}_{2,j}^{21} & 0 & 0 \\ -\frac{1}{3}\bar{A}_{2,j}^{31} & 0 & 0 \end{pmatrix} \Delta Q_{3,j}^*$$

Substituting into the original compact equation:

$$\overline{\mathbf{B}}_{2,j}^*\Delta Q_{2,j}^* + \overline{\mathbf{C}}_{2,j}^*\Delta Q_{3,j}^* = RHS$$

Where $\overline{\mathbf{B}}_{2,j}^*$ and $\overline{\mathbf{C}}_{2,j}^*$ are the modified block matrixes:

$$\overline{\mathbf{B}}_{2,j}^* = \begin{pmatrix} \frac{4}{3}\bar{A}_{2,j}^{11} + \bar{B}_{2,j}^{11} & \bar{B}_{2,j}^{12} & \bar{B}_{2,j}^{13} \\ \frac{4}{3}\bar{A}_{2,j}^{21} + \bar{B}_{2,j}^{21} & \bar{B}_{2,j}^{22} & \bar{B}_{2,j}^{23} \\ \frac{4}{3}\bar{A}_{2,j}^{31} + \bar{B}_{2,j}^{31} & \bar{B}_{2,j}^{32} & \bar{B}_{2,j}^{33} \end{pmatrix} \quad \overline{\mathbf{C}}_{2,j}^* = \begin{pmatrix} -\frac{1}{3}\bar{A}_{2,j}^{11} + \bar{C}_{2,j}^{11} & \bar{C}_{2,j}^{12} & \bar{C}_{2,j}^{13} \\ -\frac{1}{3}\bar{A}_{2,j}^{21} + \bar{C}_{2,j}^{21} & \bar{C}_{2,j}^{22} & \bar{C}_{2,j}^{23} \\ -\frac{1}{3}\bar{A}_{2,j}^{31} + \bar{C}_{2,j}^{31} & \bar{C}_{2,j}^{32} & \bar{C}_{2,j}^{33} \end{pmatrix}$$

A similar derivation for the lower wall yields:

$$\overline{\mathbf{B}}^*_{i,2}=\begin{pmatrix}\frac{4}{3}\bar{A}^{11}_{i,2}+\bar{B}^{11}_{i,2} & \bar{B}^{12}_{i,2} & \bar{B}^{13}_{i,2}\\ \frac{4}{3}\bar{A}^{21}_{i,2}+\bar{B}^{21}_{i,2} & \bar{B}^{22}_{i,2} & \bar{B}^{23}_{i,2}\\ \frac{4}{3}\bar{A}^{31}_{i,2}+\bar{B}^{31}_{i,2} & \bar{B}^{32}_{i,2} & \bar{B}^{33}_{i,2}\end{pmatrix}\qquad \overline{\mathbf{C}}^*_{i,2}=\begin{pmatrix}-\frac{1}{3}\bar{A}^{11}_{i,2}+\bar{C}^{11}_{i,2} & \bar{C}^{12}_{i,2} & \bar{C}^{13}_{i,2}\\ -\frac{1}{3}\bar{A}^{21}_{i,2}+\bar{C}^{21}_{i,2} & \bar{C}^{22}_{i,2} & \bar{C}^{23}_{i,2}\\ -\frac{1}{3}\bar{A}^{31}_{i,2}+\bar{C}^{31}_{i,2} & \bar{C}^{32}_{i,2} & \bar{C}^{33}_{i,2}\end{pmatrix}$$

And the upper wall:

$$\overline{\mathbf{A}}^*_{i,\mathrm{J}_{\max}-1}=\begin{pmatrix}\frac{4}{3}\bar{C}^{11}_{i,\mathrm{J}_{\max}-1}+\bar{A}^{11}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{12}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{13}_{i,\mathrm{J}_{\max}-1}\\ \frac{4}{3}\bar{C}^{21}_{i,\mathrm{J}_{\max}-1}+\bar{A}^{21}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{22}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{23}_{i,\mathrm{J}_{\max}-1}\\ \frac{4}{3}\bar{C}^{31}_{i,\mathrm{J}_{\max}-1}+\bar{A}^{31}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{32}_{i,\mathrm{J}_{\max}-1} & \bar{A}^{33}_{i,\mathrm{J}_{\max}-1}\end{pmatrix}$$

$$\overline{\mathbf{B}}^*_{i,\mathrm{J}_{\max}-1}=\begin{pmatrix}-\frac{1}{3}\bar{C}^{11}_{i,\mathrm{J}_{\max}-1}+\bar{B}^{11}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{12}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{13}_{i,\mathrm{J}_{\max}-1}\\ -\frac{1}{3}\bar{C}^{21}_{i,\mathrm{J}_{\max}-1}+\bar{B}^{21}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{22}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{23}_{i,\mathrm{J}_{\max}-1}\\ -\frac{1}{3}\bar{C}^{31}_{i,\mathrm{J}_{\max}-1}+\bar{B}^{31}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{32}_{i,\mathrm{J}_{\max}-1} & \bar{B}^{33}_{i,\mathrm{J}_{\max}-1}\end{pmatrix}$$

For outlet boundary conditions (i = imax-1):

$$\overline{\mathbf{A}}^*_{\mathrm{I}_{\max}-1,j}\Delta Q^*_{\mathrm{I}_{\max}-2,j}+\overline{\mathbf{B}}^*_{\mathrm{I}_{\max}-1,j}\Delta Q^*_{\mathrm{I}_{\max}-1,j}+\overline{\mathbf{C}}^*_{\mathrm{I}_{\max}-1,j}\Delta Q^*_{\mathrm{I}_{\max},j}=RHS$$

$$\begin{aligned}\Delta p^*_{\mathrm{I}_{\max},j}&=0\\ \Delta u^*_{\mathrm{I}_{\max},j}&=\frac{4}{3}\Delta u^*_{\mathrm{I}_{\max}-1,j}-\frac{1}{3}\Delta u^*_{\mathrm{I}_{\max}-2,j}\\ \Delta v^*_{\mathrm{I}_{\max},j}&=\frac{4}{3}\Delta v^*_{\mathrm{I}_{\max}-1,j}-\frac{1}{3}\Delta v^*_{\mathrm{I}_{\max}-2,j}\end{aligned}$$

$$\overline{\mathbf{C}}_{\mathrm{I_{max}}-1,j}\Delta Q^*_{\mathrm{I_{max}},j}=\begin{pmatrix}\overline{C}^{11}_{\mathrm{I_{max}}-1,j} & \overline{C}^{12}_{\mathrm{I_{max}}-1,j} & \overline{C}^{13}_{\mathrm{I_{max}}-1,j}\\ \overline{C}^{21}_{\mathrm{I_{max}}-1,j} & \overline{C}^{22}_{\mathrm{I_{max}}-1,j} & \overline{C}^{23}_{\mathrm{I_{max}}-1,j}\\ \overline{C}^{31}_{\mathrm{I_{max}}-1,j} & \overline{C}^{32}_{\mathrm{I_{max}}-1,j} & \overline{C}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}\begin{pmatrix}0\\ \frac{4}{3}\Delta u^*_{\mathrm{I_{max}}-1,j}-\frac{1}{3}\Delta u^*_{\mathrm{I_{max}}-2,j}\\ \frac{4}{3}\Delta v^*_{\mathrm{I_{max}}-1,j}-\frac{1}{3}\Delta v^*_{\mathrm{I_{max}}-2,j}\end{pmatrix}$$

$$=\begin{pmatrix}0 & \frac{4}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}\\ 0 & \frac{4}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}\\ 0 & \frac{4}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}\begin{pmatrix}\Delta p^*_{\mathrm{I_{max}}-1,j}\\ \Delta u^*_{\mathrm{I_{max}}-1,j}\\ \Delta v^*_{\mathrm{I_{max}}-1,j}\end{pmatrix}+\begin{pmatrix}0 & -\frac{1}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}\\ 0 & -\frac{1}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}\\ 0 & -\frac{1}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}\begin{pmatrix}\Delta p^*_{\mathrm{I_{max}}-2,j}\\ \Delta u^*_{\mathrm{I_{max}}-2,j}\\ \Delta v^*_{\mathrm{I_{max}}-2,j}\end{pmatrix}$$

$$\overline{\mathbf{C}}_{\mathrm{I_{max}}-1,j}\Delta Q^*_{\mathrm{I_{max}},j}=\begin{pmatrix}0 & \frac{4}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}\\ 0 & \frac{4}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}\\ 0 & \frac{4}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}\Delta Q^*_{\mathrm{I_{max}}-1,j}+\begin{pmatrix}0 & -\frac{1}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}\\ 0 & -\frac{1}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}\\ 0 & -\frac{1}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}\Delta Q^*_{\mathrm{I_{max}}-2,j}$$

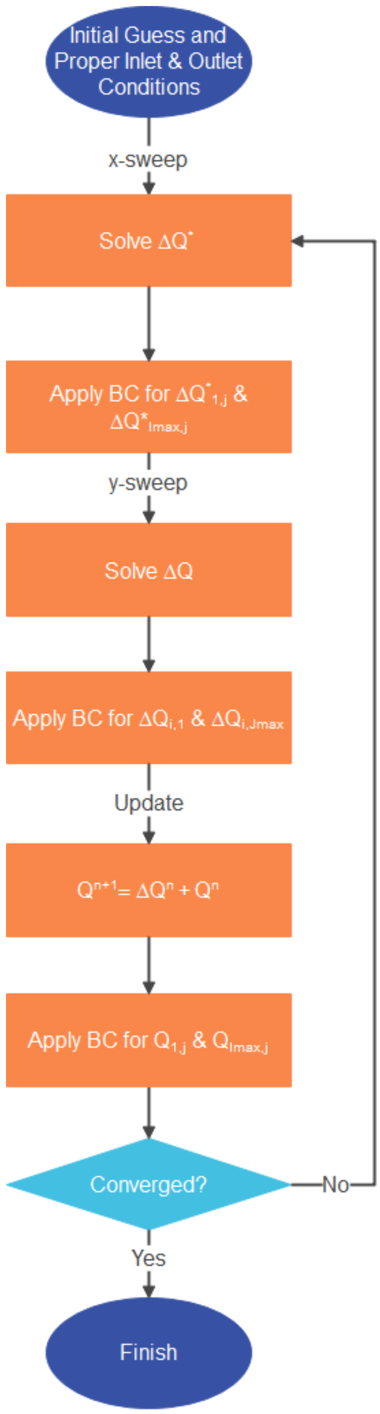
Therefore, the compact equation at the right boundary (outlet) becomes:

$$\overline{\mathbf{A}}^*_{2,j}\Delta Q^*_{\mathrm{I_{max}}-2,j}+\overline{\mathbf{B}}^*_{2,j}\Delta Q^*_{\mathrm{I_{max}}-1,j}=RHS$$

$$\overline{\mathbf{B}}^*_{\mathrm{I_{max}}-1,j}=\begin{pmatrix}\overline{B}^{11}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j}+\overline{B}^{12}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}+\overline{B}^{13}_{\mathrm{I_{max}}-1,j}\\ \overline{B}^{21}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j}+\overline{B}^{22}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}+\overline{B}^{23}_{\mathrm{I_{max}}-1,j}\\ \overline{B}^{31}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j}+\overline{B}^{32}_{\mathrm{I_{max}}-1,j} & \frac{4}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}+\overline{B}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}$$

$$\overline{\mathbf{A}}^*_{\mathrm{I_{max}}-1,j}=\begin{pmatrix}\overline{A}^{11}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{12}_{\mathrm{I_{max}}-1,j}+\overline{A}^{12}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{13}_{\mathrm{I_{max}}-1,j}+\overline{A}^{13}_{\mathrm{I_{max}}-1,j}\\ \overline{A}^{21}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{22}_{\mathrm{I_{max}}-1,j}+\overline{A}^{22}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{23}_{\mathrm{I_{max}}-1,j}+\overline{A}^{23}_{\mathrm{I_{max}}-1,j}\\ \overline{A}^{31}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{32}_{\mathrm{I_{max}}-1,j}+\overline{A}^{32}_{\mathrm{I_{max}}-1,j} & -\frac{1}{3}\overline{C}^{33}_{\mathrm{I_{max}}-1,j}+\overline{A}^{33}_{\mathrm{I_{max}}-1,j}\end{pmatrix}$$

5. Solution Algorithm



6. Generated Grids

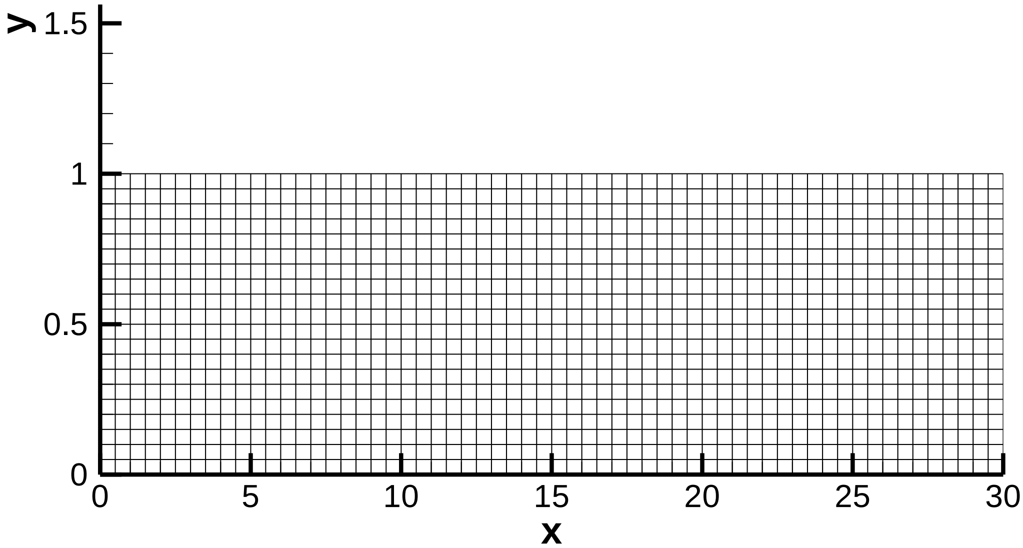


Figure 2 Debugging Grid 51x21

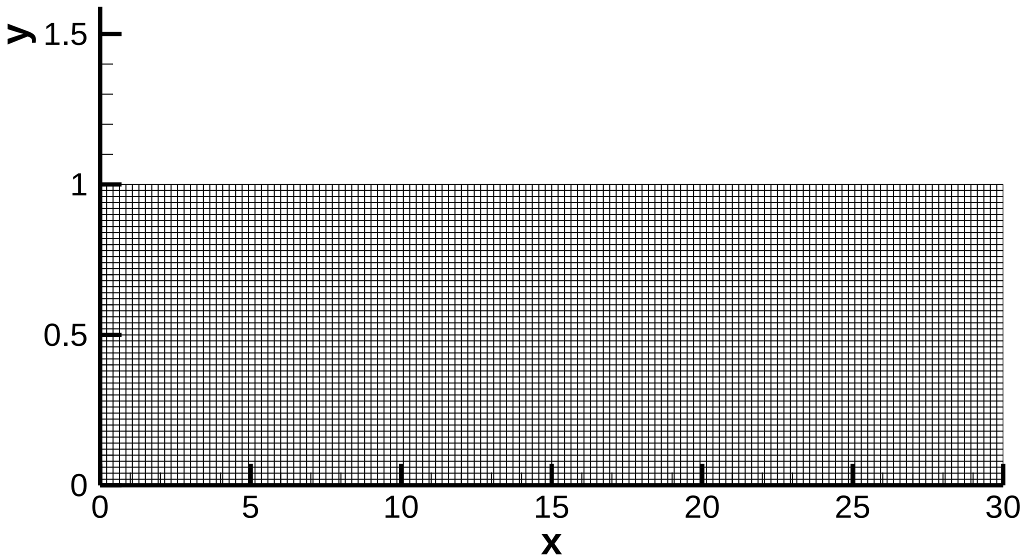


Figure 3 Final Grid 141x51

7. Results

7.1. Convergence History

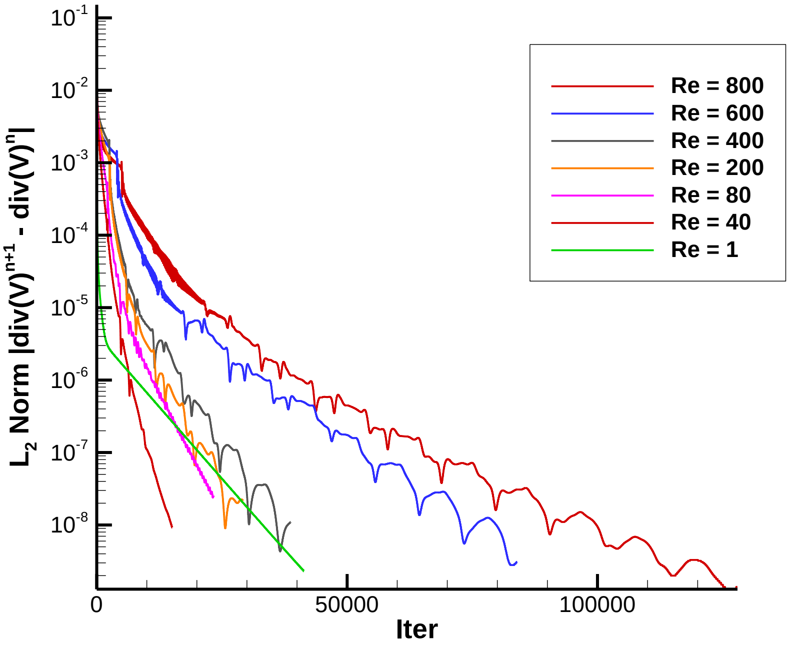


Figure 5 Convergence history of the solution based on the divergence of the velocity vector between two consequent iterations.

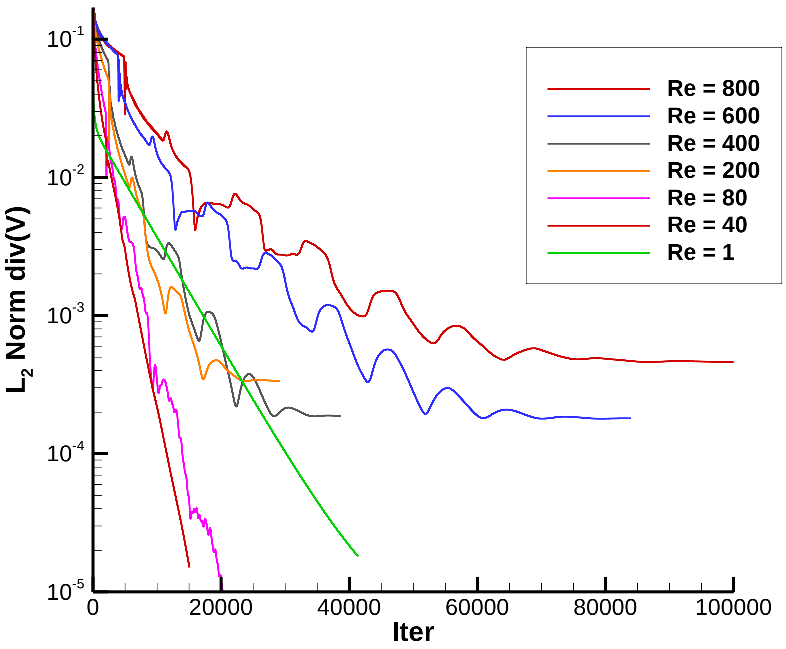


Figure 4 Convergence history of the solution based on the divergence of the velocity vector compared with exact solution (div(V) = 0).

7.2. Wall Shear

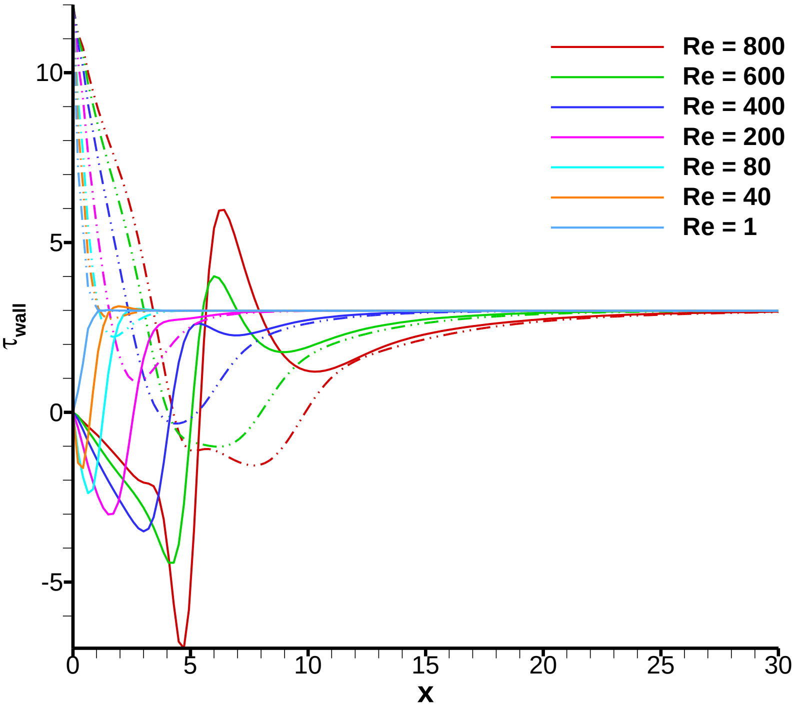


Figure 6 Wall shear for both upper and lower walls (— lower wall, —·· upper wall)

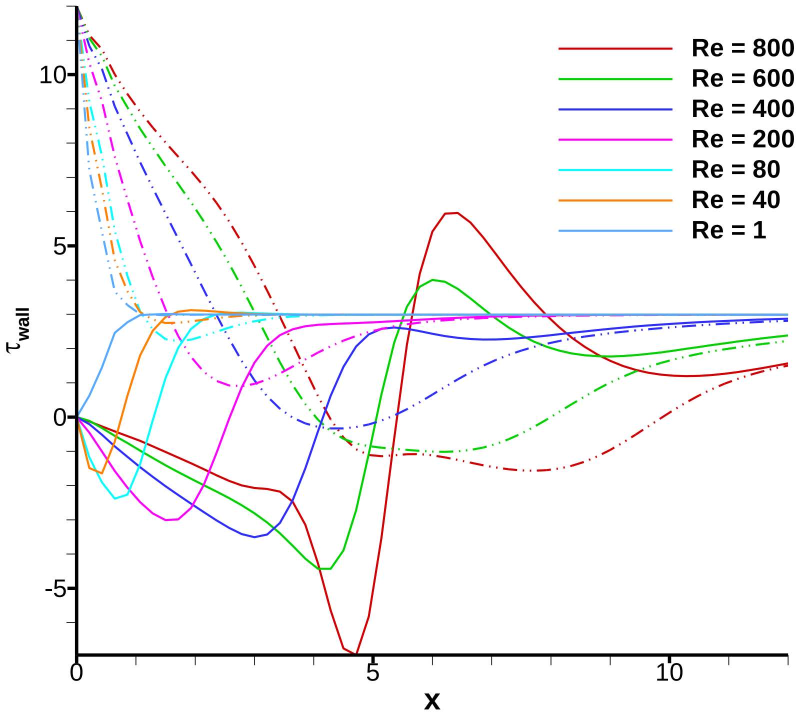
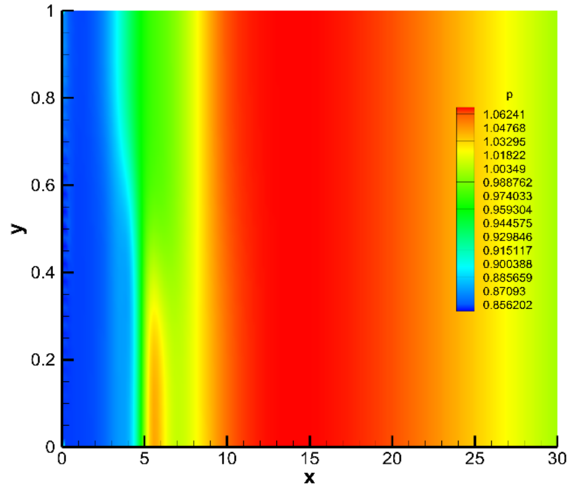
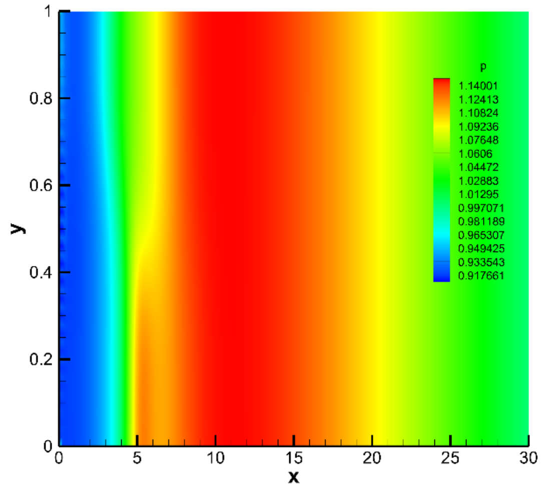


Figure 7 Zoomed view of wall shear for both upper and lower walls
(— lower wall, —·· upper wall)

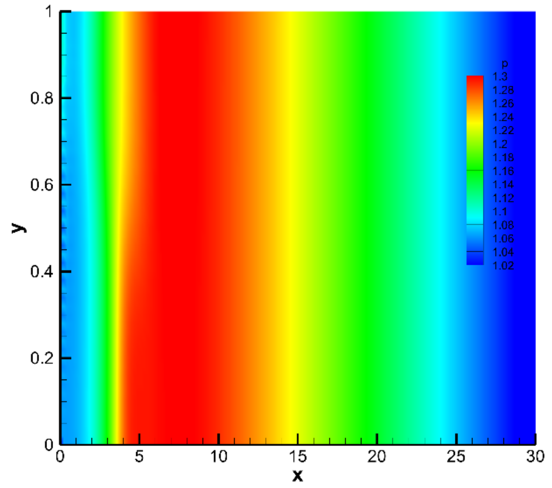
7.3. Pressure Contours



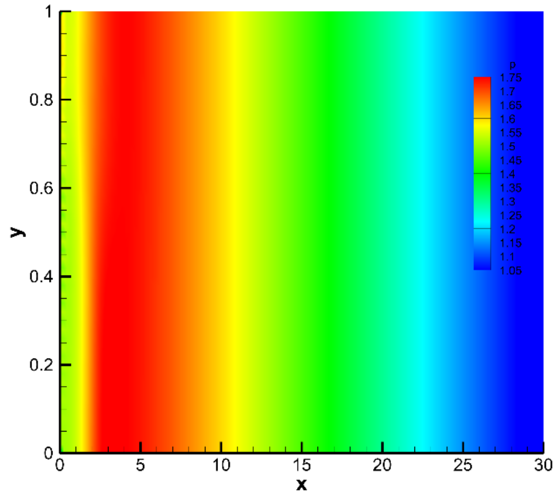
Re = 800



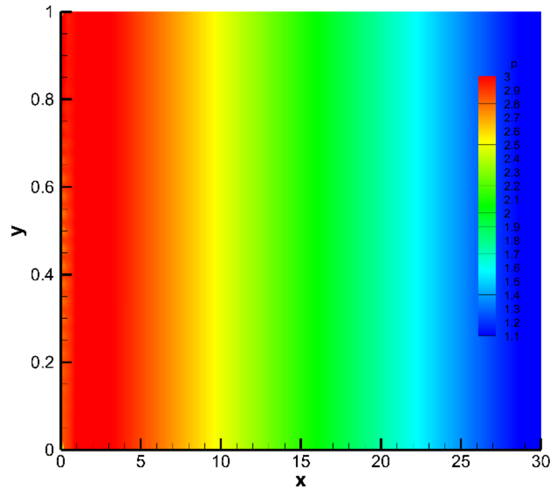
Re = 600



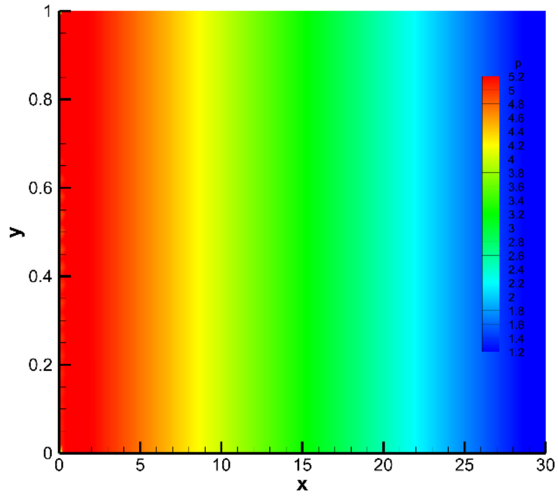
Re = 400



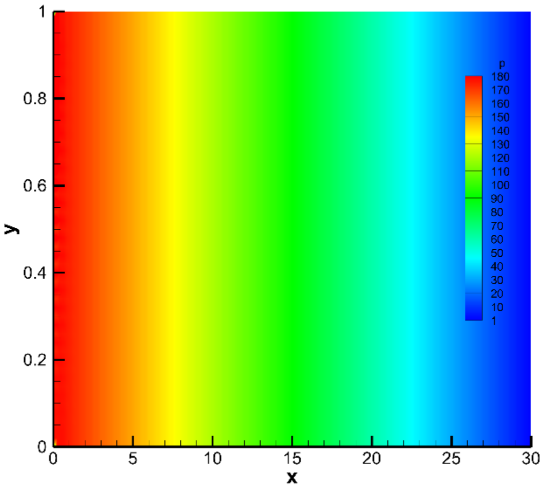
Re = 200



Re = 80



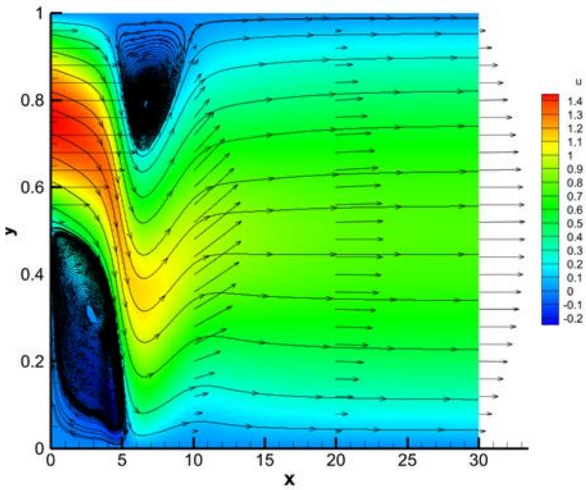
Re = 40



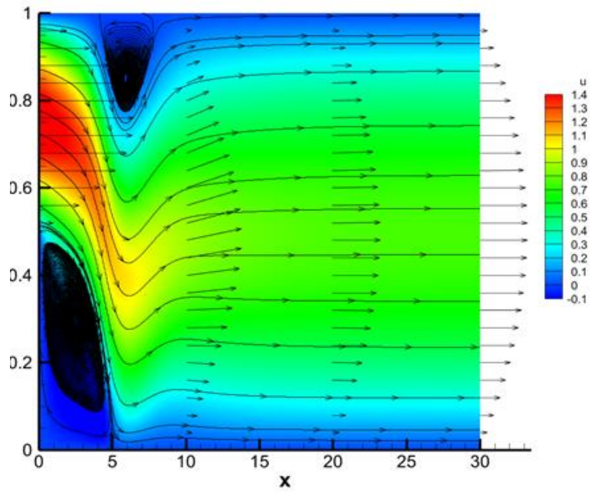
Re = 1

Figure 8 pressure contours at various Reynold numbers

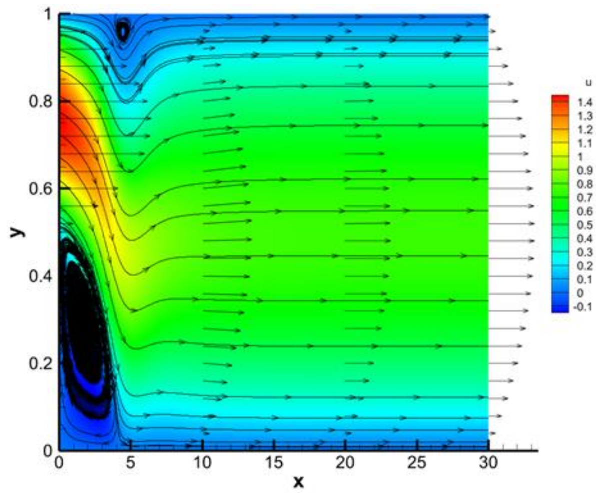
7.4. Streamlines and Velocity Vectors



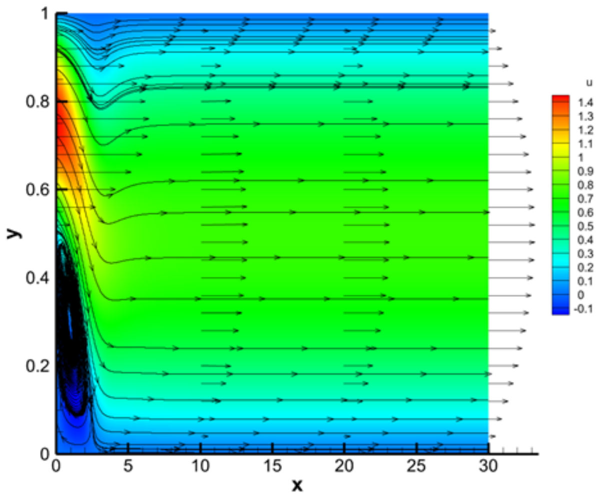
Re = 800



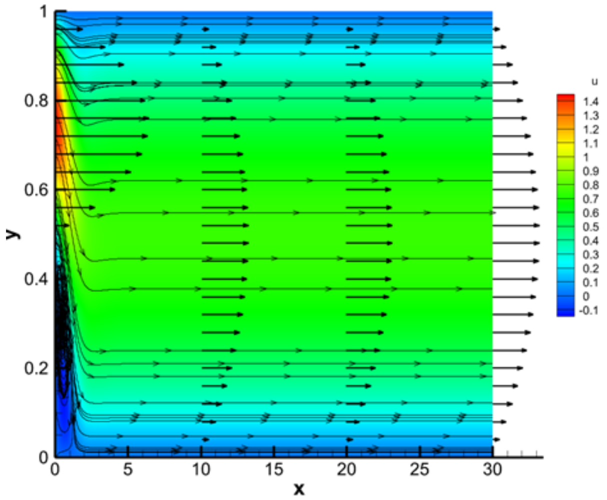
Re = 600



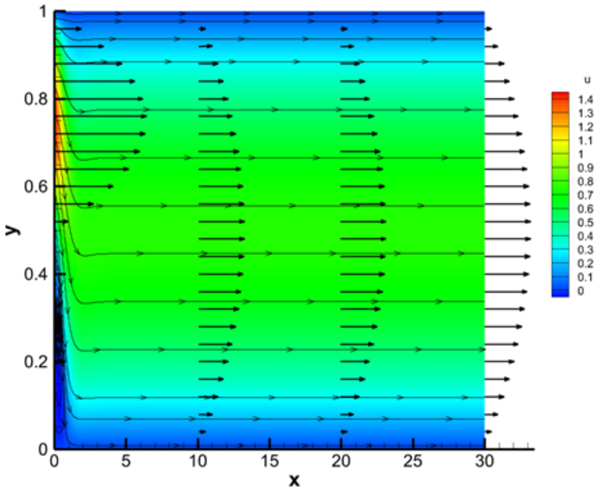
Re = 400



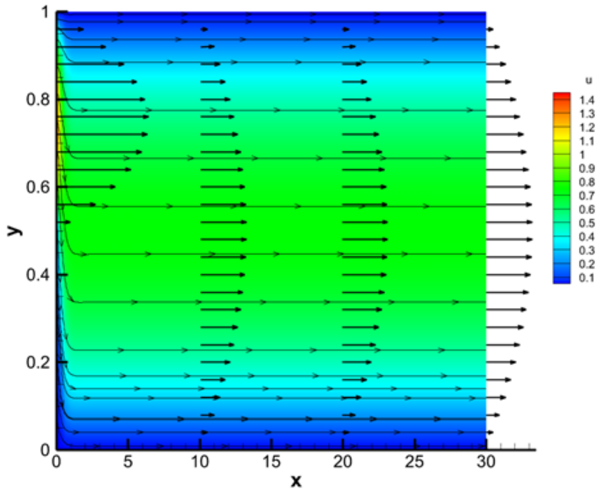
Re = 200



Re = 80



Re = 40



Re = 1

7.5. Wake Length, Separation and Reattachment Locations

Re	Wake Length (L_1)		
	Hejranfar and Khajeh-Saeed	Present Solution	Error
200	2.6660		
400	4.3221		
600	5.3725		
800	6.0953	5.3610	13.7%

Re	Separation Location (L_2)		
	Hejranfar and Khajeh-Saeed	Present Solution	Error
400	4.0084		
600	4.3748		
800	4.8530	4.2893	6.41%

Re	Reattachment Location (L_3)		
	Hejranfar and Khajeh-Saeed	Present Solution	Error
400	5.1865		
600	8.1070		
800	10.4733	9.8571	5.88%