۱. أ

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2};\quad u^*=\frac{u}{U},t^*=t\omega,y^*=y\sqrt{\frac{\omega}{\nu}}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial u^*}\frac{\partial u^*}{\partial t^*}\frac{\partial t^*}{\partial t}=U\frac{\partial u^*}{\partial t^*}\omega\to\frac{\partial u}{\partial t}=U\omega\frac{\partial u^*}{\partial t^*}$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial u^*}\frac{\partial u^*}{\partial y^*}\frac{\partial y^*}{\partial y}=U\frac{\partial u^*}{\partial y^*}\sqrt{\frac{\omega}{\nu}}\to\frac{\partial u}{\partial y}=U\sqrt{\frac{\omega}{\nu}}\frac{\partial u^*}{\partial y^*}$$
$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial\left(\frac{y^*}{\sqrt{\frac{\omega}{\nu}}}\right)}\left(U\sqrt{\frac{\omega}{\nu}}\frac{\partial u^*}{\partial y^*}\right)=\frac{U\omega}{\nu}\frac{\partial^2 u^*}{\partial y^{*2}}$$

بنابراین معادله‌ی بی‌بعد شده برابر است با

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}\to U\omega\frac{\partial u^*}{\partial t^*}=\nu\left(\frac{U\omega}{\nu}\frac{\partial^2 u^*}{\partial y^{*2}}\right)\to\frac{\partial u^*}{\partial t^*}=\frac{\partial^2 u^*}{\partial y^{*2}}$$

ب.

جمله‌ی $\cos\left(\omega t-\sqrt{\frac{\omega}{2\nu}}y\right)$ همواره بین ۱- و ۱ می‌باشد. $U\exp\left(-\sqrt{\frac{\omega}{2\nu}}y\right)$ نیز در همه‌ی yها به‌جز در بی‌نهایت دور دارای مقدار غیرصفر است. بنابراین با $t\to\infty$ حل پایا نخواهد شد و ماهیت حل نوسانی است.

۲. ت

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2};\quad u^*=\frac{u}{V},t^*=\frac{t}{H/V},y^*=\frac{y}{H},Re_H=\frac{VH}{\nu}$$

$$\frac{\partial u}{\partial t}=\frac{\partial u}{\partial u^*}\frac{\partial u^*}{\partial t^*}\frac{\partial t^*}{\partial t}=V\frac{\partial u^*}{\partial t^*}\frac{1}{H/V}\rightarrow\frac{\partial u}{\partial t}=\frac{V^2}{H}\frac{\partial u^*}{\partial t^*}$$
$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial u^*}\frac{\partial u^*}{\partial y^*}\frac{\partial y^*}{\partial y}=V\frac{\partial u^*}{\partial y^*}\frac{1}{H}\rightarrow\frac{\partial u}{\partial y}=\frac{V}{H}\frac{\partial u^*}{\partial y^*}$$
$$\frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial(Hy^*)}\left(\frac{V}{H}\frac{\partial u^*}{\partial y^*}\right)=\frac{V}{H^2}\frac{\partial^2 u^*}{\partial y^{*2}}$$

بنابراین معادله‌ی بی‌بعد شده برابر است با

$$\frac{\partial u}{\partial t}=\nu\frac{\partial^2 u}{\partial y^2}\rightarrow\frac{V^2}{H}\frac{\partial u^*}{\partial t^*}=\nu\left(\frac{V}{H^2}\frac{\partial^2 u^*}{\partial y^{*2}}\right)\rightarrow\frac{\partial u^*}{\partial t^*}=\frac{\nu}{VH}\frac{\partial^2 u^*}{\partial y^{*2}}$$
$$\frac{\partial u^*}{\partial t^*}=\frac{1}{Re_H}\frac{\partial^2 u^*}{\partial y^{*2}}$$

ث.

با اعمال $t\rightarrow\infty$ عبارت $V\left(1-\frac{y}{H}\right)$ تابعی از زمان نیست و ثابت می‌ماند. عبارت $\exp\left(-\frac{k^2\pi^2}{H^2}\nu t\right)$ نیز در تمام جملات summation به صفر میل می‌کند. بنابراین مسئله دارای حل پایا است و مقدار آن برابر است با

$$V\left(1-\frac{y}{H}\right)-\frac{2V}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{k\pi y}{H}\right)\lim_{t\rightarrow\infty}\left(\exp\left(-\frac{k^2\pi^2}{H^2}\nu t\right)\right)=V\left(1-\frac{y}{H}\right)$$

ج.

روش صریح ساده:

$$r=\frac{\nu}{\,}$$

$$u_j^{n+1}=u_j^n+r\left(u_{j-1}^n-2u_j^n+u_{j-1}^n\right)$$
$$g^{n+1}e^{ij\beta}=g^ne^{ij\beta}+r\left(g^ne^{i(j-1)\beta}-2g^ne^{ij\beta}+g^ne^{i(j+1)\beta}\right)$$

$$g=1+r\left(e^{-i\beta}+e^{i\beta}-2\right)=1+r\left(2\cos\beta-2\right)=1-2r\left(1-\cos\beta\right)=1-4r\sin^2\frac{\beta}{2}$$

$$\boxed{g=1-4r\sin^2\frac{\beta}{2}}$$

$$|g|\le 1\rightarrow\left|1-4r\sin^2\frac{\beta}{2}\right|\le 1\quad\xrightarrow{\sin^2\frac{\beta}{2}\le 1}\quad|1-4r|\le 1\rightarrow -1\le 1-4r\le 1\rightarrow 0\le r\le\frac{1}{2}$$

بنابراین به‌ازای $r\le\frac{1}{2}$ پایدار است.

روش کرنک-نیکلسون:

$$u_j^{n+1}=u_j^n+r\left(\frac{u_{j-1}^{n+1}-2u_j^{n+1}+u_{j-1}^{n+1}}{2}+\frac{u_{j-1}^{n}-2u_j^{n}+u_{j-1}^{n}}{2}\right)$$

$$u_j^{n+1}=u_j^n+\frac{r}{2}\left(u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}+u_{j-1}^{n}-2u_j^{n}+u_{j+1}^{n}\right)$$

$$g^{n+1}e^{ij\beta}=g^ne^{ij\beta}+\frac{r}{2}\left(g^{n+1}e^{i(j-1)\beta}-2g^{n+1}e^{ij\beta}+g^{n+1}e^{i(j+1)\beta}+g^ne^{i(j-1)\beta}-2g^ne^{ij\beta}+g^ne^{i(j+1)\beta}\right)$$

$$g=1+\frac{r}{2}\left((e^{-i\beta}+e^{i\beta}-2)(g+1)\right)=1-2r\left((g+1)\sin^2\frac{\beta}{2}\right)$$

$$g(1+2r\sin^2\frac{\beta}{2})=1-2r\sin^2\frac{\beta}{2}\rightarrow\boxed{g=\frac{1-2r\sin^2\frac{\beta}{2}}{1+2r\sin^2\frac{\beta}{2}}}$$

$$|g|=\left|\frac{1-2r\sin^2\frac{\beta}{2}}{1+2r\sin^2\frac{\beta}{2}}\right|=\left|1-\frac{4r\sin^2\frac{\beta}{2}}{1+2r\sin^2\frac{\beta}{2}}\right|\le 1$$

$$\xrightarrow{\sin^2\frac{\beta}{2}\le 1}\ -1\le 1-\frac{4r}{1+2r}\le 1\rightarrow 0\le\frac{4r}{1+2r}\le 2$$

$$\text{for every } r\ge 0:\frac{4r}{1+2r}\in[0,2)$$

Therefore, the scheme is unconditionally stable.

روش ضمنی ساده:

$$u_j^{n+1} = u_j^n + r\left(u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}\right)$$

$$g^{n+1}e^{ij\beta} = g^n e^{ij\beta} + r\left(g^{n+1}e^{i(j-1)\beta} - 2g^{n+1}e^{ij\beta} + g^{n+1}e^{i(j+1)\beta}\right)$$

$$g = 1 + gr\left(e^{-i\beta} + e^{i\beta} - 2\right) = 1 + gr\left(2\cos\beta - 2\right) = 1 - 2gr\left(1 - \cos\beta\right) = 1 - 4gr\sin^2\frac{\beta}{2}$$

$$\boxed{g = \frac{1}{1 + 4r\sin^2\frac{\beta}{2}}}$$

$$|g| \le 1 \to \left|\frac{1}{1 + 4r\sin^2\frac{\beta}{2}}\right| \le 1$$

$$\xrightarrow{\sin^2\frac{\beta}{2} \le 1} -1 \le \frac{1}{1+4r} \le 1$$

$$\text{for every } r \ge 0 : \frac{1}{1+4r} \in (0,1]$$

Therefore, the scheme is unconditionally stable.

حل تحلیلی:

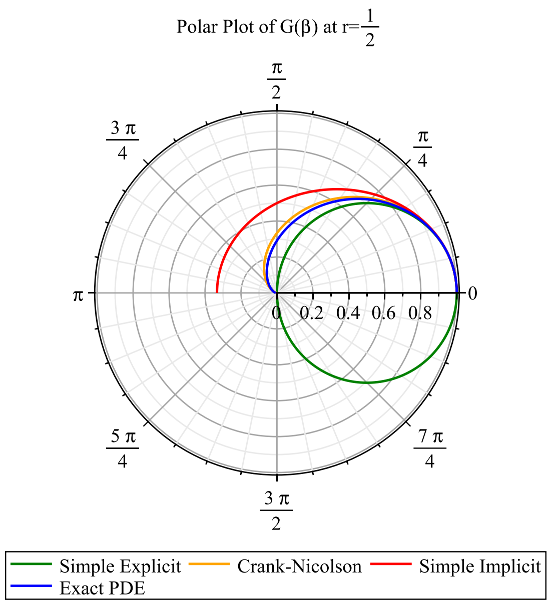
$$u = A_0 e^{at} e^{ik_m y}$$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$

$$aA_0 e^{at} e^{ik_m y} = -\nu {k_m}^2 A_0 e^{at} e^{ik_m y} \to a = -\nu {k_m}^2$$

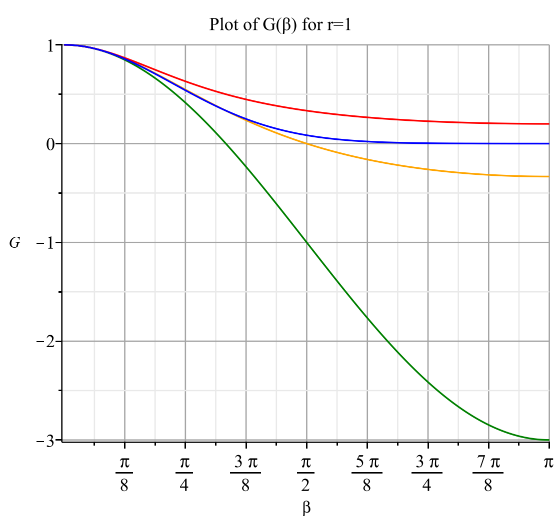
$$g_e = \frac{u^{n+1}}{u^n} = \frac{A_0 e^{a(t+\Delta t)} e^{ik_m y}}{A_0 e^{at} e^{ik_m y}} = e^{a\Delta t} = e^{-\nu {k_m}^2 \Delta t} = e^{-(\frac{\nu \Delta t}{\Delta x^2})(k_m \Delta x)^2}$$

$$\frac{\nu \Delta t}{\Delta x^2} = r, k_m \Delta x = \beta \to \boxed{g_e = e^{-r\beta^2}}$$



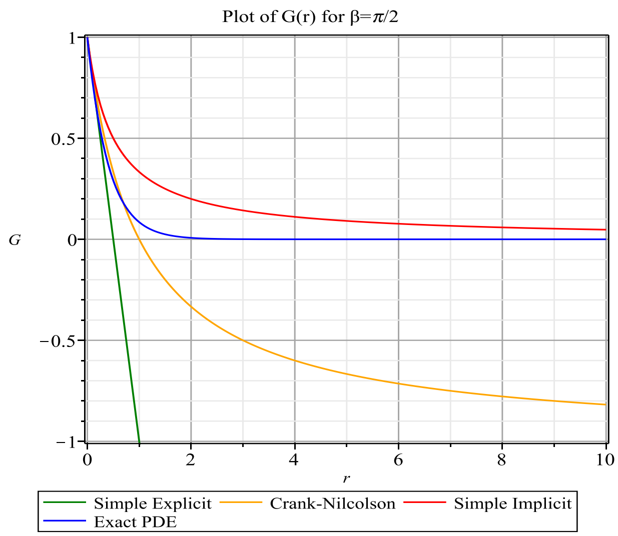
شکل ۱) مقایسه‌ی ضریب بزرگ‌نمایی بر حسب β در $r=\frac{1}{2}$

ح.



شکل ۲

خ.



شکل ۳

د. در این بخش، ابتدا معادلات گسسته‌سازی شده‌ی در هر روش ارائه می‌گردند:

- **روش صریح ساده:**

$$r=\frac{1}{Re_H}\frac{\Delta t^*}{(\Delta x^*)^2}$$
$$u_j^{n+1}=u_j^n+r\left(u_{j-1}^n-2u_j^n+u_{j-1}^n\right)$$

- **روش کرنک-نیکلسون:**

$$u_j^{n+1}=u_j^n+\frac{r}{2}\left(u_{j-1}^{n+1}-2u_j^{n+1}+u_{j+1}^{n+1}+u_{j-1}^n-2u_j^n+u_{j+1}^n\right)$$

$$\left(-\frac{r}{2}\right)u_{j-1}^{n+1}+\left(1+r\right)u_{j}^{n+1}+\left(-\frac{r}{2}\right)u_{j+1}^{n+1}=\left(\frac{r}{2}\right)u_{j-1}^{n}+\left(1-r\right)u_{j}^{n}+\left(\frac{r}{2}\right)u_{j+1}^{n}$$

$$Au_{j-1}^{n+1}+Bu_{j}^{n+1}+Cu_{j+1,j}^{n+1}=R_{j}$$

در این روش، ماتریس درنظر گرفته شده بدین صورت است:

$$\begin{pmatrix} B & C & 0 & \cdots & 0 & 0 & 0 & 0 \\ A & B & C & 0 & \cdots & 0 & 0 & 0 \\ 0 & A & B & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & A & B & C & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A & B & C & 0 \\ 0 & 0 & 0 & \cdots & 0 & A & B & C \\ 0 & 0 & 0 & 0 & \cdots & 0 & A & B \end{pmatrix}\begin{pmatrix} u_{2}^{n+1} \\ u_{3}^{n+1} \\ u_{4}^{n+1} \\ u_{5}^{n+1} \\ \vdots \\ u_{J_{max-3}}^{n+1} \\ u_{J_{max-2}}^{n+1} \\ u_{J_{max-1}}^{n+1} \end{pmatrix}=\begin{pmatrix} R_{2}-Au_{1}^{n+1} \\ R_{3} \\ R_{4} \\ R_{5} \\ \vdots \\ R_{J_{max-3}} \\ R_{J_{max-2}} \\ R_{J_{max-1}}-Cu_{J_{max}}^{n+1} \end{pmatrix}$$

- **روش ضمنی ساده:**

$$(-r)u_{j-1}^{n+1}+(1+2r)u_{j}^{n+1}+(-r)u_{j-1}^{n+1}=u_{j}^{n}$$

$$Au_{j-1}^{n+1}+Bu_{j}^{n+1}+Cu_{j+1}^{n+1}=R_{j}$$

و ماتریس سه‌قطری مربوط به آن:

$$\begin{pmatrix} B & C & 0 & \cdots & 0 & 0 & 0 & 0 \\ A & B & C & 0 & \cdots & 0 & 0 & 0 \\ 0 & A & B & C & 0 & \cdots & 0 & 0 \\ 0 & 0 & A & B & C & 0 & \vdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A & B & C & 0 \\ 0 & 0 & 0 & \cdots & 0 & A & B & C \\ 0 & 0 & 0 & 0 & \cdots & 0 & A & B \end{pmatrix} \begin{pmatrix} u_2^{n+1} \\ u_3^{n+1} \\ u_4^{n+1} \\ u_5^{n+1} \\ \vdots \\ u_{J_{max-3}}^{n+1} \\ u_{J_{max-2}}^{n+1} \\ u_{J_{max-1}}^{n+1} \end{pmatrix} = \begin{pmatrix} R_2 - Au_1^{n+1} \\ R_3 \\ R_4 \\ R_5 \\ \vdots \\ R_{J_{max-3}} \\ R_{J_{max-2}} \\ R_{J_{max-1}} - Cu_{J_{max}}^{n+1} \end{pmatrix}$$

- بی‌بعدسازی حل تحلیلی

با توجه به‌این‌که تمامی حل‌های عددی مربوط به معادله‌ی نفوذ بی‌بعدسازی شده می‌باشند، حل تحلیلی بدست آمده نیز باید بر اساس پارامترهای بی‌بعد بیان شود.

$$u_{exact}(y,t) = V\left(1-\frac{y}{H}\right) - \frac{2V}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(\frac{k\pi y}{H}\right)\exp\left(-\frac{k^2\pi^2}{H^2}\nu t\right)$$

$$\frac{u_{exact}(y,t)}{V} = \left(1-\frac{y}{H}\right) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(k\pi\frac{y}{H}\right)\exp\left(-k^2\pi^2\frac{\nu}{HV}\frac{t}{H/V}\right)$$

$$u^*_{exact}(y,t) = \left(1-y^*\right) - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{k}\sin\left(k\pi y^*\right)\exp\left(-k^2\pi^2\frac{t^*}{Re_H}\right)$$