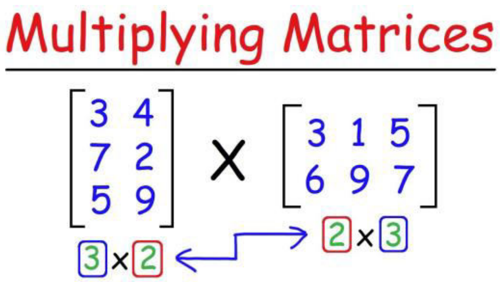
Jacobian Matrix Prerequisites

- Remember matrix multiplication = **ROWS x COLS * ROWS x COLS**
- 

- Think of matrices as transformations in space
- **Linear Transformation - a function** that takes an input **vector** and transforms one mapping to another
    - Maintains the properties of **addition and scalar multiplication** such that:
        - **T(X+Y) = T(X) + T(Y)**
        - **T(aX) = aT(X)**

          where **X** and **Y** are **vectors**, and **a** is a **scalar**
    - We can write linear transformations in a **transformation matrix**
    - Linear transformations preserve properties from one mapping to another
        - **Geometrically, grid lines remain straight, parallel and evenly spaced**
            - If you know where the (0, 1) and (1, 0) vectors land, you know where the whole grid goes
    - *Known as a **linear function**, but usually we use **linear function** to describe a straight line graph*
    - A linear transformation is when we transform each axis with a function that is linear (straight line graph)
        - Means the x transformation can be function of x and y **(e.g. $f_x = 2x + 3y$ and $f_y = 5x + ½y$)**

$$\begin{bmatrix} 2 & -3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{bmatrix} 2x + (-3)y \\ 1x + 1y \end{bmatrix}$$

- In this case, we plug in our original coordinates **(x, y)** to get new coordinates where **x'** and **y'** are functions of **x** and **y**
- The first column of the transformation matrix tells us where (1, 0) would go, and the second matrix tells us where (0, 1) would go, which we can realise if we multiply the two pairs together, and realise the 0s knock out the other column
- **Basis - set of vectors** in a **vector space** with certain **properties**
    - You get any vector in the vector space by **multiplying** the basis vectors by **different numbers**, then **adding**
    - **If any vector is removed** from the basis, the above is **no longer satisfied**
    - *For example, if you have a 2d space, a vector space can be {(1, 0), (0, 1)}*

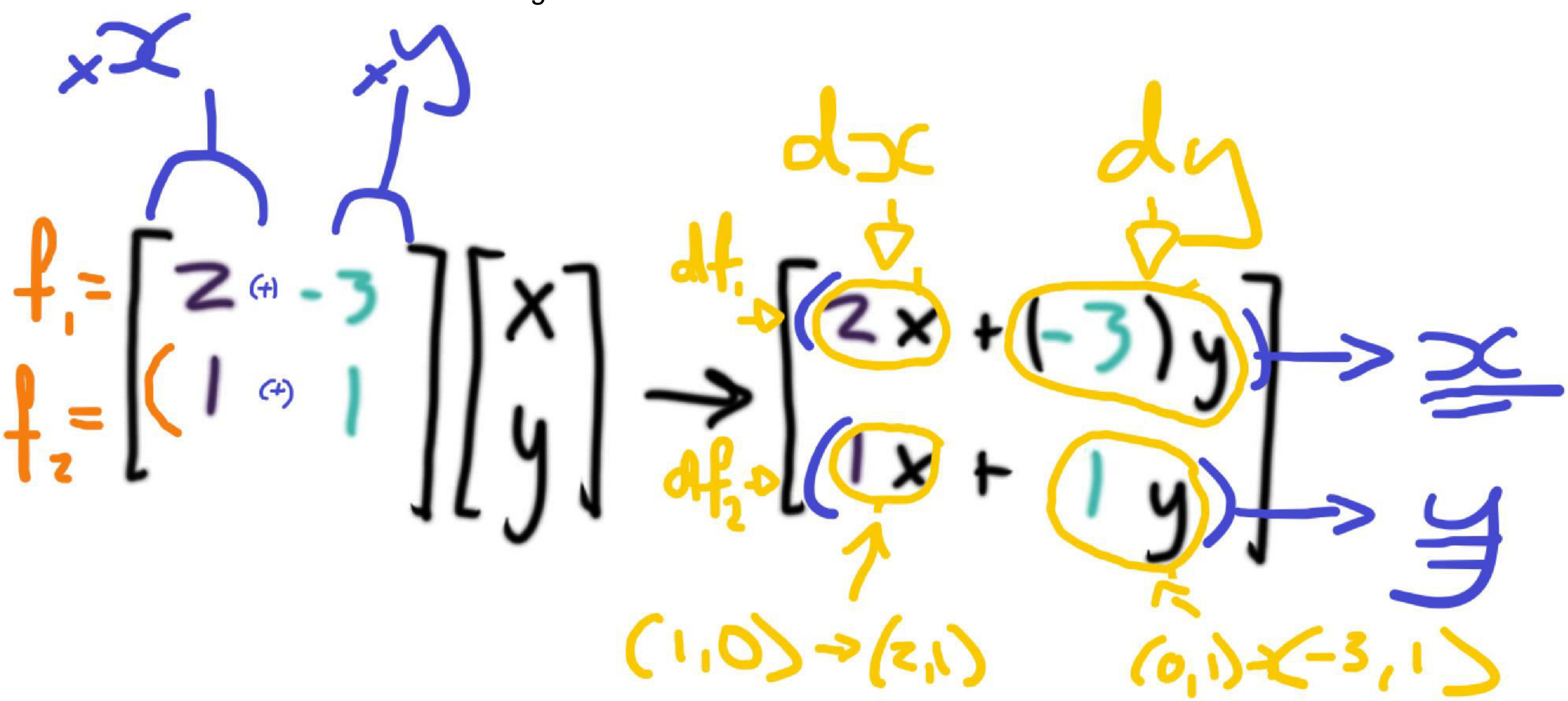
Linear Transformations as Matrices

Local Linearity for a multivariable function

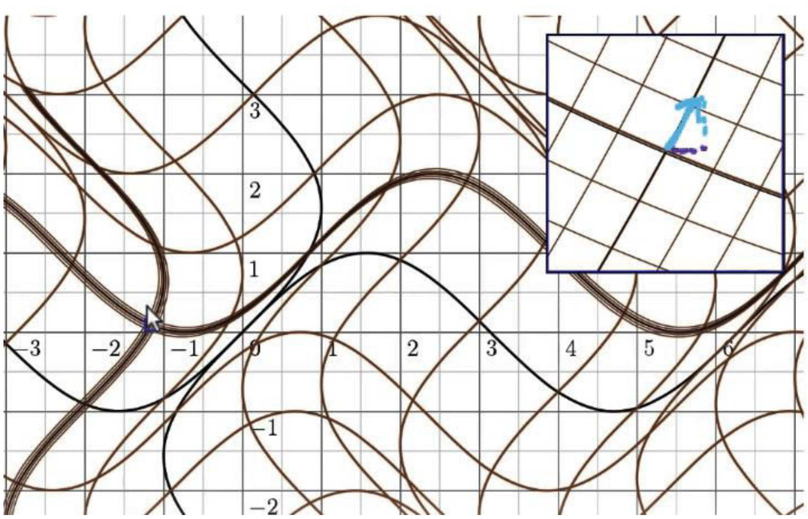
- ***Multivariate calculus** is about applying **linear algebra** principles to **non-linear problems***
- **Locally Linear Function - A nonlinear function** that is locally linear, i.e. if we zoom in far enough, appears linear
    - I.e. a transformation that is a function of sin(x) is not linear (wavy gridlines), but if we zoom in far enough, it looks like a linear transformation (straight gridlines)
    - *We can find out what linear transformation a nonlinear function looks like*

The Jacobian Matrix

- The Jacobian matrix is a matrix of partial derivatives that tells us how vectors map transformations
  - **First order derivatives**
- **The Jacobian Matrix** tells us what is the local linear function transformation approximation of a nonlinear function
  - **This is important because when we plug in coordinates into a Jacobian matrix, it gives us the transformation matrix that was applied at those coordinates**
- *The intuition is we can zoom in on a nonlinear function, make a slight move in the x direction (dx) and see where it goes after. We then look where it ended up in the x direction (i.e. due to f_1), which is equal to df_1/dx. We then look where it ended up in the y direction (i.e. due to f_2), which is f_2/dx. We then do the same by making a slight movement in y direction and calculating f_1/dy (rate of change of y vector in x direction) and f_2/dy (rate of change of y vector in y).*
  - **f_1/dx** tells us the rate of change of the **x-axis transformation** with respect to us moving in the **x direction**
  - That's why it tells us what the transformation looks like when you zoom in, because it's telling us the gradient
    - *It's kind of like doing numerical differentiation*



$$\begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix} = \begin{bmatrix} x + \sin(y) \\ y + \sin(x) \end{bmatrix}$$



(-2,1)

$$\begin{bmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{bmatrix}$$

The Jacobian (Jacobian = Jacobian Determinant)

- The determinant of a matrix tells us the scale factor by which the area defined by a vector has changed
  - When we transform an object with the matrix, how much has its area increased / decreased by
- **Jacobian Determinant -** tells us how much has the area been stretched by at a certain point in space
  - If we drew a tiny square at that point, the Jacobian matrix tells us how much that square will get squashed or stretched in or out

$$\det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = (1)(4)-(2)(3)$$
$$= 4-6$$
$$= -2 \checkmark$$

What happens in Jacobians with linear functions?

Every partial derivative will be a linear function, which means the determinant will stay constant whatever numbers you plug

Jacobian Matrices and the Chain Rule

- Compose functions **f : R^K -> R^J** with **g : R^J -> R^I**, so **(g o f) : R^K -> R^I** with **Jacobian Matrix J_a(g o f) = Jg Jf**
  - You can use the variables of both functions or you can use one or the others if you rearrange
  - At point **a**, **J_a(g o f) = $J_{f(a)}$g J_af**