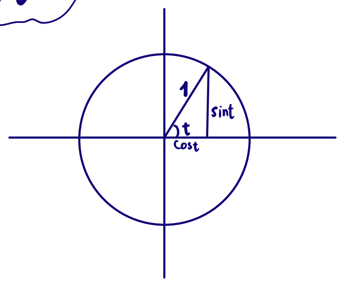
1. ZÜ Ana 3 16.10.2019

- Berechnung von Integralen EXP
- Hyperp. Funktionen sinh, cosh

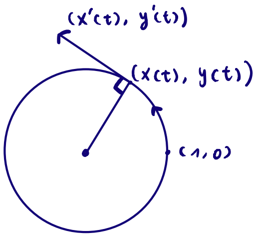
Kreisfunktionen sin, cos, tan

||

trigonom. Fkt

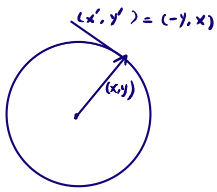


winkel ?



$$\begin{cases} x^2 + y^2 = 1 \\ x'^2 + y'^2 = 1 \end{cases}$$ ← Mit Geschw' 1 bewegen

Diff. Gleichung
$$\begin{cases} x'(t) = -y(t) \\ y'(t) = x(t) \end{cases}$$



Exponentialfkt komplex

$$\mathbb{C} \xrightarrow{exp} \mathbb{C}$$

$$z \longmapsto e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

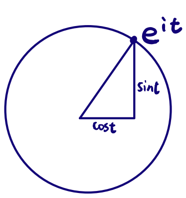
$$\boxed{\frac{e^{x+h} - e^x}{h} = e^x \cdot \underbrace{\frac{e^h - 1}{h}}_{\to 1 \text{ für } h \to 0}}$$

Funktionalgl $\quad e^{z_1 + z_2} = e^{z_1} \cdot e^{z_2}$

$\rightsquigarrow$ Ableitung $\quad (e^x)' = e^x \quad$ (x reell)

$$(e^{ix})' = i e^{ix}$$

Kreis - bzw. trign. Fkt



in $\mathbb{C} \cong \mathbb{R}^2$

$\rightsquigarrow$ Def von sin / cos

$$e^{it} =: \cos t + i \sin t$$

$$\Updownarrow$$

$\cos t = \operatorname{Re} e^{it}$ $\qquad$ $\cos t = \frac{e^{it} + e^{-it}}{2}$

$$\Big\{ \sin t = \operatorname{Im} e^{it} \quad \Longleftrightarrow \quad \Big\{ \sin t = \frac{e^{it} - e^{-it}}{2}$$

weil Exponentialreihe reele Koeff

$$e^{\bar z} = \overline{e^z},$$

denn $e^{-it} = \overline{e^{it}} = \cos t - i \sin t$

e^{it} bewegt sich mit Geschw $\equiv 1$ entlang Einheitskreis,

denn $1 = e^0 \underset{\uparrow \text{Fkt.gl}}{=} e^{it} \cdot e^{-it} = \cos^2 t + \sin^2 t$

$$\boxed{\cos^2 + \sin^2 = 1}$$

$$\text{Geschw} = |ie^{it}| = 1$$

Fktgl für exp

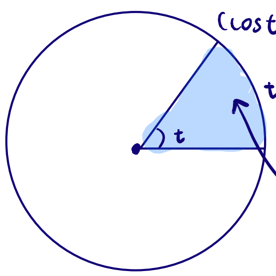
$$\rightsquigarrow \underbrace{(e^{it})'}_{\cos' t + i \cdot \sin t} = \underbrace{ie^{it}}_{-\sin t + i \cos t} \rightsquigarrow \begin{cases} \cos' = -\sin \\ \sin' = \cos \end{cases}$$

Additionstheorem für sin / cos

$$\begin{cases} \cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \\ \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha \sin\beta \end{cases}$$

$$\uparrow$$

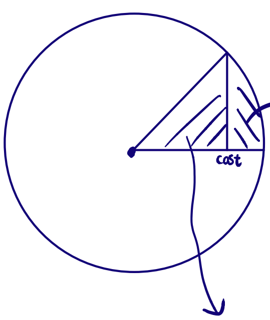
$$e^{i(\alpha+\beta)} = e^{i\alpha} \cdot e^{i\beta}$$



$(\cos t, \sin t) = (x, y)$

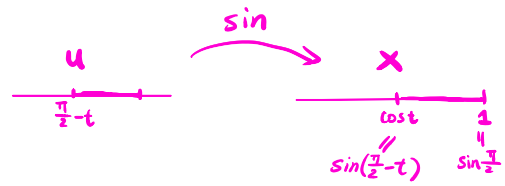
$t =$ Länge Kreisbogens $= \arccos x = \arcsin y$

Fläche $\overset{?}{=} \frac{t}{2}$



$\frac{1}{2} \cos t \sin t$

$$\text{Fläche} = \int_{\cos t = \sin(\frac{\pi}{2} - t)}^{1 = \sin\frac{\pi}{2}} \sqrt{1 - x^2}\, dx$$



Subs. Regel

substituiere $x = \sin u$

$dx = \cos u\, du$ (1 - Form)

$$\int_{\frac{\pi}{2} - t}^{\frac{\pi}{2}} \underbrace{\sqrt{1 - \sin^2 u}}_{\cos u} \underbrace{\sin' u}_{\cos u}\, du = \int_{\frac{\pi}{2} - t}^{\frac{\pi}{2}} \underbrace{\cos^2 u}_{} \, du$$

| Add. Theo

$$\frac{1 + \cos 2u}{2} = \left(\frac{u}{2} + \frac{1}{4} \sin 2u\right)'$$

$$\overset{\text{Hauptsatz}}{=} \frac{u}{2} + \frac{1}{4}\sin 2u \Big|_{\frac{\pi}{2}-t}^{\frac{\pi}{2}}$$

$$= \frac{t}{2} - \frac{1}{4}\underbrace{\sin(\pi - 2t)}_{=\underbrace{\sin 2t}_{2\sin t\cos t}}$$

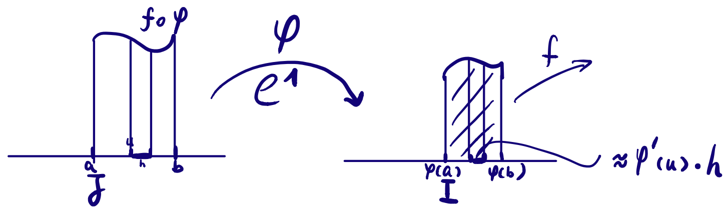
$$= \frac{t}{2} - \frac{1}{2}\sin t \cos t$$

$\Rightarrow$ Fläche des Sektors

$$= \left(\frac{t}{2} - \frac{1}{2}\sin t\cos t\right) + \frac{1}{2}\sin t \cos t$$

$$= \frac{t}{2}$$

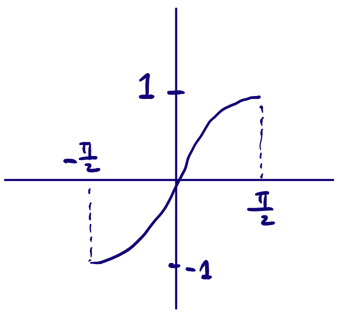
Substitutionsregel



$$\int_{\varphi(a)}^{\varphi(b)} f(x)\,dx \underset{\substack{\text{Subst} \\ x=\varphi(x)}}{=\!=} \int_a^b f(\varphi(u))\cdot\varphi'(u)\,du$$

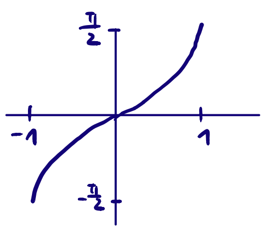
Umk. Fkt

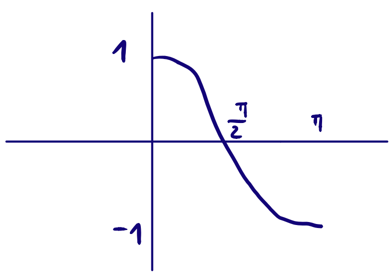
sin:



$$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \underset{\text{arccos}}{\overset{\text{sin}}{\rightleftarrows}} [-1, 1]$$

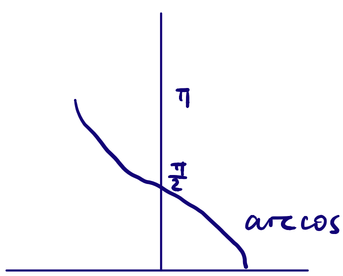
Umk fkt arcsin





$$[0, \pi] \underset{\text{arccos}}{\overset{\text{cos}}{\rightleftarrows}} [-1, 1]$$

Arcus cosinus



$$\boxed{\arccos x = \underbrace{\arcsin -x}_{-\arcsin x} + \frac{\pi}{2}} \Longleftarrow \underset{x:=}{\cos u} \overset{\text{zB Additionstheo.}}{=} \sin\left(u + \frac{\pi}{2}\right)$$

$$= \sin\left(-u + \tfrac{\pi}{2}\right)$$

$$\left(= -\sin\left(u - \tfrac{\pi}{2}\right) \right)$$

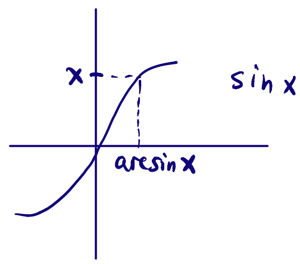
$x = \cos u$

$u = \arccos x$

$-u + \frac{\pi}{2} = \arcsin x$

Ableitungen:

wir möchen arcsin ableiten.



$$\arcsin' x = \frac{1}{\sin'(\arcsin x)} = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-\underbrace{\sin^2(\arcsin x)}_{x^2}}} = \frac{1}{\sqrt{1-x^2}}$$

$$\arccos' x = \frac{1}{\cos'(\arccos x)} = -\frac{1}{\sin(\arccos x)} = \dots = -\frac{1}{\sqrt{1-x^2}}$$

Hyperbelfkten

$$\begin{cases} \cosh x := \dfrac{e^x + e^{-x}}{2} \\ \sinh x := \dfrac{e^x - e^{-x}}{2} \end{cases}$$

$\longrightarrow$ Potenzreihenentw.

$$\leadsto \begin{cases} \sinh' = \cosh \\ \cosh' = \sinh \end{cases}$$

Additions theor.

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b$$

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b$$

$$\cos x = \frac{e^{ix} + e^{ix}}{2}$$

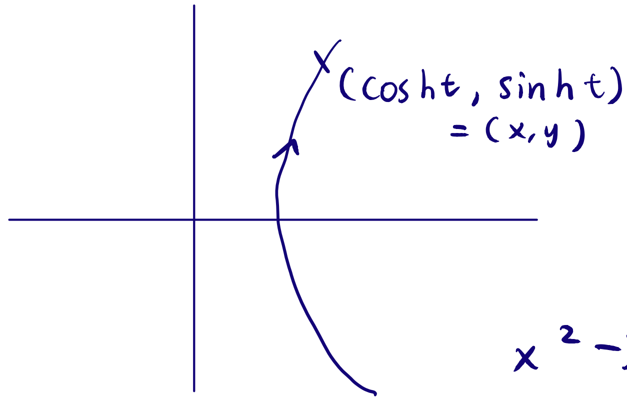
$$\sin x = \frac{e^{ix} - e^{ix}}{2}$$

$$\begin{cases} \cos x = \cosh ix \\ \sin x = -i \cdot \sinh ix \end{cases}$$

bzw $\begin{cases} \cosh x = \cos(-ix) = \cos ix \\ \sinh x = i \cdot \sin(-ix) = -i \cdot \sin ix \end{cases}$

Aus dem Additionstheorem folgt

$$1 = \cosh 0 = \cosh^2 t - \sinh^2 t$$



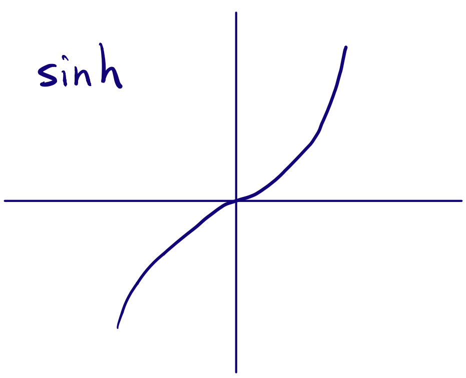
$x^2 - y^2 = 1$

Hyperbel

Geschwin. vektor $(\cosh' t, \sinh' t)$
$= (\sinh t, \cosh t)$

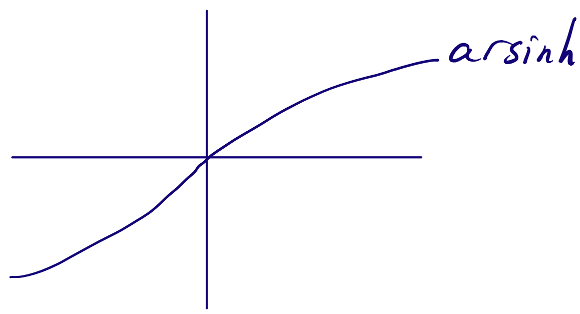
„Lorentz"- Geschw - Quadrat
$\sinh^2 t - \cosh^2 t = -1 < 0$

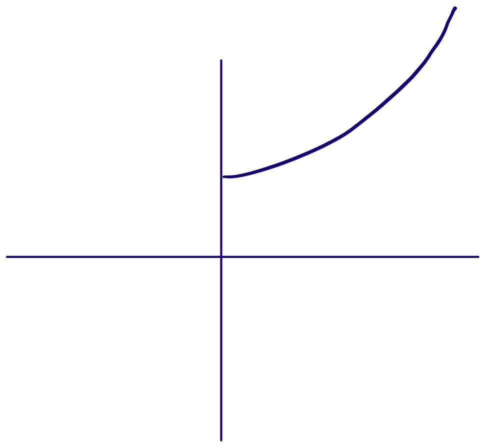
"zeitartig"



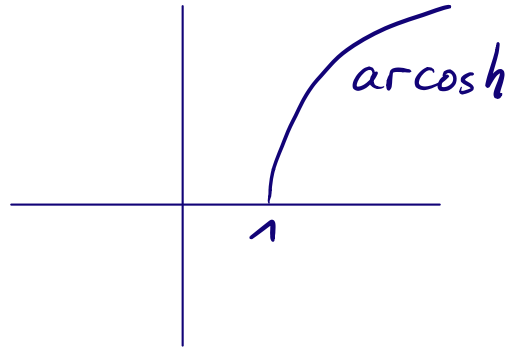
$$\mathbb{R} \underset{\operatorname{arsinh}}{\overset{\sinh}{\rightleftarrows}} \mathbb{R}$$

Area sinus hyperbolicus





$$[0,\infty) \underset{\operatorname{arcos} h}{\overset{\cosh}{\rightleftarrows}} [1,\infty)$$



2. $\sinh u = e^u - \frac{1}{e^u}$

$\Longrightarrow_{z:=} e^{\operatorname{arsinh} x}$ erfüllt die Gleichung

$$2x = z - \frac{1}{z}$$

$$\Updownarrow$$

$$(z-x)^2 = x^2+1$$

$\Rightarrow \operatorname{arsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$

analog $\operatorname{arcosh} x = \ln\left(x+\sqrt{x^2-1}\right)$

Ableitungen

$$\operatorname{arsinh}' x = \frac{1}{\sinh'(\operatorname{arsinh} x)} = \frac{1}{\cosh(\operatorname{arsinh} x)}$$

$$= \frac{1}{\sqrt{1+\sinh^2(\operatorname{arsinh} x)}}$$

$$= \frac{1}{\sqrt{1+x^2}}$$

analog $\operatorname{arcosh}' x = \frac{1}{\sqrt{x^2-1}}$