Abstract simplicial complexes

Def 1 (Abstract simplicial complex)**.** An *abstract simplicial complex* is a collection K of non-empty finite sets $\varnothing \neq S \in K$ such that every nonempty subset of S is also contained in K, i.e. $\varnothing \neq T \subseteq S \implies T \in K$.

Def 2 (Simplex)**.** Sets in K are called *simplices.*

Def 3 (Vertex)**.** An element $v \in S$ of a simplex $S \in K$ is a *vertex* (plural: vertices).

Def 4 (Dimension of a simplex and of a complex)**.**

- The *dimension* of a *simplex* $S \in K$ is $|S| - 1$.
- The *dimension* of the *complex* K is $\sup_{S\in K} \dim S$.

Def 5 (Face)**.** A *face* is a subset of a simplex, *proper* if it is a strict subset.

Def 6 (Coface)**.** A *coface* is a superset of a simplex which is in the complex.

Rem 7. *$\exists$ a bijection between the vertex set and 0-simplices.*

Def 8 (Full subcomplex)**.** A subcomplex L is full if every simplex in K whose vertices are in L belongs to L, i.e. $S \subseteq \operatorname{Vert} L, S \in K \implies S \in L$.

Vietoris-Rips complex

An abstract simplicial complex that can be defined from any metric space X and distance t by forming a simplex for every finite set of points that has diameter at most t:

Def 9 (Vietoris-Rips)**.** A *Vietoris-Rips complex* of a metric space (X, d) at scale t is defined as

$$\mathrm{Rips}_t(X) := \{\varnothing \neq Q \subseteq X \mid \operatorname{diam} Q \leqslant t\},$$

where $\operatorname{diam} Q := \sup_{x,y\in Q} d(x, y)$.

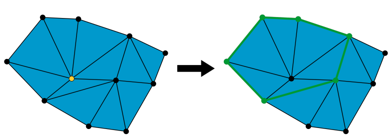
Def 10 (Closure)**.** The *closure* of a simplex is the collection of its faces, (is the smallest simplicial complex that contains all its faces).

Def 11 (Star)**.** The *star* of a simplex is the collection of its cofaces.

Rem 12. $\mathrm{St}(S) = \bigcap_{v\in S} \mathrm{St}\{v\}$.

Def 13 (Link)**.** The *link* of S is defined as

$$\mathrm{Lk}\ (S) := \{P \in K \mid P \cup S \in K, P \cap S = \varnothing\}.$$



Def 14 (Join)**.** For two *augmented* simplicial complex K and L, $\operatorname{Vert} K \cap \operatorname{Vert} L = \varnothing$, their *join* is a simplicial complex defined as

$$K * L := \{S \cup T \mid S \in K, T \in L\}.$$

Rem 15. $\mathrm{Lk}\, S * \mathrm{Cl}\, S = \mathrm{Cl}(\mathrm{St}\, S)$.

Def 16 (Cone, Apex, Base)**.** For $v \notin \operatorname{Vert} K$, $\{\{v\}\} * K$ is a *cone* with *apex* v and *base* K.

Def 17 (Suspension)**.** For $v, w \notin \operatorname{Vert} K$, $\{\{v\}, \{w\}\} * K$ is a *suspension* of K.

A *flag complex* is an abstract simplicial complex such that *every set of vertices in which all pairs are simplices in the complex* is also itself a simplex in the complex.

Def 18 (Flag complex)**.** Given a finite poset (partially order set) $P = (P, \leqslant)$, its *flag complex* (or order complex) is the Cplx that has P as vertices, and chains as simplices.

Exp 19 (Derived subdivision)**.** $\mathrm{Pos}\, K := (K, \subseteq)$ is called the *face poset* of K. Its flag complex is called the *derived subdivision* of K, denoted by $\mathrm{Sd}\, K = \mathrm{Flag}(\mathrm{Pos}\, K)$.

Maps between simplicial complexes

Let $\mathcal{A}, \mathcal{B}$ be abstract simplicial complexes.

Def 20 (Vertex map)**.** A map $\varphi : \operatorname{Vert} \mathcal{A} \to \operatorname{Vert} \mathcal{B}$ between vertex sets is a *vertex map* if the image of the vertices of a simplex always span a simplex, i.e. $S \in \mathcal{A} \implies \varphi(S) \in \mathcal{B}$.

Def 21 (Abstract simplicial map)**.** The induced map $f : \mathcal{A} \to \mathcal{B}, S \mapsto \varphi(S)$ is an *abstract simplicial map.*

Def 22 (Simplicial isomorphism)**.** If φ is a bijection and φ^{-1} is also a vertex map, then f is a *simplicial isomorphism.*

Geometric simplicial complexes

Let $V = \{v_0, ..., v_k\} \subset \mathbb{R}^d$ be a finite point set.

Def 23 (Affine combination)**.** A point $\sum_{v\in V} \lambda_v v$ with $\sum_{v\in V} \lambda_v = 1$ is an *affine combination* of V.

Def 24 (Affine hull)**.** The collection of affine combinations is the *affine hull*, denoted by $\operatorname{aff} V$.

Def 25 (Affinely independent)**.** The points in V are *affinely independent* if the coefficients of any affine combination are unique. In this case $\operatorname{aff} V$ is a k-dim affine subspace.

Def 26 (Convex combination)**.** An affine combination with $\lambda_v \geqslant 0$ is a *convex combination.*

Def 27 (Convex hull)**.** The *convex hull* is the collection of convex combinations, denoted by $\operatorname{conv} V$.

Def 28 (General linear position)**.** V is in *general linear position* if any subset of at most $d + 1$ points is affinely independent.

Exp 29 (Moment curve)**.** The moment curve

$$X = \{(t, t^2, ..., t^d)\} \mid t \in \mathbb{R}\} \subset \mathbb{R}^d$$

is in general linear position.

Rem 30. *Any finite $V \subset \mathbb{R}^d$ can be perturbed to general position, arbitrarily close to X.*

Let V be affinely independent.

Def 31 (Geometric k-simplex)**.** A *geometric k-simplex* $\sigma = \operatorname{conv} V$ is the convex hull of $k + 1$ affinely independent points. k is the *dimension* of the simplex.

Def 32 (Vertex)**.** We say σ is spanned by V. The points in V are called vertices, $V = \operatorname{Vert} \sigma$.

Rem 33. Vertex*: 0-simplex,* edge*: 1-simplex,* triangle*: 2-simplex, and* tetrahedron*: 3-simplex.*

Def 34 (Face)**.** A *(proper) face* is a simplex spanned by a non-empty (proper) subset of V.

Def 35 (Coface)**.** τ is coface of σ $:\iff$ σ is face of τ

Def 36 (Boundary)**.** The *boundary* of σ ($\mathrm{bd}\, \sigma$) is the union of all its proper faces, or equivalently, all faces of codimension 1.

Rem 37. *The boundary of a vertex is $\varnothing$.*

Def 38 (Interior)**.** $\operatorname{int} \sigma = \sigma \setminus \mathrm{bd}\, \sigma$.

We are interested in sets of simplices that are closed under taking faces and that have no improper intersections:

Def 39 (Geometric simplicial complex)**.** A *geometric simplicial complex* K in $\mathbb{R}^d$ is a collection of simplices that

(i) is closed under the face relation:
$$\sigma \in K \quad \text{and} \quad \tau \text{ face of } \sigma \implies \tau \in K,$$
(ii) intersects only in common faces: for every $\sigma, \tau \in K$ with $\sigma \cap \tau \neq \varnothing$, it is a face of both σ, τ.

Def 40 (Dimension)**.** The *dimension* of K is the supremum dimension of its simplices.

Rem 41. *Condition (ii) is equivalent to: (ii') simplices have disjoint interiors: $\sigma \neq \tau \in K \implies \operatorname{int} \sigma \cap \operatorname{int} \tau = \varnothing$.*

Def 42 (Underlying Space)**.** The *underlying space* of K (or *polyhedron*), denoted as $|K|$, is the union of its simplices together with the *coherent topology,* which can be expressed using closed sets: $U \subseteq |K|$ closed if and only if $U \cap \sigma$ is closed in subspace topology of $\sigma\ \forall \sigma \in K$.

Def 43 (Triangulation)**.** A *triangulation* of a topological space X is a homeomorphism between X and $|K|$ for some simplicial complex K. In this case, X is *triangulable.*

Simplicial maps

Let K, L be geometric simplicial complexes.

Def 44 (Simplicial map)**.** A *simplicial map* is a continuous map $f : |K| \to |L|$ mapping each simplex of K affinely onto some simplex in L. We also call $f : K \to L$ a *simplicial map.*

Rem 45. *The restriction to the vertices of K already determines f uniquely.*

Def 46 (Simplicial isomorphism)**.** If f has a simplicial inverse, it is a *simplicial isomorphism.*

Def 47 (Vertex scheme)**.** The abstract simplicial complex induced by the vertex sets of the simplices in K is called the *vertex scheme* of K.

Def 48 (Geometric realization)**.** If the vertex scheme of K is isomorphic to some abstract simplicial complex $\mathcal{A}$, then K is a *geometric realization* of $\mathcal{A}$.

Rem 49. *The geometric realization of an abstract simplicial complex is unique up to simplicial isomorphism.*

Rem 50 (Geometric Realization Theorem)**.** *An d-dim abstract simplicial complex with n vertices has geometric realizations in $\mathbb{R}^n$, $\mathbb{R}^{n-1}$, and in $\mathbb{R}^{2d+1}$.*

Gluing Constructions

Let L be an abstract simplicial complex and $f : \operatorname{Vert} L \to V$ a surjective map to some set V of vertex labels.

Def 51 (Pasting complex, map)**.** The collection $K = \{f(\sigma) \mid \sigma \in L\}$ is an abstract simplicial complex with vertices $\operatorname{Vert} K = V$, called the *pasting complex* for f.

Def 52 (Pasting map)**.** The *pasting map* for f is the simplicial map induced by f.

We want to identify only disjoint simplices in L and the identification should preserve dimension. We now phrase criterion to exclude unintended gluing of simplices.

Rem 53. *Assume that the vertices in L with the same label have disjoint closed vertex stars: for any $v \neq w \in \operatorname{Vert} L$ with $f(v) = f(w)$, we have $\operatorname{Cl}\operatorname{St} v \cap \operatorname{Cl}\operatorname{St} w = \varnothing$. Then,*

(i) the induced pasting map preserves dimension of simplices: $\dim g(\sigma) = \dim \sigma$ for all $\sigma \in L$, and

(ii) only disjoint simplices are identified: if $\sigma, \tau \in L$ with $g(\sigma) = g(\tau)$, then $\sigma \cap \tau = \varnothing$.

Homotopy

Let $f, g : X \to Y$ be two continuous maps between topological spaces.

Def 54 (Homotopic)**.** f and g are *homotopic* ($f \simeq g$) if there exists a continuous deformation of f into g, i.e. a continuous map $H : X \times [0,1] \to Y$ with $H(x,0) = f(x)$ and $H(x,1) = y(x)$ for all $x \in X$.

Def 55 (Homotopy)**.** H is called a *homotopy.*

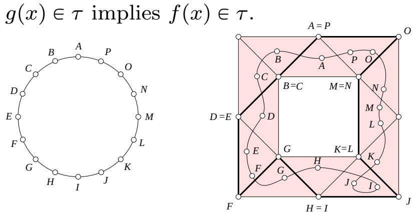
Rem 56. *Homotopy defines an equivalence relation of continuous maps.*

Simplicial approximation

We show that by subdividing, we can approximate any continuous map between triangulable spaces by a simplicial map.

Let K, L be geometric simplicial complexes and let $g : |K| \to |L|$ be a continuous map.

Def 57 (Simplicial approximation)**.** A simplicial map $f : K \to L$ is a *simplicial approximation* to g, if for all $x \in |K|$ and $\tau \in L$, $g(x) \in \tau$ implies $f(x) \in \tau$.



Rem 58. *f and g are homotopic.*

Def 59 (Subdivision)**.** A simplicial complex L is a *subdivision* of another simplicial complex K if they have the same underlying space, i.e. $|L| = |K|$, and every simplex in L is contained in a simplex in K.

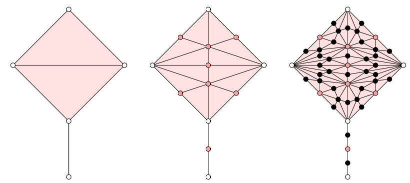
Exp 60 (Barycentric subdivision)**.** The *barycentric subdivision* of K, denoted as $\operatorname{Sd} K$, is defined as follows:
(i) The vertices are the barycenters of simplices of K:

$$\operatorname{Vert}\operatorname{Sd} K = \left\{ z(\sigma) = \textstyle\sum_{i=0}^{d} \frac{1}{d+1} v_i \;\middle|\; \sigma = \operatorname{conv}\{v_0, \ldots, v_d\} \in K \right\},$$

(ii) The k-simplices of $\operatorname{Sd} K$ are spanned by the barycenters:

$$\{z(\sigma_0), z(\sigma_1), \ldots, z(\sigma_k)\},$$

where $\sigma_0 \subsetneq \sigma_1 \subsetneq \ldots \subsetneq \sigma_k$ is a *chain* of proper faces in K.



Rem 61. *Every point $x \in |K|$ lies in the unique interior of a simplex $\sigma_x \in K$. All coefficients λ_v in $x = \sum_{v \in \sigma_x} \lambda_v v$ are strictly positive.*

Def 62 (Barycentric coordinates)**.** For each vertex $v \in \operatorname{Vert} K$, define the coordinate function $b_v : |K| \to [0,1]$

$$x \mapsto \begin{cases} \lambda_v, & \text{if } v \text{ is a vertex of } \sigma_x, \\ 0, & \text{otherwise.} \end{cases}$$

The functions b_v are the *barycentric coordinates* of K.

Rem 63. *Barycentric coordinates are continuous.*

Rem 64. *The unique simplex $\sigma_x \in K$ with $x \in \operatorname{int} \sigma_x$ is spanned by vertices $\{v \in \operatorname{Vert} K \mid b_v(x) > 0\}$.*

Def 65 (Open star)**.** The *open star* of $\sigma \in K$ is $\operatorname{st} \sigma = \bigcup_{\tau \in \operatorname{St} \sigma} \operatorname{int} \tau$.

Rem 66. *The open star is an open set: for a vertex v we have $\operatorname{st}(v) = \{x \in |K| \mid b_v(x) > 0\} = b_v^{-1}(0, \infty)$.*

Rem 67. *A vertex set S spans a simplex τ iff their open star intersection is non-empty, i.e. $\bigcap_{v \in S} \operatorname{st}(v) \neq \varnothing$, in which case we have $\operatorname{st}(\sigma) = \bigcap_{v \in S} \operatorname{st}(v)$.*

Def 68 (Star condition)**.** $g : |K| \to |L|$ satisfies the *star condition* if it maps open stars in K into open stars in L, i.e. $\forall v \in \operatorname{Vert} K\ \exists u \in \operatorname{Vert} L : g(\operatorname{st} v) \subseteq \operatorname{st} u$.

Rem 69. *Choosing such a vertex u_v for each vertex v defines a vertex map $\varphi : \operatorname{Vert} K \to \operatorname{Vert} L, v \mapsto u_v$, which extends to a simplicial approximation $f : K \to L$ of g.*

Def 70 (Mesh)**.** The *mesh* of K is the maximum length of its edges. It is also the maximum *diameter* of any simplex.

Rem 71. *Let δ be the mesh of K. Then the mesh of $\operatorname{Sd} K$ is at most $\frac{d}{d+1}\delta$, where d is the dimension of K.*

Thm 72 (Simplicial approximation)**.** *Let K, L be finite geometric simplicial complexes. Then g has a simplicial approximation $f : |\operatorname{Sd}^n K| \to |L|$ for some $n \in \mathbb{N}$.*

Complexes from geometric point sets

Homotopy equivalence

Weaker than homeomorphism. Let X, Y be topological spaces.

Def 73 (Homotopy equivalent)**.** Two spaces X, Y are *homotopy equivalent* ($X \simeq Y$) if $\exists$ continuous $f : X \to Y$, $g : Y \to X$ such that $f \circ g \simeq \mathrm{Id}_Y$ and $g \circ f \simeq \mathrm{Id}_X$.

Def 74 (Homotopy inverse)**.** g is a *homotopy inverse* to f and vice versa.

Rem 75. *Homotopy equivalence defines an equivalence relation of topological spaces.*

Consider $Y \subseteq X$. Then a continuous map $r : X \to Y$ is a *retraction* if r restricts to the identity map on Y. More generally:

Def 76 (Retraction)**.** Given $\iota : Y \to X$ continuous, a continuous $r : X \to Y$ is a *retraction* if $r \circ \iota = \mathrm{id}_Y$.

Def 77 (Deformation retraction)**.** The map $r : X \to Y$ is a *deformation retraction* if $\iota \circ r$ is homotopic to id_X.

Rem 78. *A deformation retraction can also be defined as a homotopy between a retraction and the identity map on X.*

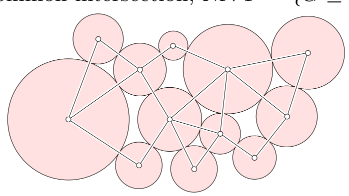
Def 79 (Contractible)**.** Assume $\exists$ deformation retraction $X \to Y$. If Y is a single point, then X is *contractible.*

Rem 80. *Two spaces X and Y are homotopy equivalent iff $\exists Z$ s.t. $\exists$ deformation retractions $Z \to X$ and $Z \to Y$.*

Nerves

We construct simplicial complexes by recording the intersection patterns of a collection of sets.

Def 81 (Nerve)**.** Let F be a finite collection of sets. The *nerve* of F consists of all subcollections whose sets have a non-empty common intersection, $\operatorname{Nrv} F = \{G \subseteq F \mid \bigcap G \neq \varnothing\}$.



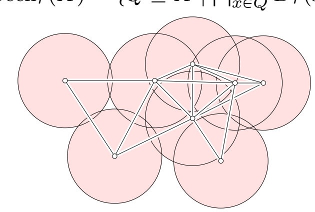
Rem 82. $\operatorname{Nrv} F$ *is always an abstract simplicial complex: if $\bigcap X \neq \varnothing$ and $Y \subseteq X$, then $\bigcap Y \neq \varnothing$ as well.*

Thm 83 (Nerve theorem for compact convex covers)**.** *Let X be compact. Assuming that F is a finite cover of X by compact convex sets. Then $|\operatorname{Nrv} F| \simeq X$.*

Čech complexes

Consider the case where F is a collection closed balls. Let $X \subset \mathbb{R}^d$ be a *finite point set.* Write $D_r(x) = \{y \in \mathbb{R}^d \mid \|y - x\| \leqslant r\}$.

Def 84 (Čech complex)**.** The Čech complex of X for radius r is defined as $\check{\mathrm{C}}\mathrm{ech}_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} D_r(x) \neq \varnothing\}$.



Rem 85. *Čech complexes are abstract simplicial complexes.*

Rem 86. *$\check{\mathrm{C}}\mathrm{ech}_r(X)$ is isomorphic to the never of the balls centered at the points in X.*

Rem 87. *If X is compact, then for every $r \geqslant \operatorname{diam} X$, the Čech complex is the full simplex, i.e.* $\operatorname{Cech}_r(X) = \operatorname{Cl} X$.

Rem 88. $Q = \{q_1, ..., q_n\} \in \operatorname{Cech}_r(X) \iff \exists z \in \bigcap_{q \in Q} D_r(q)$ $\iff Q \subseteq D_r(z) \iff$ *the smallest enclosing sphere of Q has radius $\leqslant r$.*

Enclosing spheres and circumspheres

Let $Q \subset \mathbb{R}^d$ be a finite point set and let D be a d-ball with boundary ($(d-1)$-sphere) S.

Def 89 (Enclosing sphere)**.** S is an *enclosing sphere* of Q, if $Q \subset D$. In this case, D is an *enclosing disk* of Q.

Def 90 (Circumsphere)**.** S is *circumsphere* of Q, if $Q \subset S$.

Rem 91. *Affinely independent Q has a circumsphere.*

Rem 92 (Smallest circumsphere)**.** *If $|Q| \leqslant d$, then its circumsphere is not unique. But the smallest circumsphere $S(Q)$ is unique and has center on the affine hull* $\operatorname{aff} Q$.

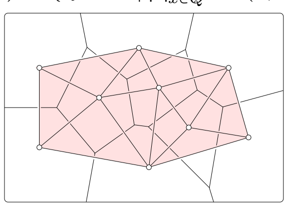
Rem 93. *Assume $B(Q, P)$ is the smallest circumsphere of P that encloses Q and $x \in Q$. If the sphere $B(Q \backslash \{x\}, P)$ encloses x, then $B(Q, P) = B(Q \backslash \{x\}, P)$, and otherwise $B(Q, P) = B(Q \backslash \{x\}, P \cup \{x\})$.*

Voronoi domains and Delaunay complexes

Let $X \subset \mathbb{R}^d$ be a finite point set.

Def 94 (Voronoi domain)**.** The *voronoi domain* of $x \in X$ is the set of points p in $\mathbb{R}^d$ that have x as the nearest neighbor in X: $\operatorname{Vor}(x, X) = \{p \in \mathbb{R}^d \mid \|p - x\| \leqslant \|p - y\| \text{ for all } y \in X\}$.

Def 95 (Delaunay complex)**.** The *Delaunay complex* of X is defined as $\operatorname{Del}(X) := \{Q \subseteq X \mid \bigcap_{x \in Q} \operatorname{Vor}(x, X) \neq \varnothing\}$.



Rem 96. *The Delaunay complex is isomorphic to the nerve of the Voronoi domains.*

Rem 97. $Q \in \operatorname{Del}(X) \implies \exists x \in \mathbb{R}^d, r > 0 : \|x - q\| = r \forall q \in Q$.

Rem 98. *$\varnothing \neq Q \subseteq X$ is a simplex of $\operatorname{Del}(X)$ iff Q has a circumsphere S ($Q \subset S$) bounding an open ball B that is empty of points in X: $X \cap B = \varnothing$.*

Def 99 (General spherical position)**.** X is in *general spherical position* if for every $Q \subseteq X$ of at most $d+1$ points, (i) Q is affinely independent; and (ii) the smallest circumsphere of Q contains no other points of X: $S(Q) \cap (X \backslash Q) = \varnothing$.

Thm 100. *Let X be in general spherical position. Then the geometric simplices* $\operatorname{GeomDel}(X) = \{\operatorname{conv} Q \mid Q \in \operatorname{Del}(X)\}$ *form a geometric simplicial complex. Its underlying space is the convex hull of the points X, i.e.* $|\operatorname{GeomDel(X)}| = \operatorname{conv} X$.

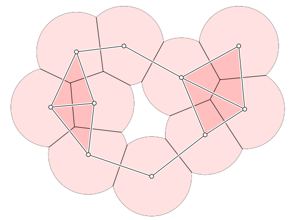
Def 101 (Lifting)**.** $l : \mathbb{R}^d \to \mathbb{R}^{d+1}, x \mapsto (x, \|x\|^2)$.

Lem 102. *A $(d-1)$-sphere $S \in \mathbb{R}^d$ corresponds to a unique hyperplane $H = \operatorname{aff} l(S) \subseteq \mathbb{R}^{d+1}$ with $l(S) = H \cap \operatorname{im} l$ (an ellipsoid). S is empty of X iff all points in $l(X)$ lie in the upper halfspace bounded by H.*

Cor 103. If $X \subset \mathbb{R}^d$ is in *general spherical position*, then $l(X)$ is in *general linear position*.

Delaunay Filtration

Def 104 (Delaunay complex)**.** The *Delaunay complex* of X *at radius* $r \geqslant 0$ is $\operatorname{Del}_r(X) = \{Q \subseteq X \mid \bigcap_{x \in Q} \operatorname{Vor}_r(x, X) \neq \varnothing\}$, where $\operatorname{Vor}_r(x, X) = D_r(x) \cap \operatorname{Vor}(x, X)$.



Rem 105. $\operatorname{Del}_r(X) \cong \operatorname{Nrv}\{\operatorname{Vor}_r(x, X) \mid x \in X\}$

Rem 106. *For sufficiently large r:* $\operatorname{Del}_r(X) = \operatorname{Del}(X)$.

Rem 107. *The Čech and Delaunay complexes for the same radius have homotopy equivalent geometric realizations:* $|\operatorname{Del}_r(X)| \simeq |\operatorname{Cech}_r(X)| \simeq \bigcup_{x \in X} D_r(x)$.

Rem 108. *$Q \subseteq X$ is a simplex of $\operatorname{Del}_r(X)$ iff the smallest circumsphere of Q excluding X has radius at most r.*

Rem 109. $\operatorname{Del}_r(X) \neq \operatorname{Del}(X) \cap \operatorname{Cech}_r(X)$.

Homology

Let K be a finite simplicial complex.

Def 110 (Euler characteristic)**.** Let k_i be the number of i-simplices. Then, the $\chi(K) = \sum_{i=0}^{\dim K} (-1)^i k_i$ is the *Euler characteristic*.

Rem 111. *If $|K| \simeq |L|$, then $\chi(K) = \chi(L)$.*

Def 112 (d-chain)**.** A *d-chain* is a *formal sum* of d-simplices $\sum_{\sigma \in K_{(d)}} \lambda_\sigma \sigma$ with coefficients in $\mathbb{Z}_2$.

Def 113 (Chain space)**.** The d-chains form a $\mathbb{Z}_2$-vector space $C_d(K)$, the *chain space*.

Rem 114. *A d-chain corresponds to a subset of d-simplices.*

Def 115 (Boundary map)**.** The *boundary* $\partial\sigma$ of a d-simplex σ is the $(d-1)$-chain of facets (faces of codimension 1) of σ. This extends linearly to a *boundary map* $\partial_d : C_d(K) \to C_{d-1}(K)$.

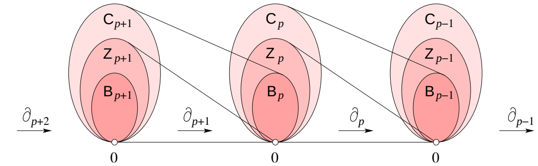
Def 116 (d-boundary)**.** A d-chain γ is called a *d-boundary* if $\gamma = \partial_{d+1}(\xi)$ for some $(d+1)$-chain ξ.

Rem 117. *The d-boundaries $B_d := \operatorname{im} \partial_{d+1}$ form a subspace of $C_d(K)$, containing boundaries of $(d+1)$ chains.*

Def 118 (d-cycle)**.** A d-chain γ is a d-cycle if $\partial_d(\gamma) = 0$.

Rem 119. *The d-cycles form a subspace $Z_d(K) := \ker \partial_d$ of , containing chains with zero boundary.*

Rem 120. *Boundaries are cycles: $B_d(K) \subseteq Z_d(K)$. In other words, $\partial_d \partial_{d+1}(\sigma) = 0$ for any $(d+1)$-chain σ.*



Def 121 (Homology)**.** The d-th *homology (space)* is the $\mathbb{Z}_2$-vector space $H_d(K) = Z_d(K)/B_d(K)$.

Def 122 (Betti number)**.** The dimension $\beta_d(K) = \dim H_d(K)$ is the d-th *Betti number*.

Rem 123. *The betti numbers in dimension 0,1 and 2 count the number of connected components, tunnels, and voids of K.*

Induced maps

A simplicial map induces a map between homology groups. Let $f : K \to L$ be a simplicial map.

Def 124 (Induced chain map)**.** The map f induces a chain map $f_\# : C_d(K) \to C_d(L)$ by specifying it on the basis, the d-simplices of K, i.e. $\sigma \mapsto f(\sigma)$ if $f(\sigma)$ is a d-simplex, otherwise 0.

Lem 125. *The map $f_\#$ induces a map $f_* : H_d(K) \to H_d(L)$ on homology.*

Rem 126. *The homology is a* functor *from simplicial complexes to vector spaces. In particular, commutative diagrams are preserved.*

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, "g \circ f"'] & L \arrow[d, "g"] \\ & M \end{tikzcd} \implies \begin{tikzcd} H_*(K) \arrow[r, "f_*"] \arrow[dr, "g_* \circ f_*"'] & L \arrow[d, "g_*"] \\ & M \end{tikzcd}$$

Thm 127. *Let f be a simplicial map and a homotopy equivalence. Then the induced homomorphism $H_*(f) : H_*(K) \to H_*(L)$ is an isomorphism.*

Computing homology

We want to find compatible basis for the subspaces.

Def 128 (Pivot)**.** Let M be a matrix. Then the *pivot index* of a column is the index of the last non-zero entry.

Def 129 (Reduced)**.** M is *reduced* if the pivots of non-zero columns are distinct.

Alg 130 (Matrix reduction)**.** Let D be the boundary matrix and V the identity matrix encoding column operations. As long as D is not reduced, we a left column to a right column of D and V simultaneously. Output: D and V.

Rem 131. *The columns $\{D_j \neq 0\}$ form a basis of $B_*(K)$.*

Rem 132. *The columns $\{V_i \mid D_i = 0\}$ form a basis of $Z_*(K)$.*

Rem 133. *The columns $\{V_i + B_d(K) \mid D_i = 0, i \notin \text{pivots } D\}$ form a basis for $H_*(K)$.*