Stochastic Calculus
Assignment III: Running Maximum

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Let B_t be a standard Brownian Motion starting from zero, and M_t be its running maximum. Calculate:

1. For a fixed level a, calculate the expectation of the first passage time T_a.

   The first passage time T_a for a fixed level a in a standard Brownian Motion B_t is defined as the first time that the process reaches or surpasses the level a. Mathematically, this is written as:

   $$T_a = \inf\{t > 0 : B_t \geq a\}$$

   The expectation of this random variable is given by:

   $$\mathbb{E}[T_a] = \int_0^\infty t\, f_{T_a}(t)\, dt$$

   where $f_{T_a}(t)$ is the probability density function of T_a, given the density function for the first passage time of a Brownian motion to level a:

   $$f_{T_a}(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right)$$

   Substituting the given density function into this integral, we have:

   $$\mathbb{E}[T_a] = \int_0^\infty t \cdot \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) dt$$

   This simplifies to:

   $$\mathbb{E}[T_a] = \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \exp\left(-\frac{a^2}{2t}\right) dt$$

   Note that since when $t \to \infty$ and $\lim_{t\to\infty} \exp(\frac{-a^2}{2t}) = 1$, there must $\exists T > 0$ s.t. $\exp(\frac{-a^2}{2t}) > \frac{1}{2}$ for all $t > T$, thus

   $$\mathbb{E}[T_a] = \frac{a}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{t}} \exp\left(-\frac{a^2}{2t}\right) dt \geq \int_T^\infty \frac{a}{\sqrt{2\pi t}} \frac{1}{2} dt = \frac{a}{\sqrt{2\pi}} \sqrt{t}|_T^\infty = \infty$$

   The expectation of the passage time T_a is infinite.

2. Calculate the quadratic variation of the running maximum M_t on the interval $[0, T]$.

   Defining the quadratic variation of a martingale M_t as:

$$\langle M_t, M_t \rangle = \lim_{|\Pi| \to 0} \sum_{i=0}^{n-1} (M(t_{i+1}) - M(t_i))^2$$

   The expression on the right hand side can be further simplified by applying an inequality:

$$\langle M_t, M_t \rangle \leq \lim_{|\Pi| \to 0} \sum_{i=0}^{n-1} (M(t_{i+1}) - M(t_i)) \max_{0 \leq i \leq n-1} |M(t_{i+1}) - M(t_i)|$$

   This can be simplified by extracting the maximum term out of the summation:

$$\langle M_t, M_t \rangle \leq \lim_{|\Pi| \to 0} M(T) \max_{0 \leq i \leq n-1} |M(t_{i+1}) - M(t_i)|$$

   As $|\Pi| \to 0$, $n \to \infty$, and given that M is a continuous process, we can conclude that:

$$\max_{0 \leq i \leq n-1} |M(t_{i+1}) - M(t_i)| \to 0$$

   Considering that $M(T)$ is known to be finite, this implies that:

$$\langle M_t, M_t \rangle = 0$$

3. Calculate the probability that $M_t > 2B_t$.

   Recall from class that the joint density of M_t and B_t, let's denote it $f_{M_t, B_t}(a, x)$, is given by:

$$f_{M_t, B_t}(a, x) = \frac{2(2a - x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a - x)^2}{2t}\right)$$

   We express this probability as a double integral over the region where a is greater than twice x, integrating the joint probability density function of M_t and B_t.

$$\mathbb{P}[M_t > 2B_t] = \sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}} \int_0^\infty \int_{-\infty}^{a/2} (2a - x) \exp\left(-\frac{(2a - x)^2}{2t}\right) dx da$$

   The inner integral is transformed from an integral over x to an integral over b using the substitution $b = x - 2a$. We also express the exponential term in terms of b.

$$\mathbb{P}[M_t > 2B_t] = \sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}} \int_0^\infty \int_{-\infty}^{-3a/2} -b \exp\left(-\frac{b^2}{2t}\right) db da$$

   Next, we simplify the double integral into a single integral over a.

$$\mathbb{P}[M_t > 2B_t] = \sqrt{\frac{2}{\pi}} t^{-\frac{3}{2}} \int_0^\infty t \exp\left(-\frac{9a^2}{8t}\right) da$$

   We then simplify the constant term in front of the integral.

$$\mathbb{P}[M_t > 2B_t] = \sqrt{\frac{2}{\pi}} t^{-\frac{1}{2}} \int_0^\infty \exp\left(-\frac{9a^2}{8t}\right) da$$

   We evaluate the remaining integral, yielding a constant.

$$\mathbb{P}[M_t > 2B_t] = \sqrt{\frac{2}{\pi}} t^{-\frac{1}{2}} \sqrt{\frac{2\pi t}{3}}$$

   Finally, we simplify the entire expression, yielding $2r/3$.

$$\mathbb{P}[M_t > 2B_t] = \frac{2}{3}$$

4. Calculate the probability density function of M_t.

Recall from class that the joint density of M_t and B_t, let's denote it $f_{M_t,B_t}(a,x)$, is given by:

$$f_{M_t,B_t}(a,x) = \frac{2(2a-x)}{\sqrt{2\pi t^3}} \exp\left(-\frac{(2a-x)^2}{2t}\right)$$

The density of M_t, denoted as $f_{M_t}(a)$, can be obtained by integrating over the joint density:

$$f_{M_t}(a) = \int_{-\infty}^{a} \frac{2(2a-x)}{t\sqrt{2\pi t}} \exp\left(-\frac{(2a-x)^2}{2t}\right) dx$$

Note that this integral is defined as such because $M_t \geq B_t$.

Substituting $u = -\frac{(2a-x)^2}{2t}$, the differential dx can be rewritten as:

$$dx = -\frac{t}{2a-x} du$$

The density of M_t can then be rewritten as:

$$\sqrt{\frac{2}{\pi t}} \int_{-\infty}^{-\frac{a^2}{2t}} \exp(u) du = \sqrt{\frac{2}{\pi t}} \left[\exp(u)\right]_{-\infty}^{-\frac{a^2}{2t}}$$

Given that $\exp(-\infty)$ is zero, the result is:

$$\sqrt{\frac{2}{\pi t}} \int_{-\infty}^{-\frac{a^2}{2t}} \exp(u) du = \sqrt{\frac{2}{\pi t}} \exp\left(-\frac{a^2}{2t}\right)$$

This gives the required density function for M_t.