# Stochastic Calculus
## Assignment II: Properties of the Brownian motion

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1. Let $B_t$ be a Brownian Motion starting from zero. Prove that for any fixed constant $\sigma > 0$, the process

$$X_t = e^{\sigma B_t - \frac{\sigma^2 t}{2}}, \quad t \geq 0$$

is a martingale with respect to the filtration generated by Brownian Motion $B_t$.

To show that a process $X_t$ is a martingale with respect to a given filtration, we need to demonstrate three conditions:

- $X_t$ is adapted to the filtration.
- $X_t$ is integrable, meaning $\mathbb{E}|X_t| < \infty$ for all $t$.
- $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $s \leq t$.

The first condition is satisfied since $X_t$ is a process generated by Brownian Motion $B_t$, thus it is adapted to the filtration generated by $B_t$.

Given $X_t = e^{\sigma B_t - \frac{\sigma^2 t}{2}}$, we have:

$$\mathbb{E}|X_t| = \mathbb{E}[X_t] = \mathbb{E}[e^{\sigma B_t - \frac{\sigma^2 t}{2}}].$$

Note that if $B_t$ is a standard Brownian motion, then $\sigma B_t - \frac{\sigma^2 t}{2}$ is normally distributed with mean $-\frac{\sigma^2 t}{2}$ and variance $\sigma^2 t$. This means that $e^{\sigma B_t - \frac{\sigma^2 t}{2}}$ is log-normally distributed.

The expected value of a log-normally distributed variable is given by $\exp\left(\mu + \frac{1}{2}\sigma^2\right)$, where $\mu$ is the mean and $\sigma^2$ is the variance. Therefore, we have:

$$\mathbb{E}[X_t] = \exp\left(-\frac{\sigma^2 t}{2} + \frac{1}{2}\sigma^2 t\right) = \exp(0) = 1.$$

Therefore, $\mathbb{E}|X_t| < \infty$ for all $t$, so $X_t$ is integrable.

Let's calculate $\mathbb{E}[X_t|\mathcal{F}_s]$. We know that given $\mathcal{F}_s$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$ and normally distributed with mean 0 and variance $t - s$. Therefore, we can write:

$$\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}\left[e^{\sigma B_s + \sigma(B_t - B_s) - \frac{\sigma^2 t}{2}}|\mathcal{F}_s\right].$$

Since $B_s$ is $\mathcal{F}_s$-measurable and $B_t - B_s$ is independent of $\mathcal{F}_s$, we have:

$$\mathbb{E}[X_t|\mathcal{F}_s] = e^{\sigma B_s - \frac{\sigma^2 s}{2}}\mathbb{E}\left[e^{\sigma(B_t - B_s) - \frac{\sigma^2(t-s)}{2}}\right].$$

Now note that $\sigma(B_t - B_s) - \frac{\sigma^2(t-s)}{2}$ is normally distributed with mean $-\frac{\sigma^2(t-s)}{2}$ and variance $\sigma^2(t-s)$. Therefore, we have:

$$\mathbb{E}[X_t|\mathcal{F}_s] = e^{\sigma B_s - \frac{\sigma^2 s}{2}} \exp\left(-\frac{\sigma^2(t-s)}{2} + \frac{1}{2}\sigma^2(t-s)\right) = X_s.$$

Therefore, the process $X_t$ is a martingale with respect to the filtration generated by the Brownian motion $B_t$.

2. The first variation of function $f(t)$ on the interval $[0, T]$ is defined as:

$$F_V(f) = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} |f(t_{i+1}) - f(t_i)|$$

Estimate what is the first variation of Brownian Motion.

Recall from class that

$$\lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 = T$$

Given that

$$\sum_{i=0}^{n-1} (B(t_{i+1}) - B(t_i))^2 \leq \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)| \max_{0<i<n-1} (B(t_{i+1}) - B(t_i))$$

Taking limits on both sides for $||\Pi|| \to 0$, we have

$$T \leq \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)| \max_{0<i<n-1} (B(t_{i+1}) - B(t_i))$$

Since Brownian motion is continuous, we have $\lim_{||\Pi|| \to 0} \max_i (B(t_{i+1}) - B(t_i)) = 0$, therefore we must have $F_V(B) = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} |B(t_{i+1}) - B(t_i)| = \infty$

3. Let $B(t)$ be a Brownian Motion and let $s < t$. Compute

$$\mathbb{E}(B(t) - B(s))^4$$

Given a Brownian Motion $B(t)$, and let $s < t$. The fourth central moment (or the fourth moment about the mean) of a normal distribution is given by $\mu_4 = 3\sigma^4$ where $\sigma$ is the standard deviation.

The increment of a Brownian motion $B(t) - B(s)$ is distributed normally with mean zero and variance $(t - s)$ (because Brownian motion has independent increments with variance proportional to the increment's length). Therefore, the fourth moment of $B(t) - B(s)$ is given by:

$$\mathbb{E}(B(t) - B(s))^4 = 3(t - s)^2$$