Stochastic Calculus
Assignment I: Symmetric Random Walk, Characteristic Function

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1. Let $M(t)$ be a symmetric random walk. Check if $M(t)^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t = \sigma(X_1, X_2, \ldots, X_t)$.

   We will check if the process $N(t) = M(t)^2 - t$ is a martingale with respect to $\mathcal{F}_t$.

   (a) $N(t)$ is $\mathcal{F}_t$-measurable for each t

   Since $N(t)$ is a function of the $\mathcal{F}_t$-measurable process $M(t)$, it is $\mathcal{F}_t$-measurable for each t.

   (b) $\mathbb{E}|N_t| < \infty$ for all t

   Since $M(t)$ is bounded by $[-t, t]$, we have $M(t)^2$ bounded by $[0, t^2]$. Hence, $\mathbb{E}[M(t)^2]$ exists and is finite.

   The value t is a deterministic number, thus $|t|$ is obviously finite.

   So, $\mathbb{E}|N_t| = \mathbb{E}|M(t)^2 - t| < \infty$ for all t, as both $M(t)^2$ and t have finite expectations.

   (c) $\mathbb{E}[N_t|\mathcal{F}_s] = N_s$ for all $s \leq t$.

   We have $N(t) = M(t)^2 - t$, so we need to compute the conditional expectation $\mathbb{E}[M(t)^2 - t|\mathcal{F}_s]$ and check if it equals $M(s)^2 - s$.

   Given that $M(s)$ is $\mathcal{F}_s$-measurable. This means that $\mathbb{E}[M(s)|\mathcal{F}_s] = M(s)$. Similarly, $M(s)^2$ is also $\mathcal{F}_s$-measurable. This means that $\mathbb{E}[M(s)^2|\mathcal{F}_s] = M(s)^2$.

   Also, we note that:

$$\mathbb{E}[(\sum_{i=s+1}^{t} X_i)^2|\mathcal{F}_s] = \mathbb{E}[\sum_{i=s+1}^{t} X_i^2 + \sum_{i\neq j} X_i X_j|\mathcal{F}_s] = \mathbb{E}[\sum_{i=s+1}^{t} 1 + 0|\mathcal{F}_s] = t - s$$

   We can then write

$$\begin{aligned}
\mathbb{E}[N(t)|\mathcal{F}_s] &= \mathbb{E}[(M(s) + \sum_{i=s+1}^{t} X_i)^2 - t|\mathcal{F}_s] \\
&= \mathbb{E}[M(s)^2 + 2M(s)\sum_{i=s+1}^{t} X_i + (\sum_{i=s+1}^{t} X_i)^2 - t|\mathcal{F}_s] \\
&= M(s)^2 + 2M(s)\mathbb{E}[\sum_{i=s+1}^{t} X_i|\mathcal{F}_s] + \mathbb{E}[(\sum_{i=s+1}^{t} X_i)^2|\mathcal{F}_s] - t \\
&= M(s)^2 - s
\end{aligned}$$

   The term $2M(s)\mathbb{E}[\sum_{i=s+1}^{t} X_i|\mathcal{F}_s]$ equals zero because each X_i for $i > s$ is independent of $\mathcal{F}_s$ and has expectation zero.

This shows that the process $N(t) = M(t)^2 - t$ is indeed a martingale with respect to $\mathcal{F}_t$.

2. In class we defined that characteristic function $\phi_X(u) = \mathbb{E}\exp(iuX)$

a. Let X_i, $1 \le i \le n$, be independent identically distributed random variables, and $M(n) = \sum_{i=1}^n X_i$. Prove that $\phi_{M(n)}(u) = (\phi_X(u))^n$.

The characteristic function of a sum of independent random variables is the product of their individual characteristic functions. So, we can write

$$\begin{aligned}
\phi_{M(n)}(u) &= \mathbb{E}[\exp(iuM(n))] \\
&= \mathbb{E}[\exp(iu\sum_{i=1}^n X_i)] \\
&= \mathbb{E}[\prod_{i=1}^n \exp(iuX_i)] && \text{(by properties of exponentials)} \\
&= \prod_{i=1}^n \mathbb{E}[\exp(iuX_i)] && \text{(by independence of } X_i\text{'s)} \\
&= \prod_{i=1}^n \phi_X(u) && \text{(by definition of } \phi_X(u)\text{)} \\
&= (\phi_X(u))^n
\end{aligned}$$

This concludes the proof.

b. Let X be a Bernoulli random variable taking values 0 and 1:

$$X = \begin{cases} 1, & \text{with probability } p \\ 0, & \text{with probability } 1-p \end{cases}$$

What is the characteristic function of X?

The characteristic function $\phi_X(u)$ of a random variable X is given by $\mathbb{E}[\exp(iuX)]$.

Given that X is a Bernoulli random variable taking values 0 and 1, we have:

$$\begin{aligned}
\phi_X(u) &= \mathbb{E}[\exp(iuX)] \\
&= \exp(iu \cdot 0) \cdot \mathbb{P}(X = 0) + \exp(iu \cdot 1) \cdot \mathbb{P}(X = 1) && \text{(by definition of expectation)} \\
&= 1 \cdot (1-p) + \exp(iu) \cdot p \\
&= (1-p) + p\exp(iu).
\end{aligned}$$

So, the characteristic function of a Bernoulli random variable X is $\phi_X(u) = (1-p) + p\exp(iu)$.

c. Assume that $p = p_n = \frac{\lambda}{n}$ for a given constant $\lambda > 0$. What is the characteristic function of $M(n)$? What is the limit of it as $n \to \infty$?

$$\lim_{n\to\infty} \phi_{M(n)}(u) = \lim_{n\to\infty} \left((1 - \frac{\lambda}{n}) + \frac{\lambda}{n}\exp(iu)\right)^n,$$

which can be rearranged into the form of the definition of the exponential function:

$$\begin{aligned}
\lim_{n\to\infty} \phi_{M(n)}(u) &= \lim_{n\to\infty} \left(1 + \frac{\lambda\exp(iu) - \lambda}{n}\right)^n \\
&= \lim_{n\to\infty} \left(1 + \frac{\lambda(\exp(iu) - 1)}{n}\right)^n \\
&= \exp\left(\lambda(\exp(iu) - 1)\right),
\end{aligned}$$

Hence, $\lim_{n\to\infty} \phi_{M(n)}(u) = \exp(\lambda(\exp(iu) - 1))$.

d. Compute the characteristic function of a Poisson random variable with parameter λ. What can you say about the limiting distribution of $M(n)$ based on this information?

The characteristic function of a Poisson random variable Y with parameter λ can be found using the definition of the characteristic function and the probability mass function of the Poisson distribution.

The probability mass function of a Poisson random variable is given by

$$P(Y = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2, \ldots.$$

Then the characteristic function of Y is given by

$$\begin{aligned}
\phi_Y(u) &= \mathbb{E}[e^{iuY}] \\
&= \sum_{k=0}^{\infty} e^{iuk} P(Y = k) \\
&= \sum_{k=0}^{\infty} e^{iuk} \cdot \frac{\lambda^k e^{-\lambda}}{k!} \\
&= e^{-\lambda} \sum_{k=0}^{\infty} \frac{(e^{iu}\lambda)^k}{k!} \\
&= e^{-\lambda} e^{\lambda e^{iu}} \\
&= \exp(\lambda(e^{iu} - 1)).
\end{aligned}$$

In the last step, we have used the power series expansion of the exponential function.

We previously found that $\lim_{n\to\infty} \phi_{M(n)}(u) = \exp(\lambda(\exp(iu) - 1))$. This matches exactly with the characteristic function of a Poisson random variable with parameter λ. This means that the limiting distribution of $M(n)$ as n goes to infinity is a Poisson distribution with parameter λ. This is a specific instance of the Poisson limit theorem.