Stochastic Calculus
Assignment IV: Stochastic Integral

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1. Let B_t be a standard Brownian motion starting from zero, ε be a number in $[0, 1]$ and $\Pi = \{t_0, t_1, \ldots, t_n\}$ be a partition of $[0, t]$ with $0 = t_0 < t_1 < \ldots < t_n = t$. Consider the approximating sum

$$S_\varepsilon^\Pi = \sum_{i=0}^{n-1}((1-\varepsilon)B(t_i) + \varepsilon B(t_{i+1}))(B(t_{i+1}) - B(t_i))$$

for the stochastic integral $\int_0^t B_s\, dB_s$. Show that

$$\lim_{||\Pi||\to 0} S_\varepsilon^\Pi = \frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$$

where the limit is in L^2. Show that the right-hand side of the above identity is a martingale if and only if $\varepsilon = 0$.

Note that the quadratic variation for Brownian motion is

$$\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 = t, \quad \text{(Lecture 3 Theorem 4)}$$

and that

$$\sum_{i=0}^{n-1} B(t_{i+1})^2 - B(t_i)^2 = B(t_n)^2 - B(t_0)^2 = B(t)^2.$$

Hence, we have

$$\begin{aligned}
\lim_{||\Pi||\to 0} S_\varepsilon^\Pi &= \lim_{n\to\infty}\sum_{i=0}^{n-1}((1-\varepsilon)B(t_i) + \varepsilon B(t_{i+1}))(B(t_{i+1}) - B(t_i)) \\
&= \lim_{||\Pi||\to 0}\sum_{i=0}^{n-1}(1-\varepsilon)B(t_i)(B(t_{i+1}) - B(t_i)) + \varepsilon B(t_{i+1})(B(t_{i+1}) - B(t_i)) \\
&= (1-\varepsilon)\lim_{||\Pi||\to 0} B(t_i)(B(t_{i+1}) - B(t_i)) + \varepsilon \lim_{||\Pi||\to 0} B(t_{i+1})(B(t_{i+1}) - B(t_i)) \\
&= (1-\varepsilon)\lim_{||\Pi||\to 0} B(t_i)B(t_{i+1}) - B(t_i)^2 + \varepsilon \lim_{||\Pi||\to 0} B(t_{i+1})^2 - B(t_i)B(t_{i+1}) \\
&= (1-\varepsilon)\left(-\frac{1}{2}\lim_{||\Pi||\to 0}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 + \frac{1}{2}\lim_{||\Pi||\to 0}\sum_{i=0}^{n-1}(B(t_{i+1})^2 - B(t_i)^2)\right) \\
&\quad + \varepsilon\left(\frac{1}{2}\lim_{||\Pi||\to 0}\sum_{i=0}^{n-1}(B(t_{i+1}) - B(t_i))^2 + \frac{1}{2}\lim_{||\Pi||\to 0}\sum_{i=0}^{n-1}(B(t_{i+1})^2 - B(t_i)^2)\right)
\end{aligned}$$

Now substitute in $\sum_{i=0}^{n-1}(B(t_{i+1})-B(t_i))^2 = t$ and $\sum_{i=0}^{n-1} B(t_{i+1})^2 - B(t_i)^2 = B(t_n)^2 - B(t_0)^2 = B_t^2$.

$$\lim_{||\Pi||\to 0} S_\varepsilon^\Pi = (1-\varepsilon)(-\frac{1}{2}(t - B(t)^2)) + \varepsilon\frac{1}{2}(t + B(t)^2) = \frac{1}{2}B(t)^2 + \varepsilon t - \frac{1}{2}t$$

Now we want to show that the right-hand side of the above identity is a martingale if and only if $\varepsilon = 0$.

($\Rightarrow$) When $\varepsilon = 0$, we want to show that the right-hand side of the above identity is a martingale.

We have shown in the previous part that if $\varepsilon = 0$, the right-hand side of the above identity is $\frac{1}{2}B(t)^2 - \frac{1}{2}t$.

We will check if the process $N(t) = \frac{1}{2}B(t)^2 - \frac{1}{2}t$ is a martingale with respect to $\mathcal{F}_t$.

(a) $N(t)$ is $\mathcal{F}_t$-measurable for each t
Since $N(t)$ is a function of the $\mathcal{F}_t$-measurable process $B(t)$, it is $\mathcal{F}_t$-measurable for each t.

(b) $\mathbb{E}|N_t| < \infty$ for all t
It is known that $\mathbb{E}[B(t)^2 = t]$, then since t is a positive finite number, we have

$$\mathbb{E}[\frac{1}{2}B(t)^2 - \frac{1}{2}t] \le \frac{1}{2}\mathbb{E}[B(t)^2 = t] = \frac{t}{2} \le \infty$$

(c) $\mathbb{E}[N(t)|\mathcal{F}_s] = N_s$ for all $s \le t$.

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[B(t)^2 - t \mid \mathcal{F}_s\right] &= \frac{1}{2}[\mathbb{E}\left[B(t)^2 \mid \mathcal{F}_s\right] - t] \\
&= \frac{1}{2}[\mathbb{E}\left[(B(t) - B(s) + B(s))^2 \mid \mathcal{F}_s\right] - t] \\
&= \frac{1}{2}[\mathbb{E}\left[(B(t) - B(s))^2 \mid \mathcal{F}_s\right] + 2B(s)\mathbb{E}\left[(B(t) - B(s)) \mid \mathcal{F}_s\right] + B(s)^2 - t] \\
&= \frac{1}{2}[(t-s) + 0 + B(s)^2 - t] \\
&= \frac{1}{2}B(s)^2 - \frac{1}{2} \\
&= N(s)
\end{aligned}$$

($\Leftarrow$) When the right-hand side of the above identity is a martingale, we want to show that $\varepsilon = 0$.

Given that $\frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$ is a martingale, we have

$$\mathbb{E}\left[\frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t\right] < \infty$$

which implies that $|\varepsilon| < \infty$.

Since it's a martingale, the expectation should not change with time, so for every $s \le t$, we have:

$$\mathbb{E}\left[\frac{1}{2}B_t^2 - \left(\varepsilon - \frac{1}{2}\right)t \mid \mathcal{F}_s\right] = \frac{1}{2}B_s^2 - \left(\varepsilon - \frac{1}{2}\right)s$$

Rewrite the above equation we can derive:

$$\frac{1}{2}\mathbb{E}\left[B_t^2 \mid \mathcal{F}_s\right] - \left(\varepsilon - \frac{1}{2}\right)t = \frac{1}{2}B_s^2 - \left(\varepsilon - \frac{1}{2}\right)s$$

Or equivalently,

$$\mathbb{E}\left[B_t^2 \mid \mathcal{F}_s\right] = 2\left(\frac{1}{2}B_s^2 - \left(\varepsilon - \frac{1}{2}\right)s\right) + 2\left(\varepsilon - \frac{1}{2}\right)t$$

Then simplifying the equation, we get

$$\mathbb{E}\left[B_t^2 \mid \mathcal{F}_s\right] = B_s^2 - (2\varepsilon - 1)(t - s)$$

The right-hand side can be rewritten as $t - s + B_s^2$. So we have:

$$t - s + B_s^2 = \mathbb{E}\left[B_t^2 \mid \mathcal{F}_s\right] = B_s^2 - (2\varepsilon - 1)(t - s)$$

By equating the terms on both sides of the equation, we get

$$2\varepsilon(t - s) = 0$$

This equation holds for every $s \leq t$ only when $\varepsilon = 0$.

Hence, we conclude that $\frac{1}{2}B_t^2 + \left(\varepsilon - \frac{1}{2}\right)t$ is a martingale if and only if $\varepsilon = 0$.

2. Let $\sigma(t)$ be a deterministic function of time, β be a constant and define

$$X(T) = \int_0^T \sigma(t)e^{-\beta t}\, dB_t.$$

Find the expectation and variance of $X(T)$. What is the distribution of $X(T)$?

Since the Itô integral is a martingale by Lecture 4, Theorem 2, one has $\mathbb{E}[X(T)] = 0$.

Given that:

$$\mathbb{E}[I_t^2(f)] = \mathbb{E}\left[\int_0^t f^2(s,\omega)\, ds\right]$$

Then, we write the variance of $X(T)$ as follows:

$$\begin{aligned}
\text{Var}[X(T)] &= \mathbb{E}[X(T)^2] - \mathbb{E}[X(T)]^2 \\
&= \mathbb{E}[X(T)^2] \\
&= \mathbb{E}\left[\left(\int_0^T \sigma(t)e^{-\beta t} dB_t\right)^2\right] \\
&= \mathbb{E}\left[\int_0^T \sigma(t)^2 e^{-2\beta t} dt\right] \\
&= \int_0^T \mathbb{E}\left[\sigma(t)^2 e^{-2\beta t}\right] dt
\end{aligned}$$

since $\sigma(t)$ is deterministic and not a random variable, it's not affected by the expectation operation, therefore, we can conclude that:

$$\int_0^T \mathbb{E}\left[\sigma(t)^2 e^{-2\beta t}\right] dt = \int_0^T \sigma(t)^2 e^{-2\beta t} dt$$

Therefore, we can finally write that:

$$\text{Var}[X(T)] = \int_0^T \sigma(t)^2 e^{-2\beta t} dt$$

Lecture 4, Equation 16 state that:

$$I(t) = \int_0^t f(s,\omega)\, dB_s = \sum_i e_i \Delta B_i$$

where e_i are elementary differentials, and ΔB_i are increments of the Brownian motion.

With this in mind, we can interpret an Itô integral as a linear combination of normal variables, due to the fact that each increment ΔB_i follows a Gaussian distribution by definition.

It is a well-established theorem in statistics that a linear combination of normally distributed random variables retains a normal distribution. Consequently, this imparts a normal distribution on the Itô integral $X(T)$. Formally, we have:

$$X(T) \sim N\left(0, \int_0^T \sigma(t)^2 e^{-2\beta t} dt\right)$$

3. Prove directly from the definition of the Itô integral that

$$\int_0^T t\, dB_t = TB_T - \int_0^T B_t\, dt.$$

Hint: Note that

$$\sum_i \Delta(s_i B_i) = \sum_i s_i \Delta B_i + \sum_i B_{i+1} \Delta s_i.$$

Let's start with a given partition $0 = t_0 < t_1 < \cdots < t_n = T$ of $[0, T]$. We express the integral as a limit of Riemann sums. The integral can be written as

$$\int_0^T t\, dB_t \approx \sum_{i=0}^{n-1} t_i (B_{t_{i+1}} - B_{t_i}),$$

If we take finer and finer grid by taking $\lim_{||\Pi|| \to 0}$, we will have:

$$\int_0^T t\, dB_t = \lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} t_i (B_{t_{i+1}} - B_{t_i}),$$

Note that the hint said that

$$\sum_i \Delta(t_i B_i) = \sum_i t_i \Delta B_i + \sum_i B_{i+1} \Delta t_i = \sum_i t_i (B_{t_{i+1}} - B_{t_i}) + \sum_i B_{i+1}(t_{i+1} - t_i)$$

Also, we note that

$$\lim_{||\Pi|| \to 0} \sum_{i=0}^{n-1} B_{i+1}(t_{i+1} - t_i) = \int_0^T B_t\, dt$$

by the definition of the Riemann integral.

Lastly, we were left to dealt with $\sum_i \Delta(t_i B_i)$ as $\lim_{||\Pi|| \to 0}$, since

$$\lim_{||\Pi|| \to 0} \sum_i \Delta(t_i B_i) = \int_0^T 1, dt B_t = tB_t \mid_0^t = TB_T$$

Substitute this back to our equation, and we will obtain that

$$\int_0^T t\, dB_t = TB_T - \int_0^T B_t\, dt.$$

Which completes the proof.