A brief transgression into NONLINEAR CONTROL THEORY

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Notes prepared for a semester course of Selected Topics of Nonlinear Systems.

This notes were prepared on-the-fly. So we ask apologies in advance for any omitted citation.

COURSE DISCLAIMER

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The objective of this course is twofold. One one hand, give a brief tour into some tools for analysis of stability and robustness of nonlinear systems. On the other hand, to introduce some well-stablished methods for control design.

We have in mind these two objectives under the premise that the student understands the importance of automatic control in modern technology. To improve your awareness of this fact, your first assignment will be to read the compendium "The impact of control technology" [1]. Please, pick 2 themes in Part 1, 4 themes in Part 2, 1 theme in Part 3 and 3 themes in Part 4 and prepare a brief report on them.

Having said this, this course will be theoretical. In each session we will cover a topic and provide some problems that you need to solve. I will also ask you to read and report on selected papers that will be considered as homework. The evaluation will be performed according to the following criterion:

2 or 3 oral examinations	50 %
Homework	20 %
Final Project	30 %

There are two kinds of final projects. In the first one, you apply some technique from Control Theory to your thesis project and prepare a presentation about it. In the second one, you pick one published research paper

[1] T. Samad and A.M. Annaswamy. The impact of control technology. IEEE Control Systems Society, available at http://ieeecss.org/general/impact-control-technology, January 2011.

and also prepare a presentation about it. In both cases you need my explicit consent about the topic you have selected.

Throughout the course, there will be one or two “extra point” problems. If you solve them correctly, prepare a report and give it to me, you will have an extra 5 % to your final grade.

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STABILITY

1.1 Brief recall on Lyapunov Stability

Consider the autonomous system

$$\dot{x} = f(x), x(0) = x_0; \quad x \in U \subseteq \mathbb{R}^n, \tag{1.1}$$

where f is a locally Lipschitz vector field in the domain U. Suppose that system (1.1) has an equilibrium position (Fig. 1) and choose coordinates x_i such that the equilibrium position is at the origin $f(0) = 0$. Let $x(t)$ denote the solution of (1.1) at time t starting at x_0.

PROBLEM 1. Show that when f is globally Lipschitz, solutions to system (1.1) are unique and exists for $t \geq 0$. HINT: see any textbook on differential equations about existence and unicity of solutions.

The solution with initial condition $x_0 = 0$ is just $x(t) = 0$, and we are interested in the behavior of solutions with neighboring initial conditions.

Definition 1.1. *An equilibrium position $x = 0$ is:*

- *stable (in Lyapunov's sense) if given any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\|x_0\| \leq \delta \Rightarrow \|x(t)\| \leq \varepsilon \text{ for all } t > 0.$$

- *unstable if it is not stable.*
- *asymptotically stable if it is stable and δ can be chosen such that*

$$\|x_0\| \leq \delta \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

Lyapunov stability can be interpreted as the continuity of the L^∞-norm of $x(t), t \geq 0$, with respect to the norm of the initial condition $\|x_0\|$, Fig 2. With these metrics, a small initial condition produces a small state trajectory.

PROBLEM 2. Investigate the Lyapunov stability of the equilibrium position of the following systems: a) $\dot{x} = 0$; b) $\dot{x}_1 = x_2, \dot{x}_2 = -x_1$; c) $\dot{x}_1 = x_1, \dot{x}_2 = -x_2$.

1.1.1 The Lyapunov function.

Theorem 1.1. *Let $x = 0$ be an equilibrium position for (1.1). Let $V : U \to \mathbb{R}$ be a continuous differentiable function such that $V(0) = 0$ and $V(x) > 0$ for all $x \in U \setminus \{0\}$. Then:*

1. *if $\dot{V}(x) \leq 0$ for all $x \in U$, then $x = 0$ is stable.*
2. *if $\dot{V}(x) < 0$ in $U \setminus \{0\}$, then $x = 0$ is asymptotically stable (AS).*

A function V, such that $V(0) = 0$ and $V(x) > 0$ is said to be positive definite. When V satisfies the conditions of the Theorem above, we say that it is "a Lyapunov function for the system in the domain U". In particular, when Condition 2 is satisfied, we say that the function is "strong".

The (Lie) derivative of the function V along the vector field f is the expression

$$L_f V(x) = \langle \nabla V(x), f(x) \rangle = \frac{\partial V(x)}{\partial x} f(x)$$

where the gradient $\frac{\partial V}{\partial x}$ is taken (by convention) as a row vector. With this notation, that the (total) derivative of $V(x(t))$ with respect to time is also given by

$$\dot{V}(x(t)) = L_f V(x(t)).$$

Proof. Given any $\varepsilon > 0$, there exists $\delta' > 0$ small enough such that the set

$$V^{-1}(\delta) = \{x \in U | V(x) = \delta\}$$

is contained inside the ball $\{x \in \mathbb{R}^n | \|x\| \leq \varepsilon\}$.

Then condition 1, implies that $\langle \nabla V(x), f(x) \rangle \leq 0$ for all $x \in V^{-1}(\delta)$. This means that the projection of f along the vector ∇V is always negative (or zero), showing that $f(x)$ (is tangent) or points towards the inside of $V^{-1}(\delta')$, see Fig. 3.

Then any trajectory that started inside $V^{-1}(\delta')$ cannot leave this set. Then, picking δ such that $\{x \in \mathbb{R}^n | \|x\| \leq \delta\}$ is inside $V^{-1}(\delta')$ completes the proof of the first claim.

The proof of the second part is analogous. □

There are several useful interpretations of Lyapunov's theorem. For the proof above, we have given a very geometric one. The essential part is the construction of an structure of rings $V^{-1}(\delta')$ that covers all $\mathbb{R}^n$ and over each ring the condition $\langle \nabla V(x), f(x) \rangle \leq 0$ implies that the vector field points inside it.

Another useful interpretation is more "energy-based". Where $V(x)$ can be thought as a potential well were the trajectories of the system slide. Then the condition $\dot{V} < 0$ means that the trajectories slide towards the point of minimal energy, see Figure 4.

PROBLEM 3. Consider $\dot{x}_1 = -x_1, \dot{x}_2 = -x_2$ with $V(x) = x_1^2 + x_2^2$. Show (and draw) that for any $x \in \mathbb{R}^2 \setminus \{0\}$, the vectors $\nabla V(x)$ and $f(x)$ points towards opposite directions. Therefore, the system is asymptotically stable and V is a Lyapunov function.

In general, one usually arrives to inequalities like $\dot{V} \leq -V$ or, more generally, to $\dot{V} \leq -V^\alpha$ for some $\alpha > 0$. When the Lyapunov function satisfies this kind of inequalities it also tells information about how fast the trajectories converge. For instance, when $\dot{V} \leq -V$, the convergence is exponential, i.e., there exists constants $C, \beta > 0$ such that

$$\|x(t)\| \leq Ce^{-\beta t}.$$

PROBLEM 4. Show that when V is a Lyapunov function for the system and satisfies $\dot{V} \leq -V^{1/2}$ the system is finite-time convergent, i.e., there exists time $T \geq 0$ such that $x(t) = 0, \forall t \geq T$. HINT: use the "comparison lemma".

Some other remarks to make. First, note that we have assumed certain differentiability of the Lyapunov function. It is possible to remove this condition (e.g., making it absolutely continuous only) at the cost of using a more general theory constructed by Zubov in the 1960.

Another important observation is that the Lyapunov functions form a family: if there is one Lyapunov function, there is an infinite number of them (why? give an example).

Finally is the question of local versus global. If the set $U = \mathbb{R}^n$ we have a global Lyapunov function, if not we have only local results. In particular, it is possible to determine the local stability of (1.1) using the linear approximation

$$\dot{x} = Ax, \quad A = \left.\frac{\partial f}{\partial x}\right|_{x=0},$$

every-time that A does not have eigenvalues in the imaginary axis. Moreover, it is always possible to do it using the Quadratic Lyapunov Function $V(x) = x^T P x$, namely, using Linear system theory.

1.1.2 The converse theorem.

The converse theorems state that, under some conditions, any Lyapunov stable system has a Lyapunov function. Apart of being a curiosity and complete the theory, these theorems are useful in proving robustness of systems that we know are stable.

Masera's Theorem states that global asymptotically stability (GAS) for system (**??**) is equivalent to the existence of smooth Lyapunov functions. Below we give (without proof) a very general converse theorem:

Theorem 1.2. *Let $x = 0$ be a asymptotically stable equilibrium position of (1.1). Let $R_A \subset U$ be its region of attraction. Then, there is a smooth positive definite function $V(x)$ and a continuous positive definite function $W(x)$ defined of R_A such that*

$$V(x) \to \infty \text{ as } x \to \partial R_A \text{ and } \quad L_f V(x) \leq -W(x), \forall x \in R_A$$

For its proof and additional details, we refer to Khalil's book Section 4.7. It is not a big surprise in geometric terms. It states that when the system is stable it is alway possible to give a (possibly non-euclidean) structure to $\mathbb{R}^n$, such that we construct rings for in which the vector field f points towards the inside.

PROBLEM 5. Show that if the linear system $\dot{x} = Ax$ is stable, then $V(x) = x^T Px$ with P solution to the Lyapunov algebraic equation $A^T P + PA = -Q$, with $Q > 0$, is a Lyapunov function for the system. HINT: see any textbook on linear system theory.

1.2 Input-to-State Stability

In the previous section we explored the stability of an autonomous systems. The theory was formulated in "The General Problem of Stability of Motion" in 1892.

In control theory, the system should have inputs that allows modifying its behavior:

$$\dot{x} = f(x, u), \quad x(0) = 0; \quad u \in \mathbb{R}^m, \tag{1.2}$$

where u is the input of the system. Let us assume that $x = 0$ is an asymptotically stable equilibrium point for the unforced system $f(x, 0) = 0$. It turns out that the stability properties of the unforced system may not tell much about the stability properties of the forced system.

EXAMPLE 1. Consider $\dot{x} = -x + (x + x^2)u$. The unforced system has a globally asymptotically stable equilibrium point $x = 0$. However, for the input $u = 1$, it results in the system $\dot{x} = x^2$ that has unbounded trajectories for any initial condition:

$$x(t) = \frac{1}{x_0^{-1} - t}.$$

The concept of input-to-state stability (ISS) attempts to capture the notion of "bounded input—bounded state". It was introduces by Eduardo Sontag in the 80's.

Definition 1.2.

- *A class $\mathcal{K}_\infty$ function is a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, strictly increasing, unbounded and satisfies $\alpha(0) = 0$, see Fig. 5.*

- *A class $\mathcal{KL}$ function is a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that $\beta(\cdot, t) \in \mathcal{K}_\infty$ for each t and $\beta(r, t) \to 0$ as $t \to \infty$ for every r.*

PROBLEM 6. Prove that the autonomous system (1.1) is (globally) Lyapunov stable if and only if there exists $\alpha \in \mathcal{K}_\infty$ such that $\|x(t)\| \le \alpha(\|x_0\|), \forall x_0, \forall t \ge 0$. and that is globally asymptotically stable if and only if there exists $\beta \in \mathcal{KL}$ such that $\|x(t)\| \le \beta(\|x_0\|, t), \forall x_0, \forall t \ge 0$.

Definition 1.3. *System (1.2) is said to be input-to-state (ISS) stable if there exists some $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that*

$$\|x(t)\| \le \beta(\|x_0\|, t) + \gamma(\|u\|_\infty).$$

Recall that the $\mathbb{L}^\infty$-norm of a signal $u : [0, t) \to \mathbb{R}^m$ is defined as

$$\|u\|_\infty = \sup_{0 \le \tau \le t} \|u(\tau)\|,$$

(where is the euclidean norm here?). It measures the "maximum amplitude" of the input signal $u(t)$.

Therefore, the ISS property means that for t large, the size of the state must be bounded by some function of the amplitude of the inputs. Moreover, the effect of the initial state is forgotten with time, and this serves to quantify the magnitude of the transient (overshoot) as a function of the size of the initial state, see Figure 6.

Corollary 1.2.1. *The zero position $x = 0$ of system (1.2) with $u = 0$ is Globally Asymptotically Stable (GAS).*

Proof. $\|x(t)\| \le \beta(\|x_0\|, t)$. □

1.2.1 Dissipative characterization of ISS.

Definition 1.4. *A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a storage function if it is positive definite ($V(0) = 0$ and $V(x) > 0$ for $x \neq 0$) and proper ($V(x) \to \infty$ as $\|x\| \to \infty$).*

Later on, we will find again storage functions to characterize the "energy" of the system. An equivalent description can be given in terms of $\mathcal{K}_\infty$ functions. A function is a storage function if and only if there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \qquad \forall x \in \mathbb{R}^n.$$

Definition 1.5. *As ISS-Lyapunov function for (1.2) is a smooth storage function V for which there exists functions $\alpha, \gamma \in \mathcal{K}_\infty$ so that*

$$\dot{V}(x,u) \le -\alpha(\|x\|) + \gamma(\|u\|), \quad \forall x, u.$$

Theorem 1.3. *If system (1.2) admits a smooth ISS-Lyapunov function, then it is ISS.*

The converse statement (any ISS system has a ISS-Lyapunov function) is also true, but much harder to prove.

Proof. By definition, the storage function satisfies $V(x) \le \alpha_2(\|x\|)$, or in other words,

$$-\|x\| \le -\alpha_2^{-1}(V(x)),$$

where the inverse function α_2^{-1} exists since it is monotonically increasing. Therefore, we have $-\alpha(\|x\|) \le -\alpha \circ \alpha_2^{-1}(V(x))$ and the inequality for the ISS-Lyapunov function can be rewritten as:

$$\dot{V}(x,u) \le -\bar{\alpha}(V(x)) + \gamma(\|u\|)$$

where $\bar{\alpha} = \alpha \circ \alpha_2^{-1}$.

We now check to complementary cases. The first when $2\gamma(\|u\|) \ge \alpha(V(x))$, the second when $2\gamma(\|u\|) \le \alpha(V(x))$. In the first case we directly obtain that $V(x) \le \alpha^{-1}(2\gamma(\|u\|))$, and taking supremum over $[0,t]$ yields

$$V(x(t)) \le \alpha^{-1}(2\gamma(\|u\|_\infty)).$$

For the second case, we arrive to the inequality $\dot{V} \le 1/2\alpha(V)$, from which we conclude asymptotic stability of V, namely, $V(x(t)) \le \beta(V(x_0), t)$, for some $\beta \in \mathcal{KL}$. Considering both cases together we arrive to

$$V(x(t)) \le \max\left\{\beta(V(x_0), t), \alpha^{-1}(2\gamma(\|u\|_\infty)\right\} \le \beta(V(x_0), t) + \alpha^{-1}(2\gamma(\|u\|_\infty).$$

since $\max\{a, b\} \leq a + b$. From here, we just have to write $V(x)$ in terms of $\|x\|$ to complete the proof. $\square$

PROBLEM 7. Finish the proof above!

An alternative characterization using $\alpha, \rho \in \mathscr{K}$ is

$$\|x\| \geq \rho(\|u\|) \Rightarrow \dot{V}(x, u) \leq -\alpha(\|x\|).$$

We now present a motivating example about the utility of having a ISS system.

EXAMPLE 2. (Modified from Sontag's "Input to State Stability" paper) We are given the scalar system

$$\dot{x} = f(x, u) = x + (x^2 + x)u.$$

In order to stabilize it, we may think of a preliminary feedback transformation $u = (x^2 + x)^{-1}v$ that renders the system linear w.r.t. the new input v:

$$\dot{x} = x + v,$$

and then we simply use $v = -2x$ to obtain the closed-loop system $\dot{x} = -x$. In other words, we have chosen the feedback law $u = k(x)$ with

$$k(x) = \frac{-2x}{x^2 + x},$$

so that $f(x, k(x)) = -x$, that is a GAS system.

Let us now consider the effect of a disturbance w matched with the control input. The system $\dot{x} = f(x, k(x) + w)$ is:

$$\dot{x} = -x + (x^2 + x)w,$$

that, as we have seen before, has solutions that explode in finite-time for $w = 1$. Thus, our selection $k(x)$ is not a good feedback law in the sense that its performance is drastically degraded once a disturbance appears. Indeed, the closed-loop system is not ISS with respect to the disturbance w.

For making the system ISS, we simply add an extra $-x$ correction term:

$$k(x) = \frac{-2x}{x^2 + x} - x,$$

so that the closed-loop system $f(x, k(x) + w)$ now becomes:

$$\dot{x} = -2x - x^2 + (x^2 + x)w$$

that is also GAS when $w = 0$ but, in addition, is also ISS w.r.t. w. Intuitively, for large x, the terms $-x^3$ dominates the term$(x^2 + x)w$ for all bounded disturbances w, and this prevents the state from getting too large, see Fig. 8.

PROBLEM 8. Prove that it is indeed ISS. HINT: consider $V(x) = |x|$. Arrive to an inequality of the form $\dot{V} = -\alpha(|x|) + (|x|^2 + |x|)w$ and use the trick for the proof of Theorem 1.3. First consider the case

$2L(|x|+|x|^2) \geq 2|x|+|x|^3$ to conclude that $|x(t)|$ is bounded by a function of $L = \sup_t w(t)$, and then consider the complementary case. Later, ask yourself if it is valid to consider this function as a storage function.

1.2.2 *Cascade interconnection.*

One of the main feature of the ISS property is that it behaves well under composition: a cascade of two ISS systems is again ISS.

Consider the cascade interconnection, see Figure 7,

$$\dot{x} = g(x,u), \quad \dot{z} = f(z,x),$$

where each of the two subsystems is assumed to be ISS. Therefore, each subsystem admits an ISS-Lyapunov function V_i:

$$\dot{V}_1 \leq -\alpha_1(\|x\|) + \gamma_1(\|u\|), \quad \dot{V}_2 \leq -\alpha_2(\|z\|) + \gamma_2(\|x\|).$$

What it is very interesting[1], is that we can redefine V_i such that both are matched in the following sense:

$$\alpha_1 = 2\theta, \quad \gamma_2 = \theta.$$

Now it is obvious why the full system is ISS: we simply use $V = V_1 + V_2$ as an ISS-Lyapunov function for the cascade

$$\dot{V}((x,z),u) \leq -\theta(\|x\|) - \alpha_2(\|z\|) + \gamma_1(\|u\|).$$

1.3 The Control Lyapunov Function

The seemingly obvious concept of a "control Lyapunov function" (CLF) introduces by Artstein and Sontag in 1983, made a tremendous impact on stabilization theory, which, at the end of the 70's was stagnant. It converted stability descriptions into tools for solving stabilization tasks.

[1] See, "Changing Supply Functions in Input/State Stable Systems" by Sontag and Teel.

Consider the problem of finding a control law $u(x)$ that stabilizes the affine nonlinear system

$$\dot{x} = f(x) + g(x)u. \tag{1.3}$$

One way to do it is to first *select* a Lyapunov function $V(x)$ and then *try to find* a feedback control $u(x)$ that renders $\dot{V}(x, u(x))$ negative definite. If one looks blindly and select an arbitrary $V(x)$, this attempt may fail. However, if $V(x)$ is a CLF, we can find a stabilizing control law $u(x)$.

Definition 1.6. *A smooth positive definite function $V(x)$ is a CLF for system (1.3) if for all $x \neq 0$*

$$L_g V(x) = 0 \Rightarrow L_f V(x) < 0.$$

PROBLEM 9. Prove that if (1.3) is stabilizable, then a CLF exists. HINT: use the converse Lyapunov theorem.

From the mere definition of a CLF, we see that the set where $L_g V(x) = 0$ is significant, because in this set the uncontrolled system has the property that $L_f V(x) < 0$. However, if $L_f V(x) > 0$ when $L_g V(x) = 0$, then $V(x)$ is not a CLF an cannot be used for a feedback stabilization design.

Once you have your CLF, Sontag's formula tells you how to find the stabilizing controller.

Theorem 1.4. *Suppose $u \in \mathbb{R}$. If $V(x)$ is a CLF for (1.3) then the origin is stabilizable by $u = \psi(x)$ where*

$$\psi(x) = \begin{cases} -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V} & if \quad L_g V(x) \neq 0 \\ 0 & if \quad L_g V(x) = 0 \end{cases}$$

Proof. $\dot{V} = L_f V + L_g V \psi$. Then if $L_g V(x) = 0$, then $\psi(x) = 0$ and $\dot{V} = L_f V(x) < 0$.

If $L_g V(x) \neq 0$, then $\dot{V} = -\sqrt{(L_f V)^2 + (L_g V)^4} < 0$, for $x \neq 0$. □

The construction of a CLF is a hard problem that has been solved for special classes of systems. For example, when the system is feedback linearizable we can construct for it a quadratic CLF.

EXAMPLE 3. Here is the example of feedback linearization. Suppose there exists a state transformation $z = T(x)$ and control $u(x)$ such that system (1.3) can be rewritten as

$$\dot{z} = (A - BK)z,$$

where $P(A - BK) + (A - BK)^T P = -Q, Q > 0$. Then $V = z^T P z = T^T(x) P T(x)$ is a CLF.

EXAMPLE 4. Consider $\dot{x} = ax - bx^3 + u$, with $a, b > 0$. One simple selection for a stabilizing controller is a linearizing one.

$$u = -ax + bx^3 - kx, \quad k > 0,$$

that renders the closed loop system $\dot{x} = -kx$.

Another selection is obtained by using a CLF and Sontag's formula. $V = 1/2x^2$ is a CLF:

$$L_g V(x) = x, \quad L_f V(x) = x(ax - bx^3)$$

the set $x \neq 0$ for which $L_g V(x) = 0$ is the void set. Then $L_f V(x)$ needs to satisfy no restriction.

Using Sontag's formula yields

$$\psi(x) = -\frac{L_f V + \sqrt{(L_f V)^2 + (L_g V)^4}}{L_g V} = -ax + bx^3 - x\sqrt{(a - bx^2)^2 + 1}$$

1.4 A brief incursion into Linear Matrix Inequalities

Linear Matrix Inequalities (LMIs) have come to be a fundamental numerical tool in modern automatic control theory. Once you express your problem as an LMI problem, there exists several efficient (numerical) methods to get a solution to your problem. However, one should not be to avid an write everything down as an LMI problem without exploring the feasibility of solving it.

Definition 1.7. *A matrix $M \in \mathbb{R}^{n \times n}$ is said to be positive definite (written as $M > 0$) if*

$$x^T M x > 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}.$$

In analogy, a positive semi-definite matrix $M \geq 0$ satisfies $x^T M x \geq 0$.

A LMI in the variable $P \in \mathbb{R}^{n\times n}$ is an expression of the form $F(P) < 0$, where function $F : \mathbb{R}^{n\times n} \to R^{m\times m}$ is linear. Solving the LMI precisely means finding the unknown matrix P that satisfies $F(P) < 0$.

The solution set of a LMI is convex, i.e. given two solutions P_1 and P_2 to the LMI, then $\alpha P_1 + (1-\alpha)P_2$ is also a solution to the $F(P) < 0$ provided that $\alpha \in [0,1]$.

EXAMPLE 5. The linear system

$$\dot{x} = Ax$$

is stable iff there exists a positive definite solution $P > 0$ to the Lyapunov equation $A^T P + PA = -Q$, for some $Q > 0$. This can be re-casted as the problem of finding a positive solution $P > 0$ to the LMI

$$A^T P + PA < 0$$

in the unknown P. Thus the linear system is stable iff the LMI above has a solution.

EXAMPLE 6. The decay rate (or largest Lyapunov exponent) of a a LTI system $\dot{x} = Ax$ is defined to be the largest α such that

$$\lim_{t\to\infty} e^{\alpha t}\|x(t)\| = 0$$

holds for all trajectories x.

We can use a quadratic Lyapunov function $V(x) = x^T P x$ to establish a lower bound on the decay rate. If

$$\dot{V} \leq -2\alpha V$$

then $V(x(t)) \leq V(x(0))e^{-2\alpha t}$ so that $\|x(t)\| \leq e^{-\alpha t}\kappa(P)^{1/2}\|x(0)\|$ for all trajectories and therefore the decay rate of the system is at least α. Next observe that the condition $\dot{V} \leq -2\alpha V$ is equivalent to

$$A^T P + PA + 2\alpha P \leq 0.$$

Therefore, the largest lower bound on the decay rate can be bound by solving the following problem:

$$\text{maximize } \alpha \text{ subject to } P > 0, \quad A^T P + PA + 2\alpha P \leq 0.$$

EXAMPLE 7. The linear system

$$\dot{x} = Ax + Bu, \quad y = Cx + Du$$

is passive if and only if it is Positive Real (PR). The Positive Real lemma states that a (controllable and observable) linear system is PR if and only if there exists a solution $P > 0$, $\varepsilon \geq 0$ to the following Kalman-Yakubovich-Popov equations:

$$\begin{pmatrix} A^T P + PA + \varepsilon P & PB - C^T \\ B^T P - C & D^T + D \end{pmatrix} \leq 0$$

EXAMPLE 8. The quadratic stabilization of a system

$$\dot{x} = Ax + Bu,$$

is the problem of finding a state feedback $u = Kx$ such that the closed loop is stable. The closed loop is stable if and only if there exists $P > 0$ such that

$$P(A + BK)^T + (A + BK)^T P < 0$$

and observe that it is not jointly convex in K nor P and, hence, it is not an LMI. However, with a simple change of variables we can obtain an equivalent condition that is an LMI.

Define $Y = KP$, so that for $P > 0$ we have $K = YP^{-1}$. Substituting in the stability condition yields

$$AP + PA^T + BY + Y^T B^T < 0$$

that is an LMI in P and Y. Thus, the closed loop system is stabilizable if and only if there exist $P > 0$ and Y such that the LMI above holds. If this LMI is feasible, then the quadratic function $V(x) = x^T P^{-1} x$ proves stability (P^{-1} since the Lyapunov function is $PA^T + A^T P$ and multiply both sides by P^{-1}) with the state feedback $u = YP{-}1x$.

Although an LMI may seem to have a specialized form, it can represent a wide variety of convex constraints on P. In particular, linear inequalities, (convex) quadratic inequalities, matrix norm inequalities, and constraints that arise in control theory, such as Lyapunov and convex quadratic matrix inequalities, can all be cast in the form of an LMI.

The following is known as Schur's complements lemma. It serves to convert quadratic inequalities into a LMIs.

Lemma 1.1. *The LMI*

$$\begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} > 0$$

is equivalent to the LMI

$$R > 0, \quad Q - SR^{-1}S^T > 0.$$

We refer to Boyd's book for a wide compendium of LMI problems.

PROBLEM 10. Show that the Ricatti matrix inequality

$$A^T P + PA - PBR^{-1}B^T P + Q < 0$$

can be written as the LMI

$$\begin{bmatrix} -A^T P - PA - Q & PB \\ B^T P & R \end{bmatrix} > 0.$$

1.4.1 Solving LMI using CVX package.

We will illustrate how to solve LMI related problem using CVX. For more information you can visit http://cvxr.com/cvx/.

Let us consider the problem of checking if the linear system $\dot{x} = Ax$ is stable. We know that this is equivalent to the existence of $P > 0$ such that the LMI

$$A^T P + PA \leq 0$$

is feasible.

We can use CVX to solve this problem using the following code

```
cvx_begin sdp
variable P(n,n) symmetric
A'*P + P*A <= -eye(n)
P >= eye(n)
cvx_end
```

where n is the dimension of the state.

You should try this small example with several A stable and not stable and see what happens. You should check, for example, that it really does satisfy the two LMIs. If there is no such P, then P's entries will be set to NaN.

You can add a (linear) objective to the problem if you like. For example, to minimize the $trace(P)$ over all P that satisfy the the two LMIs, you can use the cvx code

```
cvx_begin sdp
variable P(n,n) symmetric
minimize(trace(P))
A'*P + P*A <= -eye(n)
P >= eye(n)
cvx_end
```

2 PASSIVITY

Passivity allows studying the stability properties of interconnected systems. A passive system satisfies the following energy conservation principle:

$$\frac{d}{dt}\text{StoredEnergy} = \text{RecievedPower} - \text{DeliveredPower},$$

and it is assumed that the minimal energy is bounded from below.

There are at least two basic things that you should know about passivity. Firstly, the (negative) feedback interconnection of two passive systems is also passive. Secondly, a negative feedback of the passive output (plus some mild condition) yields the system asymptotically stable.

We the following system with inputs and outputs:

$$\dot{x} = f(x, u), \quad y = h(x), \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $u, y \in \mathbb{R}^m$ are the state, input and output respectively, see Fig. 1. The input and output of the system are "ports" that the system uses to interact with its environment.

Passivity is about energy and energy storage.

Definition 2.1. *System (2.1) is passive from u to y if there exists a positive semidefinite storage function $H(x)$ such that the energy-balance equation holds:*

$$L_f H = \dot{H} \leq u^T y.$$

When the last inequality is strict and satisfied as $\dot{H} \leq u^T y - \psi(x)$, the system is said to "strictly passive" (in the state).

As expected the energy $H(x)$ of the system can be manipulated by the control. In the geometric perspective (similar to the one given in the previous chapter), it means that in a level set $\{x|H(x) = const\}$ the projection of the vector f on the vector ∇H is less that $u^T y$. In particular, the projection is less than or equal to zero (i.e. f does not point to the outside) when $u = 0$, and is negative (it points to the inside) when $u = -K_d y$, see Figure 2.

Usually, it is assumed that u and y are conjugated, in the sense that their "product" has units of power. Integrating the energy-balance equation on a time period $[0, t]$ gives

$$H(x(t)) - H(x(0)) = \int_0^t u^T(s)y(s)ds - d(t),$$

where $d(t) \geq 0$ is the dissipation. In the particular case when $d(t) \equiv 0$, the system is said to be "lossless". The right hand side can be thought as the power given to the system and the dissipation, and the left-hand side the stored energy. Therefore, the stored energy in the system is equal to the given power minus the dissipation. In other words, the system does not produce energy by itself: it is passive.

Two obvious observations now follow. The energy $H(x)$ of the uncontrolled system (i.e. $u = 0$) is nondecreasing. In fact, it will decrease in the presence of dissipation until it reaches a point of minimal energy. In particular, the dissipation can be enhanced by extracting more energy of the system, for instance, using $u = -K_d y$.

PROBLEM 1. Assume that system (2.1) is zero-state observable, i.e., the condition $y(t) \equiv 0, u(t) \equiv 0$ implies that $x(t) \equiv 0$. Assume also that it is also passive. Show that it can be stabilized with the control $u = -\phi(y)$ such that $y^T\phi(y) > 0$, $\forall y \neq 0$. HINT: Use the storage function as a Lyapunov function and use LaSalle's lemma.

EXAMPLE 1. Consider $\dot{x}_1 = x_2, \dot{x}_2 = -x_1^3 + u$ with the storage function $H(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$. Its derivative is given by

$$\dot{H}(x) = x_2 u,$$

showing that the system is passive with the output $y = x_2$, i.e., its velocity. Moreover it is zero state observable (check it!) and then a stabilizing controller is

$$u = -kx_2 \quad \text{or} \quad u = -\frac{2k}{\pi}\arctan(x_2).$$

Definition 2.2. *A memoryless function $y = h(u)$ is said to be*

- *passive if $u^T y \geq 0$.*
- *output strictly passive if $u^T y \geq y^T \rho(y) > 0$ for some function ρ.*

See Fig.3. There is a complete zoo of variations of the passive systems (e.g., input strictly passive $\dot{V} \leq u^T y - u^T \phi(u)$, output strictly passive $\dot{V} \leq u^T y - y^T \phi(y)$) as appears in Chapter 6 of Khalil's book.

A useful picture one should have in mind is that of a scalar function on a sector.

Definition 2.3. *A scalar function $y = h(u)$ which satisfies the inequality*

$$\alpha u^2 \leq u h(u) \leq \beta u^2,$$

for some (extended) real number α, β is said to belong to the sector $[\alpha, \beta]$ and it is written as $h \in [\alpha, \beta]$.

The graph of a function which belong to the sector $[\alpha, \beta]$ is restricted to lie between the lines $y = \beta u$ and $y = \alpha u$, see Fig. 4. We will see more about these functions in the next chapter about "absolute stability".

PROBLEM 2. Show that a scalar function $y = h(u)$ is passive if and only if it belong to the sector $[0, \infty]$.

PROBLEM 3. Show that an integrator is lossless. Moreover, show that the feedback interconnection of an integrator with an $h \in (0, \infty]$ nonlinearity is output-feedback passive:

$$\dot{x} = -h(x) + u, \quad y = x.$$

HINT. Use $V = 1/2x^2$ as a storage function.

2.1 Stability and Interconnection of Passive systems.

In general, we have the following result due to Popov and Zames (1963-1966):

Theorem 2.1. *The feedback interconnection of two passive systems is passive, see Fig 5.*

Proof. Take $H_1(x_1)$ and $H_2(x_2)$ as storage functions for each system, so $\dot{H}_i \leq e_i^T y_i$. If either components is memoryless, simply take $H_i = 0$. Using $H = H_1 + H_2$ as storage function for the complete system one gets

$$\dot{H} \leq e_1^T y_1 + e_2^T y_2 = u_1^T y_1 + u_2^T y_2 = u^T y.$$

□

Note in particular that the system is also passive with respect to u_1, y_1 or u_2, y_2.

Lemma 2.1. *The origin $x = 0$ of the unforced system $\dot{x} = f(x, 0)$ is AS if it is*

- *strictly passive (in the states) or*
- *output strictly passive and zero state observable.*

Proof. See Problem 4. □

PROBLEM 4. Prove the lemma above. HINT: use the storage function as a Lyapunov function. For the second alternative, use LaSalle invariance principle.

Theorem 2.2. *Consider the feedback interconnection of two system in Fig 5. Then, the origin of the unforced ($u = 0$) closed-loop system is AS if*

- *both components are strictly passive.*

- *one component is passive, the other one strictly passive and the whole system is zero-state observable.*

Proof. For the first claim note that $\dot{H} = \dot{H}_1 + \dot{H}_2 = u^T y - \phi_1(x_1) - \phi_2(x_2)$. When $u = 0$, the origin is AS. For the second one obtain $\dot{H} \leq -\phi(x_2) \leq 0$ and apply LaSalle's invariance principle. □

EXAMPLE 2. This requires more work! Suppose we have a scalar model of the form $\dot{x} = \theta u$, where θ is an unknown (constant) parameter we want to estimate and u is a known excitation signal. Let us propose the following parameter-estimation algorithm:

$$\dot{\hat{x}} = -(\hat{x} - x) + \hat{\theta} u, \quad \dot{\hat{\theta}} = -(\hat{x} - x) u.$$

The dynamics of the error $e = \hat{x} - x$ and $\tilde{\theta} = \hat{\theta} - \theta$ are

$$\dot{e} = -e + \tilde{\theta} u, \quad \dot{\tilde{\theta}} = -eu.$$

Using $H_1 = 1/2e^2$ and $H_2 = 1/2\tilde{\theta}^2$ we obtain that the first system is strictly passive from $\theta u \to e$:

$$\dot{H}_1 = -e^2 + e\theta u$$

and the first system is passive from $e \to \theta u$:

$$\dot{H}_2 = -e\theta u.$$

The feedback interconnection of both systems is then stable.

Here is very enlightening to see what Zames wrote:

The classical definitions of gain and phase shift, in terms of frequency response, have no strict meaning in nonlinear systems. However, stability does seem to depend on certain measures of signal amplification and signal shift. (...) The inner product (x, Hx), a measure of the input-output cross-correlation, is closely related to the notion of phase shift. For example, for LTI operators the condition of positivity $(x, Hx) \geq 0$ i equivalent to the phase condition $|Arg\{H(jw)\}| \leq 90$. Theorem 1 can be viewed as a generalization to nonlinear systems of the rule that "if the open loop absolute phase shift is less than 180 then the closed loop is stable".

2.2 Passivity based control.

The passivity based approach to control relies on trying to modify the energy of the system. If the system is passive, it has its own energy $H(x)$. In the presence of dissipation, the system will arrive into the minimal point of energy. However, in many cases, we are interested in stabilizing the system in a point that is not a minimum of the original energy function.

Hence, we are led into the idea that the controller needs to reshape the energy of the system. In other words, we want to select a control $u = \beta(x)v$ so that the closed-loop dynamics satisfies the new energy-balancing equation

$$H_d(x(t)) - H_d(x_0) = \int_0^t v^T(s)z(s)ds - d_d(t),$$

where $H_d(x)$, the desired total energy function, has a strict minimum at $x^* = 0$, z (which may be equal to y) is the new passive output, and we have replaced the natural dissipation term by some function $d_d(t) \geq 0$ to increase the convergence rate.

Conceptually, the task is very simple:

Theorem 2.3. *If there exists a function $\beta(x)$ such that*

$$-\int_0^T \beta^T(x(s))y(s)ds = H_a(x(t)) - H_a(x(0)),$$

then the control $u = \beta(x) + v$ will ensure that the map $v \mapsto y$ is passive with the new energy function

$$H_d(x) = H(x) + H_a(x).$$

Proof. Simple substitution:

$$H(x(t)) - H(x_0) - \int_0^t \beta^T(s)y(s)ds = \int_0^t v^T(s)y(s)ds - d(t).$$

□

Furthermore, if $H_d(x)$ has a minimum at the desired equilibrium $x = 0$, then it will be stable.

EXAMPLE 2. Let us look at the classical example of position regulation of fully actuated mechanical systems with generalized coordinates $q \in \mathbb{R}^{n/2}$. For such system, its total energy is given by

$$H(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + V(q),$$

where $M(q) = M^T(q)0$ is the generalized mass matrix and $V(q)$ is its potential energy. Any mechanical system is passive from the input force to the generalized velocities, i.e. $y = \dot{q}$.

Suppose we want to stabilize some position $q^* \in \mathbb{R}^{n/2}$. The simplest way to satisfy the condition of the theorem is to choose

$$\beta(q) = \frac{\partial V}{\partial q}(q) - K_P(q - q^*),$$

where $K_P = K_P^T$ is a proportional gain matrix. Indeed, replacing the expression above with $y = \dot{q}$ we have

$$-\int_0^t \beta^T(q(s))\dot{q}(s)ds = -\int_0^t \beta^T(q(s))dq = -V(q(t)) + \frac{1}{2}(q(t) - q^*)^T K_P(q(t) - q^*)$$

and the new total energy for the passive closed-loop map $v \mapsto \dot{q}$ is given by

$$H_d(q, \dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + \frac{1}{2}(q(t) - q^*)^T K_P(q(t) - q^*),$$

which has a minimum at $(q^*, 0)$ as desired. To ensure that the trajectories actually converge to this minimum (i.e. that the equilibrium is AS), we add some damping $v = -K_d\dot{q}$.

2.3 Characterization of passive systems.

2.3.1 *Passivity of linear systems.*

For the linear system

$$\dot{x} = Ax + Bu, \quad y = Cx; \quad x \in \mathbb{R}^n, u \in \mathbb{R}^m, y \in \mathbb{R}^p,$$

there exists a very elegant characterization of passivity and strict passivity (in the states). Let $G(s) = C(sI - A)^{-1}B$ be its transfer function matrix. In particular, let us assume a Single-Input-Single-Output system $p = m = 1$.

The characterization is based in a fundamental result by Kalman-Yakubovich-Popov, that came to be known as the “Positive Real Lemma”. It is very important and appears in many areas outside automatic control, e.g., see "Network Analysis and Synthesis: A Modern Systems Theory Approach".

Definition 2.4. *A proper*[1] *transfer function $G(s)$ is said to be positive real (PR) if $Re\{G(s)\} \geq 0$, $\forall Re(s) \geq 0$. It is strictly positive real (SPR) if $G(s-\varepsilon)$ is PR for some $\varepsilon > 0$.*

For a MIMO system, the definition of a SPR transfer function matrix is more elaborate. The transfer function matrix $G(s)$ is SPR if $G(\sigma + jw) + G(\sigma - jw) > 0$ for all $\sigma > 0$ and real w, such that $\sigma + jw \neq \lambda(A)$. We invite the reader to see this definition for the case of MIMO systems and compare the differences. Check for instance, Chapter 6 of Khalil's book.

In essence, the Nyquist plot of a positive real transfer function is restricted into the right-side of the complex plane. Therefore, the relative degree of the system must be zero or one.

EXAMPLE 2. The transfer function

$$G(s) = \frac{1}{s+a}, \quad a > 0,$$

is PR. To see this write $s = \sigma + jw$ and

$$Re\{G(\sigma + jw)\} = \frac{\sigma + a}{(\sigma + a)^2 + w^2}$$

is positive if $\sigma > 0$.

PROBLEM 4. Show that the transfer function of Example 2 is also SPR. HINT: check if $Re\{G(\sigma-\varepsilon+jw)$ is positive for some ε.

Below is the celebrated "Positive Real Lemma" or "KYP-Lemma":

Lemma 2.2. *Let the linear system*

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad x(0) = x_0$$

be controllable and observable. Its transfer function $G(s)$ is SPR if and only if there exists $\varepsilon > 0$ and matrices $P = P^T > 0$, $L \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$ such that

$$A^T P + PA = -\varepsilon P - L^T L, \quad PB = C^T - LW, \quad D + D^T = W^T W.$$

In particular, when $\varepsilon = 0$ is allowed, $G(s)$ is just PR.

[1] A minimal realization, i.e., controllability+observability.

The Lyapunov equation guarantees that A is stable. Note that LL^T is not positive definite but necessarily semi-positive definite as long as $m < n$ (prove this!). The remarkable thing is that P also satisfies the input-output relation $PB = C^T$. The third equation above can be interpreted as the factorization of $D + D^T$. For the case $D = 0$, the above set of equations reduces to the first two equations with $W = 0$.

Proof. I still need to find a good proof for this result. Meanwhile, you can check Khalil's one. □

For linear systems, we have the following nice characterization of passivity:

Theorem 2.4. *A linear system (A, B, C) with minimal realization is passive if its is PR, and strictly passive (in the states) if it is SPR.*

Proof. Consider $H = x^T P x$ as a storage function. Its derivative is given by

$$\dot{H} = 2x^T PBu - x^T Qx = 2(Cx)^T u - x^T Qx = -2y^T u - x^T Qx,$$

where we have used the Positive Real Lemma. □

In general, a (controllable and observable) linear system (A, B, C, D) is passive if it satisfies the KYP-lemma. In an equivalent form, if there exist $\varepsilon > 0$ and a matrix $P = P^T > 0$ solution to the following Matrix Inequality

$$\begin{bmatrix} A^T P + PA + \varepsilon P & PB - C^T \\ B^T P - C & D + D^T \end{bmatrix} = -\begin{bmatrix} L \\ W^T \end{bmatrix} \begin{bmatrix} L^T & W \end{bmatrix} \leq 0.$$

In the matrix inequality above it is possible to replace the term εP by $\varepsilon I_{n \times n}$ without affecting its solvability. Hence, the problem of determining the passivity of a linear system can be translated into the solvability of a Linear Matrix Inequality (LMI).

LMIs are very useful and pervade modern control theory. The reason for this fact is that there are very efficient numerical algorithms to solve them. More on this to come in the next chapter.

2.3.2 *Passivity of affine nonlinear systems.*

For nonlinear affine systems

$$\dot{x} = f(x) + g(x)u, \quad y = h(x) \tag{2.2}$$

there exists a nonlinear version of the lemma above, presented by Hill and Moylan 1976. It is just a translation of the linear theorem:

Theorem 2.5. *The affine nonlinear system is passive from $u \mapsto y$ if and only if there exists a storage function $H(x)$ such that*

$$L_f H(x) = \frac{\partial H}{\partial x}(x) f(x) \leq 0, \quad h(x) = (L_g H(x))^T = g^T(x) \frac{\partial H}{\partial x}(x).$$

Proof. The if part is straightforward:

$$\dot{H} = L_f H + L_g H u \leq y^T u.$$

The "only if" one is not elementary and is the merit of Hill and Moylan 76 paper "The stability of Nonlinear Dissipative Systems". We refer the reader to such result. □

With the observation above, we can extend the passivity based control of the previous section from mechanical systems to general nonlinear systems.

Proposition 2.3.1. *Consider the passive system (2.2) with storage function $H(x)$ and an admissible equilibrium x^*. If we can find a function $\beta(x)$ such that the PDE*

$$\left[\frac{\partial H_a}{\partial x}(x)\right]^T [f(x) + g(x)\beta(x)] = -h^T(x)\beta(x),$$

can be solved for $H_a(x)$, then the function $H_d(x)$ has a minimum at x^* and $u = \beta(x) + v$ is an energy-balancing passivity based control.

Consequently, setting $v = 0$ we have that x^* is AS with the difference between the stored and supplied energy as Lyapunov function.

PROBLEM 4. Prove proposition above. HINT: observe that its left-hand side is $\dot{H}_a$, its right-hand side $-y^T u$ and integrate.

3 ABSOLUTE STABILITY

The framework of Lyapunov stability seems great: Lyapunov functions allows analytically determining the stability of any system. In fact, using their cousins (the Storage Function), we have see that it is possible to study much more: its robustness to disturbances and their interconnection properties.

However, there is a big (big!) problem with this approach: to study the system it in indispensable to have a Lyapunov function. The problem of finding or constructing a Lyapunov function is a hard[1].This is a serious drawback for its application in engineering problem.

To address this issue, en 1940's, Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering. Especially, to the the problem of stability of a control system with a nonlinearity in the actuator. From the introduction of Lur'e's 1951 book we find:

"This book represents the first attempt to demonstrate that the ideas expressed 60 years ago by Lyapunov, which even comparatively recently appeared to be remote from practical application, are now about to become a real medium for the examination of the urgent problems of contemporary engineering..."

In summary, Lur'e and others were the first to apply Lyapunov's methods to practical control engineering problems.

[1] In fact, it was proved by V. I. Arnold that the problem of Lyapunov Stability is algebraic unsolvable.

They restricted themselves to consider the class of linear system with a static nonlinearity in the feedback loop

$$\dot{x} = Ax + bu, \quad y = Cx, \quad u = -\phi(y); \quad x \in \mathbb{R}^n, y \in \mathbb{R}, \tag{3.1}$$

where the nonlinearity is assumed to satisfy a sector condition $\phi \in [k_1, k_2]$ for some constants, see Fig.1. The problem is to determine when the system has a unique global AS equilibrium point. With time, this came to be known as the problem of determining the "Absolute Stability of the System" or "Lure's problem".

3.1 Dreams that were not true.

Once Lure's postulated his problem, many were eager to find solutions. Some thought were evident, and postulated some conjectures:

Aizerman's conjecture (1949). $x = 0$ is a unique GAS equilibrium point if all linear systems with $\phi(y) = ky$, $k \in [k_1, k_2]$ are AS.

For $n \leq 2$, Krasovkii et.al. solved completely Aizerman conjecture in a positive manner, except when the matrix $A - k_1BC$, has multiple double zero eigenvalue. However, for $n > 2$ the same Krasovkii in 1952 showed that the system posses unstable solutions [2].

Kalman's conjecture (1957). suppose that instead $\phi' \in [k_1, k_2]$. Then $x = 0$ is the unique GAS equilibrium point if all linear systems with $\phi(y) = ky$, $k \in [k_1, k_2]$ are AS.

Markus-Yamabe conjecture (1960). Consider the system $\dot{x} = f(x)$. Suppose that the Jacobian matrix $\frac{\partial f}{\partial x}(x)$ has all eigenvalues with negative real parts for any $x \in \mathbb{R}^n$. Then the system is stable. .

[2]See, Algorithms for Finding Hidden Oscillations in Nonlinear Systems. The Aizerman and Kalman Conjectures and Chua's Circuits

Kalman conjectures turned out to be true for $n \leq 3$, but false otherwise. Markus-Yamabe's one is true for $n \leq 2$, and negative in the general case when $n \geq 3$.

EXAMPLE 1. The polynomial system

$$\dot{x}_1 = -x_1 + x_2(x_1 + x_2x_3)^2, \quad \dot{x}_2 = -x_2 - (x_1 + x_2x_3)^3, \quad \dot{x}_3 = -x_3$$

has Jacobian matrix with eigenvalues always at -1. However, it has the unbounded solution $x_1 = 18e^t$, $x_2 = -12e^{2t}$, $x_3 = e^{-t}$.

3.2 Popov's criterion.

In response to Lure's problem, many blind alleys were explored for a decade. Then suddenly, in 1960-1962, the absolute stability problem was solved with a frequency-domain criterion by Popov. Its state space form was soon stablished in a lemma by Yakuvobich and Kalman. From today's standpoint, the fundamental contribution of Popov's criterion is the introduction of the concept of passivity (positive realness) in feedback control.

Theorem 3.1. *If the SISO system*

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad u = -\phi(y),$$

satisfies:

- *A is Hurwitz and (A, B) is controllable.*
- *$\phi \in [0, k]$ for some $k > 0$.*
- *there exists $\alpha > 0$ such that $k^{-1} + (1 + \alpha s)G(s)$ is SPR.*

then $x = 0$ is GAS.

Note that if $G(s)$ is SPR it is strictly passive. Since ϕ is also passive, then its feedback interconnection generates a GAS system. Popov's criterion ask a much weaker condition: the SPRness of $(1 + \alpha s)G(s)$ for some $\alpha > 0$, knowing that ϕ is more than passive: it belong to the sector $[0, k]$.

Proof. Popov's brilliant idea was to use the sum of the two storage functions as a candidate for the Lyapunov function:

$$V = x^T P x + 2\alpha k \int_0^y \phi(\sigma) d\sigma,$$

recall that Popov made this before the concept of passivity. Its time derivative is given by

$$\dot{V} = x^T (A^T P + PA)x - 2x^T PB\phi + 2\alpha k \phi C(Ax - B\phi).$$

The sector condition $\phi(y) \leq ky$, can be rewritten as $-\phi(\phi - ky) > 0$. Adding this term to $\dot{V}$ and re-arrenging terms yields

$$\dot{V} \leq x^T (A^T P + PA)x + 2x^T (kC^T + \alpha k A^T C^T - PB)\phi - 2\phi(\alpha kCB + 1)\phi$$

If the the following KYP-equations are satisfied

$$A^T P + PA = -L^T L - \epsilon P, \quad PB = kC^T + \alpha k A^T C^T - L^T W, \quad W^2 = 2\alpha kCB + 2$$

for some $\varepsilon > 0$, then we obtain

$$\dot{V} \leq -\varepsilon x^T Px - x^T L^T x + 2x^T L^T W\phi - W^2 \phi^2 = -\varepsilon x^T Px - (Lx - W\phi)^2 < 0$$

proving that the origin is GAS.

Finally, we check that the KYP-equations above characterize the SPRness of $1 + (1 + \alpha s)kG(s)$. For this we perform some algebra:

$$\begin{aligned}
(1 + \alpha s)kG(s) &= +(1 + \alpha s)kC(sI - A)^{-1}B \\
&= kC(sI - A)^{-1}B + \alpha ksC(sI - A)^{-1}B \\
&= kC(sI - A)^{-1}B + \alpha kC(sI - A + A)(sI - A)^{-1}B \\
&= kC(sI - A)^{-1}B + \alpha kCB + \alpha kCA(sI - A)^{-1}B \\
&= k(C + \alpha CA)(sI - A)^{-1}B + \alpha kCB.
\end{aligned}$$

and hence $1 + (1 + \alpha s)kG(s)$ can be realized by $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ with

$$\bar{A} = A, \quad \bar{B} = B, \quad \bar{C} = k(C + \alpha CA), \quad \bar{D} = 1 + \alpha kCB.$$

whose KYP-equations are precisely the desired ones. $\square$

There is a nice graphical interpretation of Popov's condition. If we write

$$G(jw) = G_1(w) + jG_2(w),$$

then Popov's condition can be rewritten as

$$G_1(w) - \alpha w G_2(w) + k^{-1} > 0, \quad \forall w. \tag{3.2}$$

This follows from the definition of a SPR transfer function.

If we construct Popov's function $W(jw)$ with the same real part as $G(jw)$, but with imaginary part equal to $wIm\{G(jw)\}$, then the GAS of the origin is guaranteed if the polar plot of $W(jw)$ in the complex plane is below a line of the form $x - \alpha y + k^{-1} = 0$, see Fig 1.

PROBLEM 1. Prove that $(1+\alpha s)G(s)$ is SPR iff $Re\{(1+\alpha jw)G(w)\} > 0$ iff $G_1(w) - wG_2(w) > 0$.

Moreover, as we saw earlier, it is possible to write the SPRness in terms of the solvability of a Matrix Inequality. From this observation, we have that $x = 0$ is GAS if there exists $\varepsilon > 0$, $\alpha \geq 0$ and $P = P^T > 0$ such that

$$\begin{bmatrix} A^T P + PA + \varepsilon I & PB - k(C + CA)^T \\ B^T P - k(C + CA) & 2\alpha kCB + 2 \end{bmatrix} \leq 0, \tag{3.3}$$

that is an LMI.

Thus Popov's condition can be verified in two forms: analytically (or geometrically) by checking condition (3.2) or numerically by solving the LMI (3.3).

EXAMPLE 2. Consider

$$G(s) = \frac{s+3}{s^2 + 7s + 10}$$

and $\phi \in [0, k]$. Compute

$$G(jw) = \frac{4w^2 + 30}{-w^4 + 29w^2 + 100} + j\frac{-w(w^2 + 11)}{-w^4 + 29w^2 + 100} = G_1(w) + jG_2(w).$$

Subsituting in (3.2) we have that

$$4w^2 + \alpha w^2(w^2 + 11) + (k^{-1} - \varepsilon)(w^4 + 29w^2 + 100) > 0$$

if $\alpha \geq 0$ and $\varepsilon < 1/k$ (i.e, for any k there exists ε). Thus $x = 0$ is GAE for any $k \in [0, \infty]$.

EXTRA POINT PROBLEM 2. Consider $G(s)$ as in Example 2 an fix $k = 1$. Using MATLAB, show that the LMI (3.3) has a solution. HINT: use the CVX package, cvx.com.

3.3 The circle criterion

The circle criterion is a refinement of Popov's criterion when $\phi \in [k_1, k_2]$. Recall that Popov considered the cruder condition $\phi \in [0, k_2]$. Its proof can be constructed based on Popov's criterion and loop transformations (see Khalil, Chp 7 for many details).

Let $D(\alpha, \beta)$ be the circle connecting the points $-1/\alpha + j0$ and $-1/\beta + j0$ in the complex plane, see Fig. 2.

Theorem 3.2. *Consider the SISO system*

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \quad u = -\phi(t, y)$$

where (A, B, C, D) is a minimal realization and $\phi \in [k_1, k_2]$. Then $x = 0$ is GAS if one of the following conditions is satisfied:

- *if $0 < k_1 < k_2$, the Nyquist plot $G(jw)$ does not enter the disc $D(k_1, k_2)$ and encircles it ρ-times in the counterclockwise direction, where ρ is the number of unstable poles of $G(s)$, Fig 3.*
- *if $0 = k_1 < k_2$, $G(s)$ is stable (Hurwitz) and its Nyquist plot $G(jw)$ lies to the right of the vertical line defines by $Re\{s\} = -1/k_2$, Fig 4.*
- *if $k_1 < 0 < k_2$, $G(s)$ is stable (Hurwitz) and $G(jw)$ lines in the interior of the disk $D(k_1, k_2)$, Fig. 5.*

Recall that $\phi \in [k_1, k_2]$ means that

$$k_1 y \leq \phi(y) \leq k_2 y.$$

PROBLEM 3. Show that the sector condition can be rewritten as

$$(\phi(y) - k_1 y)(\phi(y) - k_2 y) \leq 0, \forall y \in \mathbb{R}$$

The circle criterion has also a LMI equivalent. In fact it derived using only the quadratic storage function as a Lyapunov function.

Theorem 3.3. *Consider the system* (A, B, C) *with* $u = +\phi(y)$. *Suppose that* $\phi \in [k_1, k_2]$. *Then* $x = 0$ *is GAE if there exists a solution* $P = P^T > 0$ *and* $\varepsilon > 0$ *to the following LMI:*

$$\begin{bmatrix} A^T P + PA - k_1 k_2 C^T C + \varepsilon I & PB + \frac{1}{2}(k_1 + k_2) C \\ B^T P + \frac{1}{2}(k_1 + k_2) C^T & 0 \end{bmatrix} \leq 0.$$

Proof. $V = x^T P x$ has the derivative $\dot{V} = x^T(A^T P + PA)x + 2x^T PB\phi$. Add the positive term $-(\phi - k_1 Cx)(\phi - k_2 Cx) > 0$, add an subtract $\varepsilon x^T P x$, and write the resulting quadratic form in the variables $[x, \phi]$. Replace εP by εI. $\square$

DISSIPATIVE SYSTEMS

4

Dissipative systems are of particular interest in engineering and physics. The dissipation hypothesis, which distinguishes such systems from general dynamical systems, results in a fundamental constraint on their dynamic behavior. Typical examples of dissipative systems are electrical networks in which part of the electrical energy is dissipated in the resistors in the form of heat, viscoelastic systems in which viscous friction is responsible for a similar loss in energy, and thermodynamic systems for which the second law postulates a form of dissipation leading to an increase in entropy.

In this part we review the theory as introduced by Jan Willems[1] in 1972. Our interest is in that it provides a unified framework from most concepts we have seen so far: Lyapunov stability, ISS, passivity, the circle criterion, etc. Its particularization to Quadratic "supply rates" provides a computationally efficient framework and will be used for design.

Consider the dynamical system

$$\dot{x} = f(x,u), \quad y = h(x), \tag{4.1}$$

where $x \in \mathbb{R}^n, u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output of the system. Note that in contrast to passivity, here we allow $m \neq p$.

Definition 4.1. *A function $s : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ is called a supply rate.*

The supply rate $s(u,y)$ models something like power delivered to the system when input value is u and the output value is y, see Fig. 1.

[1] DissipativeDynamicalSystems Part 1: General Theory. Dissipative Dynamical Systems Part II: Linear Systems with Quadratic Supply Rates.

Definition 4.2. *The system is said to be dissipative w.r.t. the supply rate s if there exists a storage function $V(x)$ such that*

$$\dot{V} \le s(u, y).$$

The inequality above is called the dissipation inequality. It can be rewritten in the form

$$L_f V(x, u) \le s(u, h(x)),$$

By an appropriate translation of coordinates, we can assume that $V(x(0) = 0$, and integrating

$$V(x(t)) - V(x(0)) \le \int_0^t s(u(\tau), y(\tau)) d\tau$$

and since V is positive, definite, we have the equivalent formulation

$$\int_0^t s(u(\tau), y(\tau)) d\tau \ge 0$$

along the trajectories of the system as a characterization of a dissipative system.

Dissipativity means that the increase in storage is bounded by the supply, see Fig 2. In particular, if the equality holds, the system is conservative.

When the system is closed, it does not interchange energy with the ambient so its supply is zero: $s = 0$. Then, the dissipativity condition is equivalent to the existence of a Lyapunov function V. Thus, dissipativity is a natural generalization of Lyapunov theory to open systems, i.e., stability is for closed systems and dissipativity for open system.

Some notions we have encountered so far can be re-stated as dissipativity w.r.t. a special supply rate:

- (strict) passivity: $\dot{V} \le y^T u - \alpha(\|x\|)$, $\alpha \in \mathcal{K}$.
- input-to-state stability: $\dot{V} \le -\alpha(\|x\|) + \gamma(\|u\|)$.
- input-output stability $\dot{V} \le -\alpha_1(\|x\|) - \alpha_2(\|y\|) + \gamma(\|u\|)$.

To see the generality that the dissipative framework may provide, we offer the following example.

EXAMPLE 1. Consider a thermodynamic system at uniform temperature T on which mechanical work is being done at a rate W en which is being heated at rate Q. Let (T, Q, W) be the external variables of such system.

The first and the second law of thermodynamics may then be formulated by saying that the system is conservative (or lossless) with respect to the supply $s_1 = W + Q$ and strictly dissipative with respect to the supply $s_2 = -Q/T$.

Indeed, the first law of thermodynamic states that the energy is conserved, i.e.

$$E(x(t_1)) = E(x(t_0)) + \int_{t_0}^{t_1} (Q(t) + W(t))dt.$$

This means that the absorbed energy is equal to the heat absorbed and the mechanical work.

The second law states that the entropy S (i.e. measure of disorder) increases

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} -\frac{Q(t)}{T(t)}dt.$$

4.1 Characterization of storage functions.

A crucial role is played by a quantity termed the available storage

Definition 4.3. *The available storage, V_a of a system with supply rate s is the function from $\mathbb{R}^n$ into $\mathbb{R} \cup \infty$ defined by*

$$V_a(x) = \sup_{u(\cdot)\in\mathscr{U}, x\to} -\int_0^T s(u(t), y(t)dt,$$

where the notation $x \to$ denotes the supremum over all motions starting in state x at time 0 and where the supremum is taken over all admissible control $u \in \mathscr{U}$.

Taking $T = 0$, we see that $V_a(x) > 0$. The available storage is the maximum amount of storage which may at any time have been extracted from the system. It is something like "recoverable work".

Theorem 4.1. *$V_a(x) < \infty, \forall x \in \mathbb{R}^n$ if an only if the system is dissipative.*

Sufficiency. If $V_a(x) < \infty$ we need to show that it is a storage function, i.e.,

$$V_a(x_1) \leq V_a(x_0) + \int_0^T s(t)dt,$$

for a control u transfering $x_0 \to x_1 = x(T)$. The proof is simply conceptual

$$V_a(x_0) = \sup_{u(\cdot)\in\mathscr{U}, x\to} -\int_0^T s(u(t), y(t)dt \geq -\int_0^T s(u(t), y(t))dt + \sup_{u(\cdot)\in\mathscr{U}, x_1\to} -\int_0^T s(u(t), y(t))dt$$

That is that in extracting the available storage from x_0 (second expression from left to right), we could first take $x_0 \to x_1$ using some given u (third expression) and then extract the available storage from x_1 (fourth expression), see Fig 5. This combined process is clearly suboptimal (we can do it probably better) which yields the $\geq$ symbol.

[*Necessity*] If the system is dissipative then there exists a storage function $V(x)$ such that

$$V(x_1) \leq V(x_0) + \int_0^t s(t)dt$$

since $V(x)$ is positive definite, then $0 \leq V(x_1)$ so subtracting $\int s$ from the inequality yields

$$V(x_0) \geq -\int_0^t s(t)dt.$$

Taking supremum over $x_0 \to$ in both sides completes the proof.

□

Above we have emphasizes what happens when the system starts off in a particular state. One may similarly examine what happens when the system ends up in a particular state.

Definition 4.4. *The required supply V_r of a system with supply rate s is the function from $\mathbb{R}^n$ into $\mathbb{R} \cup \infty$ defined by*

$$V_r(x) = \inf_{u(\cdot)\in\mathscr{U}, x^*\to x} \int_{t_-}^0 s(t)dt$$

where the notation $\inf_{u(\cdot)\in\mathscr{U}, x^*\to x}$ *denotes the infimum over all admisible control and $t_- \leq 0$ such that $x = \phi(0, t_-, x^*, u)$*

For the required supply V_r we let the system start in a given state and bring it to its presents state in the most efficient manner, i.e., by using no more supply from the outside than is absolutely necessary.

In analogy to the last theorem, we have the following result

Theorem 4.2. *Assume that the system is dissipative. If the state space is reachable from x^* then $V_r < \infty$ and the required supply V_r is a possible storage function.*

Proof. Since the system starts at x^*, we have $V_r(x^*) = 0$. Now, if $V_r(x) < \infty$ we have

$$V_r(x) \leq \int_{t_{-1}}^{t} s(t)dt = V_r(x^*) + \int_{t_{-1}}^{t} s(t)dt$$

where $s(t) = s(u(t), y(t))$ is such that $u(t)$ transfers $x^* \to x$. The inequality follows since $V_r(x)$ is the infimum of the right-hand side. This proves that V_r is a storage function so the system is dissipative. □

Note also that if the system is dissipative there exists a storage function V (taking the normalization $V(x^*) = 0$) such that

$$V(x) \leq \int_{t_{-1}}^{0} s(t)dt$$

for any u that transfers x^* at t_{-1} to x at 0. Taking the infimum in both sides yields $V_r(x) \geq V(x)$. So we have the following parametrization of the storage functions

Corollary 4.1.1. *For a dissipative system, any storage function V satisfies $V_a \leq V \leq V_r$*

4.2 Interconnection and stablity of dissipative systems

One of the more interesting properties of dissipative systems is that, under the correct interconnection, interconnected dissipative systems remain dissipative, see Fig 3. We have seen particular examples of this property: the series interconnection of ISS systems is ISS and the feedback connection of passive systems is passive. Dissipative theory allows going further in this direction. This is relevant since it simplifies the analysis of complex systems: first analyze the components and then their interconnection.

Another nice property of dissipative system is that closing two complementary dissipative system produce a stable system, see Fig. 4. One example of this is when one system is unstable and the other system represents a controller. The controller is then designed to correctly dissipate the energy produced by the first system in order to produce a stable system.

Theorem 4.3. *Suppose that we have two dissipative systems with ports* (u_1, y_1) *and* (u_2, y_2) *and they are interconnected using the rule*

$$u_2 = y_1, \quad u_1 = -y_2$$

Assume further that they are complementary dissipative: $s_1(u, y) + \alpha s_2(y, -u) = 0$. *Then the origin is stable, Fig 6.*

PROBLEM 1. Give a proof of the theorem above. HINT: use the sum of storage functions as a candidate Lyapunov function.

To illustrate a representative use of the theory of dissipative systems, we present an alternative proof of Popov's criterion. In this proof, the conceptual interchange of energy is more evident.

We shall explore these (and more) results in the particular case of dissipative LTI systems with dissipative memoryless nonlinearities. We refer to the original papers (quite abstract) or the book by Brogliato for details about the genera case.

4.2.1 A short proof of Popov's criterion

Consider the MIMO Lure's problem depicted in Fig 7. It is direct to show that it is equivalent to Fig 8 for an arbitrary matrix $Q \in \mathbb{R}^{n \times n}$.

It can be seen as the interconection of two system Σ_1 and Σ_2 described by

$$\Sigma_1 : \begin{cases} \dot{x}_1 = Ax_1 + Bu_1, \\ y_1 = CAx_1 + CQx_1 + CBu_1 \end{cases}$$

with $x_1(0) = x(0)$, and

$$\Sigma_2 : \begin{cases} \dot{x}_1 = -Qx_2 + u_2, \\ y_2 = f(x_2), \end{cases}$$

with $x_2(0) = Cx(0)$. The interconnection is given by the rules

$$u_1 = -y_2, \quad u_2 = y_1.$$

Popov's criterion postulates the following conditions for stability:

i) (A, B, C) is a minimal realization of $G(s) = C(sI - A)^{-1}B$,

ii) $(sI + Q)G(s)$ is Positive Real,

iii) the path integral $\int_0^{x_2} f'(\sigma)d\sigma \geq 0$ for all x_2.

iv) $f^T(\sigma)Q\sigma \geq 0, \forall\sigma$.

Indeed, from condition (i) and (ii) we know that Σ_1 is passive, i.e. there exists $V_1(x_1)$ such that $\dot{V} \leq u_1^T y_1$.

For Σ_2 we propose

$$V_2(x_2) = \int_0^{x_2} f'(\sigma)d\sigma$$

that is positive definite by condition (iii) and satisfies

$$\dot{V}_2 = -f^T(x_2)Qx_2 + u_2^T y_2 \leq u_2^T y_2$$

by condition (iv). Hence the total energy $V = V_1 + V_2$ satisfies

$$\dot{V} \leq u_1^T y_1 + u_2^T y_2 = 0$$

by the interconnection rule. This shows that the system is stable.

4.3 Linear Systems with Quadratic Supply Rates

Consider the LTI system

$$\dot{x} = Ax + Bu \quad y = Cx + Du, \quad x(0) = x_0, \tag{4.2}$$

and let us restrict ourselves to quadratic supply rates

$$s(u, y) = \begin{bmatrix} y & u \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}.$$

In analogy to the Lyapunov theory for linear systems, the quadratic dissipativity of linear systems can be studied using quadratic storage functions.

Theorem 4.4. *System (4.2) is (Q, S, R)-dissipative w.r.t. $s(u, y)$ above if and only if there exists a quadratic storage function $V = x^T P x$ such that $\dot{V} \le s(u, y)$.*

Proof. If the system is dissipative then V_a and V_r are storage functions. They are quadratic, since they are defined using a optimal control problem with quadratic cost subject to a linear system. □

With this characterization, a (linear) system is (Q, S, R)-dissipative w.r.t. the quadratic supply-rate if and only there exists a positive solution $P = P^T > 0$ and $\varepsilon > 0$ such that

$$\begin{pmatrix} A^T P + PA + \varepsilon P - C^T Q C & PB + C^T S \\ B^T P + S^T C & R \end{pmatrix} < 0$$

where again we can always change εP by εI.

PROBLEM 2. Prove the claim above!

Definition 4.5. *A memoryless nonlinearity $y = \psi(t, u)$ with $\psi(t, 0) = 0, \forall t \ge 0$, is (Q, S, R)-dissipative if $s(u, y) \ge 0$.*

EXAMPLE 1. If a nonlinearity satisfies $\phi(y) \in [0, k]$ then it is $(Q, S, R) = (-1, k/2, 0)$ dissipative. For seeing this, no that the sector condition can be written as $ky - \phi > 0$ and multiplying by ϕ in both sides as

$$ky\phi - \phi^2 > 0.$$

This last inequality can be rewritten as

$$\begin{bmatrix} \phi & y \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & k/2 \end{bmatrix} \begin{bmatrix} \phi \\ y \end{bmatrix} \geq 0$$

The stability of (closed) interconnected dissipative system is related to its complementary dissipativity, as the following theorem illustrates:

Theorem 4.5. *Consider the feedback interconnection of a (Q, S, R)-strictly dissipative system with a $(-R, S^T, -Q)$-strictly dissipative system, see Fig. 9. Then $x = 0$ is GExpS.*

Proof. Let V_1 and V_2 be storage functions for each system. Then, $V = V_1 + V_2$ satisfies

$$\dot{V} = s_1(u_1, y_1) + s_2(u_2, y_2) - \varepsilon_1 V_1 - \varepsilon V_2 = \begin{bmatrix} y & u \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} -u & y \end{bmatrix} \begin{bmatrix} -R & S^T \\ S & -Q \end{bmatrix} \begin{bmatrix} -u \\ y \end{bmatrix} - \varepsilon_1 V_1 - \varepsilon V_2 \leq -\varepsilon_1 V_1 - \varepsilon V_2.$$

□

where $u = u_1$ and $y = y_1$.

In particular, Lure's problem looks much simpler:

Theorem 4.6. *Consider the feedback interconnection of a (Q, S, R)-strictly dissipative system with a $(-R, S^T, -Q)$-dissipative nonlinearity as shown in Fig 10. Then $x = 0$ is GExpS.*

Proof. Use $V_2 = 0$. Compute $\dot{V} = \dot{V}_1 + \dot{V}_2$ and then add s_2 that is positive.

□

CONTROLLABILITY AND OBSERVABILITY FOR LINEAR SYSTEMS

5

Now that we are well versed in the stability-like of closed and open system we shall discuss two other main properties of a system.

The first one is called controllability and is the possibility of drive the system from one state to another by means of the control input. The second one is the observability and is the possibility of reconstructing the state trajectory from the output trajectory.

Both properties have a pretty nice and complete characterization for linear time invariant system. However, for nonlinear system, there exists only sufficient conditions as always.

Here we will follow Trentelman. Let us consider

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du, \end{cases} \quad x(0) = x_0, \tag{5.1}$$

where we will assume:

- u takes values in a m-dimensional space $\mathscr{U}$, which we often identify with $\mathbb{R}^m$.
- x takes values in a n-dimensional space $\mathscr{X}$, which we often identify with $\mathbb{R}^n$.
- y takes values in a p-dimensional space $\mathscr{Y}$, which we often identify with $\mathbb{R}^p$.

The class of admissible input functions will be denoted as $\mathbb{U}$, it will often be the space of locally integrable function. The space $\mathscr{X}$ will be called the state space. The explicit solution for the linear system above can be obtained by the variation-of-constants formula as follows:

$$x_u(t, x_0) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau, \tag{5.2}$$

and the corresponding output given by $y_u(t, x_0) = Cx_u(t, x_0)$.

PROBLEM 1. Prove that the formula above is a solution of the system Σ.

HOMEWORK 1. Study sections 2.1 and 2.2 from Trentelman.

5.1 Controllability.

In this section we concentrate in the relation between u and x. We are interesting in what extent one can influence the state by a suitable choice of control.

For this purpose, we introduce the (at time T) the reachable space $\mathscr{W}_T$:

$$\mathscr{W}_T = \{x^* \in \mathscr{X} | u \in \mathbb{U} \text{ such that } x_u(T, 0) = x^*\},$$

and, from the variation-of-constants formula (5.2) we see that

$$\mathscr{W}_T = \left\{ \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau | u \in \mathbb{U} \right\}$$

Definition 5.1. *Call system Σ reachable at time T if $\mathscr{W}_T = \mathscr{X}$.*

I.e., if every state can be reached from the origin.

Definition 5.2. *Σ is controllable (at time T) if every point is reachable from any point in a given time interval $[0, T]$*

It be seen that, for a linear system, reachability implies controllability [1]. This is because, using (5.2), the condition for point x_1 to be reachable from x_0 is that

$$x_1 - e^{AT}x_0 \in \mathscr{W}_T$$

that is always satisfied if $\mathscr{W}_T = \mathscr{X}$.

Lemma 5.1. *Let $\eta \in \mathbb{R}^n$ be a row vector and $T > 0$. Then the following statements are equivalent:*

(i) $\eta \perp \mathscr{W}_T$, *i.e.* $\eta x = 0$ *for all* $x \in \mathscr{W}_T$.

(ii) $\eta A^k B = 0$ *for* $k = 0, 1, 2, \ldots,$

(iii) $\eta \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = 0.$

Proof. Each point

(i) $\Leftrightarrow$ (ii) If $\eta \perp \mathscr{W}_T$ then

$$\int_0^T \eta e^{A(t-\tau)} B u(\tau) d\tau = 0, \quad \text{for every} u \in \mathcal{U}$$

in particular, choosing $u(t) = B^T e^{A^T(T-\tau)} \eta^T$ yields

$$\int_0^T \|\eta e^{A(t-\tau)} B u(\tau)\|^2 d\tau = 0$$

and hence $\eta e^{At} B = 0$ for each $t \geq 0$. Using the series expansion for e^{At} proves sufficiency. For necessity, assuming (ii) implies that $\eta e^{At} B = 0$ for each t, then

$$\int_0^T \eta e^{A(t-\tau)} B u(\tau) d\tau = 0, \quad \text{for every} u \in \mathcal{U}$$

and hence (i) follows.

[1] This is not true for general nonlinear systems

(ii) $\Leftrightarrow$ (iii) Sufficiency follows by evaluating the vector-matrix product. Necessity comes from the Cayley-Hamilton theorem. According to this theorem

$$A^n = \text{linear combination of } I, A, \cdots, A^{n-1}$$

Therefore, $\eta A^k B = 0$ for $k = 0, \dots, n-1$ implies that $\eta A^k B = 0$ for all $k \in \mathbb{N}$.

□

From the equivalence of (i) and (iii) of the lemma above we have the following result:

Theorem 5.1. *$\mathscr{W}_T$ is independent of T for $T > 0$. Specifically:*

$$\mathscr{W}_T = \mathscr{W} = \operatorname{im}\begin{pmatrix} B & AB & \cdots & A^{n-1}B \end{pmatrix}$$

Denoting $\mathscr{B} = \operatorname{im} B$, let us denote the as A-invariant subspace generated by $\mathscr{B}$ as

$$\langle A|\mathscr{B}\rangle = \mathscr{B} + A\mathscr{B} + \cdots A^{n-1}\mathscr{B}.$$

Then we can write $\mathscr{W} = \langle A|\mathscr{B}\rangle$.

Moreover, from the Lemma above we can also obtain the following result:

Theorem 5.2. *The following statements are equivalent:*

(i) *the system Σ if controllable at T for all $T > 0$.*

(ii) $\langle A|\mathscr{B}\rangle = \mathscr{X}$.

(iii) $\operatorname{rank}\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n$

Condition (iii) is known as the Kalman rank-criterion and the matrix $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$ is known as the Controllability Matrix of the system.

EXAMPLE 1. Consider

$$A = \begin{bmatrix} -2 & -6 \\ 2 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

Then, we can compute

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 2 & 4 \end{bmatrix}$$

which clearly has rank equal to 1. Hence, the system is not controllable.

The reachable space is the span of $\begin{bmatrix} B & AB \end{bmatrix}$, i.e., the line with parametric representation $x = \alpha(-3, 2)^T$ or equivalently, the line given by the equation $2x_1 + 3x_2 = 0$. This can also be seen as follows: introducing $z = 2x_1 + 3x_2$ yields $\dot{z} = z$. Hence if $z(0) = 0$ we must have $z(t) = 0$ for all $t \geq 0$.

5.1.1 Controllable eigenvalues.

Definition 5.3. *An eigenvalue $\lambda \in \lambda(A)$ is called (A, B)- controllable if*

$$\operatorname{rank}\begin{pmatrix} A - \lambda I & B \end{pmatrix} = n.$$

Note that instead of the rank condition, we can write:

"for every row vector η we have: $\eta A = \lambda \eta$ and $\eta B = 0 \Longrightarrow \eta = 0$"

in other words, there does not exists a left eigenvector of A corresponding to the eigenvalue λ which is orthogonal to $\operatorname{im} B$. Still another description is given in terms of subspaces as follows:

$$(A - \lambda I)\mathscr{X} + B\mathscr{U} = \mathscr{X} \quad \Leftrightarrow \quad \operatorname{im}(A - \lambda I) + \operatorname{im} B = \mathscr{X}.$$

An evident result is the following one:

Theorem 5.3. (A, B) *is controllable if and only if every eigenvalue of A is (A, B)-controllable.*

5.1.2 Stabilization by state feedback

Given system Σ, our objective is to construct a static full state feedback controller $u = Kx$ such that the system behaves as desired. A usual requirement is that the closed-loop system is stable, i.e. all its eigenvalues belong to $\mathbb{C}_-$. However, we can also ask for certain transient characteristics that can be described by placing the closed-loop eigenvalues in certain domain of interest denoted by $\mathbb{C}_g$.

When our feedback controller is applied we obtain a closed loop system $\dot{x} = (A + BK)x$. Therefore, the problem of stabilization by static state feedback reads as follows:

"given a stability domain $\mathbb{C}_g$ and maps A and B, determine a map $K : \mathscr{X} \to \mathscr{U}$ such that $\sigma(A + BK) \subset \mathbb{C}_g$. "

The solution to this problem is provided by the following celebrated result known as "pole-placement theorem".

Theorem 5.4. *Let A and B be given. Then there exists for every monic polynomial $p(s)$ of degree n a map $K : \mathscr{X} \to \mathscr{U}$ such that the characteristic polynomial of $A + BK$ is equal to $p(s)$, if and only if (A, B) is controllable.*

Necessity. If (A, B) is not controllable, there exists an uncontrollable eigenvalue λ. For this eigenvalue there exists a row vector η different from zero such that $\eta A = \lambda\eta$ and $\eta B = 0$.

Then we have that $\eta(A + BK) = \lambda A$ for all K and hence $\lambda \in \sigma(A + BK)$ for all K. So if $p(s)$ is a polynomial such that $p(\lambda) \neq 0$, there does not exists a feedback K such that $p(s) = p_{A+BK}(s)$.

[Sufficiency] The reader is referred to [Trentelman, pp. 59]. □

A corollary of this result is as follows.

Corollary 5.1.1. *Let (A, B) be controllable. Then there always exists a matrix K such that $(A - BK)$ is stable.*

5.2 Observability.

The idea to investigate to what extent it is possible to reconstruct the state x when the input u and output y are known.

Definition 5.4. *Call system Σ observable if for any pair of states $x_0, x_1 \in \mathscr{X}$ and input $u \in \mathbb{U}$ the condition $y_u(t, x_0) = y_u(t, x_1)$ for all $t \in [0, T]$ implies that $x_u(t, x_0) = x_u(t, x_1)$ for all $t \in [0, T]$.*

In particular, the definition above implies that $x_0 = x_1$. A pair of states x_0 and x_1 are indistinguishable if they give rise to the same output values for every input u. In other words, a system is observable if any two distinct (initial) states are distinguishable.

According to the variation-of-constants formula, we can write $y_u(t, x_0) = y_u(t, x_1)$ as follows

$$Ce^{At}x_0 + \int_0^t K(t-\tau)u(\tau)d\tau + Du(t) = Ce^{At}x_1 + \int_0^t K(t-\tau)u(\tau)d\tau + Du(t)$$

where $K(t-\tau) = e^{A(t-\tau)}B$.

From this we see that the condition $y_u(t, x_0) = y_u(t, x_1)$ is equivalent to $Ce^{At}x_0 = Ce^{At}x_1$. Hence, the observability is independent of the input u. Hence, x_0 and x_1 will be indistinguishable if and only if $v = x_0 - x_1$ and 0 are indistinguishable.

We apply Lemma 5.1 with $\eta = v^T$ and transpose the equations. Then we have the following chain of equivalences

$$\left\{Ce^{At}x_0 = Ce^{At}x_1\right\} \Leftrightarrow \{Ce^{At}v = 0\} \Leftrightarrow \{CA^k v = 0, k \in \mathbb{Z}\}$$

and, finally, using Cayley-Hamilton theorem, we can show that we only need the first n terms, i.e.

$$Ov = 0$$

where the matrix O is known as the Observability matrix

$$O = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}.$$

PROBLEM 2. Prove the claims above.

As a consequence, the distinguishability of two vectors does not depend on T. The space of vectors v for which $Ov = 0$ is denoted as $\langle \ker C | A \rangle$ and is called the unobservable space.

From the discussion above we have the following characterization:

Theorem 5.5. *The following statements are equivalent:*

(i) Σ *is observable,*

(ii) every state is distinguishable from the origin,

(iii) the unobservable space is trivial $\langle \ker C | A \rangle = 0$,

(iv) the observability matrix satisfies $\operatorname{rank} O = n$.

Also in analogy to the pole-placement result of Theorem 5.4 and Corollary 5.1.1, we have the following result

Theorem 5.6. *If the pair* (A, C) *is observable, then there exists a matrix* L *such that* $\sigma(A - LC) \in \mathbb{C}_g$.

PROBLEM 3. Let

$$A = \begin{pmatrix} -11 & 3 \\ -3 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 \end{pmatrix}$$

Show that (C, A) is not observable and that the unobservable space $\langle \ker C | A \rangle$ is the span of $(1, 1)^T$.

5.2.1 State observers

When the state of Σ is not directly measured, one often tries to reconstruct them using another dynamic system called "observer". It takes the input and output of the system in question and provides an estimate $\hat{x}$ of its internal state x.

An observer is another dynamic system

$$\Xi : \begin{cases} \dot{\hat{x}} = A\hat{x} + Bu - L(\hat{y} - y) \\ \hat{y} = C\hat{x}, \end{cases} \quad \hat{x}(0) = \hat{x}_0, \tag{5.3}$$

that is excited by the input u of system Σ and an the injection term $(\hat{y} - y)$. This last terms quantifies the difference between the output of the real system and that of the observer. The objective of the designer is to appropriately choose L such that $\hat{x}(t) \to x(t)$.

The dynamics of the observation error $e = \hat{x} - x$ is given by

$$\dot{e} = (A - LC)e$$

and hence we have the following result.

Theorem 5.7. *Σ is observable, then there exists a gain L such that Ξ is an observer for it in the sense that $\hat{x}(t) \to x(t)$ exponentially fast.*

NONLINEAR CONTROL DESIGN

6

The first and obvious question to ask is: why is it necessary to use nonlinear controllers. Let us argue by means of two examples.

Consider the problem of regulating the velocity of a AC motor. In principle, the functioning of any electrical machine can be described by means of Maxwell's equations. These equation are indeed linear Partial Differential Equations. Hence, to avoid these complications, experience have shown that it is better to replace them by a no-linear model in ordinary derivatives.

Other important examples comes from optimal decision making. For instance, consider the problem of driving from City A to City B in minimal time. What is the best possible acceleration profile given that it is bounded by $|\tau_{motor}| \leq K$? It turns out that the optimal strategy is the so-called bang-ban control which is also nonlinear (see Fig 1).

In this chapter we will Review some standard methods for the control of nonlinear systems. In fact, there is a broad family of design methods. Here we present the following taxonomy:

1. By approximation (Fig 2).
   a) Taylor linearization.
   b) Gain-scheduling method.
2. By transformation (Fig 3).
   a) controller canonical form + pole placement.
   b) exact linearization.

c) differential flatness.

3. By energy functions (Fig 4)

   a) Lyapunov/Lyapunov-control functions.

   b) Storage functions.

6.1 Taylor linearization

We start with a nonlinear controlled model of our system

$$\dot{x} = f(x, u), \quad y = h(x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$ are the state, input and output respectively. Let us assume that we are interested in keeping the system near an operating point x^* and furthermore, that we can make it an equilibrium point by a suitable choice of control u^*, i.e.

$$f(x^*, u^*) = 0.$$

Let us introduce the deviation variables $\bar{x} = x - x^*$m $\bar{u} = u - u^*$ and $\bar{y} = y - y^*$, they measure the deviations from the equilibrium point. The (first-order) Taylor linearization of the nonlinear system near x^* is given by

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}, \quad \bar{y} = C\bar{x},$$

where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x^*, u^*}, \quad B = \left.\frac{\partial f}{\partial u}\right|_{x^*, u^*}, \quad C = \left.\frac{\partial h}{\partial x}\right|_{x^*}$$

Here it is important to recall that the actual control is composed as $u = u^* + \bar{u}$, i.e., the nominal control u^* plus small corrections.

Assume that that (A, B) is controllable. Then it is possible to find a matrix K such that the eigenvalues of $A + BK$ are at the desired locations in the left-half complex plane. Since $A + BK$ is Hurwitz, then it follows via Lyapunov first method that $x = x^*$ is locally assymptotically stable.

Applying the linear state feedback, the closed loop system is

$$\dot{x} = f(x, u^* + Kx)$$

EXAMPLE 1. (Inverted pendulum). Consider the following model for an inverted pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a\sin(x_1) - bx_2 + cu,$$

where x_1 is the angle subtended by the rod an the vertical axis and u is the applied torque. Our objective is to stabilize the pendulum at an angle $x_1 = x_1^*$, i.e. $x_2^* = 0$. The steady state component of the control needs to satisfy

$$0 = -a\sin(x_1^*) - b0 + cu^*, \quad \Rightarrow u^* = \frac{a}{c}\sin(x_1^*)$$

Now introduce the deviation variables $\bar{x}_1 = x_1 - x_1^*$, $\bar{x}_2 = x_2^*$ and $\bar{u} = u - u^*$. Then we have

$$\dot{\bar{x}}_1 = \bar{x}_2, \quad \dot{\bar{x}}_2 = -a[\sin(\bar{x}_1 + x_1^*) - \sin(x_1*)] - b\bar{x}_2 + c\bar{u}$$

Linearizing the system above near $\bar{x} = 0$ gives

$$A = \begin{pmatrix} 0 & 1 \\ -a\cos(\bar{x}_1 + x_1^*) & -b \end{pmatrix}_{\bar{x}_1 = 0} = \begin{pmatrix} 0 & 1 \\ -a\cos(x_1^*) & -b \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ c \end{pmatrix}.$$

Let us denote $K = [k_1, k_2$, then it can be easily verified (do it) that $A + BK$ is Hurwitz provided that

$$k_1 > \frac{a}{c}\cos(x_1^*), \quad k_2 = \frac{b}{c}$$

and the final applied torque is given by

$$u = \frac{a}{c}\sin(x_1^*) + k_1(x_1 - x_1^*) + k_2 x_2.$$

PROBLEM 1. Complete the example above using a Lyapunov analysis.

6.1.1 *Gain-scheduling*

The method above is very sound since it allows constructing a controller that locally stabilizes any nonlinear system. The main problem, however, is such locality: what happens if we want to stabilize the system in a whole region of the state space?

One somehow natural way to do it is to linearize the system in several operating points, construct its corresponding controller and switch between controllers according to where the system currently is, see Fig 5. This is the main idea of the Gain-Scheduling approach. Nevertheless, note that the problem is still nonlinear! The approach has been considerably successful in flight application: there are several models of the vessel according to the mach number of speed.

6.2 Feedback linearization

Consider an affine control system

$$\dot{x} = f(x) + g(x)u,$$

the main idea is to find a controller

$$u = \alpha(x) + \beta(x)v, \quad v \in \mathbb{R}^m$$

and a change of coordinates $z = T(x)$ such that the closed loop system with v as a new inputs looks linear.

EXAMPLE 2. (Inverted pendulum). Consider again the model for an inverted pendulum

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = -a\sin(x_1) - bx_2 + cu,$$

The controller

$$u = \frac{a}{c}[\sin(x_1 + x_1^*) - \sin(x_1^*)] + \frac{1}{c}v,$$

transforms the system into

$$\dot{x}_1 = x_1, \quad \dot{x}_2 = -bx_2 + v$$

that is linear. Now using $v = k_1x_1 + k_2x_2$ yields

$$\dot{x} = A_c x, \quad A_c = \begin{pmatrix} 0 & 1 \\ k_1 & k_2 - b \end{pmatrix}$$

and any selection $k_1 < 0$ and $k_2 - b < 0$ implies that A_c is Hurwitz. The final controller is

$$u = \frac{a}{c}[\sin(x_1 + x_1^*) - \sin(x_1^*)] + \frac{1}{c}[k_1x_1' + k_2x_2]$$

The example above illustrates the particular case when no transformation is needed, i.e. when $z = x$. In such a case, we are interested in a system of the form

$$\dot{x} = Ax + B\beta^{-1}(x)[u - \alpha(x)]$$

since when the control $u = \alpha + \beta v$ is applied yields

$$\dot{x} = Ax + Bv$$

so we can design $v = Kx$.

In the general case, we look for something quite similar.

Definition 6.1. *The system*

$$\dot{x} = f(x) + g(x)u$$

with $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$ is said to be input-to-state *linearizable if there exists a diffeomorphism $T : D_x \to \mathbb{R}^n$ with $z(0) = 0$ such that the changes of coordinates*

$$z = T(x)$$

transforms the system into the form

$$\dot{z} = Az + B\beta^{-1}(x)[u - \alpha(x)]$$

with (A, B) controllable and $\beta(x)$ non-singular in D_x.

Let us do some arithmetic now. Using the diffeomorphism $z = T(x)$ the original system is (dynamically) equivalent to

$$\dot{z} = \frac{\partial T}{\partial x} f(x) + \frac{\partial T}{\partial x} g(x)u.$$

On the other hand, from the definition we also have

$$\dot{z} = Az + B\beta^{-1}(x)[u - \alpha(x)] = AT(x) + B\beta^{-1}(x)[u - \alpha(x)]$$

Comparing term by term we see that

$$\frac{\partial T(x)}{\partial x} f(x) = AT(x) - B\beta^{-1}(x)\alpha(x), \tag{6.1}$$

$$\frac{\partial T(x)}{\partial x} g(x) = B\beta^{-1}(x) \tag{6.2}$$

that are two PDEs known as the *matching conditions*. The existence of $\{T, \alpha, \beta, A, B\}$ satisfying these equations are thus a necessary and sufficient condition for the system to the input-to-state linearizable.

Let us do even more arithmetics for the case of a single input $m = 1$. If (A, B) is controllable, it is always possible to write then in the *canonical controller form*

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -a_0 & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

where a_i are the coefficients of the characteristic polynomial of the system.

Let us write also

$$T(x) = \begin{pmatrix} T_1(x) \\ \vdots \\ T_n(x) \end{pmatrix}$$

then we have

$$A - B\beta^{-1}(x)\alpha(x) = \begin{pmatrix} T_2(x) \\ T_3(x) \\ \vdots \\ T_n(x) \\ -\alpha(x) \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1/\beta(x) \end{pmatrix} \alpha(x)$$

since for $m = 1$ both α and β are scalars.

From the computation above, we can rewrite the matching conditions as follows

$$\frac{\partial T_1(x)}{\partial x} f(x) = T_2(x), \cdots, \quad \frac{\partial T_{n-1}(x)}{\partial x} f(x) = T_n(x), \quad \frac{\partial T_n(x)}{\partial x} f(x) = -\frac{\alpha(x)}{\beta(x)}$$

$$\frac{\partial T_1(x)}{\partial x} g(x) = 0, \cdots, \quad \frac{\partial T_{n-1}(x)}{\partial x} g(x) = 0, \frac{\partial T_n(x)}{\partial x} g(x) = -\frac{1}{\beta(x)}$$

From the expressions above, we see that in this particular case everything is parametrized by the selection of $T_1(x)$ in a recursive way. Indeed, once $T_1(x)$ is given, the second component $T_2(x)$ is computed using

$$\frac{\partial T_1(x)}{\partial x} f(x) = T_2(x)$$

and so on until $T_n(x)$ is computed. Finally, α and β are computed from the last expressions.

Recall that $T(x^*) = 0$.

EXAMPLE 3. Consider the system

$$\dot{x} = \begin{pmatrix} a \sin x_2 \\ -x_1^2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Observe that when $u = 0$, then $x = 0$ is an equilibrium of the system however it is not stable. Since the system has a single input, recall that the objective is to find $T_1(x)$. From the matching equations we have the following conditions:

$$\frac{\partial T_1}{\partial x} g = 0, \quad \Rightarrow \frac{\partial T_1}{\partial x_2} = 0$$

that means that T_1 does not depend on x_2, and

$$\frac{\partial T_1}{\partial x} f = T_2, \quad \Rightarrow \frac{\partial T_1}{\partial x_1} a \sin(x_2) = T_2$$

Thus, the only condition (apart form being a diffeomorphism) for $T_1(x)$ is that it does not depend on x_2 and that $T_1(0) = 0$. Let us pick for instance $T_1(x) = x_1$ that implies that

$$T_2(x) = a \sin(x_2).$$

Observe that in fact this is the right-hand side for $\dot{x}_1$. Letting $z_1 = T_1(x)$ and $z_2 = T_2(x)$ yields

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = -a x_1^2 \cos(x_2) + a \cos(x_2) u$$

finally, let

$$u = \frac{1}{a \cos(x_2)} (a x_1^2 \cos(x_2) + v)$$

yields the linear closed loop system

$$\dot{z}_1 = z_2, \quad \dot{z}_2 = v.$$

Input-output linearization

Let us again consider an affine control system but with outputs

$$\dot{x} = f(x) + g(x)u, \quad y = h(x),$$

and for convenience of presenting the main ideas let us assume that it is SISO, namely $p = m = 1$.

In contrast to our previous discussion, our goal now is to linearize only its input-output behavior. Suppose that we have chosen

$$T_1(x) = h(x).$$

To linearize the system, we view the matching conditions as restrictions in the form of which the derivatives of y depend on u. Making $\phi_1(x) = h(x)$ yields

$$\dot{y} = \frac{\partial \phi_1}{\partial x}[f + gu]$$

if $\frac{\partial \phi_1}{\partial x} g = 0$ we get

$$\dot{y} = \frac{\partial \phi_1}{\partial x} f =: \phi_2(x)$$

Taking another derivative

$$\ddot{y} = \frac{\partial \phi_2}{\partial x}[f + gu]$$

and if again $\frac{\partial \phi_2}{\partial x} g = 0$ yields

$$\ddot{y} = \frac{\partial \phi_2}{\partial x} f =: \phi_3(x)$$

If this process is repeated we will get

$$\frac{\partial \phi_i}{\partial x} g = 0, \quad i = 1, \cdots, n-1$$

$$\frac{\partial \phi_n}{\partial x} g \neq 0$$

where $\phi_{i+1}(x) = \frac{\partial \phi_i}{\partial x} f$ for $i = 1, \ldots, n-1$ and it holds that

$$y = \phi_1(x), \quad \dot{y} = \phi_2(x), \cdots$$

Taking the n-th derivative we obtain

$$y^{(n)} = \frac{\partial \phi_n}{\partial x} f + \frac{\partial \phi_n}{\partial x} gu$$

and from here we see that the system would be input-output linearizable by picking

$$u = \frac{1}{\frac{\partial \phi_n}{\partial x} g} \left[-\frac{\partial \phi_n}{\partial x} f + v \right]$$

since we obtain

$$y^{(n)} = v.$$

Definition 6.2. *The system above has relative degree r with respect to $y = h(x)$ if*

$$\frac{\partial \phi_i}{\partial x} g = 0, \quad i = 1, \ldots, r-1$$

and $\frac{\partial \phi_r}{\partial x} g \neq 0$, with $\psi_1(x) = h(x)$ and $\phi_{i+1}(x) = \frac{\partial \phi_i}{\partial x} f$.

In other words, the system has relative degree r if the control inputs appears for the first time in the r-th derivative of the output. Another equivalent statement of this definition can be given in terms of Lie derivatives:

$$L_g L_f^{i-1} h(x) = 0 \quad \text{for } i = 1, \ldots, r-1 \quad \text{and } L_g L_f^{r-1} h(x) \neq 0.$$

With the construction above, in fact we have proposed the following change of coordinates

$$T(x) = \begin{pmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1} h(x) \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

Form the discussion above we can obtain the following two conclusions:

Theorem 6.1. *If the system has relative degree r then it is input-output linearizable. In addition, if it happens that its relative degree is n then it is also input-to-state linearizable.*

The question is now: what happens to the rest of the $(n-r)$ coordinates in the case $r < n$? Are they linear or not? The dynamics for such variables receive the name of zero-dynamics. The answer is that the zero dynamics were not linearized, but they remain hidden from the input-output response as we shall discuss. They are dangerous since they might be unstable despite the io-behaviour is stable.

Suppose that the system has relative degree r. Then we can take $\{h(x), \cdots, L_f^{r-1}h(x)\}$ as part of the transformation $T(x)$ we need. Afterwards, a least locally, we can always pick another $n-r$ functions ϕ_i such that

$$T(x) = \begin{pmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-r}(x) \\ \hline h(x) \\ L_f h(x) \\ \vdots \\ L_f^{r-1}h(x) \end{pmatrix} = \begin{pmatrix} \phi(x) \\ \hline \psi(x) \end{pmatrix} = \begin{pmatrix} \eta \\ \hline \xi \end{pmatrix}$$

is a local diffeomorphism (see Khalil pp. 516) and such that all of them are orthogonal to g:

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } i = 1, \ldots, n-r.$$

One can think that these functions complement the relative degree of $h(x)$.

Observe that one has that

$$\dot{\eta} = \frac{\partial \phi}{\partial x}[f + gu] = \frac{\partial \phi}{\partial x} f$$

does not depend on u.

Then we have

$$\begin{aligned}\dot\eta &= f_0(\eta,\xi),\\ \dot\xi &= A\xi + B\gamma(x)[u-\alpha(x)],\\ y &= C\xi\end{aligned}$$

where (A, B) is in the canonical controller form and $C = [1, 0, \cdots, 0]$ and

$$f_0(\eta,\xi) = \frac{\partial\phi}{\partial x}f(x)\bigg|_{x=T^{-1}(z)}, \quad \gamma(x) = L_g L_f^{r-1}h(x), \quad \alpha(x) = -\frac{L_f^r h(x)}{L_g L_f^{r-1}h(x)}.$$

Remark. These equations are said to be in the *normal form*. This form decomposes the state into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$u = \alpha(x) + \beta(x)v$$

while the internal part is made *unobservable* from y using the same control. The internal dynamics are described by η.

Definition 6.3. *The zero dynamics is $\dot\eta = f_0(\eta, 0)$. The system is said to be minimum-phase if the zero dynamics are stable.*

The name zero dynamics comes from the fact that for linear systems $\dot\eta = A_0\eta$ where A_0 corresponds to the zeros of the transfer function.

EXAMPLE 4. Consider the system

$$\begin{aligned}\dot x_1 &= -x_1 + \frac{x_3^2}{1+x_3^2}\\ \dot x_2 &= x_3\\ \dot x_3 &= x_1\sin(x_3) + u\\ y &= x_2\end{aligned}$$

The relative degree is $r = 2$. What is the zero dynamics in this case? The coordinates ψ are given by

$$\psi_1 = y = x_2, \quad \psi_2 = \dot y = x_3$$

The zero dynamics is then orthogonal to these two variables: it is x_1. It is given by setting $y \equiv 0$ in the dynamics for x_1:

$$\dot x_1 = -x_1$$

that is asymptotically stable. Then the system is of minimum phase.

6.3 Energy-based methods

The basic idea is illustrated by the following basic example:

EXAMPLE 5. Consider the system

$$\dot{x} = ax - bx^3 + u$$

and suppose we want to stabilize $x = 0$.

Propose the energy function

$$V = \frac{1}{2}x^2$$

and assume that $u = k(x)$. Then we have

$$\dot{V} = ax^2 - bx^3 + xk(x) < 0$$

and if $k(x) = -ax - \gamma x$, $\gamma > 0$ then

$$\dot{V}(x) = -bx^4 - \gamma x^2 < 0$$

and $x = 0$ is GAS. Observe that we do not cancel all the nonlinearities, we kept the ones that "help" our Lyapunov function.

In this section, we shall discuss two methods: Lyapunov redesign and Backstepping.

6.4 Lyapunov redisign

Here we address the following question: given a nominal design, how to modify it in the presence of uncertainties. This will naturally lead us to consider our first sliding mode controller.

Consider the perturbed system

$$\dot{x} = f(x) + g(x)u + w(t, x)$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a disturbance or perturbation. If nothing is known about the perturbation it is impossible to do anything. Henceforth, we shall assume

Definition 6.4. *The disturbance $w(t, x)$ is said to be matched with the control if there exists $\delta(t, xu)$ such that*

$$w(t, x) = g(x)\delta(t, x, u)$$

In this form, if the disturbance is matched then the system can be rewritten as follows:

$$\dot{x} = f(x) + g(x)[u + \delta(t, x, u)] \tag{6.3}$$

Let us assume that we now a nominal control law $u = \psi(x)$ such that it stabilizes uniformly the nominal system. What this means is that

$$\dot{x} = f(x) + g(x)\psi(x)$$

has $x = 0$ as a GAS equilibrium and moreover we know a certificate for this in the form of a Lyapunov function:

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \qquad \frac{\partial V}{\partial x}[f + g\psi] \leq -\alpha_3(\|x\|)$$

where $\alpha_i \in \mathscr{K}$.

Now we will make the following sensible assumption about the perturbation:

$$\|\delta(t, x, \psi + v)\| \leq \rho(t, x) + k_0\|v\|, \quad 0 \leq k_0 < 1$$

where ρ is a known function that measures the size of the uncertainty and k_0 characterizes the growth of the uncertainty with respect to the control. The assumption tell us that it grows slower than a linear function and that its gain is less than one.

We shall show that with the knowledge of V, the function ρ and the constant k_0 we can design an additional feedback control v such that the overall control $u = \psi(x) + v$ stabilizes the actual system in the presence of uncertainties. The main idea is to design v such that V is a Lyapunov function for the perturbed system.

Let us compute the derivative of V along the perturbed system and substitute the nominal control:

$$\dot{V} = \frac{\partial V}{\partial x}[f(x) + g(x)\psi] + \frac{\partial V}{\partial x}g(x)[v + \delta] \leq -\alpha_3(\|x\|) + \frac{\partial V}{\partial x}g(x)[v + \delta]$$

and hence the goal is to select v such that $\frac{\partial V}{\partial x}g(x)[v + \delta] \leq 0$.

Let us denote $w^T = \frac{\partial V}{\partial x} g(x)$ and observe that this quantity is known. Hence, we can rewrite

$$\frac{\partial V}{\partial x} g(x)[v + \delta] = w^T v + w^T \delta \leq w^T v + \|w\| \big[\rho(t, x) + k_0 \|v\|\big].$$

From here we propose the following control:

$$v = -\eta(t, x) \frac{w}{\|w\|_2}$$

with $\eta(t, x) \geq 0$, that is discontinuous at $x = 0$ since $\frac{\partial V}{\partial x}|_{x=0} = 0$ since $x = 0$ is a minimum of the Lyapunov function. With this we observe that

$$\frac{\partial V}{\partial x} g(x)[v+\delta] \leq w^T \left(-\eta \frac{w}{\|w\|_2}\right) + \|w\| \rho + k_0 \|w\| \eta = \|w\| \left(-\eta + \rho + k_0 \eta\right) \leq 0$$

if η is selected such that

$$\eta(t, x) \geq \frac{1}{1 - k_0} \rho(t, x)$$

6.5 Backstepping

In this section we will use Lyapunov redesign in the special case of uncertainties that represent additional systems.

Let us consider a system in the following form:

$$\dot{\eta} = f(\eta) + g(\eta)\xi, \quad \dot{\xi} = u, \tag{6.4}$$

where $[\eta, \xi] \in \mathbb{R}^n$, $\xi \in \mathbb{R}$ see Fig. 1. Suppose that the system

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$

can be stabilized with the control law

$$\dot{\xi} = \phi(\eta), \quad \phi(0) = 0.$$

This means that with $\xi = \phi(\eta)$, the system

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

has $\eta = 0$ is an asymptotically stable system. We shall assume that we know a Lyapunov function $V(\eta)$ such that

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -w(\eta)$$

with $w(\eta) > 0$ a positive definite function. We can rewrite the original system (6.4) as

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)], \quad \dot{\xi} = u$$

depicted in Figure 2.

Let us define $z = \xi - \phi(\eta)$ that denotes the tracking error in the control for the first state. In this coordinates we have

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z, \quad \dot{z} = v$$

where

$$v = u - \dot{\phi} = u - \frac{\partial \phi}{\partial \eta}[f + g\xi]$$

and the signal ϕ “back-step”, see Fig 3. For this new system, we propose the following Lyapunov function

$$V_1(\eta, \xi) = V(\eta) + \frac{1}{2}z^2$$

and then

$$\dot{V}_1 = \frac{\partial V}{\eta}[f + g\phi] + \frac{\partial V}{\eta}g(\eta)z + zv \leq -w(\eta) + \frac{\partial V}{\eta}g(\eta)z + zv$$

choosing

$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz, \quad k > 0$$

we obtain that

$$\dot{V}_1 \leq -w(\eta) - kz^2$$

that shows that $\eta = 0, z = 0$ is asymptotically stable. In summary, the following control law stabilizes the whole system given that we know $\phi(\eta)$ stabilizing the first subsystem:

$$u = \frac{\partial \phi}{\partial \eta}[f + g\xi] - \frac{\partial V}{\partial \eta} g(\eta) - kz.$$

In the general case we have a system in the so-called strictly feedback form:

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x) z_1, \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1) z_2 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \cdots, z_{k-1}) + g_{k-1}(x, z_1, \cdots, z_{k-1}) z_k \\
\dot{z}_k &= f_k(x, z_1, \cdots, z_k) + g_k(x, z_1, \cdots, z_k) u
\end{aligned}$$

In this system, each variable z_i acts as a virtual control for the state z_{i-1}. We suppose that

$$g_i(x, z_1, \cdots z_i) \neq 0, \quad i = 1, \ldots, k,$$

in a domain of interest

EXAMPLE 5. This last assumption is not necessary for the stabilization of a system. For instance, we can stabilize at $x = 0$ the following system

$$\dot{x} = -x + xu,$$

and $g(x) = 0$ and $x = 0$. .

The general procedure is as follows. We start considering

$$\dot{x} = f_0(x) + g_0(x) z_1$$

and we shall assume that we know a virtual control $z_1 = \phi_0(x)$ such that stabilizes $x = 0$ with a known Lyapunov function

$$V_0(x) \text{ such that } \frac{\partial V_0}{\partial x}[f_0 + g_0 \phi_0] \leq -w(x).$$

Now we take the following subsystem

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1,\\ \dot{z}_1 &= f_1(x,z_1) + g_1(x,z_1)z_2.\end{aligned}$$

By comparing terms with respect to our motivating example, we see that the following virtual control

$$z_2 = \phi_1(x,z_1) = \frac{1}{g_1}\left[-f_1 + \frac{\partial\phi_0}{\partial x}(f_0 + g_0 z_1) - \frac{\partial V}{\partial x}g_0 - k_1(z_1 - \phi_0)\right]$$

with $k_1 > 0$, and the following Lyapunov function proves it

$$V_1(x,z_1) = V_0(x) + \frac{1}{2}[z_1 - \phi_0(x)]^2$$

Now we take one subsystem more:

$$\begin{aligned}\dot{x} &= f_0(x) + g_0(x)z_1,\\ \dot{z}_1 &= f_1(x,z_1) + g_1(x,z_1)z_2\\ \dot{z}_2 &= f_2(x,z_1,z_2) + g_2(x,z_1,z_2)z_3,\end{aligned}$$

and we consider z_3 as virtual control. Considering

$$\eta = \begin{pmatrix} x \\ z_1 \end{pmatrix}, \quad \xi = z_2, \quad u = z_3$$

we have the same form of our prototipic example. Hence we can repeat the same procedure

$$z_3 = \phi_2(x,z_1,z_2) = \frac{1}{g_2}\left[-f_2 + \frac{\partial\phi_1}{\partial x}(f_0 + g_0 z_1) + \frac{\partial\phi_1}{\partial z_1}(f_1 + g_1 z_2 - \frac{\partial V}{\partial z_1}g_1 - k_2(z_2 - \phi_1)\right]$$

We repeat the process until we compute

$$u = \phi_k(x, z_1, \cdots, z_k)$$

with a Lyapunov function $V_k(x, z_1, \ldots, z_k)$.

EXAMPLE 6. Consider the system

$$\begin{aligned}\dot{x} &= -x + x^2 z,\\ \dot{z} &= u.\end{aligned}$$

We start with

$$\dot{x} = -x + x^2 z$$

one possibility is to choose $z = \phi_0(x) = 0$ and propose $V = 0.5x^2$. Then

$$\dot{V}_0 = \frac{\partial V_0}{\partial x}[-x + x^2\phi_0(x)] = -x^2 < 0$$

From here we simply compute our control:

$$u = -\frac{\partial V}{\partial x} g(x) - kz = -x^3 - kz$$

and we have the Lyapunov function

$$V = 0.5x^2 + 0.5z^2$$

to certify its stability. Indeed, we have

$$\dot{V} = -x^2 - kz^2 < 0.$$

.

PROBLEM 2. Consider the same system

$$\begin{aligned} \dot{x} &= -x + x^2 z, \\ \dot{z} &= u. \end{aligned}$$

but use $\phi_0(x) = x + x^2$. Compute the resulting control. .

SLIDING MODE CONTROL

7

Sliding modes is a technique of robust control. It produces controllers that are insensitive (more than robust) to matched disturbances.

We shall begin by reviewing the classical method of design using first order sliding modes and then complement it with the use of second order sliding mode algorithms. For the first part we recommend Utkin's book.

7.1 First-order sliding mode controllers

Consider a single input nonlinear system with matched perturbations

$$\dot{x} = f(x) + g(x)[u + w],$$

where $u \in \mathbb{R}$. The objective is to drive $x(t)$ to zero despite the presence of disturbances.

For doing so we suppose the existence of the sliding variable:

Definition 7.1. *A sliding variable $\sigma(t) = \sigma(x(t))$ is virtual output such that*

$$\dot{\sigma} = a_0(x) + b_0(x)[u + w]$$

and $\sigma(t) \equiv 0$ implies that $x(t) \to 0$ as $t \to \infty$. In other words it a minimum phase virtual output with relative degree one.

The sliding surface is defined by as the manifold $\{x|\sigma(x) = 0\}$. In this form, the control objective can be restated as making the sliding variable vanish in finite-time and keep it afterwards. In this form, the system reaches the sliding surface and slides towards the origin of the state space.

The requirement of relative degree one can be relaxed by using higher-order sliding mode algorithms.

Thus there are two problems: how to design the sliding surface and how to pick the control to ensure that the sliding surface is attained in finite-time despite perturbations.

The design of the sliding surface is very easy when the system is written in correct coordinates:

Theorem 7.1. *Suppose that we write the original system in the canonical controller form:*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= x_3,\\ &\vdots\\ \dot{x}_n &= a(x) + b(x)[u + w].\end{aligned}$$

Then the sliding surface can be constructed as

$$\sigma = x_n + c_{n-1}x_{n-1} + \cdots + c_1 x_1$$

where the coefficients c_i are selected such that the polynomial

$$p(s) = s^n + c_{n-1}s^{n-1} + \cdots + c_1$$

is stable.

Indeed, once the system reaches the sliding surface its behavior will be given by the linear system $x_n + c_{n-1}x_{n-1} + \cdots + c_1 x_1 = 0$. This can be designed to have certain additional performance measure like settling time or overshot.

PROBLEM 1. For which kind of systems can we find a sliding variable with relative degree one? For instance, it happens for mechanical systems!

EXAMPLE 1. Consider the uncertain mechanical system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = f(x, u, w).$$

The sliding variable can be designed as $\sigma = x_2 + cx_1$ with $c > 0$, see Fig 1. Indeed, once $\sigma = 0$ we have that $x_2 = -cx_1$ or equivalently $\dot{x}_1 = -cx_1$ that implies

$$x_1(t) \propto e^{-ct}, \quad x_2(t) \propto -ce^{-ct}.$$

The second problem is how to drive the system into the sliding surface and maintain it there. Since the sliding surface has relative degree one we have that

$$\dot{\sigma} = a_0(x) + b_0(x)[u + w]$$

where the functions $a_0(x)$ and $b_0(x)$ are known.

Theorem 7.2. *Suppose that*

$$|w(t)| \leq L, \forall t \geq 0.$$

with $L \geq 0$ *a known number. Then the discontinuous control*

$$u = \frac{1}{b_0(x)}[-\eta(x)\,\mathrm{sign}(\sigma) - a_0(x)], \quad \eta(x) > |b_0(x)|L,$$

guarantees that $\sigma(t) = 0$ *is stablished in finite-time and kept afterwards.*

For Multi Input systems the idea is different and relies in the so-called Normal Form, see Utkin's book. Consider the controllable linear system

$$\dot{x} = Ax + B(u + w)$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Let us transform it into the Regular Form by introducing the following transformation

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = Tx, \quad T = \begin{bmatrix} B^{\perp} \\ B^{+} \end{bmatrix}$$

where $B^{\perp}$ is the orthogonal to matrix B (i.e. $B^{\perp}B = 0$) and B^{+} is the pseudo-inverse of B (i.e. $B^{+}B = I_{m\times m}$). Under this coordinates the system takes the following form

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2, \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + u + w, \end{aligned}$$

If the pair (A, B) was controllable then also the pair (A_{11}, A_{12}) is controllable. What this means is that if we could set the virtual control

$$x_2 = -Cx_1$$

then there exists a matrix C such that $A_{11} + A_{12}C$ is Hurwitz so $x_1 \to 0$ and $x_2 \to 0$. To provide the desired virtual control, let us force a sliding motion on

$$\sigma = x_2 + Cx_1 = 0$$

in such a way that in the sliding surface we have

$$\dot{x}_1 = (A_{11} + A_{12}C)x_1$$

that is stable.

For forcing the sliding motion consider using the unit controller:

$$u = -\eta(x)\frac{\sigma}{\|\sigma\|_2}.$$

Propose the Lyapunov function

$$V = \frac{1}{2}\sigma^T\sigma$$

whose derivative

$$\dot{V} = \sigma^T\dot{\sigma} = \sigma^T\left[(A_{21} + CA_{11})x_1 + (A_{22} + CA_{12})x_2 + w\right] - \eta(x)\|\sigma\|$$

Choosing the gain function

$$\eta(x) = (\alpha\|x\| + L)$$

with α and L sufficiently large, it follows that $\dot{V} \leq -\varepsilon\sqrt{V}$ for some $\varepsilon > 0$ and hence $\sigma = 0$ will be stablished in finite-time.

7.2 Second-order sliding modes: the Super-Twisting Algorithm

The main problem with relay controllers is that they introduce a large amount of chattering, i.e., undesired oscillations. To alleviate this problem, second order algorithms were introduced. A particular successful algorithm is the Super-Twisting Algorithm.

The Super-Twisting Algorithm is useful to control sliding surfaces with relative degree one but using a continuous control.

Let us consider again the single input system with relative degree one surface σ written as

$$\dot{\sigma} = u + w$$

where we have previously linearized everything we know. The goal is to steer σ to zero in finite-time. Let us assume that $w(t)$ is not necessarily bounded but that its derivative is:

$$|\dot{w}(t)| \leq L.$$

The Super-Twisting control is a dynamic one:

$$u = -k_1\phi_1(\sigma) + v, \quad \dot{v} = -k_2\phi_2(\sigma)$$

where

$$\phi_1(\sigma) = |\sigma|^{1/2}\operatorname{sign}(\sigma) + \sigma, \quad \phi_2(\sigma) = 0.5\operatorname{sign}(\sigma) + 1.5|\sigma|^{1/2}\operatorname{sign}(\sigma) + \sigma$$

The algorithm contain continuous (linear) and discontinuous terms. If we let $\tilde{w} = w + v$ we obtain

$$\dot{\sigma} \;=\; -k_1\phi_1(\sigma) + \tilde{w}, \quad \dot{\tilde{w}} = -k_2\phi_2(\sigma) + \dot{w}.$$

In this new extended system what enters as perturbation is not w itself but its derivative $\dot{w}$. Since we know it is bounded, the discontinuous term $\operatorname{sign}(\sigma)$ present in ϕ_2 can be used to eliminate it. It can be proved that if

$$k_2 > 2L \quad \text{and } k_1 > 2\sqrt{L}$$

then the identities $\sigma(t) = 0$ and $\tilde{w}(t) = 0$ will be stablished.

7.3 The Super-Twisting Algorithm as a Differentiator

Let us consider the problem of estimating the derivative of a signal $\sigma(t)$ under the assumption that its second derivative is uniformly bounded by a known number:

$$|\ddot{\sigma}(t)| \leq L, \quad \forall t \geq 0.$$

Let us assume that we directly measure the signal σ and introduce the following state space model with $x_1 = \sigma$, $x_2 = \dot{\sigma}$:

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = w; \quad y = x_1$$

where $w = \ddot{\sigma}$. The objective is thus to estimate x_2 based on the measurement y.

For solving such problem, one can use the following Super-Twisting Observer (or differentiator):

$$\dot{\hat{x}}_1 = -k_1 \phi_1(y - \hat{x}_1) + \hat{x}_2, \quad \dot{\hat{x}}_2 = -k_2 \phi_2(y - \hat{x}_1)$$

where $k_1, k_2 > 0$ are the gains of the differentiator. Observe that in this problem the second derivative of the signal to differentiate acts as a perturbation. Hence, under the conditions

$$k_2 > 2L \quad \text{and } k_1 > 2\sqrt{L}$$

the identity $\hat{x}_2(t) = x_2(t) = \dot{\sigma}(t)$ will be stablished after a finite-time transient. What is most important is that under the presence of measurement noises uniformly bounded by δ, the precision of the differentiator will be proportional to $\sqrt{L\delta}$. This last expression shows that this differentiator has the best possible order of accuracy according to Kolmogorov's computations (See Kolmogorov 62).

7.4 Second-Order Sliding Mode Controllers

Here we study the case of how to control a surface with relative degree 2. For this consider the case when

$$\ddot{\sigma} = u + w$$

where again $|w(t)| \leq L$. In this case one option would be to introduce another sliding variable $\theta = \dot{\sigma} + \lambda\sigma$ that has relative degree one again. However, there are other options to directly tackle this problem and this is the main original motivation for higher-order sliding modes.

Chronologically, the first algorithm was the Twisting Algorithm:

$$u = -k_1 \operatorname{sign}(\sigma) - k_2 \operatorname{sign}(\dot{\sigma})$$

where one can observe that the derivative of σ is required to implement this algorithm. It can be shown that under the assumption that

$$k_2 > k_1 > L$$

then the identities $\sigma(t) = \dot{\sigma}(t) = 0$ will be stablished and kept after a finite-time transient. With this algorithm it is no longer necessary to design the sliding surface, one can have a plug-and-play controller.

The Twisting chatters a lot since it is discontinuous on both axis $\{\sigma = 0\}$ and $\{\dot{\sigma} = 0\}$. The so-called Quasi-Continuous algorithms alleviates the chattering a little more since it is only discontinuous at the intersection of both axis. It takes the following form:

$$u = -\alpha \frac{\dot{\sigma} + |\sigma|^{1/2} \operatorname{sign}(\sigma)}{|\dot{\sigma}| + |\sigma|^{1/2}}$$

It can be shown that under the condition that $\alpha > L$, then the identities $\sigma(t) = \dot{\sigma}(t) = 0$ will be stablished and kept after a finite-time transient.

OBSERVABILITY AND OBSERVER DESIGN

8

APPENDICES