Mixed-Integer Conic Programming

Fantastic problems and how to solve them

Mathieu Tanneau

GERAD

November 15, 2018

GERAD
GROUP FOR RESEARCH IN DECISION ANALYSIS

CANADA EXCELLENCE RESEARCH CHAIR
DATA SCIENCE FOR REAL-TIME DECISION-MAKING

POLYTECHNIQUE MONTRÉAL

"Almost all convex problems that arise in practice are representable using only 5 types of cones."
- S. Boyd

1. Conic Programming

2. Mixed-Integer Conic Programming

Definition (Cone)

A set $\mathcal{K} \subset \mathbb{R}^n$ is a *cone* if

$$\forall x \in \mathcal{K}, \lambda \geq 0, \lambda x \in \mathcal{K}$$

Definition

A cone $\mathcal{K}$ is

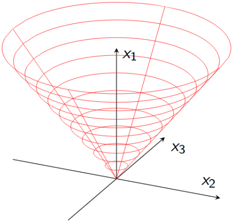
- *convex* if $\mathcal{K}$ is convex
- *closed* if $\mathcal{K}$ is closed
- *pointed* if
$$[(x \in \mathcal{K}) \wedge (-x \in \mathcal{K})] \Rightarrow x = 0$$

The *non-negative orthant*:

$$\mathbb{R}^n_+ = \left\{ x \in \mathbb{R}^n \mid x_i \geq 0, \ \forall i \right\}$$

The *Lorentz cone* (aka *second order cone*):

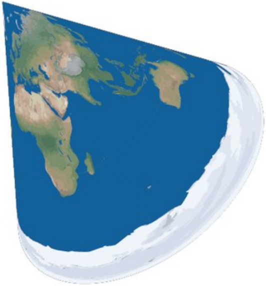
$$L^n = \left\{ x \in \mathbb{R}^n \,\middle|\, x_n \geq \sqrt{x_1^2 + \cdots + x_{n-1}^2} \right\}$$



The cone of *positive semi-definite* matrices:

$$S^n_+ = \left\{ S \in \mathbb{S}^n \mid \forall x \in \mathbb{R}^n, x^T S x \geq 0 \right\}$$

The MOSEK cone



Definition (dual cone)

Let $\mathcal{K} \subset \mathbb{R}^n$ and define the *dual cone* of $\mathcal{K}$:

$$\mathcal{K}_* := \left\{ s \in \mathbb{R}^n \mid s^T x \geq 0, \ \forall x \in \mathcal{K} \right\}$$

$\mathcal{K}_*$ is a *closed convex cone*

- If $int(\mathcal{K}) \neq \emptyset$, then $\mathcal{K}_*$ is pointed
- If $\mathcal{K}$ is a closed convex pointed cone, then $int(\mathcal{K}_*) \neq \emptyset$
- If $\mathcal{K}$ is a closed convex cone, so is $\mathcal{K}_*$ and

$$(\mathcal{K}_*)_* = \mathcal{K}$$

- If $\mathcal{K} = \mathcal{K}_*$ then $\mathcal{K}$ is *self-dual* (e.g. $\mathbb{R}^n_+, L^n, S^n_+$)

Linear programming

$$
(P) \quad \begin{array}{rll} \min\limits_{x} & c^T x & \\ s.t. & Ax & = b \\ & x & \geq 0 \end{array}
\qquad\qquad
(D) \quad \begin{array}{rll} \min\limits_{y,s} & b^T y & \\ s.t. & A^T y + s & = c \\ & s & \geq 0 \end{array}
$$

Linear programming

$$(P)\ \min_{x}\ c^T x \quad \text{s.t.}\ \ Ax = b,\ \ x \geq 0$$

$$(D)\ \min_{y,s}\ b^T y \quad \text{s.t.}\ \ A^T y + s = c,\ \ s \geq 0$$

Conic Programming

$$(P)\ \min_{x}\ c^T x \quad \text{s.t.}\ \ Ax = b,\ \ x \in \mathcal{K}$$

$$(D)\ \min_{y,s}\ b^T y \quad \text{s.t.}\ \ A^T y + s = c,\ \ s \in \mathcal{K}_*$$

and strong duality holds if primal is strictly feasible

Refs: `www.mosek.com`, [Ben-Tal and Nemirovski, 2001]

Interior-Point is the default choice for (general) Conic programs.
https://docs.mosek.com/8.1/capi/solving-conic.html

IPM solves a perturbed version of:

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ x &\in \mathcal{K} \\ s &\in \mathcal{K}_* \\ x^T s &= 0 \end{aligned}$$

(with tons of implementation details)

Also possible to use simplex algorithm in some cases (LP, convex QP)

1. Conic Programming
2. Mixed-Integer Conic Programming

Mixed-Integer Conic Program

$$
\begin{aligned}
(MICP) \quad \min_{x} \quad & c^T x \\
s.t. \quad & Ax = b \\
& x \in \mathcal{K} \cap \mathbb{X}
\end{aligned}
$$

Two (main) families of solution methods:

- Conic B&B
- Outer Approximations

Same as LP-based B&B, but node relaxations are NLPs [Gupta and Ravindran, 1985], [Borchers and Mitchell, 1994], [Bonami et al., 2013]

Pros:

- Easy to embed in an existing branch-and-bound framework
- Finite-time convergence (under some conditions)

Cons:

- Node relaxations can take time, hard to warmstart
- Slater condition may no longer hold in nodes
- Cuts?

Build (and refine) a polyhedral approximation of the cone. [Vielma et al., 2017] [Lubin et al., 2018]

Pros:

- Use existing (and fast!) MILP tools
- Can embed in a B&C, no NLP needed

Cons:

- Introduces additional variables/cuts (possibly a lot)
- Underlying MILPs may still be hard to solve
- Still have to check incumbent for conic feasibility

Solvers for Mixed-Integer Conic/Convex Programming

Solver	Problems	B&B	OA
BONMIN	MI-Convex	✓	✓
Pajarito	MI-Conic	X	✓
SCIP	MI-Convex	✓	✓
CPLEX	MISOCP	?	?
Gurobi	MISOCP	?	?
KNITRO	MI-Convex	?	?
MOSEK	MI-Convex	✓	✓

Ben-Tal, A. and Nemirovski, A. (2001).
Lectures on Modern Convex Optimization.
Society for Industrial and Applied Mathematics.

Bonami, P., Lee, J., Leyffer, S., and Wächter, A. (2013).
On branching rules for convex mixed-integer nonlinear optimization.
J. Exp. Algorithmics, 18:2.6:2.1–2.6:2.31.

Borchers, B. and Mitchell, J. E. (1994).
An improved branch and bound algorithm for mixed integer nonlinear programs.
Computers & Operations Research, 21(4):359 – 367.

Gupta, O. K. and Ravindran, A. (1985).
Branch and bound experiments in convex nonlinear integer programming.
Management Science, 31(12):1533–1546.

Lubin, M., Yamangil, E., Bent, R., and Vielma, J. P. (2018).
Polyhedral approximation in mixed-integer convex optimization.
Mathematical Programming, 172(1):139–168.

Vielma, J. P., Dunning, I., Huchette, J., and Lubin, M. (2017).
Extended formulations in mixed integer conic quadratic programming.
Mathematical Programming Computation, 9(3):369–418.