Basics of Fluid Mechanics

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I. Some Useful Mathematical Tools

A fundamental understanding of fluid motion begins with the Navier–Stokes equations, which describe the conservation of mass, momentum, and energy in a fluid. Deriving these equations requires a solid foundation in mathematical tools such as vector calculus, tensor notation, and differential operators. These tools allow us to rigorously express physical laws—like Newton's second law and the principle of conservation—in a form suitable for continuous media. This lecture note introduces the essential mathematical framework needed to transition from physical intuition to formal derivation, enabling a deeper appreciation of the structure and complexity of the governing equations of fluid dynamics.

A. Material Derivative

In fluid dynamics, the **material derivative** (also called the substantial or total derivative) is a mathematical tool used to describe how a quantity changes *as we follow a fluid particle* in motion.

Suppose a scalar field $\phi(\vec{x}, t)$ (such as temperature, pressure, or velocity component) varies in both space and time. The material derivative of ϕ is defined as:

$$\frac{D\phi}{Dt} = \frac{\partial \phi}{\partial t} + \vec{u} \cdot \nabla \phi$$

Here:

- $\frac{\partial \phi}{\partial t}$ is the *local derivative*, representing how ϕ changes at a fixed point in space.
- $\vec{u} \cdot \nabla \phi$ is the *convective derivative*, representing the change due to the movement of the fluid carrying the quantity ϕ through space.

Together, the material derivative tells us how ϕ changes for a fluid particle as it moves along its path. It is essential in expressing the laws of conservation (mass, momentum, energy) in a form that follows the motion of the fluid.

B. Reynolds Transport Theorem

We need to describe the laws governing fluid motion using both system concepts and control volume concepts.

- **System** represents a given mass of fluids.
- **Control volume** represents a given volume in space.

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The Reynolds Transport Theorem(TRTT) provides an analytical tool to shift from one representation to another. Let B represent any of fluid parameters and b represent the amount of that parameter per unit mass. That is,

$$B = mb \tag{1}$$

where m is the mass of the portion of fluid of interest. The parameter B is termed an *extensive property*, and the parameter b is termed an *intensive property*. It should be noted that the value of B is dependent of fluid mass, meanwhile the value of b is independent of fluid mass.

The amount of an extensive property that a system possesses at a given instant, B_{sys}, can be determined by adding up the amount associated with each fluid particle in the system. For infinitesimal fluid particles of size $\delta\forall$ and mass $\rho\delta\forall$, this summation in the limit of $\delta\forall$ takes the form of an integration over all the particles in the system and can be written as

$$B_{sys} = \lim_{\delta\forall\to 0} \sum_i b_i(\rho_i \delta\forall_i) = \int_{sys} \rho b d\forall \tag{2}$$

When the control volume and the system momentarily occupy the same volume in space, the rate of change of B in the system (dB_{sys}/dt) and the rate of change of B in the control volume (dB_{CV}/dt) need not to be same. The Reynolds Transport Theorem provides the relationship between the time rate of change of an extensive property for a system and that for a control volume.

1. Deriving The Reynolds Transport Theorem

Assuming property B only changes with time:

$$B_{sys}(t) = B_{CV}(t) \tag{3}$$

since the system and the fluid within the control volume assumed to coincide at this particular moment. Its value at time $t + \delta t$ is

$$B_{sys}(t + \delta t) = B_{CV}(t + \delta t) - B_{inflow}(t + \delta t) + B_{outflow}(t + \delta t) \tag{4}$$

The change in the amount of B in the system, δB_{sys}, can be calculated within the time interval, δt :

$$\frac{\partial B_{sys}}{\partial t} = \frac{B_{sys}(t + \delta t) - B_{sys}}{\delta t} = \frac{B_{CV}(t + \delta t) - B_{inflow}(t + \delta t) + B_{outflow}(t + \delta t) - B_{sys}(t)}{\delta t} \tag{5}$$

Using $B_{CV}(t) = B_{sys}(t)$:

$$\frac{\partial B_{sys}}{\partial t} = \frac{B_{CV}(t + \delta t) - B_{CV}(t)}{\delta t} - \frac{B_{inflow}(t + \delta t)}{\delta t} + \frac{B_{outflow}(t + \delta t)}{\delta t} \tag{6}$$

This is an important relationship between system and the control volume and we will be exploring that relationship below:

1) In the limit of $\delta t \to 0$, the left handside of Eq. (6) is equal to the time rate of change of B for the system and donated as $\frac{DB_{sys}}{Dt}$.
2) In the limit of $\delta t \to 0$, the first term on the right handside of Eq. (6) is seen to be the time rate of change of the amount B within the control volume:

$$\lim_{\delta t \to 0} \frac{B_{CV}(t + \delta t) - B_{CV}(t)}{\delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t}\left(\int_{CV} \rho b d\forall\right) \tag{7}$$

3) The third term on the right handside of Eq. (6) represents the rate at which the extensive parameter B flows from the control volume across the control surface. In other words, it represents the fluid flows through output surface. During the time interval $t = 0$ to $t = \delta t$ the volume of fluid that flows across is given by

$$\delta\forall_{output} = A_2 \delta l_2 = A_2(V_2 \delta t) \tag{8}$$

assuming 1 dimensional control volume. Thus, the amount of B within the outflow region is its amount per volume, ρb, times the volume:

$$B_{outflow}(t + \delta t) = (\rho_2 b_2)(\delta\forall_2) = \rho_2 b_2 A_2 V_2 \delta t \tag{9}$$

where b_2 and ρ_2 are the constant values of b and ρ across outflow section. Therefore, the rate at which this property flows from the control volume, $\dot{B}_{out}$, is given by:

$$\dot{B}_o ut = \lim_{\delta t \to 0} \frac{B_{outflow}(t + \delta t)}{\delta t} = \rho_2 A_2 V_2 b_2 \tag{10}$$

4) Same assumptions can be made for inflow. Hence, the rate of inflow of the property B into the control volume, $\dot{B}_i n$, is given by:

$$\dot{B}_i n = \lim_{\delta t \to 0} \frac{B_{inflow}(t + \delta t)}{\delta t} = \rho_1 A_1 V_1 b_1 \tag{11}$$

If we combine Eqs. (6), (7), (10), and (11); we see that the relationship between the time rate of change of B for the system and that for the control volume is given by:

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \dot{B}_{out} - \dot{B}_{in} \tag{12}$$

It should be noted that the time derivative associated with a system may be different from that for control volume.

The term $\dot{B}_{out}$ represents the exit flowrate of the property B from the control volume. Its value can be thought of as arising from the addition of the contributions through each infinitesimal area element of size δA on the portion of the control surface dividing the control volume from the out of the control volume. This surface is denoted as CS_{out}. In the time δt the volume of fluid that passes across each area element is given by $\delta\forall = \delta l_n \delta A$ where $\delta l_n = \delta l cos(\theta)$ is the height of the small volume element, and θ is the angle between the velocity vector and the outward pointing normal to the surface, $\hat{n}$. Thus, since $\delta l = V\delta t$ the amount of property B carried across the area element δA in the time interval δt is given by

$$\delta B = b\rho\delta\forall = b\rho(Vcos(\theta)\delta t)\delta A \tag{13}$$

The rate at which B is carried out of the control volume across the small area element δA, denoted $\delta\dot{B}_{out}$, is

$$\delta\dot{B}_{out} = \lim_{\delta t\to 0}\frac{\rho b\delta\forall}{\delta t} = \lim_{\delta t\to 0}\frac{(\rho bVcos(\theta)\delta t)\delta A}{\delta t} = \rho bVcos(\theta)\delta A \tag{14}$$

By integrating over the entire outflow portion of the control surface CS_{out}, we obtain

$$\dot{B}_{out} = \int_{CS_{out}} d\dot{B}_{out} = \int_{CS_{out}} \rho bVcos(\theta)dA \tag{15}$$

The quantity $Vcos(\theta)$ is the component of the velocity normal to the area element δA. From the definition of the dot product, this can be written as

$$Vcos(\theta) = \vec{V}\cdot\hat{n} \tag{16}$$

Hence, an alternate form of the outflow rate is:

$$\dot{B}_{out} = \int_{CS_{out}} \rho b\vec{V}\cdot\hat{n}dA \tag{17}$$

In a similar fashion, by considering the inflow portion of the control surface, CS_{in}, we find that the inflow rate of B into the CV is:

$$\dot{B}_{in} = -\int_{CS_{in}} \rho bVcos(\theta)dA = -\int_{CS_{in}} \rho b\vec{V}\cdot\hat{n}dA \tag{18}$$

We use the standard notation that the unit normal vector to the control surface, $\hat{n}$, points out from the control volume. The value of $cos(\theta)$ is positive on CV_{out} and negative on CV_{in}.

The net flux of parameter B across the entire control surface is:

$$\dot{B}_{out} - \dot{B}_{in} = \int_{CS_{out}} \rho b\vec{V}\cdot\hat{n}dA - \left(-\int_{CS_{in}} \rho b\vec{V}\cdot\hat{n}dA\right) = \int_{CS} \rho b\vec{V}\cdot\hat{n}dA \tag{19}$$

where the integration is over the entire control surface. By combining Eqs. (12) and (19):

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CS} \rho b \vec{V} \cdot \hat{n} dA \tag{20}$$

Assuming

$$B_{CV} = \int_{CV} \rho b d\forall \tag{21}$$

Eq. (20) can be written as:

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\left(\int_{CV} \rho b d\forall\right) + \int_{CS} \rho b \vec{V} \cdot \hat{n} dA \tag{22}$$

which Eq. (22) is **the Reynolds Transport Theorem**.

2. Physical Interpretation

- The left handside of Eq. (22) is the time rate of change of an arbitrary extensive parameter of a system. This may represent the rate of change of mass, momentum, energy, or angular momentum.
- Because the system is moving and the control volume if stationary, the time rate of change of the amount B within the control volume is not necessarily equal to that of the system. The first term on the right handside of Eq. (22) represents the rate of change of B within the control volume as the fluid flows through it.
  - Recall that b is the amount of B per unit mass, so that the $\rho b d\forall$ is the amount of B in a small volume $d\forall$.
  - Thus, the time derivative of the integral of ρb throughout the control volume is the time rate of change of B within the control volume at a given time.
- The last term in Eq. (22) represents the net flowrate of the parameter B across the entire control surface.
  - Each fluid particle or fluid mass carries a certain amount of B with it.
- The Reynolds Transport Theorem is the integral counterpart of the material derivative.
  - Both the material derivative and the Reynolds Transport Theorem represent ways to transfer from the Lagrangian viewpoint(follow a particle or follow a system) to the Eulerian viewpoint(observe what happens in the fixed control volume).
- Consider a steady flow ($\partial()/\partial t = 0$) so that Eq. (22) reduces to:

$$\frac{DB_{sys}}{Dt} = \int_{CS} \rho b \vec{V} \cdot \hat{n} dA \tag{23}$$

  - If there is to be a change in the amount of B associated with the system, there must be a net difference in the rate that B flows into the control volume compared with the rate that it flows out of the control volume.
  - For steady flows the amount of the property B within the control volume does not change with time. The

amount of the property associated with the system may or may not change with time. The difference between that associated with the control volume and that associated with the system is determined by the rate at which B is carried across the control surface.

- For the special unsteady sitations in which the rate of inflow of parameter B is exactly balanced by its rate of outflow. Eq. (22) reduces to:

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t} \int_{CV} \rho b d\forall \tag{24}$$

  In that case, any rate of change in the amount of B associated with the system is equal to the rate of change of B within the control volume.

References

[1] Munson, B. R., Okiishi, T. H., Huebsch, W. W., and Rothmayer, A. P., *Fundamentals of Fluid Mechanics*, 7th ed., John Wiley & Sons Inc., 2012.