9.3 Representing Graphs and Graph Isomorphism

- Sometimes, two graphs have exactly the same form, in the sense that there is a one-to-one correspondence between their vertex sets that preserves edges. In such a case, we say that the two graphs are isomorphic.

Representing Graphs

- One way to represent a graph without multiple edges is to list all the edges of this graph. Another way to represent a graph with no multiple edges is to use adjacency lists, which specify the vertices that are adjacent to each vertex of the graph.

Representing Graphs

- Example 1: Use adjacency lists to describe the simple graph given in Figure 1.

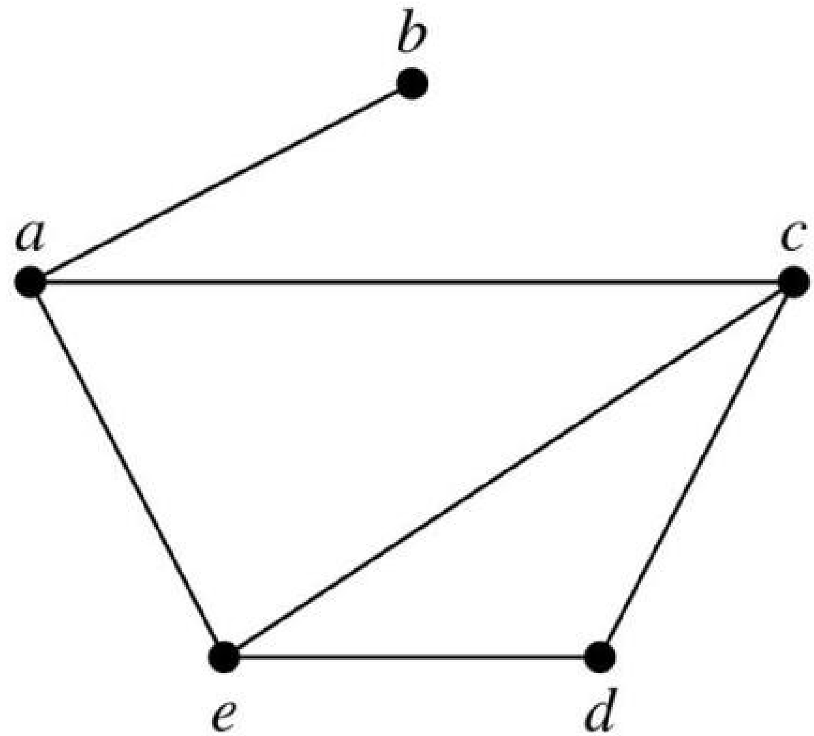

FIGURE 1 A Simple Graph.

Representing Graphs

- Example 2: Represent the directed graph shown in Figure 2 by listing all the vertices that are the terminal vertices of edges starting at each vertex of the graph.

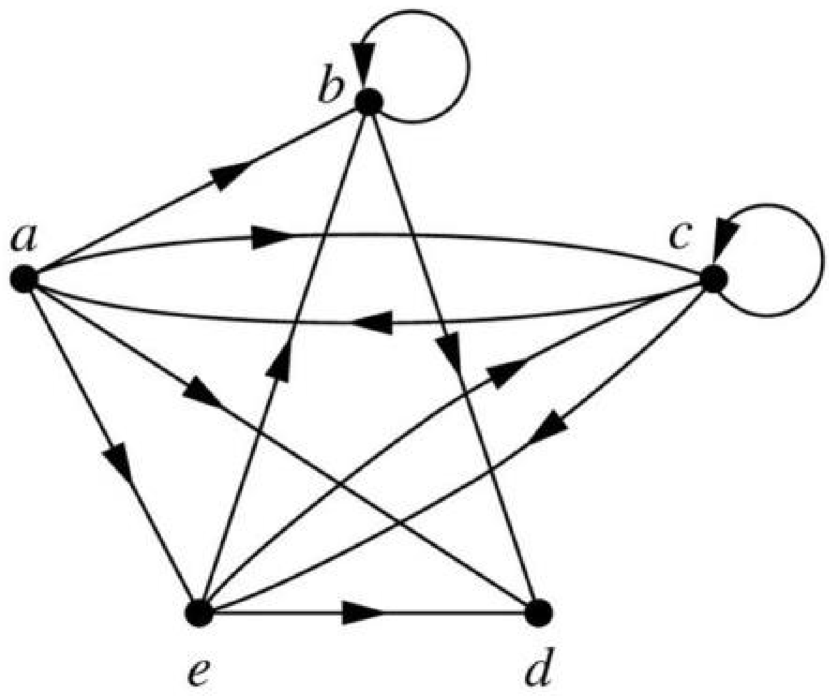

FIGURE 2 A Directed Graph.

Adjacency Matrices

- Suppose that G=(V, E) is a simple graph where |V|=n . Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The adjacency matrix **A** (or $\mathbf{A_G}$) of G, with respect to this listing of the vertices, is the n x n zero-one matrix with 1 as its (i, j)th entry when v_i and v_j are adjacent, and 0 as its (i, j)th entry when they are not adjacent. In other words , if its adjacency matrix is $A=[a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of G,} \\ 0 & \text{otherwise.} \end{cases}$$

Adjacency Matrices

- Example 3: Use an adjacency matrix to represent the graph shown in below.

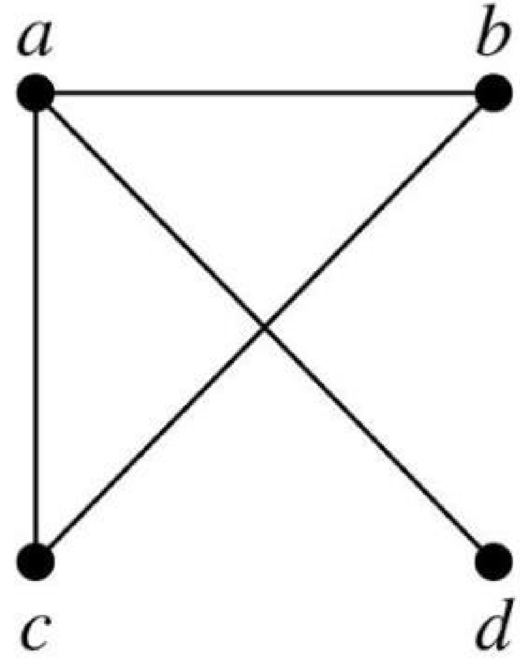

FIGURE 3 Simple Graph.

Adjacency Matrices

- Example 4: Draw a graph with the adjacency matrix with respect to the ordering of vertices a, b, c, d.

$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

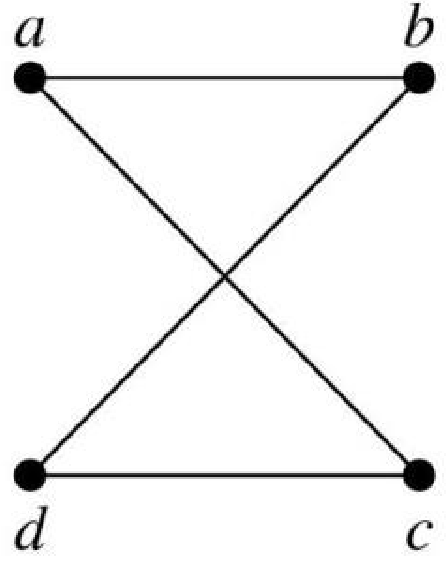

FIGURE 4 A Graph with the Given Adjacency Matrix.

Adjacency Matrices

- Example 5: Use an adjacency matrix to represent the pseudograph shown in Figure 5.

$$\begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

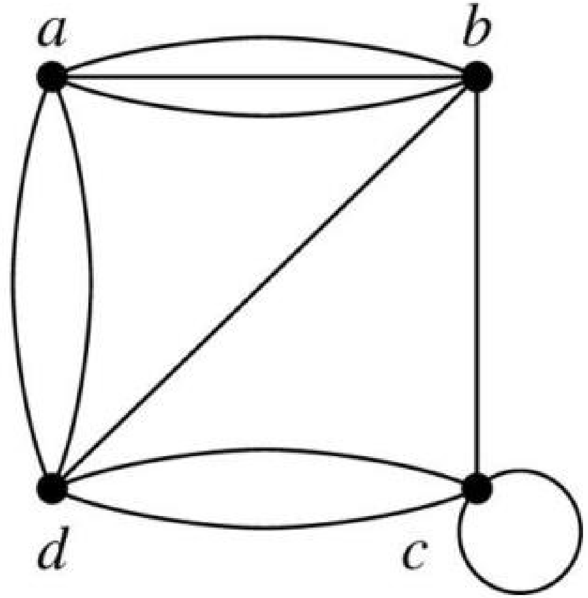

FIGURE 5 A Pseudograph.

Adjacency Matrices

- TRADE-OFFS BETWEEN ADJACENCY LISTS AND ADJACENCY MATRICES
- Sparse : A simple graph contains relatively few edges, it's usually preferable to use adjacency lists rather than an adjacency matrix to represent the graph.
- Sparse matrix : The adjacency matrix of a sparse graph is a sparse matrix, it is a matrix with few nonzero entries.
- Dense : suppose that it contains many edges , such as a graph that contains more than half of all possible edges.

Incidence Matrices

- Another common way to represent graphs is to use incidence matrices. Let G =(V, E) be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G . Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M=[m_{ij}]$, where

$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Incidence Matrices

- Example 6: Represent the graph shown in below with an incidence matrix.

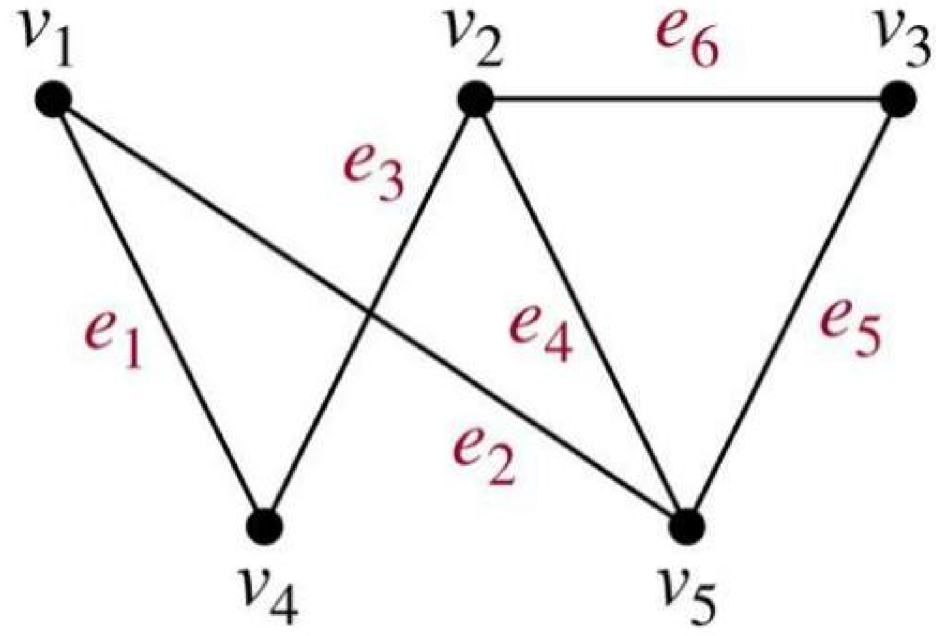

FIGURE 6 An Undirected Graph.

Incidence Matrices

- Example 7: Represent the pseudograph shown in Figure 7 using an incidence matrix.

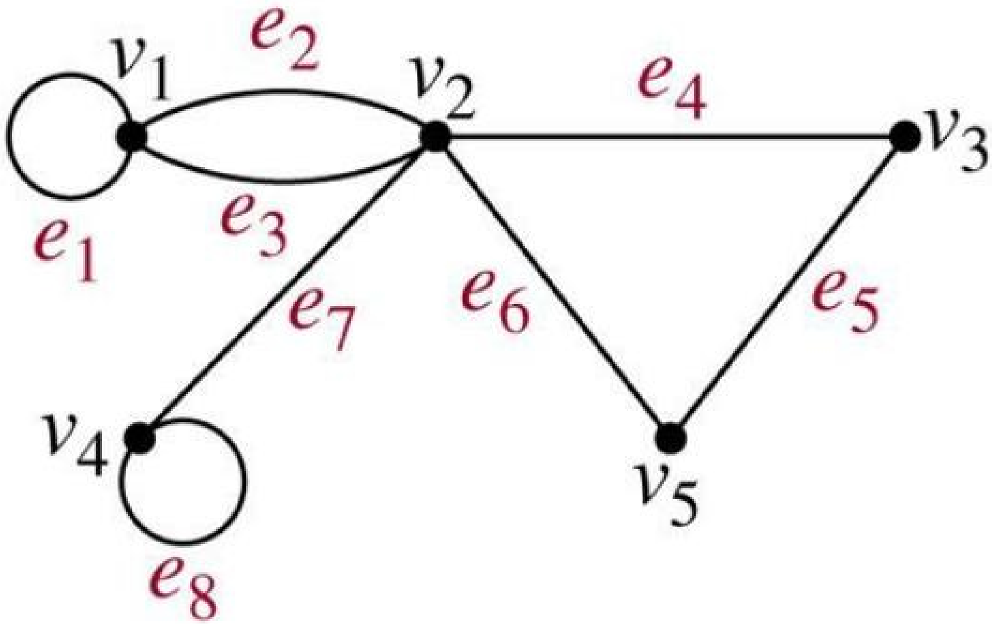

FIGURE 7 A Pseudograph.

Isomorphism of Graphs

- Definition 1: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with the property that *a* and *b* are adjacent in G_1 if and only if

  $f(a)$ and $f(b)$ are adjacent in G_2, for all *a* and *b* in V_1. Such a function *f* is called an isomorphism.*

Isomorphism of Graphs

- Example 8: Show that the graphs G=(V, E) and H=(W, F), displayed in below , are isomorphic.

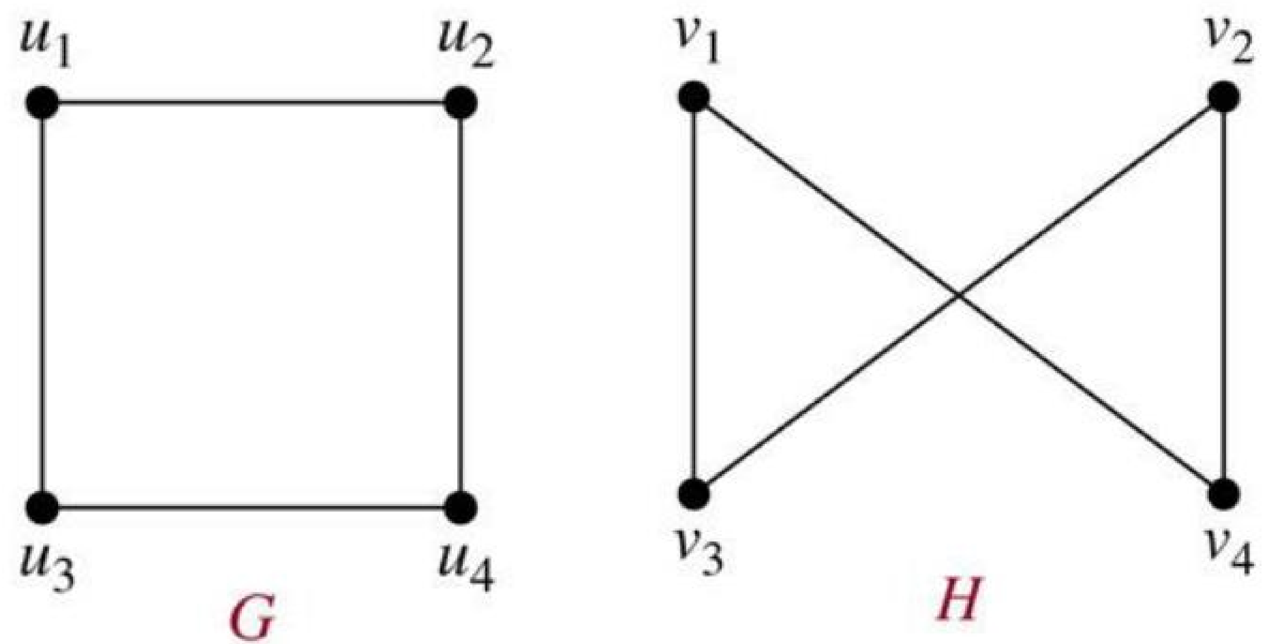

FIGURE 8 The Graphs G and H.

Isomorphism of Graphs

- Example 9: Show that graphs displayed in below are not isomorphic.

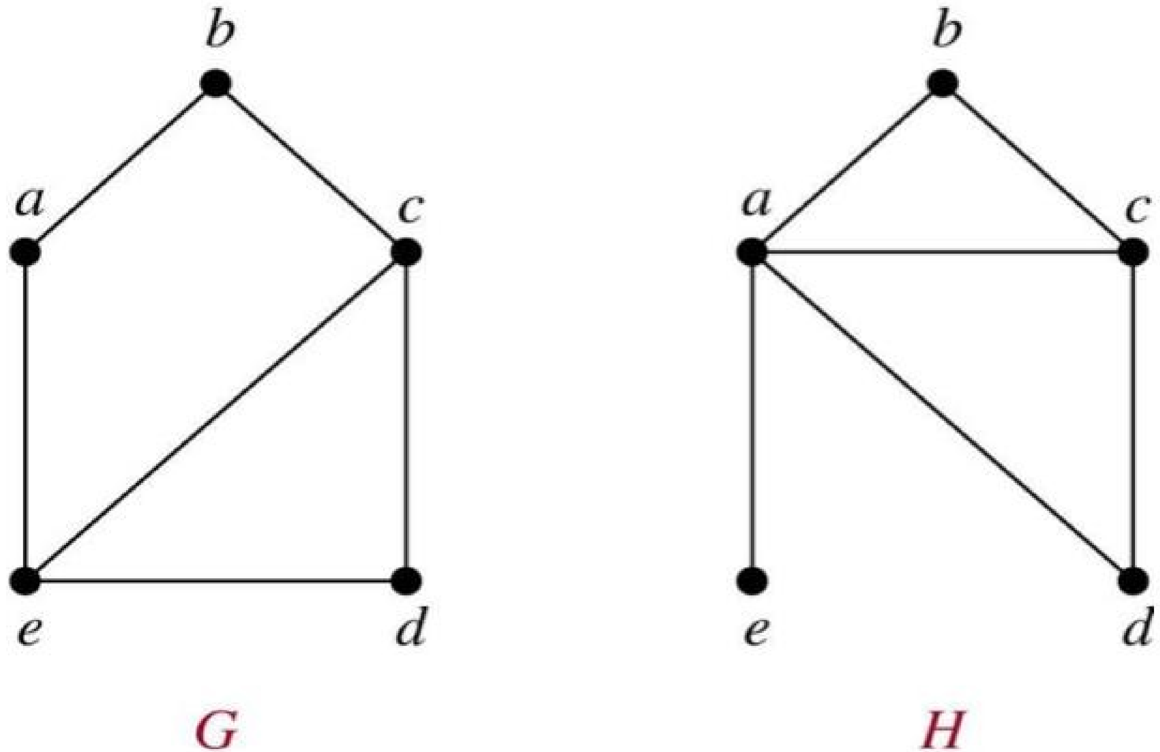

FIGURE 9 The Graphs G and H.

Isomorphism of Graphs

- Example 10: Determine whether the graphs shown in below are isomorphic.

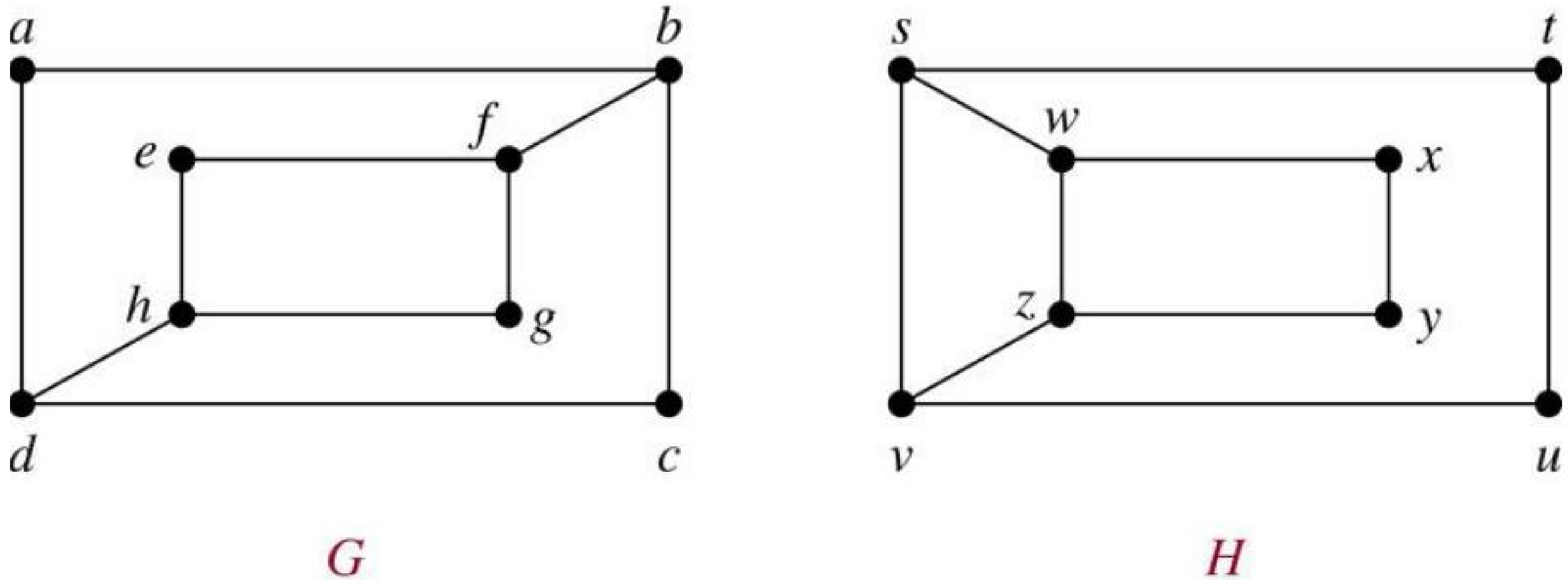

FIGURE 10 The Graphs G and H.

Isomorphism of Graphs

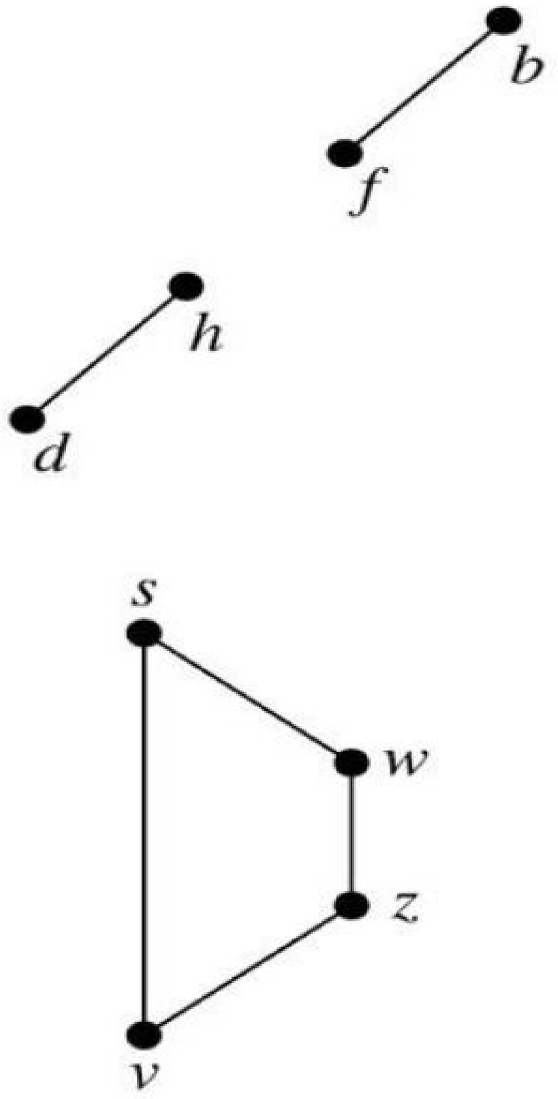

FIGURE 11 The Subgraphs of G and H Made Up of Vertices of Degree Three and the Edges Connecting Them.

Isomorphism of Graphs

- Example 11: Determine whether the graphs G and H displayed in below are isomorphic.

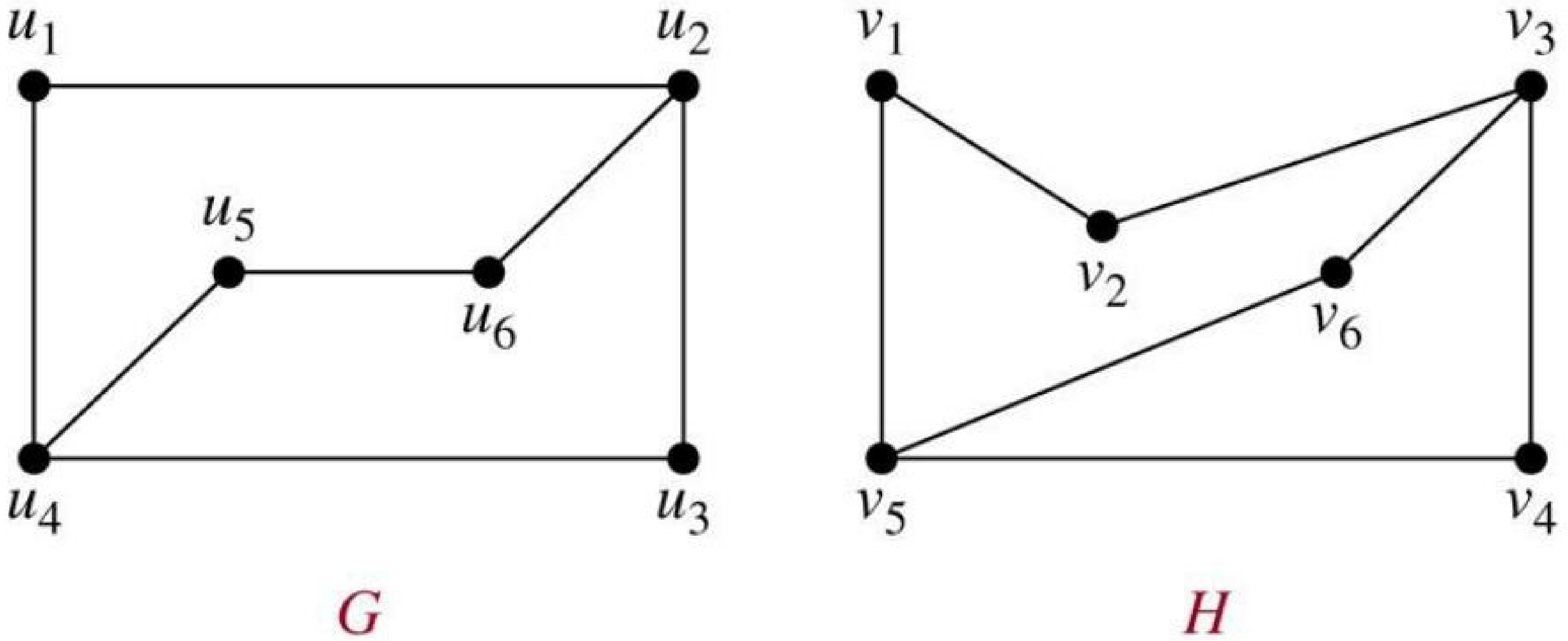

FIGURE 12 Graphs G and H.