A Markov Model For The Spread Of Hepatitis C Virus

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Objectives

We provide a Markov model for the propagation of the Hepatitis C virus in an endogenous population made up of susceptible and infected drug injecting users.

- Our model can be interpreted via a **deterministic differential system**. We show this system has a unique solution and simulate it.
- We prove that the asymptotic behaviour of the Markov process is close to the solution of the deterministic differential system.
- We apply a **functional Law of Large Numbers** to estimate the stationary distribution of the random process.
- We plot the **asymptotic prevalence** of the Hepatitis C Virus as a function of a specific parameter

Markov Model

We model the population with a random process $X(t) = (X_1(t), X_2(t))$. $X_1(t)$ represents the number of infected individuals at time t and $X_2(t)$ represents the susceptible individuals at time t. After each change of state (i.e every time X changes), the individuals stay in their current state during a time determined by an exponential distribution of parameter q_i determined by the graph below.

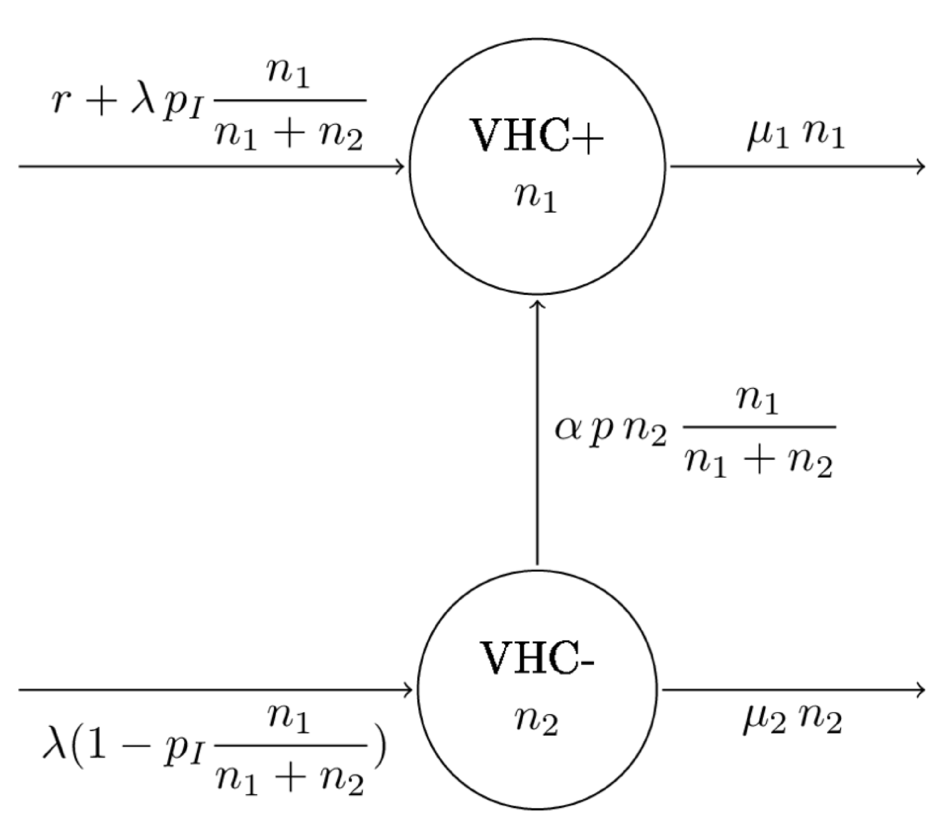

Figure 1: Transition Graph of the Markov Model

Transition Rates

Parameters described in Figure 1:

- λ, r : Arrival rate of susceptible ($\mathcal{P}(\lambda)$) and infected ($\mathcal{P}(r)$) individuals
- p_I : Probability of a susceptible $\to$ infected transition
- p : Probability of a susceptible $\to$ infected transition caused by injection using contaminated material
- α : Injection rate
- μ_1, μ_2 : Rates of exit from the environment (individual stopped injecting drugs)

Deterministic Model

The Mean Field Approximation applied to our Markov Model leads to a differential system:

$$(S_r(x^0)) \Leftrightarrow$$

$$\begin{cases} \psi_1'(t) = r + \lambda p_I \frac{\psi_1(t)}{\psi_1(t)+\psi_2(t)} - \mu_1\psi_1(t) + \alpha p \frac{\psi_1(t)\psi_2(t)}{\psi_1(t)+\psi_2(t)}, \\ \psi_1(0) = x_1^0, \\ \psi_2'(t) = \lambda(1 - p_I \frac{\psi_1(t)}{\psi_1(t)+\psi_2(t)} - \mu_2\psi_2(t) - \alpha p \frac{\psi_1(t)\psi_2(t)}{\psi_1(t)+\psi_2(t)}), \\ \psi_2(0) = x_2^0. \end{cases}$$

with $x_0 = (x_1^0, x_2^0) \in (\mathbb{R}_+ \times \mathbb{R}_+)\ (0,0)$

$(S_r(x^0))$ has a unique solution $\Psi(t) = (\Psi_1(t), \Psi_2(t))$. Furthermore, this solution converges towards a fixed point ξ.

$$\Psi(t) \longrightarrow (\xi_1, \xi_2)$$

Mean Field Approximation

We will now consider the sequence $(X^N(t) = (X^N(t), X^N(t))$ of Markov Processes with the same transitions as in Figure 1, but with rates r_N and λ_N such that :

$$\frac{1}{N} r_N \xrightarrow{N\to+\infty} r,\ \frac{1}{N}\lambda_N \xrightarrow{N\to+\infty} \lambda$$

Mean Field Approximation

Important Result

Let $\psi(x^0,.) = (\psi_1(x^0,.), \psi_2(x^0,.))$ be the solution of the differential system $(S_r(x^0))$.

Assume that $\mathbb{E}[\|\frac{1}{N}X^N(0) - x^0\|^2] \xrightarrow{N\to+\infty} 0$.

Then for any T > 0,

$$\mathbb{E}[\|\frac{1}{N}X^N(t) - \psi(x^0,t)\|^2] \xrightarrow{N\to+\infty} 0$$

For a large population, the X process is close to the solution ψ of the deterministic differential system.

Results

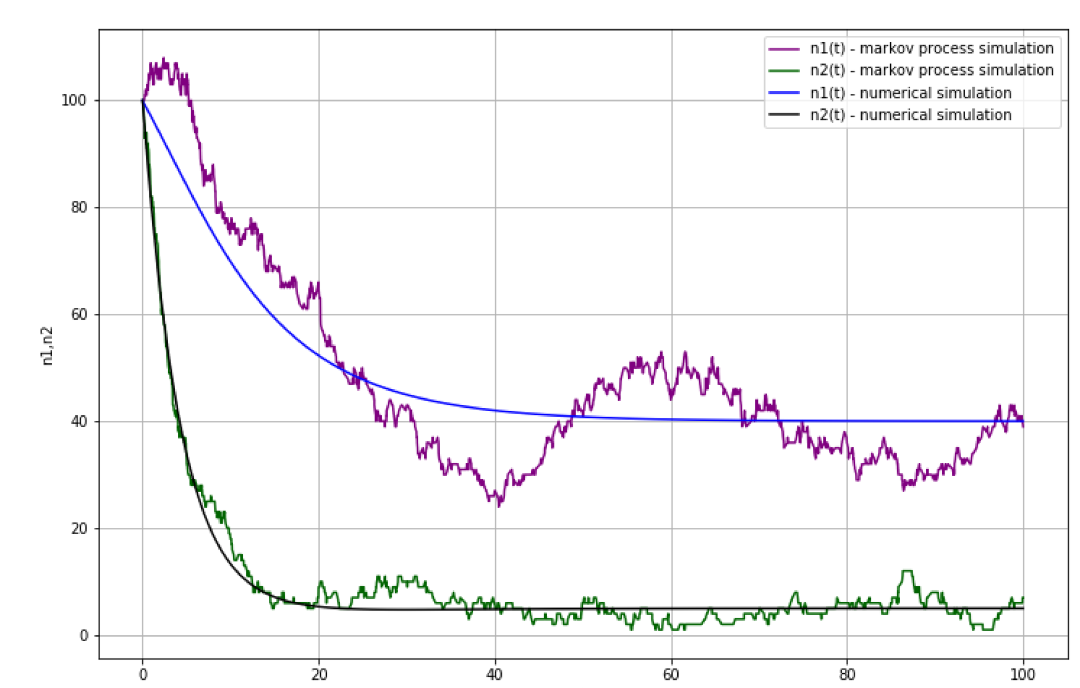

Figure 2: Superposition of the Markov process and numerical solution over time. p = 0.1 Initial $(n1, n2) = (100, 100)$

Stationary Regime

$Y^N = \frac{1}{N}X^N$ verifies : $P_{Y^N(\infty)} \xrightarrow{N\to+\infty} \delta_\xi$

For large populations, the stationary distribution of the process **converges towards a dirac distribution on point** ξ, with ξ the fixed point of the deterministic model

Asymptotic Prevalence

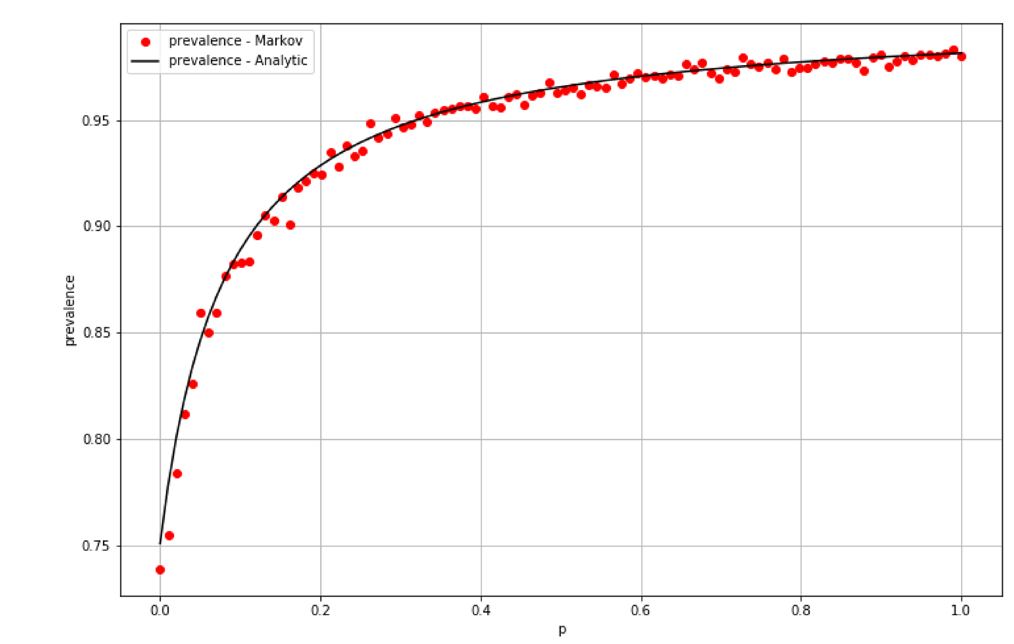

Figure 3: Evolution of the prevalence of the virus as a function of p, $(n1, n2) = (100, 100)$, N=100

References

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[1] Decreusefond L. Coutin, L. and J.S. Dhersin. A Markov Model For The Spread Of Hepatitis C Virus. September 2008.

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