

Sufficient criterion for guaranteeing that a two-qubit state is unsteerable

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Quantum steering can be detected via the violation of steering inequalities, which provide sufficient conditions for the steerability of quantum states. Here we discuss the converse problem, namely, ensuring that an entangled state is unsteerable and hence Bell local. We present a simple criterion, applicable to any two-qubit state, that guarantees that the state admits a local hidden state model for arbitrary projective measurements. Specifically, we construct local hidden state models for a large class of entangled states, which thus cannot violate any steering or Bell inequality. In turn, this leads to sufficient conditions for a state to be only one-way steerable and provides the simplest possible example of one-way steering. Finally, by exploiting the connection between steering and measurement incompatibility, we give a sufficient criterion for a continuous set of qubit measurements to be jointly measurable.

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I. INTRODUCTION

Entanglement lies at the heart of quantum physics. Notably, correlations arising from local measurements performed on separated entangled systems can exhibit nonlocal correlations [1,2]. Specifically, the observed statistics cannot be reproduced using a local hidden variable model, as witnessed by violation of a Bell inequality.

Recently, the effect of Einstein-Podolsky-Rosen steering has brought novel insight into quantum nonlocality. Originally discussed by Schrödinger [3] and used in quantum optics [4], the effect was recently formalized in a quantum information-theoretic setting [5]. Considering two distant observers sharing an entangled state, steering captures the fact that one observer, by performing a local measurement on one's subsystem, can nonlocally steer the state of the other subsystem. Steering can be understood as a form of quantum nonlocality intermediate between entanglement and Bell nonlocality [5,6] and is useful to explore the relation between these concepts. It was demonstrated experimentally (see, e.g., [7]) and finds application in quantum information processing [8–10].

Steering can be detected via steering inequalities (analogous to Bell inequalities) [11], the violation of which provides a sufficient condition for a given quantum state to be steerable. Derived for both discrete and continuous variable quantum systems [11–13], such inequalities can be obtained using semidefinite programming [14–17].

Interestingly, whereas the effect of steering implies the presence of entanglement, the converse does not hold [5]. Specifically, there exist entangled states that provably cannot give rise to steering (and hence are referred to as unsteerable) [5,18], even when general measurements are considered [6]. The correlations of such states can in fact be reproduced without entanglement, using a so-called local hidden state (LHS) model [5], and therefore can never violate any steering inequality. Since a LHS model is a particular case of a local hidden variable model, any unsteerable state is Bell local.

Determining which entangled states are steerable and which ones are not is a challenging problem in general. This is mainly due to the fact that, when constructing a LHS model, one must ensure that the model reproduces the desired quantum correlations for any possible measurements. Local hidden state

models have been constructed for entangled states featuring a high level of symmetry [18–22] (see [23] for a review). For more general states very little is known, even for the simplest case of two-qubit states. Based on the concept of the steering ellipsoid [24], Ref. [25] derived a condition guaranteeing unsteerability of Bell diagonal two-qubit state. This method, however, is not applicable to general two-qubit states, for which unsteerability conditions are still missing.

Here, via the construction of a class of LHS models, we derive a simple criterion sufficient for guaranteeing that a two-qubit state is unsteerable, considering arbitrary projective measurements. In turn, this criterion can also be used to guarantee one-way steerability [22,26], a weak form of steering where only one of the observers can steer the state of the other. We illustrate the relevance of the criterion with examples, providing in particular the simplest possible example of one-way steering. Finally, by exploiting the strong connection between steering and measurement incompatibility [27,28], we provide a sufficient condition for a continuous set of dichotomic qubit positive-operator-valued measures (POVMs) to be jointly measurable.

II. PRELIMINARIES

Consider two distant parties, Alice and Bob, sharing an entangled quantum state ρ . On her subsystem, Alice makes measurements, described by operators $\{M_{a|x}\}$, with $M_{a|x} \geq 0$ and $\sum_a M_{a|x} = \mathbb{1}$, where x denotes the measurement setting and a its outcome. The possible states of Bob's subsystem, conditioned on Alice's measurement x and her output a , are characterized by a collection of (subnormalized) density matrices $\{\sigma_{a|x}\}_{a,x}$, called an assemblage, with

$$\sigma_{a|x} = \text{Tr}_A(M_{a|x} \otimes \mathbb{1})\rho. \quad (1)$$

Note that it also includes Alice's marginal statistics $p(a|x) = \text{Tr} \sigma_{a|x}$. The assemblage $\{\sigma_{a|x}\}$ is called unsteerable if it can be reproduced by a LHS model, i.e., it admits a decomposition

$$\sigma_{a|x} = \sigma_{a|x}^{\text{LHS}} = \int \sigma_\lambda p(a|x\lambda) d\lambda \quad \forall a, x, \quad (2)$$

where $\{\sigma_\lambda\}$ is a set of positive matrices such that $\int \text{Tr} \sigma_\lambda d\lambda = 1$ and the $p(a|x, \lambda)$'s are probability distributions. The

right-hand side of (2) can be understood as follows. Alice sends the quantum state $\sigma_\lambda / \text{Tr} \sigma_\lambda$ to Bob with probability density $\text{Tr} \sigma_\lambda$. Given her measurement input x , she then outputs a with probability $p(a|x, \lambda)$. In this way, Alice can prepare the same assemblage for Bob as if the state ρ had been used, without the need for entanglement. Bob will thus be unable to distinguish whether he and Alice share the entangled state ρ or if the above LHS strategy were used. In contrast, if a decomposition of the form (2) does not exist, which can be certified, for example, via violation of a steering inequality [11], the use of an entangled state is certified. In this case, ρ is termed steerable from Alice to Bob.

Interestingly, not all entangled states are steerable. That is, there exist entangled states, called unsteerable, which admit a LHS model for all projective measurements [5, 18] and even considering general POVMs [6]. A natural question is thus to determine which entangled states are steerable and which ones are not. This is a challenging problem, mainly due to the difficulty of constructing LHS models for a continuous set of measurements.

III. SUFFICIENT CRITERION FOR UNSTEERABILITY

Our main result is a simple criterion, sufficient for a two-qubit state to admit a LHS model for arbitrary projective measurements. Consider a general two-qubit state, expressed in the local Pauli basis

$$\rho_0 = \frac{1}{4} \left(\mathbb{1} + \vec{a}_0 \cdot \vec{\sigma} \otimes \mathbb{1} + \mathbb{1} \otimes \vec{b}_0 \cdot \vec{\sigma} + \sum_{i,j=x,y,z} T_{ij}^0 \sigma_i \otimes \sigma_j \right), \quad (3)$$

where \vec{a}_0 and \vec{b}_0 are Alice and Bob's local Bloch vectors and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ is the vector of Pauli matrices. Our criterion for unsteerability is simply based on the local Bloch vectors and the correlation matrix T_{ij}^0 .

The first step consists in converting the state ρ_0 into a canonical form. For this, based on previous work [6, 29], we make the following observation.

Lemma 1. Let Λ be a positive linear map on the set of quantum states and

$$\rho_\Lambda = \mathbb{1} \otimes \Lambda(\rho) / \text{Tr}[\mathbb{1} \otimes \Lambda(\rho)] \quad (4)$$

be a valid bipartite quantum state. If ρ is unsteerable from Alice to Bob, then ρ_Λ is also unsteerable from Alice to Bob. Furthermore, if Λ is invertible and its inverse map positive, then ρ is unsteerable from Alice to Bob if and only if ρ_Λ is unsteerable from Alice to Bob.

Note that Λ does not have to be completely positive and may therefore correspond to a nonquantum operation. For a proof see, e.g., Lemma 2 of [6], where the condition of complete positivity can simply be relaxed.

Let us now consider the positive linear map

$$\mathbb{1} \otimes \Lambda(\rho_0) = \mathbb{1} \otimes \rho_B^{-1/2} \rho_0 \mathbb{1} \otimes \rho_B^{-1/2}, \quad (5)$$

where $\rho_B = \text{Tr}_A[\rho_0]$. This map is invertible as long as ρ_B is mixed with the inverse (positive) map given by

$$\mathbb{1} \otimes \Lambda^{-1}(\rho_0) = \mathbb{1} \otimes \rho_B^{1/2} \rho_0 \mathbb{1} \otimes \rho_B^{1/2}. \quad (6)$$

The interesting property of this map is that when applied to an arbitrary state ρ_0 , the resulting state has $\vec{b} = 0$, i.e., Bob's reduced state is maximally mixed [30]. Given the above lemma, the application of the map preserves the steerability (or unsteerability) of ρ_0 .

Finally, we apply local unitaries (which also cannot change the steerability of the state) so that our state has a diagonal T matrix, giving us the canonical form

$$\rho = \frac{1}{4} \left(\mathbb{1} + \vec{a} \cdot \vec{\sigma} \otimes \mathbb{1} + \sum_{i=x,y,z} T_i \sigma_i \otimes \sigma_i \right), \quad (7)$$

where \vec{a} and T_i are in general different from the original \vec{a}_0 and T_{ij}^0 . Below we give a sufficient criterion for the unsteerability of any state ρ expressed in the canonical form. In turn this provides a sufficient criterion for unsteerability of any two-qubit state.

Theorem 1. Let ρ_0 be a two-qubit state with corresponding canonical form ρ as given in Eq. (7). If

$$\max_{\hat{x}} [(\vec{a} \cdot \hat{x})^2 + 2\|T\hat{x}\|] \leq 1, \quad (8)$$

where \hat{x} is a normalized vector and $\|\cdot\|$ the Euclidean vector norm, then ρ is unsteerable from Alice to Bob, considering arbitrary projective measurements.

Proof. We first characterize the assemblage resulting from projective measurements on a state in the canonical form ρ . Alice's measurement is given by a Bloch vector \hat{x} and output $a = \pm 1$, corresponding to operators $M_{\pm|\hat{x}} = (\mathbb{1} \pm \hat{x} \cdot \vec{\sigma})/2$. For $a = +1$, the steered state is (see, for example, [25])

$$\sigma_{+|\hat{x}} = \text{Tr}_A(M_{+|\hat{x}} \otimes \mathbb{1} \rho) = \frac{1}{4}[(1 + \vec{a} \cdot \hat{x})\mathbb{1} + T\hat{x} \cdot \vec{\sigma}]. \quad (9)$$

Notice that the above state is diagonal in the basis $\{|\hat{s}\rangle, |-\hat{s}\rangle\}$, with Bloch vector $\hat{s} = \frac{T\hat{x}}{\|T\hat{x}\|}$; we omit the \hat{x} dependence to simplify notation. The corresponding eigenvalues are

$$\alpha(\hat{x}) = \frac{1}{4}(1 + \vec{a} \cdot \hat{x} + \|T\hat{x}\|), \quad \beta(\hat{x}) = \frac{1}{4}(1 + \vec{a} \cdot \hat{x} - \|T\hat{x}\|). \quad (10)$$

Note that by construction $\alpha(\hat{x}) \geq \beta(\hat{x})$.

Our goal is now to construct a LHS model for this assemblage. First, the local hidden states σ_λ are taken to be pure qubit states and hence are represented by unit Bloch vectors $\hat{\lambda}$ and uniformly distributed over the sphere

$$\sigma_{\hat{\lambda}} = \frac{|\hat{\lambda}\rangle \langle \hat{\lambda}|}{4\pi}. \quad (11)$$

Normalization is ensured as $\int \text{Tr}[\sigma_{\hat{\lambda}}] d\hat{\lambda} = 1$. This ensures that we obtain the correct reduced state for Bob:

$$\frac{1}{4\pi} \int |\hat{\lambda}\rangle \langle \hat{\lambda}| d\hat{\lambda} = \frac{\mathbb{1}}{2} = \rho_B. \quad (12)$$

Next we define Alice's response function to be given by the distribution

$$p(\pm|\hat{x}, \hat{\lambda}) = \frac{1 \pm \text{sgn}[\hat{s} \cdot \hat{\lambda} - c(\hat{x})]}{2}, \quad (13)$$

parametrized by the function $-1 \leq c(\hat{x}) \leq 1$, with \hat{s} the Bloch vector of the eigenvector of $\sigma_{+|\hat{x}}$ with the largest eigenvalue. The function (13) can be understood as follows (see Fig. 1). If $\hat{\lambda}$ is in the spherical cap centered on \hat{s} such that

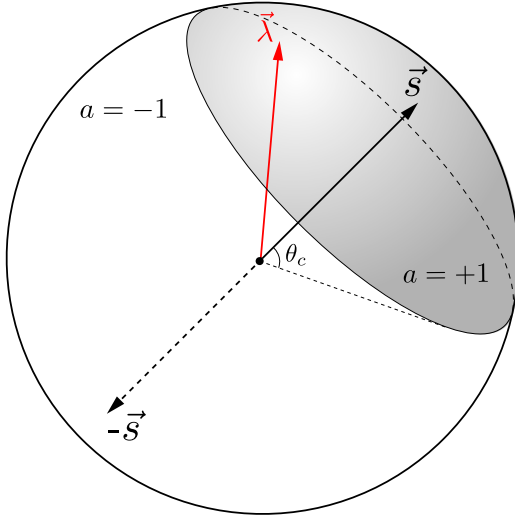


FIG. 1. Illustration of Alice's response function (13) in our LHS model. If $\text{sgn}[\hat{s} \cdot \hat{\lambda} - c(\hat{x})] \geq 0$ then $a = +1$ (shaded spherical cap, with angle $\theta_c = \arccos[c]$); otherwise $a = -1$. The assemblage (14) then corresponds to the average (subnormalized) density matrix obtained by integrating pure qubit states $|\hat{\lambda}\rangle$ over the shaded region.

$\hat{\lambda} \cdot \hat{s} \geq c(\hat{x})$, the output is $a = +1$; otherwise $a = -1$. Note that we need only concentrate on the case $a = +1$; the case $a = -1$ is automatically satisfied from $\sigma_{+1|\hat{x}} + \sigma_{-1|\hat{x}} = \rho_B$ and (12). We now calculate the assemblage predicted by this model, given by

$$\sigma_{+1|\hat{x}}^{\text{LHS}} = \int \sigma_{\hat{\lambda}} p(+|\hat{x}, \hat{\lambda}) d\hat{\lambda} = \frac{1}{4\pi} \int |\hat{\lambda}\rangle \langle \hat{\lambda}| p(+|\hat{x}, \hat{\lambda}) d\hat{\lambda}.$$

We parametrize the state $|\hat{\lambda}\rangle$ using the Bloch decomposition in the basis $\{|\hat{s}\rangle, |-\hat{s}\rangle\}$:

$$|\hat{\lambda}\rangle = |\hat{\lambda}(\theta, \phi)\rangle = \cos \frac{\theta}{2} |\hat{s}\rangle + \sin \frac{\theta}{2} e^{i\phi} |-\hat{s}\rangle. \quad (14)$$

Working in this basis and integrating over the spherical cap for which $a = +1$ (see Fig. 1), (14) becomes

$$\int_0^{2\pi} \int_0^{\theta_c} \begin{pmatrix} \cos^2 \frac{\theta}{2} & \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{-i\phi} \\ \cos \frac{\theta}{2} \sin \frac{\theta}{2} e^{i\phi} & \sin^2 \frac{\theta}{2} \end{pmatrix} \frac{\sin \theta d\phi d\theta}{4\pi},$$

where $\theta_c = \arccos[c(\hat{x})]$ is the angle of the spherical cap. Since $\int_0^{2\pi} e^{i\phi} d\phi = 0$, the off-diagonal components will be zero and $\sigma_{+1|\hat{x}}^{\text{LHS}}$ is therefore diagonal in the $\{|\hat{s}\rangle, |-\hat{s}\rangle\}$ basis, as desired. From this, the eigenvalues of $\sigma_{+1|\hat{x}}^{\text{LHS}}$, i.e., $\alpha'(\hat{x})$ and $\beta'(\hat{x})$, are given by

$$\alpha'(\hat{x}) + \beta'(\hat{x}) = \frac{1}{2} \int_0^{\theta_c} \sin \theta d\theta = \frac{1 - \cos \theta_c}{2}, \quad (15)$$

$$\alpha'(\hat{x}) - \beta'(\hat{x}) = \frac{1}{2} \int_0^{\theta_c} \cos \theta \sin \theta d\theta = \frac{1 - \cos^2 \theta_c}{4}. \quad (16)$$

Upon using $\theta_c = \arccos[c(\hat{x})]$ one then finds

$$\alpha'(\hat{x}) + \beta'(\hat{x}) = \frac{1}{2} [1 - c(\hat{x})], \quad (17)$$

$$\alpha'(\hat{x}) - \beta'(\hat{x}) = \frac{1}{4} [1 - c^2(\hat{x})], \quad (18)$$

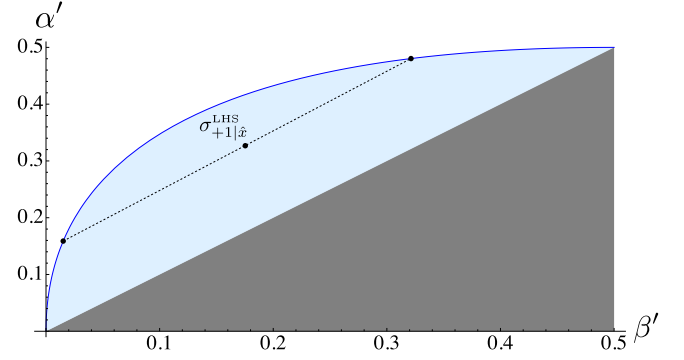


FIG. 2. Plot of the achievable range of eigenvalues (α', β') in our LHS model (for a fixed direction \hat{s}). The upper blue curve corresponds to the condition $\alpha' = \sqrt{2\beta'} - \beta'$ and is achieved by the response functions (13); any point in the light blue area below may be achieved by taking a suitable convex combination of these functions (e.g., the dashed line). Since we have $\alpha' \geq \beta'$, the gray area is not of interest.

from which we get the eigenvalues as a function of $c(\hat{x})$ as

$$\alpha'(\hat{x}) = \sqrt{2\beta'(\hat{x})} - \beta'(\hat{x}), \quad \beta'(\hat{x}) = \frac{1}{8} [1 - c(\hat{x})]^2, \quad (19)$$

corresponding to the curve of Fig. 2. Since this curve is concave, by fixing \hat{s} and taking convex combinations of the response functions (13) with different $c(\hat{x})$, we may prepare any steered states corresponding to (α', β') below this curve, leading finally to

$$\beta'(\hat{x}) \leq \alpha'(\hat{x}) \leq \sqrt{2\beta'(\hat{x})} - \beta'(\hat{x}). \quad (20)$$

This corresponds to the blue area in Fig. 2. We thus conclude that the model reproduces the assemblage of any canonical state ρ , as long as its eigenvalues satisfy the above relation, i.e., $\alpha(\hat{x}) \leq \sqrt{2\beta(\hat{x})} - \beta(\hat{x})$, for any measurement vector \hat{x} , or equivalently

$$\max_{\vec{x}} \{[\alpha(\vec{x}) + \beta(\vec{x})]^2 - 2\beta(\vec{x})\} \leq 0. \quad (21)$$

Using (10) to convert this maximization into Bloch vector notation, we arrive at (8). ■

A natural question is whether condition (8) is also necessary for unsteerability. Unfortunately, this is not the case. Consider the state $\rho_c = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, which does not satisfy (8) [choose, e.g., $\hat{x} = (0, 0, 1)$], but is separable and hence clearly unsteerable. Note however that condition (8) can in fact be strengthened by considering convex combinations with separable states (see Appendix A). An interesting open question is then whether there exist unsteerable states, which cannot be written as convex combinations of unsteerable states satisfying condition (8) and separable states. Nevertheless, condition (8) turns out to be useful for proving the unsteerability of interesting classes of states, as we illustrate below.

IV. APPLICATIONS

We now illustrate the relevance of the above result with some applications. We consider the class of states

$$\rho(p, \chi) = p |\psi_\chi\rangle \langle \psi_\chi| + (1 - p) \rho_\chi^A \otimes \mathbb{1}/2, \quad (22)$$

where $|\psi_\chi\rangle = \cos \chi |00\rangle + \sin \chi |11\rangle$ is a partially entangled two-qubit state, $\rho_\chi^A = \text{Tr}_B |\psi_\chi\rangle \langle \psi_\chi|$, $p \in [0, 1]$, and

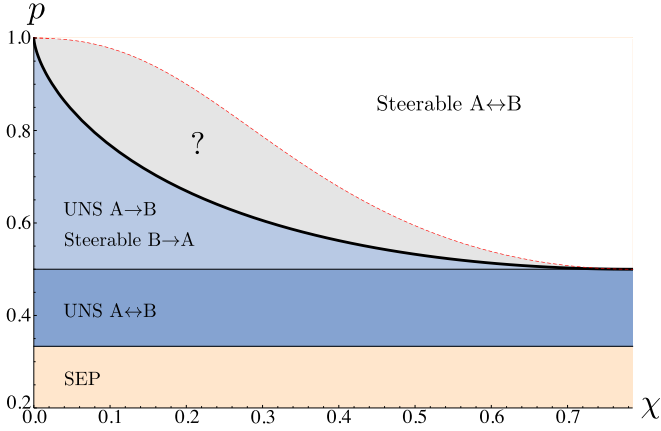


FIG. 3. Characterization of entanglement and steering for states $\rho(p, \chi)$. The solid black curve corresponds to (23), obtained from our unsteerability criterion. The state is separable in the light orange region, unsteerable (in both directions) in the dark blue region, unsteerable only from Alice to Bob (hence one-way steerable) in the light blue region, and two-way steerable in the white region [obtained from Eq. (19) of Ref. [25]]. What happens in the gray region is an interesting open question.

$\chi \in]0, \pi/4]$. The state is entangled for $p > 1/3$. From Theorem 1 it follows that $\rho(p, \chi)$ is unsteerable from Alice to Bob if

$$\cos^2(2\chi) \geq \frac{2p-1}{(2-p)p^3}, \quad (23)$$

as we show in Appendix B. This result is illustrated in Fig. 3 (black solid line). Note that our result recovers the case of a two-qubit Werner state $\rho(1/2, \pi/4)$, which admits a LHS model [18] (in both directions).

A. One-way steering

Alice and Bob play different roles in the steering scenario. Hence steerability in one direction (say from Alice to Bob) does not necessarily imply steerability in the other direction (from Bob to Alice). This effect of one-way steering was first observed in the context of continuous variable systems and Gaussian measurements [26]. More recently, an example of a two-qubit one-way steerable state was presented considering arbitrary projective measurements [22]. That is, while Alice can steer Bob using a finite number of measurements, it would be impossible for Bob to steer Alice as the state admits a LHS model (Bob to Alice). Moreover, a qutrit-qubit state was shown to be one-way steerable considering POVMs [6].

Clearly, our results are also useful for capturing one-way steering. Consider a given state ρ , the canonical form of which is found to satisfy condition (8). From Theorem 1 it follows that ρ is unsteerable from Alice to Bob. Moreover, if steerability from Bob to Alice can be verified using standard methods, e.g., via violation of a steering inequality or using semidefinite programming methods [14, 15], one-way steerability of ρ is proven.

We present different examples of one-way steering. Our states of interest will be the states $\rho(p, \chi)$ defined above. This state is unsteerable from Alice to Bob for projective measurements when (23) is satisfied, corresponding to the

area below the thick black line of Fig. 3. The steerability from Bob to Alice of the above state was discussed in previous works. In particular it was shown that $\rho(p, \chi)$ is unsteerable if $p \leq 1/2$ for all χ [20]. However, for $p > 1/2$, the state becomes steerable from Bob to Alice for all χ . This can be seen as follows. By applying on Alice's side the filter $F_\chi = \text{diag}(1/\cos \chi, 1/\sin \chi)$, we obtain the state

$$\frac{1}{2} F_\chi \otimes \mathbb{1} \rho(p, \chi) F_\chi \otimes \mathbb{1} = \rho(p, \pi/4), \quad (24)$$

which is simply a Werner state with visibility p . Since this state is steerable for $p > 1/2$ [5], it follows from Lemma 1 that all states $\rho(p, \chi)$ with $p > 1/2$ and satisfying (23) are one-way steerable from Bob to Alice for projective measurements.

Simplest one-way steering

A relevant question to ask is how many measurements are needed in order to demonstrate one-way steering. So far, the only known examples for a two-qubit state required as many as 13 measurements [22] and considered only projective measurements and similarly for the qutrit-qubit example of Ref. [6]. Here we present the simplest possible example of one-way steering, that is, a two-qubit state such that Alice cannot steer Bob even with POVMs, although Bob can steer Alice using only two measurement settings.

We start with the case of projective measurements. We show that the states $\rho(p, \chi)$ with $p > 1/\sqrt{2}$ and satisfying (23) are one-way steerable and only two measurements are required for demonstrating steering from Bob to Alice. To prove this we proceed as follows. First, from Lemma 1 it is sufficient to consider the state $\rho(p, \pi/4)$, i.e., a Werner state [see Eq. (24)]. Since this state violates the Clauser-Horne-Shimony-Holt (CHSH) Bell inequality for $p > 1/\sqrt{2}$ [2], it is nonlocal and thus steerable from Bob to Alice with two measurements.

Next we move to the case of POVMs, building on the above example. Following protocol 2 of [31], we construct the state

$$\rho_{\text{POVM}}(p, \chi) = \frac{1}{2} \rho(p, \chi) + \frac{1}{2} |0\rangle\langle 0| \otimes \rho_B, \quad (25)$$

where $\rho_B = \text{Tr}_A \rho(p, \chi)$, which is now unsteerable from Alice to Bob for POVMs, for p and χ satisfying (23). We now show that steering from Bob to Alice is possible using only two measurements. From Lemma 1 we can focus our analysis on the state

$$\begin{aligned} \rho_F &= \frac{F_\chi \otimes \mathbb{1} \rho_{\text{POVM}} F_\chi \otimes \mathbb{1}}{\text{Tr}(F_\chi \otimes \mathbb{1} \rho_{\text{POVM}} F_\chi \otimes \mathbb{1})} \\ &= \frac{\cos^2 \chi (p |\phi^+\rangle\langle \phi^+| + (1-p) \mathbb{1}/4) + \frac{1}{2} |0\rangle\langle 0| \otimes \rho_B}{\cos^2 \chi + 1/2}. \end{aligned} \quad (26)$$

Using the CHSH violation criterion [32], one can find the range of parameters such that ρ_F violates the CHSH inequality and is thus steerable from Bob to Alice with two measurements. We find a parameter range $p > 0.83353$ and corresponding χ given by condition (23).

B. Sufficient condition for joint measurability

Theorem 1 also finds application in quantum measurement theory. This follows from the direct connection existing

between steering and the notion of joint measurability of a set of quantum measurements [27,28], which has already found applications (see, e.g., [33]). This allows us to convert our sufficient condition for unsteerability into a sufficient condition for joint measurability of a set of qubit dichotomic POVMs. Notably, this condition is applicable to continuous sets of POVMs.

A set of measurements $\{M_{a|x}\}$ is said to be jointly measurable [34] if there exists a joint POVM $\{G_\lambda\}$ with outcomes λ and probability distributions $p(a|x, \lambda)$, from which the statistics of any of the measurements $\{M_{a|x}\}$ can be recovered by a suitable postprocessing, that is,

$$M_{a|x} = \int G_\lambda p(a|x, \lambda) d\lambda \quad \forall a, x. \quad (27)$$

Let $\{M_{\pm|x}\}$ be a set of dichotomic qubit POVMs

$$M_{\pm|x} = \frac{1}{2}(k_x \mathbb{1} + \vec{m}_x \cdot \vec{\sigma}), \quad (28)$$

with $\|\vec{m}_x\| \leq k_x \leq 2 - \|\vec{m}_x\|$ and $M_{-|x} = \mathbb{1} - M_{+|x}$. Then the set $\{M_{\pm|x}\}$ is jointly measurable if

$$k_x(k_x - 2) + 2\|\vec{m}_x\| \leq 0 \quad (29)$$

for all x . This can be seen as follows. A set of measurements $\{M_{\pm|x}\}$ is jointly measurable if and only the assemblage given by $\sigma_{\pm|x} = \rho^{1/2} M_{\pm|x} \rho^{1/2}$, where ρ is a full-rank quantum state [35], is unsteerable. Choosing $\rho = \mathbb{1}/2$ we get the corresponding assemblage $\sigma_{\pm|x} = \rho^{1/2} M_{\pm|x} \rho^{1/2} = \frac{1}{2} M_{\pm|x}$. Following Theorem 1, condition (29) ensures the unsteerability of $\sigma_{\pm|x}$ and consequently the joint measurability of $\{M_{\pm|x}\}$.

V. CONCLUSION

We have presented a simple criterion sufficient for a qubit assemblage to admit a LHS model. Notably, our method can guarantee the unsteerability of a general two-qubit state and should thus find applications. We have shown that the criterion allows one to detect entangled states that are only one-way steerable and provides the simplest such examples. Moreover, the criterion is relevant to quantum measurement theory, as it provides a sufficient condition for a continuous set of dichotomic qubit POVMs to be jointly measurable. Further to this, our criterion has also found applications in the connection between measurement incompatibility and nonlocality [36] and multipartite nonlocality [37].

It would be interesting to extend this criterion in several directions. First, can the criterion be strengthened, e.g., by considering convex combinations, in order to become necessary and sufficient? Also, while we focused here on projective measurements, generalizing the method to POVMs would be useful.¹ Whether the present ideas can be adapted to the case of higher-dimensional systems (beyond qubits) is also a natural question. In particular, a natural case to consider is that of entangled states of dimension $d \times 2$, where our method should be directly applicable. Applications to multipartite steering [38,39] would also be interesting.

¹Note that, starting from our result, one can construct new entangled states admitting a LHS model for all POVMs, using Protocol 2 of Ref. [31].

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APPENDIX A: CONVEX COMBINATIONS OF UNSTEERABLE STATES

Since our criterion (8) is not linear and since it does not detect all separable states, it can be useful to consider convex combination of states. Specifically, consider an entangled unsteerable state of the form

$$\rho = p\sigma + (1 - p)\rho_{\text{SEP}}, \quad (A1)$$

where ρ_{SEP} is a separable (hence unsteerable) state and σ is an unspecified state. If σ is unsteerable, then it follows that ρ is unsteerable. However, it could be that, while ρ violates condition (8), σ does not. In this case, the unsteerability of ρ can be shown by finding suitable p and ρ_{SEP} such that

$$\sigma = \frac{\rho - (1 - p)\rho_{\text{SEP}}}{p} \quad (A2)$$

satisfies condition (8).

As a simple example, consider the state

$$\rho = \frac{1}{2}\sigma + \frac{1}{2}\left(\frac{1}{2}|00\rangle\langle 00| + \frac{1}{2}|11\rangle\langle 11|\right), \quad (A3)$$

where σ is the two-qubit isotropic state $\sigma = (|\phi^+\rangle\langle\phi^+| + \mathbb{1}/4)/2$. Hence ρ is an equal mixture of σ and the separable classically correlated state. One finds that for ρ , $T_z = 3/4$ and so ρ violates (8) for $\hat{x} = (0, 0, 1)$. However, the state σ has $T = \mathbb{1}/2$ and $\vec{a} = \vec{0}$ and therefore satisfies (8), hence proving the unsteerability of ρ .

APPENDIX B: PROOF OF UNSTEERABILITY OF $\rho(p, \chi)$

Here we show that for the class of states (22), Theorem 1 implies that the $\rho(p, \chi)$ is unsteerable if

$$\cos^2 2\chi \geq \frac{2p - 1}{(2 - p)p^3}. \quad (B1)$$

To do this, we first consider states in canonical form (7), which satisfy $\vec{a} = (0, 0, a_z)$ and $|T_x| = |T_y|$. In order to perform the maximization of Theorem 1, we parametrize \hat{x} using spherical coordinates $\hat{x} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Our criterion (8) may now be written as

$$\max_{\theta, \phi} F(\theta, \phi) \leq 1,$$

$$F(\theta, \phi) = (\vec{a} \cdot \hat{x})^2 + 2\|\vec{T}\hat{x}\| \\ = \cos^2 \theta a_z^2 + 2\sqrt{T_x^2 + \cos^2 \theta (T_z^2 - T_x^2)}. \quad (B2)$$

Unsurprisingly, F depends only on θ since the problem is symmetric with respect to the x and y directions and we may ignore the maximization over ϕ . Note that if $|T_z| = |T_x|$ then the maximization occurs at $\theta = 0$ and our condition for unsteerability becomes

$$a_z^2 + 2|T_z| \leq 1. \quad (B3)$$

In the case $|T_z| \neq |T_x|$, one should find the extremal points of $F(\theta)$ and prove that they do not exceed 1. To find these

extrema we solve

$$\frac{dF}{d\theta} = -\sin 2\theta \left(a_z^2 + \frac{T_z^2 - T_x^2}{\sqrt{T_x^2 + \cos^2 \theta (T_z^2 - T_x^2)}} \right) = 0. \quad (\text{B4})$$

From $\sin 2\theta = 0$ we have solutions $\theta = 0, \pi/2, \pi$ and possibly other solutions given by

$$a_z^2 + \frac{T_z^2 - T_x^2}{\sqrt{T_x^2 + \cos^2 \theta (T_z^2 - T_x^2)}} = 0. \quad (\text{B5})$$

We now derive conditions such that (B5) has no solution. After rearranging (B5) we have

$$\cos^2 \theta = \frac{T_x^2}{T_x^2 - T_z^2} - \frac{T_x^2 - T_z^2}{a_z^4}. \quad (\text{B6})$$

This has no solution if the right-hand side is greater than 1 or less than 0. Hence we have two conditions

$$\frac{T_x^2}{T_x^2 - T_z^2} < \frac{T_x^2 - T_z^2}{a_z^4} \quad \text{or} \quad \frac{T_z^2}{T_x^2 - T_z^2} > \frac{T_x^2 - T_z^2}{a_z^4}. \quad (\text{B7})$$

If one of the above conditions is fulfilled we therefore have extrema for $\theta = 0, \pi/2, \pi$ only. In this case, since $F(0) = F(\pi)$, our condition for unsteerability becomes

$$\max_{\theta} F(\theta) = \max \{a_z^2 + 2|T_z|, 2|T_x|\} \leq 1. \quad (\text{B8})$$

We now move to the explicit case of $\rho(p, \chi)$. We find a canonical state with $|T_x| = |T_y|$, $\vec{a} = (0, 0, a_z)$, and

$$\begin{aligned} a_z &= \frac{(1 - p^2) \cos 2\chi}{1 - p^2 \cos^2 2\chi}, \\ T_z &= \frac{p(1 - \cos^2 2\chi)}{1 - p^2 \cos^2 2\chi}, \\ T_x &= \sqrt{\frac{p^2(1 - \cos^2 2\chi)}{1 - p^2 \cos^2 2\chi}}. \end{aligned} \quad (\text{B9})$$

We now introduce the ansatz (for $p \geq \frac{1}{2}$)

$$\cos^2 2\chi = \frac{2p - 1}{(2 - p)p^3}. \quad (\text{B10})$$

Eliminating the variable χ we find

$$\begin{aligned} a_z^2 &= \frac{(2 - p)(2p - 1)}{p}, \\ T_z &= \frac{(1 - p)^2}{p}, \\ T_x &= 1 - p. \end{aligned} \quad (\text{B11})$$

For the case $p = \frac{1}{2}$ we have $|T_z| = |T_x|$ and we find that (B3) is satisfied. For $p > \frac{1}{2}$ we show that the second condition of (B7) holds. To this end, we calculate

$$\frac{T_z^2}{T_x^2 - T_z^2} - \frac{T_x^2 - T_z^2}{a_z^4} = \frac{(3 - p)(1 - p)^3}{(p - 2)^2(2p - 1)}. \quad (\text{B12})$$

This is easily seen to be positive for $p \in [\frac{1}{2}, 1]$ and so $F(\theta)$ has extrema at $\theta = 0, \pi, \pi/2$ only. It therefore remains to prove (B8). We find

$$a_z^2 + 2|T_z| = 1, \quad 2|T_x| = 2(1 - p) \quad (\text{B13})$$

and so (B8) is satisfied for $p > \frac{1}{2}$. This proves that the state $\rho(p, \chi)$ is unsteerable if $p \geq \frac{1}{2}$ and p and χ satisfy (B10), which corresponds to the black curve of Fig. 3. Finally, we note that for a fixed χ , lowering p amounts to putting more weight on the separable part of the state. Since a convex combination of an unsteerable state with a separable state is also unsteerable, all points below the curve of Fig. 3 are also unsteerable. Hence, we arrive at (23).

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