Title: Robust Feasibility of Systems of Quadratic Equations Using Topological Degree Theory
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The manuscript studies the margin of robust feasibility of a given solution to a set of quadratic equations. To be specific, it considers the problem of determining the margin $r \geq 0$ such that the system $F(x) - u = 0$ has a feasible solution $x \in \Omega := \{x \in \mathbb{R}^n : Ax \leq b\}$ for all $u \in \Omega_u := \{u \in \mathbb{R}^n : u_i^* - r \leq u_i \leq u_i^* + r, i = 1, \ldots, n\}$, where F is a quadratic transformation. Given any $u \in \Omega_u$, by using the topological degree theory, it is sufficient for such x to exist if there is no solution on the boundary of Ω, assuming the degree is non-zero. The manuscript then exploit this fact to derive optimization calculating lower bounds and upper bounds on the robustness margin r. While the original optimization problems are nonconvex, linear programming relaxations are proposed for computational efficiency. Illustration and computational results are presented in the end of the manuscript.

Compared with the existing literature (e.g., [1]), the manuscript has the following contributions:

1. It first uses topological degree theory (or known as the mapping degree theory, in a sense of generalization of the Brower fixed point theorem) to study the robust feasibility of the quadratic equations.
2. While the paper [1] focuses on the inner approximation (or lower bound on the robustness margin), this manuscript also tries to develop upper bounds for the robustness feasibility problem.
3. It presents tractable formulation for the bounding problem.

The manuscript is overall very interesting in both the topic and its methodology. However, the reviewer believes that the following question must be addressed to avoid confusion or potential gaps in the reasoning.

Question: On page 6, it says that the robust solvability problem can be determined by validating or *invalidating* the statement (8), which is the hypothesis of property (5) (homotopic invariance of topological degree). However, according to the reviewer's understanding, the hypothesis is only a sufficient condition to guarantee the invariance of degree. Invalidating the hypothesis does not necessarily disprove the statement. If the authors intend to show that the statement (8) is indeed a necessary and sufficient condition for the system to be solvable, then a theorem/proposition should be given with detailed proof.

The above question is related to the bounds on the robustness margin discussed in the manuscript.

1. In the proof of Theorem 4.2, the argument "Since $\partial F(X) = F(\partial X)$, there cannot exist an $\hat{\mathbf{x}}' \in \partial\Omega$ such that $F(\hat{\mathbf{x}}') = \hat{\mathbf{u}}$, and hence by definition the system is not robust feasible" is not clear. The author should elaborate on the non-existence of such $\hat{\mathbf{x}}'$ and the reason why the system is not robust feasible.
2. In the proof of Theorem 5.1, it argues by invalidating the statement (8) to show the system is not robust feasible. The argument is not clear to the reviewer.

The reviewer finds that Section 3 Theoretical Results lies in the center of the contribution of this paper, and thus suggests that the authors expand the discussion in this section, compared to those well-known results in the development of the relaxations. Below are some minor comments or suggestions on the manuscript.

Minor comments:

1. In Section 3.1, the topological degree theory is briefly introduced based on the Jacobian of the mapping. However, only continuity, not differentiability, of the mapping F is assumed. In fact, the degree is only well-defined due to the property (5), invariance under homotopy. Therefore, it is suggested that the authors could directly assume differentiability since the main question in the manuscript is to discuss the robust solvability of a quadratic mapping F.
2. The proof of Lemma 3.1 and Theorem 3.2 can be shortened. Lemma 3.1 is equivalent to that a linear optimization over a compact set attains its optimum at a boundary point and Theorem 3.2 uses this fact, together with the assumption $\partial F(X) = F(\partial X)$, and swaps the max and min. It is also suggested that instead of assuming the injectivity of F which is not used later, the condition $\partial F(X) = F(\partial X)$ can be directly assumed.
3. In the formulation of the lower bounding problem in Section 4, the authors claim that "Note that if we impose $\mathrm{rank}(X) = 1$, we do get an exact reformulation." This is not true since the reformulation is not the standard Shor relaxation of QCQPs (see for example [2]). For instance, if $A = 0$ and $bb^T \geq 0$ elementwise, then the variables x and X are decoupled, leading to a potential relaxation gap.

References

[1] Dvijotham, Krishnamurthy, Hung Nguyen, and Konstantin Turitsyn. "Solvability regions of affinely parameterized quadratic equations." *IEEE Control Systems Letters* 2.1 (2017): 25-30.

[2] Boyd, Stephen, and Lieven Vandenberghe. "Semidefinite programming relaxations of non-convex problems in control and combinatorial optimization." *Communications, Computation, Control, and Signal Processing*. Springer, Boston, MA, 1997. 279-287.