Review report of “Robust feasibility of systems of quadratic equations using topological degree theory”

The paper studies the problem of robust feasibility of systems of quadratic equations under perturbation of right hand side constants. The paper uses degree theory to derive upper and lower bounds on the robustness margin. These bounds are given through optimal objective values of nonlinear nonconvex programs. The paper then proposes convex relaxations of them using SDP, LP, and MIP. Computational experiments are conducted for power flow equations of small power systems.

Comments:

The paper is poorly written. Many statements are difficult to understand and some proofs may not be correct. Here are some examples.

1. In Definition 2.1 of robust feasibility, the sentence “The largest r such that $e_i \geq r\ \forall i$ such that $e_i > 0$ in (3) such that the system is robust feasible …” is very cumbersome and should be rewritten, as this is an important definition of the paper.

2. Eq. (4) is cited as a property of the degree of a mapping. The left-hand side of the equality is the degree $d(\Omega, F, y)$ of a mapping $F: \Omega \to R^n$ at a point $y \in R^n$. The right-hand side is the sum of the sign of the Jacobian of F at the preimage $F^{-1}(y)$ of y. Should there be an assumption that $F^{-1}(y)$ is not empty or a definition of the degree $d(\Omega, F, y)$ when $F^{-1}(y)$ is empty? Otherwise, Eq. (6) seems to follow trivially from Eq. (4).

3. Lemma 3.1 is a key result in the paper. However, its statement and proof are both quite poorly written. In the statement, it does not assume that there is a unique solution to $F(x) = u^*$ for the forecast system. However, this seems to be needed in the proof. The proof starts with “Since $d(\Omega, F(x), u^*) \neq 0$ is verified by property (4)”. It is not clear how this is verified by property (4) without such an assumption.

   Then the proof states “the problem is robust solvable, it follows by the fact that …” This sentence is not justified in the proof. The paragraphs before the lemma give some intuition. But a rigorous argument is needed.

4. In Lemma 3.2’s proof, it states that since $F(X)$ is compact, hence $F(X) \setminus \partial F(X)$ contains a ball of a positive radius. This may not be true, because a compact set in R^n may not have an interior, i.e. $F(X) \setminus \partial F(X)$ may be empty. The argument in the proof needs the assumption that $F(X)$ has an interior, which is not a trivial assumption. First, X may not have an interior; second, even if X has an interior, the image $F(X)$ under a continuous mapping may not have an interior. Lemma 3.2 is used in the following result such as Theorem 5.1. The issue with compactness should be checked there as well.

   In the second last line of the proof, it says “such an x^* exists since $F(X)$ is compact”. It is not clear why this is the case or why it is needed. If y is contained in the ball that is contained in $F(X)$, then would this mean y has a preimage in X? It is actually not clear what the use of compactness in the assumption of the lemma is.

5. Theorem 3.3 is the main theorem of the paper. In the second line, it uses the Invariance of Domain Theorem to show that $\partial F(X) = F(\partial X)$ when X is a compact set in R^n, when F is injective and continuous. This argument does not seem to be correct. The Invariance of Domain Theorem states that if $F: U \rightarrow R^n$ is an injective and continuous mapping of an open set U in R^n, then its image $F(U)$ is open in R^n and $F: U \rightarrow F(U)$ is a homeomorphism. It is critical that the set U is an open set in R^n, not just an open set in the subspace topology. Therefore, to apply this theorem to the setting of Theorem 3.3, it seems that X should have a nonempty interior, otherwise the interior of X may not be an open set in R^n and the Invariance of Domain Theorem may not be applied.

   The assumption F is an injective mapping seems to be quite strong. In the case of quadratic mappings defined in Eq. (1), how to verify this?

6. In Theorem 4.1, "e_i denotes the error bounds associated with u_i^{min} and u_i^{max}. " Here, error bounds just mean $u_i^{max} - u_i^{min}$? Error bounds have some specific meaning in optimization theory. Please clarify this and how e_i is used in the proof (it does not directly appear in the proof).

7. In Theorem 4.2, it defines $\partial\Omega_i = \{x|(Ax)_i = b_i, Ax \leq b\}$ and uses it as a nonempty set without qualification. However, it should be noted that $\partial\Omega_i$ may be empty, for example when the constraint $(Ax)_i \leq b_i$ is redundant.

8. The computational results on power flow equations show the computation times are quite large already for the 30-bus system. The scalability of the proposed approach seems to be quite limited.