Image Compression using Singular Value Decomposition (SVD)

- Introduction

- Problem Statement: How can we transmit 1 million numbers (1000 x 1000 matrix) from a satellite to a laboratory on Earth?

- Approach/Methods: Singular Value Decomposition

Singular Value Decomposition:

- Effective tool for minimizing data storage and data transfer.
- SVD exists for *any* and *all* matrices: square, non-square, large, small. So, no matter what kind of term by document matrix the internet yields, we know it has a singular value decomposition.
- Takes a matrix and divides it into two orthogonal matrices and a diagonal matrix. This allows us to rewrite our original matrix as a sum of much simpler rank one matrices.

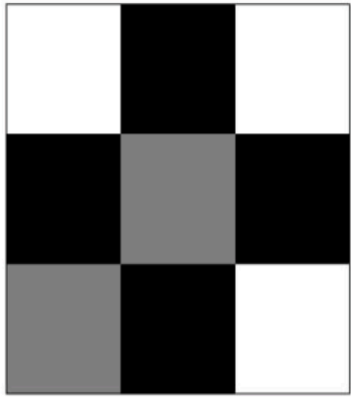
$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = USV^T$$

$$= \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{6}}{6} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \\ -\frac{\sqrt{6}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{2}}{2} & \frac{\sqrt{3}}{3} \end{bmatrix}^T$$

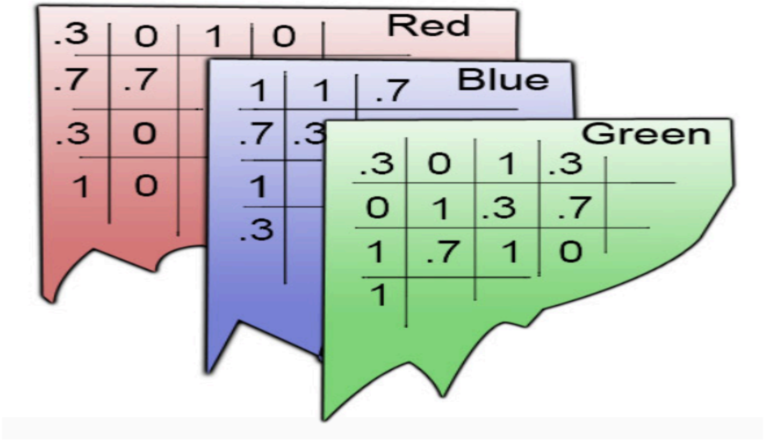
$$A = USV^T = s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{k,k}\mathbf{u}_k\mathbf{v}_k^T.$$

Images as matrices

- We can represent images as matrices. Once it's understood how an image is represented as a matrix, it can be understood how useful **SVD** is
  - Gray scale images (one number per pixel) can be represented as a n x m matrix.
  - Every subsequent value inside the matrix tells the computer how bright to display the corresponding pixel
  - *Color images*, in turn, can be represented by three matrices. Each matrix specifies the amount of **Red**, **Green** and **Blue** (RGB) that makes up the image.
  - Each color can be represented as a n x 3m matrix   A = [Ared , Agreen, Ablue]



$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & .5 & 0 \\ .5 & 0 & 1 \end{bmatrix}$$



- Problem Statement :Transmit 1 million numbers:
  - A grayscale image of size 1000 x 1000, must be transmitted from a satellite to a laboratory on Earth which equal to **1 million numbers** (one for each pixel).
  - SVD decomposition $A = USV^T$

    It's enough that the satellite sends;
    - 20 first columns of *U*
    - 20 first **numbers** $s_{i,i}$
    - 20 first columns of *V*

U S V

(Totaling 20x1000 + 20 + 20x1000 = **40,020 numbers**)

  - By receiving 40,020 numbers data, the laboratory on Earth calculates the matrix

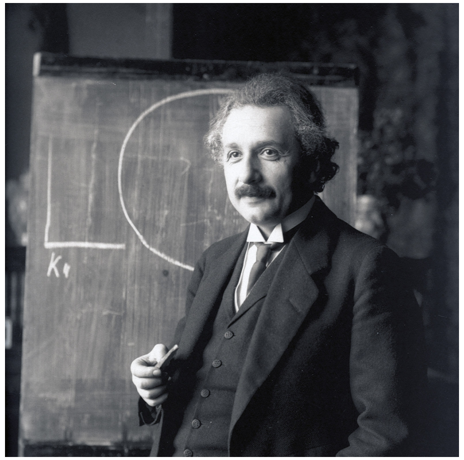
$$s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{20,20}\mathbf{u}_{20}\mathbf{v}_{20}^T$$

that will give an approximation of the original image

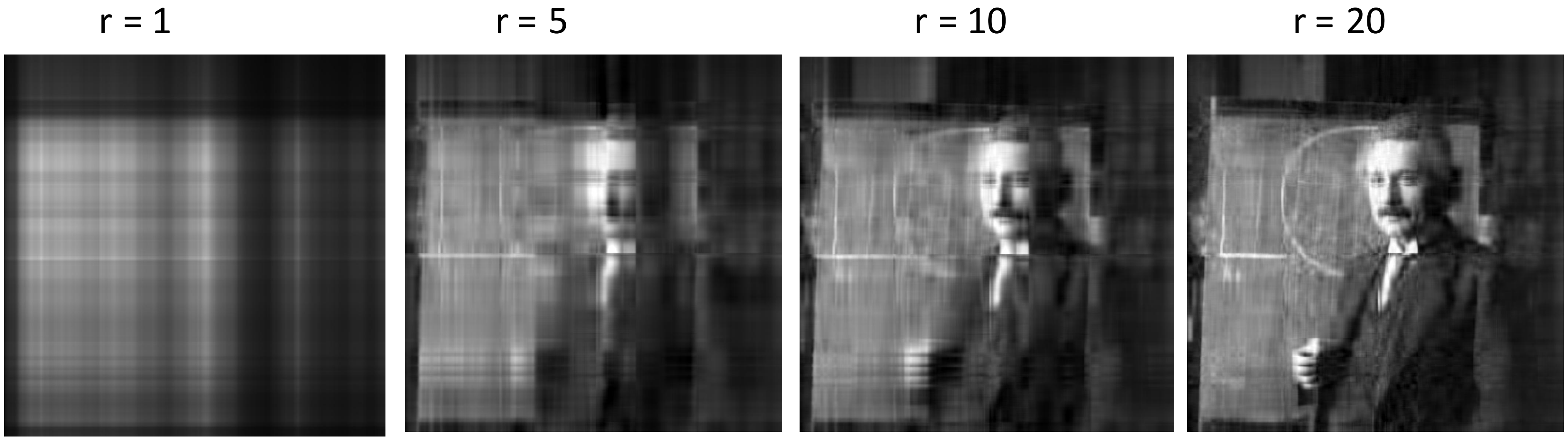
SVD Application: Image Compression

$$s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{r,r}\mathbf{u}_r\mathbf{v}_r^T \text{ for } r = 1, 5, 10 \text{ and } 20.$$

- Generate approximations to the original image
- Original image corresponds to the case r = 1024
- 1024x1024 = **1,048,579**
- r=20 ➔ 20x1024 + 20 + 20x1024 = **40,980** **(%3.9)**



Albert Einstein (1879 – 1955)
1024 x 1024 x 3



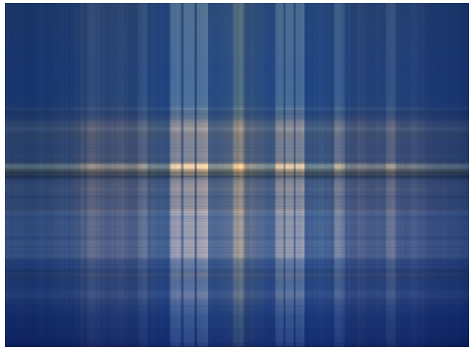
SVD Application: Image Compression

$$s_{1,1}\mathbf{u}_1\mathbf{v}_1^T + s_{2,2}\mathbf{u}_2\mathbf{v}_2^T + \cdots + s_{r,r}\mathbf{u}_r\mathbf{v}_r^T \text{ for } r = 1, 5, 10 \text{ and } 20.$$

- Generate approximations to the original image
- Original image corresponds to the case

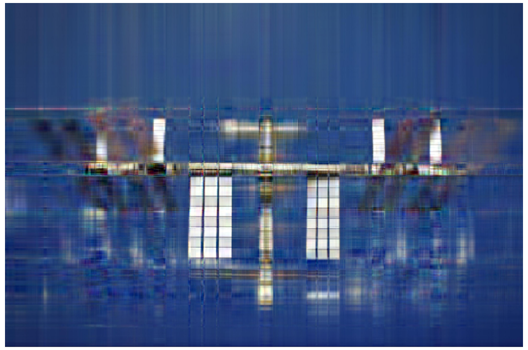
International Space Station
2849 x 4288 x 3

r = 1

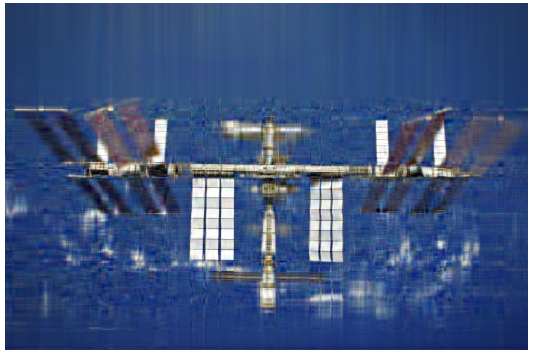


r = 5

r = 10



r = 20



r = 50

SVD: Other Application

Rank, Range, and Null space

- The range of matrix A is The left singular vectors of U corresponding to the non-zero singular values.
- The rank of matrix A can be calculated from SVD by the number of nonzero singular values.
- The null space of matrix A is The right singular vectors of V corresponding to the zeroed singular values.

$$A = UWV^T$$

Range Rank Null Space

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}$$

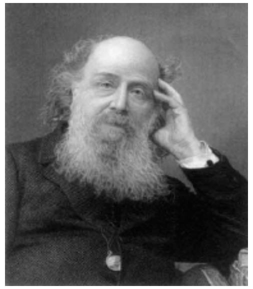
Digital Signal Processing (DSP)

- SVD is used as a method for noise reduction.
- Let a matrix A represent the noisy signal:
  - compute the SVD,
  - and then discard small singular values of A.
- It can be shown that the small singular values mainly represent the noise, and thus the rank-k matrix A_k represents a filtered signal with less noise.

SVD was discovered by the following people:

E. Beltrami
(1835 – 1900)

M. Jordan
(1838 – 1922)



J. Sylvester
(1814 – 1897)

E. Schmidt
(1876-1959)

H. Weyl
(1885-1955)

Question & Answer

Thank you