1. a. $\sum_{i=1}^{N} 2i-1 = N^2$

Inductive basis

$N = 1$

$\sum_{i=1}^{1} 2i-1 = 1^2$ $1 = 1$ True

Inductive hypothesis

If $N=k$ we can assume that

$$\sum_{i=1}^{k} (2i-1) = k^2$$

Inductive step aims to prove that $N = k+1$ also holds true

$$\sum_{i=1}^{N} (2i-1) = N^2$$

$$\sum_{i=1}^{k+1} (2i-1) = (k+1)^2$$

Summation of K+1 Should be equal to the Summation of the previous value + K+1

$$\sum_{i=1}^{k+1}(2i-1) = \sum_{i=1}^{k}(2i-1) + (2(k+1)-1)$$

By induction, $\sum_{i=1}^{K}(2i-1)$ can be replaced by K^2

So,

$$\sum_{i=1}^{k+1}(2i-1) = K^2 + (2(k+1)-1)$$

$$= K^2 + 2k + 2 - 1$$

$$\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$$

Since $\sum_{i=1}^{k}(2i-1) = k^2$, then $\sum_{i=1}^{k+1}(2i-1) = (k+1)^2$

$\sum_{i=1}^{N}(2i-1) = N^2$ is true

1. b.

$$\sum_{i=1}^{N} i^3 = \left(\sum_{i=1}^{N} i\right)^2$$

Inductive basis

$N=1$

$$\sum_{i=1}^{1} 1^3 = \left(\sum_{i=1}^{N} 1\right)^2$$

$1 = 1$ ✓ True

Inductive Hypothesis

$\left(\sum_{i=1}^{N} i\right)^2$ can be rewritten as

$$\frac{1}{4} N^2 (N+1)^2$$

If $N=k$ we can assume that

$$\sum_{i=1}^{k} i^3 = \frac{1}{4} k^2 (k+1)^2$$

Inductive Step

Aims to prove that $N = k+1$ holds true

$$\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$$

$\sum_{i=1}^{k} i^3$ can be replaced by $\frac{1}{4}k^2(k+1)^2$

$$\sum_{i=1}^{k+1} i^3 = \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$\sum_{i=1}^{k+1} i^3 = \frac{1}{4}k^2(k^2+2k+1) + (k^2+2k+1)(k+1)$$

$$= \frac{1}{4}\left(k^4+2k^3+k^2\right) + k^3 + k^2 + 2k^2 + 2k + k + 1$$

$$= \frac{k^4+2k^3+k^2+4k^3+4k^2+8k^2+8k+4k+4}{4}$$

$$= \frac{1}{4}(k^2+3k+2)^2$$

$$\sum_{i=1}^{k+1} i^3 = \frac{1}{4}(k+1)^2((k+1)+1)^2$$

$$\sum_{i=1}^{k} i^3 = \frac{1}{4}k^2(k+1)^2$$

$$\sum_{i=1}^{N} i^3 = \frac{1}{4}(N)^2(N+1)^2 \text{ is true}$$

2.

$$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

Inductive basis

$n=1$

$$\sum_{i=1}^{1} \frac{1}{1^2} \leq 2 - \frac{1}{1}$$

$1 \leq 1$ True

Inductive hypothesis

If $n=k$ we can assume that

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

Inductive step

Aims to prove $n=k+1$ holds true

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq \sum_{i=1}^{k} i^2 + \frac{1}{(k+1)^2}$$

$\sum_{i=1}^{k} \frac{1}{i^2}$ can be replaced by $2 - \frac{1}{k}$ by applying the induction hypothesis

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq \left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2}$$

$$\left(2 - \frac{1}{k}\right) + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k+1}$$

Since $\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$ then $\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k+1}$

$\sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$ is true

3.

Consider the series, 1,2,3,4,5,10,20,40,80 …

The series starts as an arithmetic series and becomes a geometric series starting with 10. Prove by induction that any positive integer can be written as a sum of distinct members of this series.

Inductive basis

Any positive integer ≤ 10 can be written as a sum of distinct members of the series

$1 = 1$ $\qquad$ $6 = 5 + 1$

$2 = 2$ $\qquad$ $7 = 5 + 2$

$3 = 3$ $\qquad$ $8 = 5 + 3$

$4 = 4$ $\qquad$ $9 = 5 + 4$

$5 = 5$ $\qquad$ $10 = 5 + 4 + 1$

Any positive integer > 10 can be written as a sum of distinct members of the series

$11 = 10 + 1$

Inductive hypothesis

n can be written as a sum of distinct members of the series

Inductive step

Aims to prove that $n+1$ can be written as a sum of distinct members of the series

$n+1-a$ with a being the largest term in the series such that $a \leq n+1$

d = distinct member

$$a + (a+d) + (a+2d) \dots (a+(n-1)d)$$

$$a = \frac{n}{2}\left(2a+(n-1)d\right)$$

$\downarrow$

0

for $n=1$

$$a = \frac{1}{2}\left(2a + 0\right)$$

<u>for $n = k+1$</u>

$$= \frac{k}{2}\left(2a + (k-1)d\right) + a + kd$$

$$= (k+1)a + \left(\frac{k+1}{2}\right)kd$$

$$\left(\frac{k+1}{2}\right)\left(2a + (k+1-1)d\right) = \frac{n}{2}\left(2a + (n-1)d\right)$$

is true

4. Assume $n = 2^m$

<u>Second iteration</u>

$$T(n) = 4T\left(\frac{n}{2}\right) + n^2$$

$$T\left(\frac{n}{2}\right) = 4T\left(\frac{n/2}{2}\right) + \left(\frac{n}{2}\right)^2$$

$$T(n) = 4\left(4T\left(\frac{n}{4}\right) + \left(\frac{n}{2}\right)^2\right) + n^2$$

$$T(n) = 16T\left(\frac{n}{4}\right) + 4\left(\frac{n}{2}\right)^2 + n^2$$

<u>Third iteration</u> $i = 3$

$$T\left(\frac{n}{4}\right) = 4T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2 \qquad 2$$

$$T(n) = 16\left(4T\left(\frac{n}{8}\right) + \left(\frac{n}{4}\right)^2\right) + 4\left(\frac{n}{2}\right)^2 + n^2$$

$$= 64T\left(\frac{n}{8}\right) + 16\left(\frac{n}{4}\right)^2 + 4\left(\frac{n}{2}\right)^2 + n^2$$

<u>Fourth Iteration</u>

$$T\left(\frac{n}{8}\right) = 4T\left(\frac{n}{16}\right) + \left(\frac{n}{8}\right)^2$$

$$T(n) = 64\left(4T\left(\frac{n}{16}\right) + \left(\frac{n}{8}\right)^2\right) + 16\left(\frac{n}{4}\right)^2 + 4\left(\frac{n}{2}\right)^2 + n^2$$

$$T(n) = 256T\left(\frac{n}{16}\right) + 64\left(\frac{n}{8}\right)^2 + 16\left(\frac{n}{4}\right)^2 + 4\left(\frac{n}{2}\right)^2 + n^2$$

Pattern

$$T(n) = 4^k T\left(\frac{n}{2^k}\right) + \sum_{i=0}^{k-1} 4^i \left(\frac{n}{2^i}\right)^2$$

Prove using induction

$k = m \quad 2^k = 2^m = n$

$$T(n) = 4^m T\left(\frac{n}{2^m}\right) + \sum_{i=0}^{m-1} 4^i \left(\frac{n}{2^i}\right)^2$$

Simplifies to n^2

$$T(n) = 4^m T\left(\frac{n}{2^m}\right) + \sum_{i=0}^{m-1} n^2$$

Inductive basis

$m = 1$

$$T(n) = 4T\left(\frac{n}{2}\right) + \sum_{i=0}^{1-1} n^2$$

Inductive step

$$T(n) = 4^m T\left(\frac{n}{2^m}\right) + \sum_{i=0}^{m-1} n^2$$

$$T(n) = 4^{m+1} T\left(\frac{n}{2^{m+1}}\right) + \sum_{i=0}^{m} n^2$$

$$T\left(\frac{n}{2^m}\right) = 4^{m+1} T\left(\frac{n}{2^{m+1}}\right) + \sum_{i=0}^{m} n^2$$

$$T(n) = \left(4^{m+1} T\left(\frac{n}{2^{m+1}}\right) + \sum_{i=0}^{m} n^2 = \right.$$

$$\left(4^m T\left(\frac{n}{2^m}\right) + \sum_{i=0}^{m-1} n^2 + n^2\right)$$

✓ True