

## Problem 9

**Theorem 1.1.** *A hypercube  $Q_n$  is Hamiltonian. It has a girth of 4 for  $n \geq 2$  and  $\infty$  otherwise. Its diameter is  $n$ , its order is  $2^n$  and it has a size of  $2^{n-1} \cdot n$ .*

*Proof.* Let  $S$  be a set of cardinality  $|S| = n$ . We construct  $Q_n = (V_Q, E_Q)$  by creating a vertex for each subset of  $S$  and moreover add edges between those subsets which differ by only one element. In the following, we may use binary representations of the vertices of  $Q_n$  since  $V_Q = \mathcal{P}(S) \cong (\mathbb{Z}/2\mathbb{Z})^n$  (we can denote a 1 for including an element and a 0 for excluding an element in a subset).

**Order:** Since  $V_Q = \mathcal{P}(S)$  and  $|\mathcal{P}(S)| = 2^n$ , the order of  $Q_n$  is  $2^n$ .

**Size:** Each of the  $2^n$  vertices is adjacent to  $n$  other vertices since we can insert / remove each of the  $n$  elements of  $S$ . For undirected edges, we have  $\frac{2^n \cdot n}{2} = 2^{n-1} \cdot n$  edges. Thus, the size of  $Q_n$  is  $2^{n-1} \cdot n$ .

**Girth:** We differ between two cases.

- **Case 1:**  $n = 1$ . Our graph contains exactly one edge and is therefore acyclic. Hence, the girth is  $\infty$  for  $n = 1$ .

- **Case 2:**  $n \geq 2$ . Our graph contains the cycle  $(\emptyset, \{a\}, \{a, b\}, \{b\}, \emptyset)$  ( $a, b \in S$ ) which has length 4.

A shorter cycle  $(A, B, C, A)$  ( $A, B, C \in V_Q$ ) does not exist due to the property that two adjacent vertices differ by exactly one element. For such a cycle,  $B$  and  $A$  differed by one element, and hence  $A$  and  $C$  differed by two or are equal. However, a difference of zero or two elements between two consecutive elements renders any walk invalid. The edge  $\{C, A\}$  could not be contained in  $Q_n$ .

From these considerations, for  $n \geq 2$ , the girth is 4.

**Diameter:** For any set  $A \in V_Q$ , we are able to get to any other element  $B \in V_Q$  by inserting or removing a maximum of  $n$  elements. Thus, a path of length  $n$  is sufficient to walk from any  $A$  to any  $B$ . Furthermore, there exist  $A$  and  $B$  such that a path of length  $n$  is the shortest path between them. E.g.  $A = \emptyset \in V_Q$ ,  $B = S \in V_Q$ . Thus, the diameter of  $Q_n$  is  $n$ .

**Hamiltonian:** A Hamiltonian cycle is equivalent to an enumeration of  $(\mathbb{Z}/2\mathbb{Z})^n$  in which consecutive elements differ by exactly one element. We provide such an enumeration: the *Gray Code*<sup>a</sup>. Thus, there exists a Hamiltonian cycle and  $Q_n$  is Hamiltonian.  $\square$

<sup>a</sup>For  $n = 2$ : 00,01,11,10,00. Generally, the  $k$ 'th vertex in the Hamiltonian cycle is  $k \otimes \lfloor \frac{k}{2} \rfloor$  whereby  $\cdot \otimes \cdot$  denotes the exclusive or.

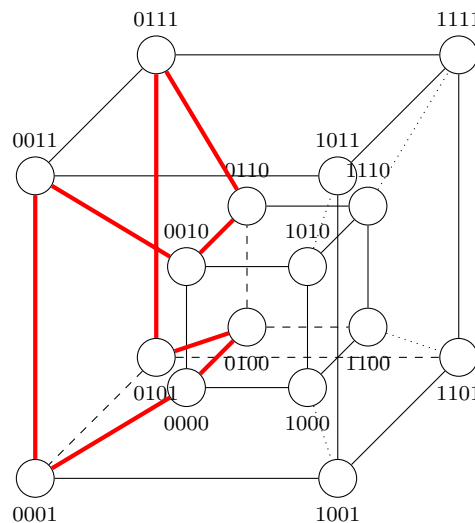


Figure 1: The hypercube  $Q_4$  and the Hamiltonian cycle of  $Q_3$  as a subgraph of  $Q_4$ .

**Theorem 1.2.** *A complete bipartite graph  $K_{m,n}$  is Hamiltonian iff  $m = n$ . It's girth is 4 for  $m, n \neq 1$  and  $\infty$  otherwise. It's diameter is 1 for  $m = n = 1$  and 2 otherwise. The graph's order is  $m + n$  and it's size is  $m \cdot n$ .*

*Proof.* Let  $V = \{v_1, \dots, v_m\}$  and  $W = \{w_1, \dots, w_n\}$  denote the two partitions of  $K_{m,n} = (V_K, E_K)$ .

**Order:** The first partition has  $m$  elements, the second  $n$  elements. Thus,  $K_{m,n}$  has an order of  $m + n$ .

**Size:** Each of the  $m$  elements of the first partition are connected to each of the  $n$  elements in the second partition. Thus,  $K_{m,n}$  has a size of  $m \cdot n$ .

**Girth:** If either  $m = 1$  or  $n = 1$ , then all vertices of one partition are incident to and only to the single vertex of the other partition. Hence, there is no cycle in  $K_{1,n}$  or  $K_{m,1}$  and the girth of  $K_{m,n}$  is  $\infty$  if  $n = 1$  or  $m = 1$ .

If  $m, n \neq 1$ , each cycle must have even length since any two consecutive vertices in a path of  $K_{m,n}$  are in different partitions. Thus, we require an even amount of edge-crossings to enclose a walk. Any cycle has a length of at least 3, thus the girth of  $K_{m,n}$  has to exceed or be equal to 4.

Furthermore, we find such a cycle of length 4 easily since both partitions  $V, W$  have at least 2 vertices:  $(v_1, w_1, v_2, w_2, v_1)$ . From these considerations, the girth of  $K_{m,n}$  must be equal to 4.

**Diameter:** For  $m = n = 1$ , there are exactly two vertices in different partitions. They have a distance of 1 and thus, the diameter of  $K_{1,1}$  is 1.

Since two consecutive vertices in a path of  $K_{m,n}$  are in different partitions  $V, W$  the distance between two vertices in the same partition has to be at least 2. Moreover, we find a path of distance 2 between  $v_1 \in V$  to  $v_2 \in V$ :  $(v_1, w, v_2)$  for any  $w \in W$ . Thus, the diameter does not exceed 2.

For any two vertices in different partitions  $V, W$ , they are directly connected by a path of length 1.

Hence, the diameter of  $K_{m,n}$  is 2 if not  $m = n = 1$ .

**Hamiltonian:** For  $m = n$ , we find always find a Hamiltonian cycle:  $(v_1, w_1, v_2, w_2, \dots, v_n, w_n, v_1)$ .

However, for  $m \neq n$ , there can not exist a Hamiltonian cycle.

We assume such a Hamiltonian cycle  $c = (u_1, u_2, \dots, u_{n+m}, u_1)$  existed for  $m \neq n$ .

- **Case 1:**  $m + n$  odd. Then,  $u_{n+m}$  and  $u_1$  were in the same partition which rendered the edge  $\{u_{n+m}, u_1\}$  invalid. Hence,  $K_{m,n}$  is not Hamiltonian for an odd  $n + m$ .
- **Case 2:**  $m + n$  is even. Then, there was an  $n_0 < n + m$  such that  $(u_1, \dots, u_{n_0})$  is the shortest sub walk that covers one partition but not both. Again, we inspect two cases.
  - $n_0 = m + n - 1$ . Then, both partitions have the same number of vertices which is contradictory to our precondition that  $m \neq n$ .
  - $n_0 < m + n - 1$ . Then, we are trapped in one partition for we are not able to cover two vertices of the same partition consecutively.

Hence,  $K_{m,n}$  ( $m \neq n$ ) is not Hamiltonian for an even  $m + n$ .

All in all,  $K_{m,n}$  is Hamiltonian if and only if  $m = n$ . □

**Theorem 1.3.** *The Petersen graph is not hamiltonian, it has a girth of 5, a diameter of 2, an order of 10 and a size of 15.*

*Proof.* **Order:** The graph has 10 vertices and thus, it's order is 10.

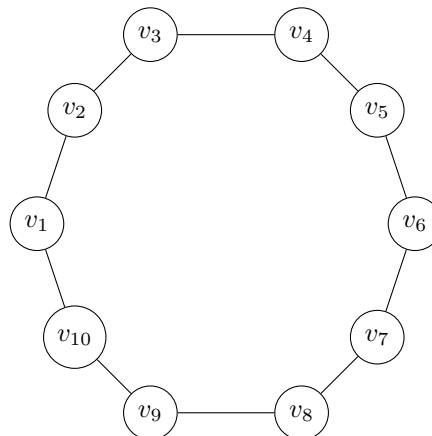
**Size:** The graph has 15 edges and thus, it's size is 15.

**Girth:**

**Diameter:**

**Hamiltonian:**

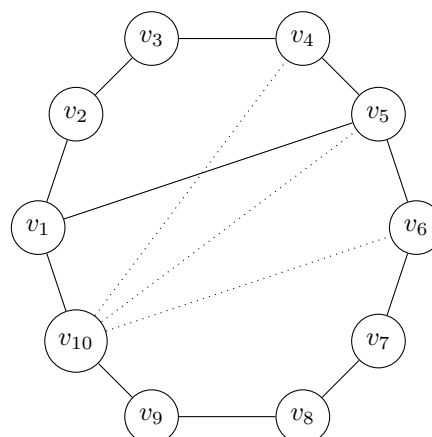
Let's assume the Petersen graph is Hamiltonian, then there would be a cycle containing all 10 nodes. Now we can draw the graph by starting off by drawing the hamilton cycle.



Now in order for  $v_1$  to have degree 3 it has to have another edge. We already know that the minimum cycle in the Peterson graph is 5 so  $v_1$  can only be connected to  $v_5, v_6$  or  $v_7$ . Because of symmetry reasons we only need to consider the following two cases.

- $(v_1, v_5)$  is an edge of Peterson Graph

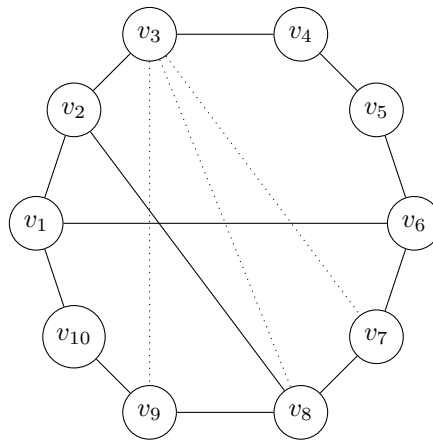
Then either  $(v_{10}, v_4)$ ,  $(v_{10}, v_5)$  or  $(v_{10}, v_6)$  has to be part of the Peterson graph. In all three cases a cycle is created with size smaller than 5. ( $v_1, v_5, v_4, v_{10}, v_1$  with edge  $(v_{10}, v_4)$ ,  $v_1, v_5, v_{10}, v_1$  with edge  $(v_{10}, v_5)$  or  $v_1, v_5, v_6, v_{10}, v_1$  with edge  $(v_{10}, v_6)$ ) Concluding, the edge  $(v_1, v_5)$  cannot be part of the Peterson graph.



- $(v_1, v_6)$  is an edge of Peterson Graph

Examining  $v_2$  we see that  $v_2$  can only be connected to  $v_8$  because all with all other vertices there would be a cycle of size smaller than 5. Then examining  $v_3$  we see that  $v_3$  can neither be connected

to  $v_7$ ,  $v_8$  nor  $v_9$  because in every case there would be a cycle of size smaller than 5. Concluding, the edge  $(v_1, v_6)$  cannot be part of the Peterson graph.



Because of the symmetry of the cycle we can conclude that there is no way to construct the Peterson graph from a cycle with 10 vertices. Thus, we know that the Peterson graph cannot be Hamiltonian.  $\square$

## Problem 11

In the following I will show how to construct for each  $k > 1$  a  $k$ -regular graph with no 1-factor. The basic idea is that for even  $k$ , simply  $K_{n+1}$  is such a graph. For odd  $k$ , I will show a way how to build such a  $k$ -regular graph by connecting one vertex to  $k$  components with odd number of vertices.

For each even integer  $k > 1$ , the complete graph  $K_{k+1}$  is a  $k$ -regular graph without a 1-factor.  $K_{k+1}$  is by definition  $k$ -regular and because  $K_{k+1}$  has a odd number of components if  $k$  is even, it is obviously impossible to find a 1-factor. For each odd  $k > 1$ , we are able to construct a  $k$ -regular graph without a 1-factor in the following way.

In order to guarantee that the graph has no 1-factor, we can use Tutte's theorem. We construct the graph by starting with a single vertex  $v \in V$  connected to  $k$  subgraphs  $S$  which are not inter-connected. Then, we

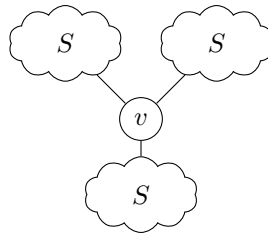


Figure 2: Example with  $k=3$

construct  $S = (V, E)$  in such a way that  $|V|$  is odd and exactly one vertex  $u \in V$  has degree  $k - 1$  while all other vertices have degree  $k$ .

If we then connect  $u$  to  $v$ , we obtain a  $k$ -regular graph. If we removed  $v$  we would have  $k$  components  $S$  with an odd number of vertices. Furthermore,  $k > 1$  and thus, by Tutte's theorem, we know that the resulting graph has no 1-factor.

In order to construct  $S$  we first need the following lemma.

**Lemma 2.0.1.** *For any odd integer  $k > 1$  it is possible to construct a  $(k - 1)$ -regular graph  $G = (V, E)$  with  $k + 1$  vertices.*

*Proof.* We can obtain  $G$  from  $K_{k+1}$  by removing all edges from  $K_{k+1}$  which are contained in a perfect matching of  $K_{k+1}$ .

By removing those edges, the degree of every vertex decreases exactly by one.  $K_{k+1}$  is by definition  $k$ -regular and hence  $G$  is  $k - 1$ -regular.

A perfect matching in  $K_{k+1}$  exists, because  $k + 1$  is even, and the conditions of Tutte's theorem are always satisfied in a complete graph. Moreover, we find such a matching by randomly choosing edges  $\{u, v\}$  and removing  $u$  and  $v$  from  $K_{k+1}$ . □

*Constructing a connected graph  $S = (V, E)$  with  $|V| = k + 2$  and the degree sequence  $(k, k, \dots, k, (k - 1))$ .* First we construct a  $(k - 1)$ -regular graph  $S' = (V', E')$  with  $k + 1$  vertices as described in the aforementioned way. Then, we can add one vertex to  $S'$  and connect it to all except one vertices in  $V'$ . Thus, we have a new graph  $S$ . Because we have added only one vertex  $|V| = |V'| + 1 = k + 2$  and the degree of the newly added vertex is  $k$ , the degree of all the other vertices except of the last one is increased by one.

Hence,  $S$  has the degree sequence  $(k, k, \dots, k, (k - 1))$ . By connecting one vertex to  $k$  subgraphs isomorph to  $S$ , like described above, we have constructed a  $k$ -regular graph without a perfect matching.

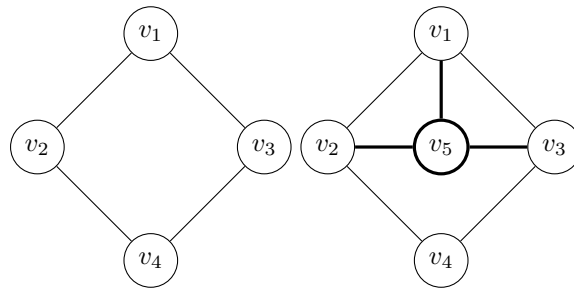


Figure 3:  $S'$  and  $S$  for  $k=3$

## Problem 12

**Theorem 3.1.** Any graph  $G$  with  $2n$  vertices and  $\delta(G) \geq n$  has a 1-factor.

*Proof.* Let  $G = (V, E)$  be a graph with  $|V| = 2n$  and  $d(v) \geq n \forall v \in V$ .

In the following, we will prove for nontrivial  $G$  that the number of odd components in a graph  $G - S$  does not exceed the number of vertices in  $S$ . By *Tutte's Matching Theorem*,  $G$  then has a 1-factor.

**$n = 1$**  Then,  $G$  is a simple graph with two vertices that are connected by one edge. This is of course a perfect matching of  $G$ .

**$n \geq 2$**  Let  $S \subseteq V$  be a set of vertices,  $G' = (V', E') := G - S$  and  $k := |S|$ .

As  $G'$  is created by removing all vertices of  $S$  and their incident edges from  $G$ , we obtain the following properties:

- $\forall v \in V : d(v) \geq n - k$
- For any component  $C$  of  $G'$ ,  $|V(C)| \geq n - k + 1$
- the order of  $G$  is  $2n - k$

In the following cases, we prove that  $\lambda := \# \text{odd components} \leq k$ .

**$k = 0$  :**

Because  $G$  consists of one even component,  $\lambda = 0 \leq k$

**$k = 1$  :**

After removing any vertex of  $G$ , the degree of a vertex in  $G'$  is reduced by one or less. Hence,  $\forall v \in V : d(v) \geq n - 1$ . This implies that the size of any component in  $G'$  is at least  $n$ . As far as  $|V'| = 2n - 1$  there can only exist one component in  $G'$  with order  $2n - 1$  (which is odd).

All in all, we have shown that  $\lambda = 1 \leq k$ .

**$2 \leq k \leq n$  :**

As the minimum size of a component in  $G'$  is  $n - k + 1$  and  $|V'| = 2n - k$ , we can bound the amount of components by the following term:  $\frac{2n-k}{n-k+1}$ .

We now have to prove that  $\frac{2n-k}{n-k+1} \leq k \iff 2n-k \leq k(n-k+1) \iff 0 \leq (k-2)n - k^2 + 2k =: f(k)$ .

To prove this inequality, we have to determine the minimum value of  $f$  in the defined boundaries.

$$* f'(k) = n - 2k + 2 \stackrel{!}{=} 0 \iff k = \frac{n}{2} + 1$$

$$* f''(k) = -2$$

So  $f$  has a maximum but no local minimum. To find the minimum value within the given range, we have to check the borders:

$f(2) = 0 = f(n)$ . Thus,  $\min(f) = 0 \geq f(k)$  which proves the inequality.

All in all, we have shown that the number of components does not exceed or is equal to  $k$ . This implies that  $\lambda \leq k$ .

$n \leq k \leq 2n$  :

Because the number of components can not exceed the number of vertices and  $V' = 2n - k \leq n \leq k$ , there can not be more than  $k$  odd components.

Hence,  $\lambda \leq k$ .

Finally, we have shown that the number of resulting components in  $G'$  is bounded by the order of  $S$ . In other words:  $\forall S \subseteq V(G) : \# \text{odd components of } G - S \leq |S|$ .

Thus, we have shown that all conditions for *Tutte's Matching Theorem* are satisfied. Hence,  $G$  has a perfect matching aka 1-factor.

□