

## Problem 13

**Theorem 1.1.** *If a graph has an ear-decomposition, then it is 2-connected.*

*Proof.* By *Menger's Theorem*, a graph  $G$  is  $k$ -connected if and only if for any two vertices  $a, b$  in  $G$  there exist  $k$  independent  $a$ - $b$ -paths. We find those 2 paths for any ear-composable graph.  $\square$

## Problem 14

For  $0 < l < m \leq d$ , we will construct a graph  $F(d, l, m)$ .

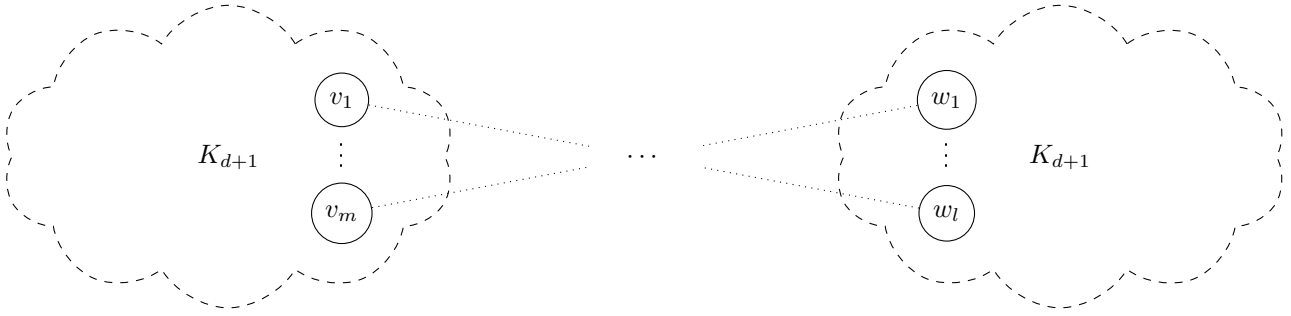


Figure 1:  $F(d, l, m)$

First, we construct two complete graphs on  $d + 1$  vertices.  $(V, E) \simeq K_{d+1}$ ,  $(W, E') \simeq K_{d+1}$ .

Then, we join  $m$  vertices  $v_1, \dots, v_m \in V$  of the first complete graph and  $l$  vertices  $w_1, \dots, w_l \in W$  of the second such that each  $v_i$  has a degree of exactly  $d + 1$  and each  $w_j$  of at least  $d + 1$  ( $i \in [m], j \in [l]$ ).

Formally, for our constructed graph  $F(d, l, m) := (V_F, E_F)$ , the vertex set is the union of both complete graphs ( $V_F = V \cup W$ ) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid \delta_{ij} = 1 \ (i, j \in \mathbb{N})\} \quad (1)$$

for a delta function  $\delta_{ij}$  ( $i, j \in \mathbb{N}$ )

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i > l, j = l \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{d}$

No degree of a vertex of the complete graphs has been decreased. Thus,  $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$ .

Moreover, we have increased the degree of exactly  $l + m < 2(d + 1)$  vertices. Indeed, the complete graph on  $d + 1$  vertices is  $d$ -regular and hence there is at least one vertex of degree  $d$  in  $F(d, l, m)$ . Thus,  $\delta(F(d, l, m)) \leq d$ .

From these considerations,  $\delta(F(d, l, m)) = d$ .

$\kappa(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{l}$

In  $F(d, l, m)$ , the two complete graphs are only joined by edges between  $l$  vertices of one and  $m$  vertices of another complete graph. The graph obviously disconnects by removing those first  $l$  vertices. Thus,  $\kappa(F(d, l, m)) \leq l$ .

Moreover, a complete graph on  $d + 1$  vertices is internally connected with  $\kappa(K_{d+1}) = d > l$ .

Thus, it is neither possible to disconnect one of the complete graph by removing less than  $l$  vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than  $l$  vertices.

From these considerations,  $\kappa(F(d, l, m)) = l$ .

$\kappa'(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{m}$

In  $F(d, l, m)$ , the two complete graphs are only joined by exactly  $m$  edges and a removal of those  $m$  vertices obviously disconnects  $F(d, l, m)$ . Thus,  $\kappa'(F(d, l, m)) \leq m$ .

Moreover, a complete graph on  $d + 1$  vertices is internally connected with  $\kappa'(K_{d+1}) = d > m$ .

Thus, it is neither possible to disconnect one of the complete graph by removing less than  $m$  edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than  $m$  edges.

From these considerations,  $\kappa'(F(d, l, m)) = m$ .

## Problem 15

I will prove that any block-cut-vertex graph is a tree, by showing by contradiction that any block-cut-vertex graph is acyclic and connected.

**Theorem 3.1.** *The block-cut-vertex graph  $G = (V, E)$  of any connected graph  $G' = (V', E')$  is a tree.*

*Proof. Acyclic*

Let's assume for the sake of contradiction that  $G$  has a cycle  $C = (b_1 b_2 \dots b_1)$ . Let's denote the subgraphs  $B_1, B_2, \dots, B_n$  of  $G'$  which are the 2-connected components and bridges corresponding to the nodes  $b_1, b_2, \dots, b_n$  of  $G$ . Let  $B_1$  and  $B_2$  be as stated above two different subgraphs of  $G'$ . Because the corresponding nodes  $b_1$  and  $b_2$  are adjacent in  $G$ ,  $B_1$  and  $B_2$  have to share a vertex  $x \in V(B_1) \cap V(B_2)$ . We can use the same argument for each pair  $B_i, B_{i+1}$ . Additionally, we know that each component  $B_j$  is either 2-connected or a bridge. Thus we can find a circle through all the components  $B_1, B_2, \dots, B_n$  which is, like every cycle, 2-connected. This is a contradiction to  $B_1, B_2, \dots, B_n$  being the blocks of an block-cut-vertex graph, because by definition these blocks are either bridges or maximal 2-connected components. In this case though, we found a larger two connected component. Thus  $G$  has to be acyclic.

*Connected*

Let's assume for the sake of contradiction that  $G$  is not connected (but  $G'$  is connected). If  $G$  is not connected we have at least two not connected components in  $G$ . Because  $G$  is the block-cut-vertex graph of  $G'$  each node in  $G'$  is represented by at least one component of  $G$  because each vertex in  $G'$  is either part of a 2-connected component or is incident to a bridge. Additionally because  $G'$  is connected, there is at least one edge  $e = (uv) \in G'$  with  $u$  and  $v$  being in a 2-connected component or bridge which are in  $G$  represented in different not connected components. Now there are two cases:

Either  $u$  and  $v$  are connected by an additional path, not using  $e$ , then  $u$  and  $v$  would be 2-connected and would lie in the same 2-connected component. Or there is no additional path connecting  $u$  to  $v$ , hence  $e$  is a bridge. Either way,  $u$  and  $v$  would be represented in  $G$  by one and the same node which is a direct contradiction with  $u$  and  $v$  being represented in different components in  $G$ . Thus  $G$  has to be connected if  $G'$  is connected.  $\square$

## Problem 16