Problem 21

Lemma 1.0.1. *In every planar triangulation G on atleast four vertices, there exists a vertex v which does not lie on the outer bound of G*

Proof. For the sake of contradiction let's assume there is no such vertex v. Then all the vertices lie on the outer bound of G forming a cycle with atleast four vertices. This is a contradiction with G being a planar triangulation, because we can add a edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be atleast one vertex v not on the outer bound of G. □

Theorem 1.1. *Every planar triangulation G on atleast four vertices contains a vertex whose neighbourhood induces a cycle.*

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G. Let $N(v) = p_1, p_2, ..., p_n$ be the neighbourhood of v. $N(v)$ induces a cycle if all p_i are connected to a cycle and if there are no additionally edges between p_i and p_j with $|i - j| > 1$. We first will proof that all p_i form a cycle, hence the induced subgraph $N(v)$ has a cycle as a subgraph. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G, or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v,p_i,p_{i+1} and atleast one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence G is no triangulation. Now we will proof that either $N(v)$ has no additional edges and thus is a cycle or that one of the vertices adjacent to v induces a cycle.

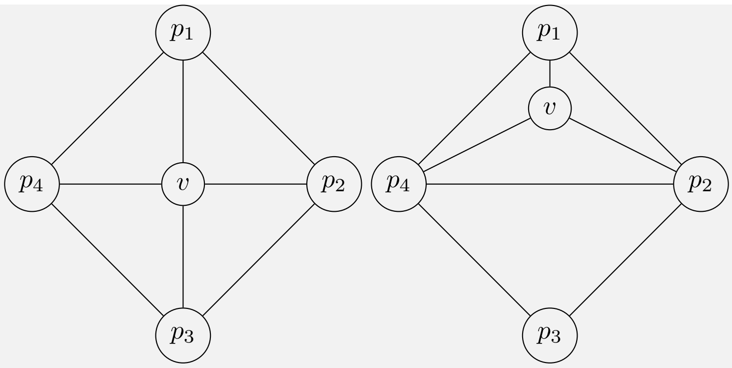
- *There are no additional edges connecting two vertices p_i and p_j with $|i - j| > 1$.*
  In this case $N(v)$ is a cycle and we found a vertex whose neighbourhood induces a cycle.
- *There is an edge $p_i p_{i+2}$*
  In this case the neighbourhood induces graph of p_{i+1} is a cycle namely $p_i p_{i+1} v$.
- *There is an edge between to vertices p_i and p_j with $|i - j| > 2$ and without loss of generality $i < v$.*
  This case there is a bounded face in G which is not a triangle. This face is the face bounded by at least the edges $p_i p_{i+1}$, $p_j p_{j-1}$, $p_i p_j$ and atleast one or more edges forming a path from p_{i+1} to p_{j-1}. Hence this case can never occur in a planar triangulation.

In summary we now that we can always find a vertex v whose neighbourhood induces a cycle or one of the neighbours of v here called p_{i+1} has a neighbourhood inducing a cycle. □

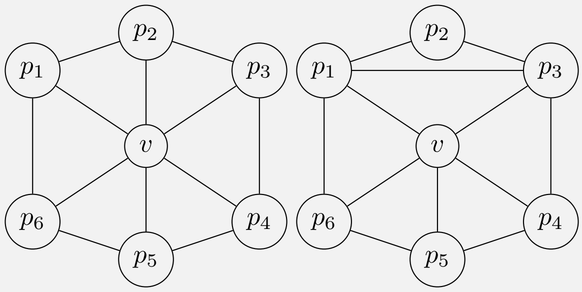
Lemma 1.1.1. *Every planar graph G with a maximal number of triangles and atleast 4 vertices has a vertex of degree 3.*

Proof. Using theorem 1.1 we can find a vertex v whose neighbourhood induces a cycle $p_1 p_2 ... p_n$. In the following we will show by induction over the degree of v that if G has a maximal number of triangles, v cannot have a degree greater than 3.

Basis: v has a degree of 4 If v has a degree 4, then the subgraph with the nodes v, p_1, p_2, p_3, p_4 only has 4 triangles. We can change the edge set of this subgraph to get a subgraph with the same vertices and the same outer structure but with containing 6 triangles. We can achieve this by removing edge vp_3 and adding the edge $p_2 p_4$. Thus we know that if G has a maximal number of triangles v cannot have degree 4.



Induction step Let the degree of v be $n > 3$. We now can again change the edges set without changing the outer structure of the induced subgraph $N(v)$ so that v has degree $n-1$ and the number of triangles in $N(v)$ stays the same. We can achieve this by removing the edge vp_2 and adding the edge p_1p_3. By induction we can repeat this operation until $n = 4$ and have the base case. Thus we can increase the number of triangles in the subgraph without changing the vertex count or the outer structure. Hence, the degree of v cannot be greater than 3 if G has a maximal number of triangles.



$\square$

Theorem 1.2. *Every n-vertex planar graph has at most $3n-8$ triangles.*

Proof. Let G be a planar graph with a maximal number of triangles. In the following we will show by induction that a planar graph G with atleast 3 vertices with a maximum number of triangles has exactly $3n-8$ triangles.

Base: $n = 3$ The planar graph with maximum number of triangles is $K3$ which has one triangle. $3n - 8 = 3 * 3 - 8 = 1$ so for the base case the formula is correct.

Induction Step Let G be such a graph with n nodes. By using lemma 1.1.1 we can find a vertex v with degree 3 whose neighbourhood induces a cycle. If we remove this node we decrease the number of nodes by one and the number of triangles by 3. By induction we know that the resulting graph has no more than $3*(n-1)-8$ triangles. Hence, we know that G has no more than $3*(n-1)-8+3 = 3*n-8$ triangles. $\square$

Problem 22

Theorem 2.1. *Any TK_3-free graph G on n vertices contains a maximum of $n-1$ edges.*

Proof. First, K_3 is the triangle C_3. Subdividing any edge of C_i results in C_{i+1}. Moreover, any cycle has a $TC_3 = TK_3$.

Hence, a graph G is TK_3-free if and only if it is acyclic. Further we assume that G is connected (since joining two disjoint acyclic components will not create a cycle but increase the edge count).

From these considerations, the maximum number of edges of an n-vertex, TK_3-free graph equals the maximum number of edges in an n-vertex tree. Any n-vertex tree contains a maximum of $n-1$ edges. □

Theorem 2.2. *If a graph G is 3-connected then $TK_4 \subseteq G$.*

Proof. By Tutte (1961), any 3-connected graph has a construction sequence $G_0, G_1, ..., G_n$ whereby $G_0 = K_4$ and $G_n = G$.

For any $i < n$, $G_i = (V_i, E_i)$ can be constructed by contracting an edge $e = \{x, y\}$ of G_{i+1} ($x, y \in V_{i+1}$, $d(x), d(y) \geq 3$).

Since $d(y) \geq 3$ and contracting e results in G_i, we can effectively say that there is a third vertex z in G_{i+1} for which $\{\{x, y\}, \{y, z\}\} \subset E_{i+1}$ is a subdivision of $\{x, z\} \in E_i$.

Thus, G_{i+1} has a TG_i and inductively, by the transitivity of topological minority, G_{i+1} has a $TG_0 = TK_4$. □

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Problem 24

Theorem 3.1. *Every outerplanar graph is 3-colorable.*

Proof. Let $G = (V_G, E_G)$ be a outerplanar graph and $G_1, ..., G_k$ the components of G. If $\forall i \in \{1, .., k\}$ G_i is 3-colorable, then G itself is 3-colorable. Thus, because $\forall i \in \{1, .., k\}$ G_i is outerplanar, we have to proof, that a connected, outerplanar graph is 3-colorable.

Let $H = (V_H, E_H)$ be a connected, outerplanar graph.
If $|V_H| <= 2$, H can be colored with 3 colors by giving each vertex a different color.
Else, if $|V_H| > 2$, it can be proofed by induction:

- *Base:* $|V_H| = 3$
  Because H contains exactly three vertices, H can be simply colored by giving each vertex a different color. So H is 3-colorable.
- *Step:* $|V_H| > 3$
  Let E_N be the edges of H which are not in the border of the unbounded face. Then two cases have to be considered:
  - $|E_N| = 0$
    In this case, all edges are in the border of the unbounded face. Hence, H is either a circle, thus, is 3-colorable, or H contains a cut vertex. If H contains a cut vertex v, $H - \{v\}$ consists of at least two connected components. These components merged with $\{v\}$ and the related edges in H, named $C_1, ..., C_n$ ($V = C_1 \cup ... \cup C_n$), are induced subgraphs of H, hence, they are outerplanar. So, by induction, $\forall i \in \{1, ..., n\}$ C_i is 3-colorable. Because colors can be renamed, the color of v can be set and still each C_i is 3-colorable. Moreover, there is no edge between two components of H, because all components of $H - \{v\}$ are connected and maximal.
    By these conditions, H is 3-colorable.
  - $|E_N| > 0$
    In this case, there is at least one edge not laying in the border of the unbounded face, hence, laying in another face.
    Let $e = (u, v) \in E_N$. Because H is outerplanar, e cuts H into two connected parts H_1 and H_2 ($V(H_1) \cap V(H_2) = \{u, v\}$ and $E(H_1) \cap E(H_2) = \{e\}$). Assume there is an edge e_2 between $H_1 \setminus \{u, v\}$ and $H_2 \setminus \{u, v\}$. Then either e_2 have to cross e, or e_2 cut w.l.o.g u from the border of the unbounded face. A contradiction in both cases, hence, there is no such edge, and $H_1 \cup H_2 = H$.
    As before, H_1 and H_2 are 3-colorable by induction (both are induced subgraphs of H). So H_1 and H_2 can still be colored with 3 colors after setting the color of u and v ($c(u) \neq c(v)$).
    By these conditions (all vertices were colored with 3 colors), H is 3-colorable.

All in all, a connected outerplanar graph is 3-colorable.
Hence, because a outerplanar graph consists of connected, outerplanar components, each outerplanar graph is 3-colorable.

□