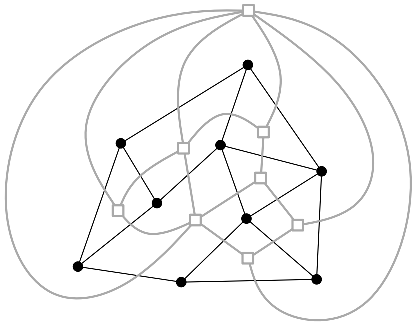
Solution sheet 5

Date: November 28. Discussion of solutions: November 30.

Let G be a plane embedded graph. Further assume that G is 3-connected. Then we define the *plane dual of* G, denoted by G^*, as the graph whose

- vertices correspond to the faces of G, and
- edges correspond to pairs of faces that share an edge of G.



A plane graph (black) and its plane dual (gray).

Problem 17. **5 points**

In a planar triangulation let n_i be the number of vertices of degree i. Prove that

$$\sum_{i\in\mathbb{N}}(6-i)n_i = 12.$$

Solution.

Let G be a planar triangulation with n vertices. It is known from the lecture that the number m of edges in G is exactly $3n-6$.

Now the problem can be solved using the vertex count $n = \sum_{i\in\mathbb{N}} n_i$, the edge count $2m = \sum_{i\in\mathbb{N}} i \cdot n_i$ and the equation $m = 3n-6$ above:

$$\sum_{i\in\mathbb{N}}(6-i)n_i = 6\sum_{i\in\mathbb{N}} n_i - \sum_{i\in\mathbb{N}} i\cdot n_i = 6n - 2m = 6n - 2(3n-6) = 12.$$

$\square$

Problem 18. **5 points**

Prove or disprove each of the following for every natural number n.

(a) There exists a connected 4-regular planar graph with at least n vertices.

(b) There exists a connected 5-regular planar graph with at least n vertices.

(c) There exists a connected 6-regular planar graph with at least n vertices.

Justify your answers!

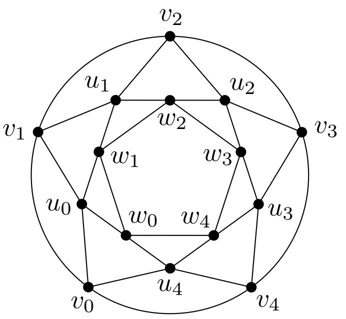
Solution.

We shall construct two infinite families of 4-regular planar graphs and one infinite family of 5-regular planar graphs. Since for every natural number n there is only a finite number of graphs with less than n vertices, the existence of these infinite families proves for every natural number n the existence of a 4-regular planar graph and a 5-regular planar graph on at least n vertices.

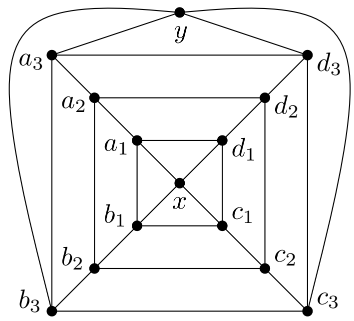
Furthermore, we shall argue that there is no 6-regular planar graph.

(a) **First construction.**

For every natural number $k \geq 3$ construct the graph $G_k^{(1)}$ as the disjoint union of two cycles $(v_0, \dots, v_{k-1})$ and $(w_0, \dots, w_{k-1})$ of length k each, and k additional vertices $\{u_0, \dots, u_{k-1}\}$ with edges between u_i and v_i, w_i, v_{i+1} and w_{i+1} for $i = 0, \dots, k-1$, where all indices are considered modulo k.



The graph $G_5^{(1)}$

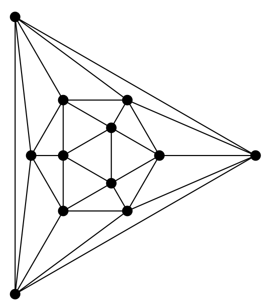


The graph $G_3^{(2)}$

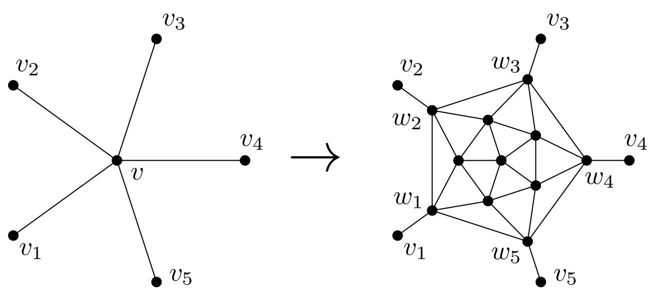
Second construction.

For every natural number $k \geq 1$ construct the graph $G_k^{(2)}$ as the disjoint union of k cycles $\{(a_i, b_i, c_i, d_i)\}_{i=1,\dots,k}$ of length 4 each with edges $a_i a_{i+1}$, $b_i b_{i+1}$, $c_i c_{i+1}$, $d_i d_{i+1}$ for $i = 1, \dots, k-1$, and two vertices x and y with edges xa_1, xb_1, xc_1, xd_1, and ya_k, yb_k, yc_k, yd_k.

(b) An infinite class of 5-regular planar graphs can be constructed recursively. We start with G_1 being the icosahedron graph I and then define G_k for $k \geq 2$ recursively as follows. Consider the 5-regular planar graph G_{k-1} and the icosahedron graph I, a vertex v in G_{k-1} and a vertex w in I. Let $v_1, \dots, v_5$ be the five neighbors of v in G_{k-1} in clockwise order and $w_1, \dots, w_5$ be the five neighbors of w in I in clockwise order. Then define G_k to be the disjoint union of $G_{k-1} - v$ and $I - w$, together with the edges $v_i w_i$ for $i = 1, \dots, 5$.



The icosahedron graph I



Replacing v by $I - w$

(c) Since every planar graph G on n vertices has at most $3n - 6$ edges, the average degree of G is at most $\frac{6(n-12)}{n}$, which is strictly less than 6. Since every 6-regular graph clearly has average degree 6, there is no 6-regular planar graph.

□

Problem 19. **5 points**

Prove that if a plane embedded graph on n vertices has no triangular face, then it has at most $2n-4$ edges.

Solution.

Let $G=(V,E)$ be any plane embedded graph with no triangular face. Let F be the set of faces in G and $\deg(f)$ be the degree of a face $f \in F$, i.e., the number of "sides of edges" bounding f. Then, since every edge has two sides and every side bounds exactly one face we have

$$\sum_{f\in F}\deg(f)=2|E|.$$

On the other hand we have $\deg(f)\geq 4$ for every $f\in F$ by assumption, and $|V|-|E|+|F|=2$ by Euler's formula. Putting things together we obtain

$$\begin{aligned}|E|-(2|V|-4)=2|F|-|E|&=\frac{4|F|}{2}-|E|\\&\leq\frac{\sum_{f\in F}\deg(f)}{2}-|E|=\frac{2|E|}{2}-|E|=0,\end{aligned}$$

and thus $|E|\leq 2|V|-4$. $\square$

Problem 20. **5 points**

Let G be a plane embedded graph on n vertices such that $G^*\cong G$. Prove that G has $2(n-1)$ edges. For all $n\geq 4$ find such a graph.

Solution.

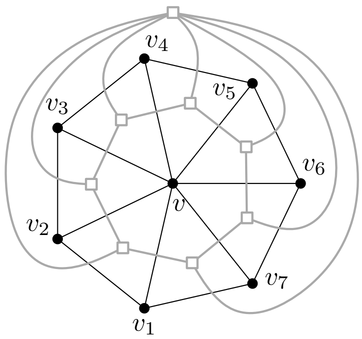
Let G be such a graph. From $G=G^*$ we immediately get that $|V(G)|=|F(G)|$ and hence by Euler's formula

$$|E(G)|=|V(G)|+|F(G)|+2=2|V(G)|+2.$$

For the second part, let $n\geq 4$. We consider the graph $G=(V,E)$ that is the $(n-1)$-wheel, i.e.,

- $V=\{v,v_1,\ldots,v_{n-1}\}$ and
- $E=\{vv_i \mid i=1,\ldots,n-1\}\cup\{v_iv_{i+1}\mid i=1,\ldots,n-2\}\cup\{v_1v_{n-1}\}$.

Then G has $n-1$ triangular faces whose incidences form a $(n-1)$-cycle, and one face of degree $n-1$ that is adjacent to all triangular faces. Thus $G^*\cong G$.



The 7-wheel and its plane dual.

Remark: In case $n = 4k + 1$ *for some natural number* $k \geq 1$*, we could also take the graph* $G_k^{(2)} - y$ *from the second construction for Problem 18 (a).* $\square$

Open Problem.

Without using the 4-Color-Theorem, prove that every n-vertex planar graph has an independent set of size at least $n/4$.