Solution sheet 4

Date: November 21. Discussion of solutions: November 22.

Problem 13. **5 points**

Prove that if a graph has an ear-decomposition, then it is 2-connected.

Solution.

Let G be a graph that admits an ear-decomposition, i.e., G can be constructed from a cycle be successively adding H-paths to graphs H already constructed. We claim that every intermediate graph H is 2-connected. Indeed, we shall show that G has no cut vertex.

This clearly holds in the very beginning, where H is a cycle. Now let H be a 2-connected graph and P be an H-path. Let $H' = H + P$ be the next intermediate graph and let v be any vertex of H'. If v was already a vertex in H then v is no cut vertex in H and thus no cut vertex in H' either. On the other hand, if v lies on P, then we consider $H' - v = H + (P - v)$. This graph consists of H, which is connected, and up to two paths (the connected components of $P - v$) each of which has an endpoint in H. In particular, $H' - v$ is connected and hence v is no cut vertex of H'.

We have shown that no intermediate graph in the ear-decomposition has a cut vertex and thus is 2-connected. Since this also holds for the last graph, which is G itself, this concludes the proof. $\square$

Problem 14. **5 points**

For all natural numbers ℓ, m, d with $0 < \ell \leq m \leq d$ construct a graph with minimum degree d that is ℓ-connected (and not $(\ell+1)$-connected) and m-edge-connected (and not $(m+1)$-edge-connected).

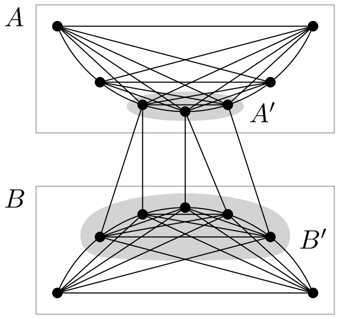
Justify your answer.

Solution.

First we deal with the case that $\ell = m = d$. Here it suffices to consider the complete graph $G = K_{d+1}$ on $d+1$ vertices. Clearly, G has minimum degree d and the removal of any $d-1$ vertices or $d-1$ edges does not disconnect the graph. Hence G is d-connected and d-edge-connected.

Moreover, G is not $(d+1)$-connected because G has order $d+1$, and G is not $(d+1)$-edge-connected because G has minimum degree d.

Now assume in the following that $\ell < d$. The graph G can be constructed as follows: The vertex set is the disjoint union of sets A and B with $|A| = |B| = d+1$. Let $A' \subset A$ and $B' \subset B$ with $|A'| = \ell$ and $|B'| = m$. The edges of G are such that $G[A]$ and $G[B]$ are complete induced subgraphs and further every vertex in A' is connected to at least one vertex in B', while every vertex in B' has exactly one neighbor in A', but no further edges outside B.



The graph constructed for $\ell = 3$, $m = 5$, $d = 6$.

The minimum degree of G is d because every vertex has at least d neighbors in his complete subgraph, and there is at least one vertex in $A \setminus A'$ with exactly d neighbors. (Here we use the fact that $\ell < d$.)

The graph G is not $(\ell + 1)$-connected, as removing A' disconnects $A \setminus A'$ (which is non-empty) from B. (Recall that $|A'| = \ell$.) Similarily, removing the A'-B' edges, of which there are exactly m, disconnects A from B, so G is not $(m + 1)$-edge-connected.

Now let us show that G is ℓ-connected. Let X be a vertex set with $|X| < \ell$. Because A and B are complete, both $G[A] - X$ and $G[B] - X$ remain connected subgraphs. Let $n = |A' \cap X|$. There are $\ell - n > 0$ vertices in A' that are not contained in X. Each of these has at least one neighbor in B'; of these at most $|X| - n < \ell - n$ were removed by X. So we find adjacent vertices in A' and B', not removed by X. This shows that $G - X$ is connected.

Finally, we show that G is m-edge-connected. Let Y be an edge set with $|Y| < m$. The subgraphs $G[A] - Y$ and $G[B] - Y$ stay connected. Furthermore, of the m edges between A' and B', at least one remains after removing Y. This shows that $G - Y$ is connected. $\square$

Problem 15. **5 points**

Prove that the block-cut-vertex graph of any connected graph is a tree.

Solution.

First let us prove the following claims.

Claim. Any two different blocks B, B' of G share at most one vertex.

Assume B and B' share two vertices x, y. Then the blocks cannot be bridges or isolated vertices. Pick an arbitrary vertex $v \in B \cup B'$ and, without loss of generality, let $v \neq x$. Then $B - v$ and $B' - v$ are connected, because B and B' are 2-connected subgraphs. But they are still connected via x, so $B \cup B'$ is also a 2-connected subgraph of G, in contradiction to the maximality of the blocks. This also implies that every edge of G lies in exactly one block.

Claim. A vertex v shared by two blocks B, B' is a cutvertex.

Let w, w' be neighbors of v in B and B', respectively. If $G - v$ is connected, then there is a w'-w path P avoiding v. Then $vw'Pw$ contains a B-path P', so by the ear-decomposition result, $B \cup P'$ is a 2-connected subgraph of G, in contradiction to the maximality of B.

Now let H be the block-cut-vertex graph of G. First we show that H is connected. Because every cut vertex is an element of a block, and hence adjacent to it in H, it is sufficient to show that any two blocks B, B' of G are linked in H. Let v, v' be vertices in B, B', and let $P = v_0 \dots v_n$ be a path linking them in G, $v_0 = v$ and $v_n = v'$. (Here we use that G is connected.) By our first claim every edge $v_{i-1}v_i$ lies in a unique block B_i. But note that may be $B_i = B_j$ for $i \neq j$. The sequence $Bv_0B_1v_1, \dots v_{n+1}B_nv_nB'$ alternates between blocks of G and vertices, contained in both blocks next to it. If a vertex v_i is not a cut vertex, the two blocks next to it are actually the same block, so the vertex and the block following it can be removed from the sequence. This turns it into a walk in H linking B and B'.

Now we show that H is acyclic. Assume there is a cycle $C = v_1B_1v_2B_2 \dots v_nB_nv_1$, $n > 1$, in H such that the v_i's are cut vertices in G and the B_i's are blocks of G. Moreover assume that C is a shortest such cycle. For notational convenience, let $v_{n+1} = v_1$. Let P_i be a path in B_i linking v_i with v_{i+1}.

Claim. The P_i's are pairwise disjoint.

Assume for the sake of contradiction, that an inner vertex w of P_i occurs in P_j, $j \neq i$, and without loss of generality, let $i = n$. Then $j \neq 1$ and $j \neq n-1$ as otherwise B_n would share two vertices with the block B_1 respectively B_{n-1}, which would contradict our first claim. But by our second claim, a vertex w that is shared by two blocks is a cutvertex, so $wB_j \dots B_nw$ would be a cycle in H shorter than C – a contradiction to the minimality of C.

Therefore, the concatenation of the paths $P := P_2, \dots, P_n$ is a B_1-path, so by the ear-decomposition result, $B_1 \cup P$ is a 2-connected subgraph of G, a contradiction to the maximality of B_1. □

Problem 16. **5 points**

Prove each of the following statements for any graph G.

(a) If G is 3-regular, then $\kappa'(G) = \kappa(G)$.

(b) If G is 4-regular, then $\kappa'(G) \leq \kappa(G) + 2$.

(c) If G is the d-dimensional hypercube, then $\kappa'(G) = \kappa(G) = d$.

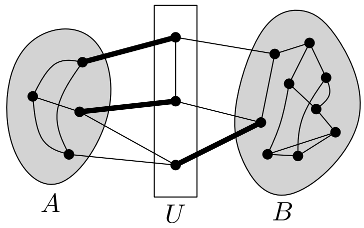
Solution.

From the lecture we know that in any graph G we have

$$\kappa(G) \leq \kappa'(G) \leq \delta(G). \tag{1}$$

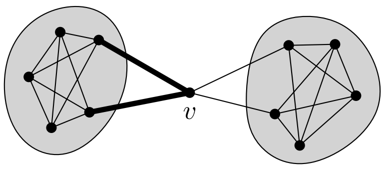
(a) By (1) it suffices to prove that if G is 3-regular, then $\kappa'(G) \leq \kappa(G)$. In particular, when G is disconnected after the removal of k vertices, then there exists a set S of k edges such that $G - S$ is also disconnected.

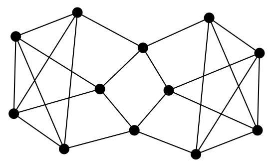
So let $G = (V, E)$ be 3-regular and U be a minimal set vertices such that $G - U$ is disconnected. Let A be a connected component of $G - U$ and let $B = V \setminus (U \cup A)$. Since every vertex $u \in U$ has degree three, this vertex has at most one neighbor in A or at most one neighbor in B. Indeed by the minimality of U every $u \in U$ has precisely one neighbor in A or precisely one neighbor in B. Consider the incident edge at u that ends in A in the former case, respectively ends in B in the latter case. Note that it might be the case that u has one edge to A *and* only one edge to B, and that in this case we choose the edge to A.



Removing the set S of these edges from G disconnects the graph. Indeed, there is no path starting in A and ending in B, as every such path would enter the set U from A and leave it towards B.

(b) Let G be any 4-regular graph. From (1) we get $\kappa'(G) \leq \delta(G) = 4$, and hence it suffices to consider the case that $\kappa(G) = 1$. In other words, we consider the case that G has a cut vertex v. Then, similarly to (a) we see that it suffices to remove at most two of the incident edge sat v in order to disconnect the graph.





A 4-regular graph G with $\kappa'(G) = \kappa(G) + 2$.

(c) Now consider Q_d, the d-dimensional hypercube, $d \geq 1$. Again using (1) we see that it suffices to prove that $\kappa'(Q_d) \geq d$. By Menger's Theorem this is equivalent to finding d vertex-disjoint paths between any two vertices in Q_d.

So let $u \neq v$ be any two vertices in Q_d. We identify the vertices of Q_d with the d-digit binary numbers and assume without loss of generality that u corresponds to $0\cdots 0$ and v corresponds to $0\cdots 01\cdots 1$. More precisely, for u we have the all-0 binary number while for v we have k leading 0's followed by $d-k$ 1's, $k > 0$.

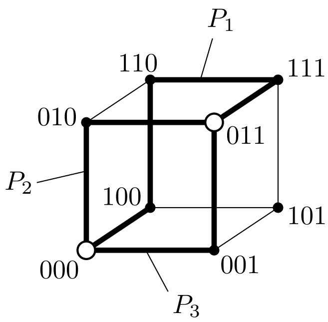
Now we define d disjoint paths $P_1, \ldots, P_d$ from u to v by specifying for each path P_i the digits that are flipped in the binary numbers and in which order they are flipped. For $i = 1, \ldots, k$ the path P_i has the flipping sequence

$$i \to k+1 \to k+2 \to k+3 \to \cdots \to d \to i.$$

For $i = k+1, \ldots, d$ the path P_i has the flipping sequence

$$i \to i-1 \to i-2 \to \cdots \to k+1 \to d \to d-1 \to \cdots \to i+1.$$

Since each P_i ($i = 1, \ldots, d$) flips the digits $k+1, \ldots, d$ exactly once and every other digit an even number of times, it is indeed a path from u to v. Moreover, it is easily seen that P_i and P_j do not share inner vertices for $i \neq j$ ($i, j \in \{1, \ldots, k\}$).



The paths P_1, P_2, P_3 in Q_3 from 000 to 011.

$\square$

Open Problem.
Let G be an $(a+b+2)$-edge-connected graph. Does there exist a partition $\{A, B\}$ of $E(G)$ so that (V, A) is a-edge-connected and (V, B) is b-edge-connected?