Problem sheet 8

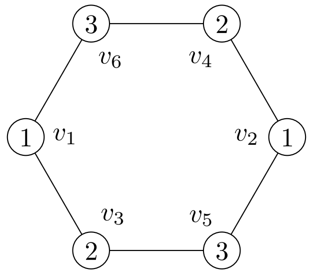
Due date: **December 18, 12:00 am.** Discussion of solutions: December 20.

(Please prepare solutions for **at most three** problems.)

A proper vertex coloring c of a graph $G = (V, E)$ is called a *greedy coloring* if there is an order $v_1, \ldots, v_n$ of V such that $c(v_1) = 1$ and for all $i = 2, \ldots, n$ we have

$$c(v_i) = \min\{c \in \mathbb{N} \mid c(v_j) \neq c \; \forall v_j \in N(v_i) \text{ with } j < i\}.$$

The *grundy number* of G, denoted by $\Gamma(G)$, is the maximum k for which there exists a greedy k-coloring of G.



A greedy 3-coloring.

Problem 29. **5 points**

(a) Determine $X = \min\{k \in \mathbb{N} \mid \Gamma(T) \leq k \text{ for all trees } T\}$.

(b) Prove for any graph G that $\Gamma(G) \leq \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$.

Problem 30. **5 points**

Show that a planar triangulation is 4-colorable if and only if its plane dual is 3-edge-colorable.

Problem 31. **5 points**

Show each of the following for any graph G.

(a) $\operatorname{tr}(A(G)) = 0$

(b) $\operatorname{tr}(A(G)^2)$ equals twice the number of edges of G

(c) $\operatorname{tr}(A(G)^3)$ equals six times the number of triangles in G

Where for any $n \times n$ matrix A, $\operatorname{tr}(A) = \sum_{i=1}^{n} A[i, i]$ denotes the *trace of* A.

Problem 32. **5 points**

Prove that if a graph G is d-regular, then d is an eigenvalue of $A(G)$, and that if G is additionally bipartite, then $-d$ is also an eigenvalue of $A(G)$.

Open Problem.

Prove or disprove that for every connected graph G and every $k \geq 1$ we have

$$t_k \geq d_k - k + 2,$$

where d_k denotes the k-th largest degree in G and t_k denotes the k-th largest eigenvalue of $D - A(G)$, with $D[i, i] = \deg(v_i)$ and $D[i, j] = 0$ for all $i \neq j$.