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Problem 1

Theorem 1.1. *Any tree T has at least $\Delta(T)$ leaves.*

Lemma 1.1.1. *In a tree $T = (V, E)$, every path that cannot be appended ends with a leaf.*

Proof. Let $v \in V$ be the last vertex of such a path.
Assume v is not a leaf then $d(v) > 1$, because v has an incoming edge and is no leaf. According to the degree of v and the fact there is no cycle in the path, we are able to increase the length of the path by inserting the non-used edge of v, a contradiction to the path's property. Therefore, v has no more edges left which are not already included in the path. In other words, v is a leaf. □

Lemma 1.1.2. *In a tree, any two vertices in a graph are joined by a unique path.*

Proof. For any two vertices within a tree, there is exactly one path joining them. Otherwise, there was a cycle which contradicted our definition of a tree. □

Proof. Let $T = (V, E)$ be a tree and $v \in V$ be a vertex with $d(v) = \Delta(T)$. We will show that there is at least one leaf in T for any vertex $v' \in V$ adjacent to v.
We simply create a path $(v, v', ...)$ that cannot be appended. As already proven in *Lemma 1.1.1*, the path has to end in a leaf. Now we have to prove that for distinct v-adjacent vertices, the paths end with different leaves. We define $(v, v', ..., l_1)$, $(v, v'', ..., l_2)$ to be such non appendable paths. Considering that the leaves l_1, l_2 were equal for $v' \neq v''$, then T would not be a tree (*Lemma 1.1.2*).
Therefore, there is at least one leaf for any edge indicent to v. Thereby, the number of leaves exceeds or is equal to $\Delta(T)$. □

Problem 2[1]

Theorem 2.1. *If any removal of an edge increases the number of connected components of a graph G, then G is acyclic.*

Proof. Let S be a connected component of G. If S contained any cycle $C = (v_0, ..., v_i, v_j, ..., v_0)$, then the removal of an edge $\{v_i, v_j\}$ would still leave a complete walkthrough $(v_j, ..., v_0, ..., v_i)$ of S and therefore maintain the component's connectivity. But - as our preconditions state - the removal of any edge increases the number of connected components (disconnects a component).
Thus, a component of G does not contain any cycles. Considering that none of the graph's connected components contains a cycle, G is acyclic as well. □

Theorem 2.2. *If adding any edge introduces a cycle in an acyclic graph $G = (V, E)$, then any two vertices in G are joined by a unique path.*

Proof. If adding an edge $\{v_0, v_1\}$ joining two non-adjacent vertices $v_0, v_1 \in V$ introduces a cycle $(v_0, ..., v_1, v_0)$, then there had to be *at least* one path from v_0 to v_1.
Furthermore, if there was *more than one* path joining v_0 and v_1, then there would have already been a cycle (but G is acyclic). $\Rightarrow$ Any vertex had to be joined by a unique path. □

Theorem 2.3. *If any two vertices in a graph are joined by a unique path, then any removal of an edge increases the number of connected components.*

Proof. Let $G = (V, E)$ be a graph in which all vertices are joined by a unique path. Let $e = \{v_0, v_1\} \in E$ be an edge. Thus, the unique path from v_0 to v_1 runs over (and is exactly) e. From these considerations, removing e would make v_1 inaccessible from v_0 and would thereby increase the number of connected components. □

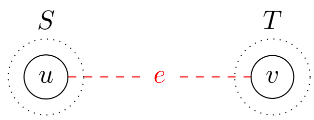
[1] Bonus

Problem 3

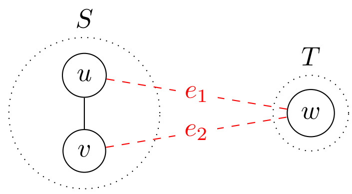
Theorem 3.1. *Either a graph or its complement is connected.*

Proof. For a *connected* graph, we're done.
Let $G = (V, E)$ be disconnected.
Claim: Any two vertices $u, v \in V$ are connected in $\bar{G}$.
Proof: There are only two cases to distinguish, either u and v lie in the same component or in different components.

- **Case 1:** u and v are in different components $S = (V_S, E_S)$ and $T = (V_T, E_S)$ with $u \in V_S$ and $v \in V_T$.
  Then G does not contain the edge $e = \{u, v\}$. Otherwise, S and T were interconnected. From these considerations, the graph's complement $\bar{G}$ does contain e.



- **Case 2:** u and v are in the same component $S = (V_S, V_T)$, $u, v \in V_S$:
  G is disconnected, hence there is atleast another not empty component $T = (V_T, E_T)$ with $S \neq T$. Considering that V_T is not empty, then there is a vertex $w \in V_T$ such that the edges $e_1 = \{u, w\}$ and $e_2 = \{v, w\}$ exist in $\bar{G}$ (see Case 1). From these considerations, $\bar{G}$ also contains the path (u, w, v) .



Therefore, any two vertices in a *disconnected graph* G are connected in $\bar{G}$ either by one or two edges and hereby $\bar{G}$ is connected.

□

Problem 4

I will prove the theorem that if u and v are the only vertices with odd degree in a graph G, then there is a path connecting u and v. Our assumption is going to be that there is no path connecting those odd-degree vertices - which we will prove to be a contradiction.

Theorem 4.1. *If u and v are the only vertices of odd degree in a graph then there is a u-v-path.*

Proof. Let $G = (V, E)$ be a graph with vertices $u, v \in V$ and let u and v be the only vertices with odd degree in G.
Assuming that there is no path connecting u and v, then u and v have to be in different components. If they were in the same component, then there had to be a path connecting u and v.
Let A be the component of u, then u is the only odd-degree vertex in A (as seen above, v is not in A). Because u is the only vertex with odd degree in A, the sum over the degree of all vertices in A is odd. However, the sum over the degrees over all vertices in a graph has to be even and this leads to the conclusion that A is no valid graph.
This is a contradiction to A being a valid component of G, hence the assumption that there is no path connecting u and v must be wrong leaving the only conclusion that there is a u-v-path. $\square$