Problem 1

Theorem 1.1. *Any tree T has at least $\Delta(T)$ leaves.*

Lemma 1.1.1. *Any connected subgraph of a tree is a tree as well.*

Proof. If a graph $G = (V, E)$ is acyclic, then E does not contain any cyclic subset and hence there is no cyclic subgraph of G. From these considerations, any connected subgraph of G is acyclic and therefore a tree. □

Lemma 1.1.2. *Any tree T with $\Delta(T) + 1$ vertices has $\Delta(T)$ leaves ($\Delta(T) \geq 1$).*

Proof. Let $v_0 \in V_T$ denote the vertex of $T = (V_T, E_T)$ with $d(v_0) = \Delta(T)$.
For any $v, w \in V_T \setminus \{v_0\}$ ($v \neq w$):

- $d(v) \geq 1$: v_0 is adjacent to v since $|V_T| = \Delta(T) + 1$. $\Rightarrow d(v) \geq 1$
- $d(v) \leq 1$: Any edge $\{v, w\} \in E_T$ would create a cycle (v_0, v, w, v_0) which would render T invalid as a tree.

$\Rightarrow \forall v \in V_T \setminus \{v_0\} : d(v) = 1 \Leftrightarrow v$ is a leaf. Notably, $|V_T \setminus \{v_0\}| = \Delta(T)$. □

Lemma 1.1.3. *For any tree $T = (V_T, E_T)$ with $|V_T| > \Delta(T) + 1 \geq 3$, there exists a partition (S, S') of T with $\Delta(S) + \Delta(S') = \Delta(T)$.*

Proof. Let $v_0 \in V_T$ denote a vertex with $d(v_0) = \Delta(T)$. Since $|V_T| > \Delta(T) + 1$, $v0$ has at least one non-leaf adjacent vertex $v_1 \in V_T$. Now, let (S_0, S_1) denote the partition at the edge $\{v_0, v_1\}$ whereby S_0 contains v_0 and (S_1, v_1 respectively).
As seen in Lemma 1.1, S_0 and S_1 are trees.

- case $d(v_1) = 2$: In this case, a cut at the edge $\{v_0, v_1\}$ would make

□

Proof by induction. Let $T = (V_T, E_T)$ be a tree.

Basis: $\Delta(T) = 1$
Inherently, $V_T =^! \{v_1, \ldots, v_m\}$ ($v_i = v_j \Rightarrow i = j$) and $E_T =^! \{\{v_1, v_2\}, ..., \{v_{m-1}, v_m\}\}$ ($m, i, j \in \mathbb{N}$). Any modification would violate our preconditions. v_1, v_m are the only vertices with degree 1. Therefore, the number of leaves is $2 \geq \Delta(T)$.

Step: For some $n \in \mathbb{N}:$ Any $T = (V_T, E_T)$ with $\Delta(T) = n$ has at least n leaves .
Let $T' = (V_{T'}, E_{T'})$ be a tree with $\Delta(T') > n$. □

Problem 2

Theorem 2.1. *If any removal of an edge increases the number of connected components of a graph G, then G is acyclic.*

Proof. Let S be a connected component of G. If S contained any cycle $C = (v_0, ..., v_i, v_j, ..., v_0)$, then the removal of an edge $\{v_i, v_j\}$ would still leave a complete walkthrough $(v_j, ..., v_0, ..., v_i)$ of S and therefore maintain the component's connectivity. But - as our preconditions state - the removal of any edge increases the number of connected components (disconnects a component).
Thus, a component of G does not contain any cycles. Considering that none of the graph's connected components contains a cycle, G is acyclic as well. □

Theorem 2.2. *If adding any edge introduces a cycle in an acyclic graph $G = (V, E)$, then any two vertices in G are joined by a unique path.*

Proof. If adding an edge $\{v_0, v_1\}$ joining two non-adjacent vertices $v_0, v_1 \in V$ introduces a cycle $(v_0, ..., v_1, v_0)$, then there had to be *at least* one path from v_0 to v_1.
Furthermore, if there was *more than one* path joining v_0 and v_1, then there would have already been a cycle (but G is acyclic). $\Rightarrow$ Any vertex had to be joined by a unique path. □

Theorem 2.3. *If any two vertices in a graph are joined by a unique path, then any removal of an edge increases the number of connected components.*

Proof. Let $G = (V, E)$ be a graph in which all vertices are joined by a unique path. Let $e = \{v_0, v_1\} \in E$ be an edge. Thus, the unique path from v_0 to v_1 runs over (and is exactly) e. From these considerations, removing e would make v_1 inaccessible from v_0 and would thereby increase the number of connected components. □

Problem 3

Theorem 3.1. *Either a graph or its complement is connected.*

Proof. For a *connected* graph, we're done.

Let $G = (V, E)$ be disconnected.
Claim: Any two vertices $u, v \in V$ are connected in $\bar{G}$.
Proof: There are only two cases to distinguish, either u and v lie in the same component or in different components.

- **Case 1:** u and v are in different components $S = (V_S, E_S)$ and $T = (V_T, E_S)$ with $u \in V_S$ and $v \in V_T$.
  Then G does not contain the edge $e = \{u, v\}$. Otherwise, S and T were interconnected. From these considerations, the graph's complement $\bar{G}$ does contain e.

- **Case 2:** u and v are in the same component V_S, $u, v \in V_S$:
  G is disconnected, hence there is atleast another not empty component V_T with $V_T \neq V_S$. Considering that V_T is not empty, then there is a vertex $w \in V_T$ such that the edges $\{u, w\}$ and $\{v, w\}$ exist in $\bar{G}$ (see Case 1). From these considerations, $\bar{G}$ also contains the path (u, w, v) .

Therefore, any two vertices in a *disconnected graph* G are connected in $\bar{G}$ either by one or two edges and hereby $\bar{G}$ is connected.

$\square$

Problem 4

I will prove the theorem that if u and v are the only vertices with odd degree in a graph G, then there is a path connecting u and v. Our assumption is going to be that there is no path connecting those odd-degree vertices - which we will prove to be a contradiction.

Theorem 4.1. *If u and v are the only vertices of odd degree in a graph then there is a u-v-path.*

Proof. Let $G = (V, E)$ be a graph with vertices $u, v \in V$ and let u and v be the only vertices with odd degree in G.
Assuming that there is no path connecting u and v, then u and v have to be in different components. If they were in the same component, then there had to be a path connecting u and v.
Let A be the component of u, then u is the only odd-degree vertex in A (as seen above, v is not in A). Because u is the only vertex with odd degree in A, the sum over the degree of all vertices in A is odd. However, the sum over the degrees over all vertices in a graph has to be even and this leads to the conclusion that A is no valid graph.
This is a contradiction to A being a valid component of G, hence the assumption that there is no path connecting u and v must be wrong leaving the only conclusion that there is a u-v-path. □