

## Problem 21

**Lemma 1.0.1.** *In every planar triangulation  $G$  on at least four vertices, there exists a vertex  $v$  which does not lie on the outer bound of  $G$*

*Proof.* For the sake of contradiction let's assume there is no such vertex  $v$ . Then all the vertices lie on the outer bound of  $G$  forming a cycle with at least four vertices. This is a contradiction with  $G$  being a planar triangulation, because we can add an edge between two not adjacent vertices to  $G$  and the result is still planar. Hence, there has to be at least one vertex  $v$  not on the outer bound of  $G$ .  $\square$

**Theorem 1.1.** *Every planar triangulation  $G$  on at least four vertices contains a vertex whose neighbourhood induces a cycle.*

*Proof.* After lemma 1.0.1 there exists a vertex  $v$  which does not lie on the outer bound of  $G$ . Let  $N(v) = p_1, p_2, \dots, p_n$  be the neighbourhood of  $v$ .  $N(v)$  induces a cycle if all  $p_i$  are connected to a cycle and if there are no additional edges between  $p_i$  and  $p_j$  with  $|i - j| > 1$ . We first will proof that all  $p_i$  form a cycle, hence the induced subgraph  $N(v)$  has a cycle as a subgraph. For the sake of contradiction let's assume this is not the case, hence there is  $p_i$  and  $p_{i+1}$  which are not connected. This is either a contradiction with  $v$  not lying on the outer bound of  $G$ , or with  $G$  being a maximal planar graph. Because if  $v_i$  and  $v_{i+1}$  are not connected and  $v$  is not on the outer bound of  $G$  there is a face bounded by  $v, p_i, p_{i+1}$  and at least one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence  $G$  is no triangulation. Now we will proof that either  $N(v)$  has no additional edges and thus is a cycle or that one of the vertices adjacent to  $v$  induces a cycle.

- *There are no additional edges connecting two vertices  $p_i$  and  $p_j$  with  $|i - j| > 1$ .*  
In this case  $N(v)$  is a cycle and we found a vertex whose neighbourhood induces a cycle.
- *There is an edge  $p_i p_{i+2}$*   
In this case the neighbourhood induces graph of  $p_{i+1}$  is a cycle namely  $p_i p_{i+1} v$ .
- *There is an edge between two vertices  $p_i$  and  $p_j$  with  $|i - j| > 2$  and without loss of generality  $i < v$ .*  
This case there is a bounded face in  $G$  which is not a triangle. This face is the face bounded by at least the edges  $p_i p_{i+1}, p_j p_{j-1}, p_i p_j$  and at least one or more edges forming a path from  $p_{i+1}$  to  $p_{j-1}$ . Hence this case can never occur in a planar triangulation.

In summary we now that we can always find a vertex  $v$  whose neighbourhood induces a cycle or one of the neighbours of  $v$  here called  $p_{i+1}$  has a neighbourhood inducing a cycle.  $\square$

**Lemma 1.1.1.** *Any planar triangulation  $G$  has at most  $2n - 4$  faces and  $2n - 5$  triangles*

*Proof.* Every inner face of  $G$  is a triangle, hence every face is adjacent to at least 3 edges. Additionally, every edge is adjacent to exactly two faces, hence  $2e \geq 3 * f \Leftrightarrow e \geq \frac{3}{2}f$  ( $e$ : number of edges,  $f$ : number of faces). By Euler's formula we know  $2 = n - e + f \Leftrightarrow e - f = n - 2 \Rightarrow \frac{1}{2}f \leq n - 2 \Leftrightarrow f \leq 2n - 4$ . Now we know that  $G$  has at most  $2n - 4$  faces and because the unbounded face cannot be a triangle we know that  $G$  has at most  $2n - 5$  triangles.  $\square$

**Theorem 1.2.** *Every  $n$ -vertex planar graph has at most  $3n - 8$  triangles.*

*Proof.* A planar graph which  $G$  is maximal in the number of triangles has to be a triangulation, because if there would be a inner face which is adjacent to more than 3 edges, we could divide the face into triangles thus forming a graph with same number of vertices but more triangles. Now we can apply lemma 1.1.1 and we know that  $G$  has at most  $2n - 5$  triangles. Now we know that  $2n - 5 \leq 3n - 8$  for every  $n > 2$ , hence we have proven an even stronger theorem.  $\square$

## Problem 22