Solution sheet 3

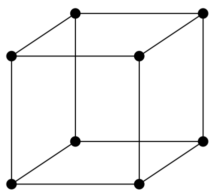
Date: November 14. Discussion of solutions: November 15.

Let $G = (V, E)$ be a graph. Then we define:

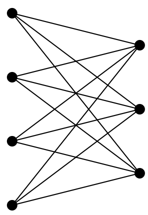
- The *diameter of G* is the length of a longest shortest path in G (counted by number of edges). That is, if $\text{dist}(u, v)$ denotes the distance of two vertices u and v in G, then the diameter of G is given by $\max_{u,v \in V} \text{dist}(u, v)$. The diameter of a disconnected graph is defined to be infinity.
- The graph G is called *Hamiltonian* if there exists a cycle in G that contains all vertices of G. Such a cycle is then called a *Hamiltonian cycle*.
- The *girth of G* is the length of a shortest cycle in G. The girth of an acyclic graph is defined to be infinity.
- The *chromatic index of G*, denoted by $\chi'(G)$, is the minimum number of colors needed to color the edges of G so that no two edges of the same color have a common endpoint. Such colorings are also called *proper edge-colorings of G*.

Problem 9. **5 points**

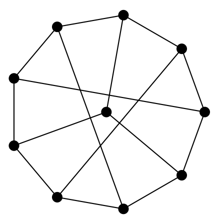
Find order, size, girth and diameter of the following graphs and figure out whether they are Hamiltonian: Q_n for all integer $n \geq 1$, $K_{m,n}$ for all integer $m, n \geq 1$, Petersen's graph. Justify your answer.



hypercube Q_3



complete bipartite graph $K_{4,3}$



Petersen graph

Solution.

We go through the three graphs/graph classes one by one.

The n-dimensional hypercube Q_n: Since the vertices of Q_n are in bijection with the n-digit binary numbers, we have

$$|V(Q_n)| = 2^n.$$

Moreover, every edge of Q_n corresponds one-to-one to two binary numbers that differ in exactly one position. Hence, Q_n is n-regular, which implies

$$|E(Q_n)| = \frac{1}{2} \sum_{v \in V(Q_n)} d(v) = \frac{1}{2} \cdot 2^n \cdot n = n2^{n-1}.$$

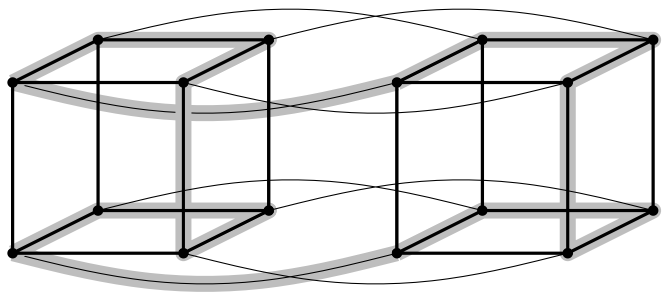
Further note that Q_n is bipartite; the bipartition classes correspond to those binary numbers with an even, respectively odd, number of 1's. Thus the girth of Q_n is at least 4. Because Q_n contains the 4-cycle $100\cdots 0 \to 000\cdots 0 \to 010\cdots 0 \to 110\cdots 0 \to 100\cdots 0$, provided $n \geq 2$ we conclude

the girth of Q_n is 4, if $n \geq 2$ and ∞, otherwise.

For two vertices in Q_n the length of a shortest path between these two vertices is given by the number of digits in which the two corresponding numbers differ. Indeed every path is at least that long because going along an edge changes only one digit, and in order to find a path of that length it suffices to flip the digits in which the binary numbers differ one by one. Considering for example the vertices corresponding to the all-0-number and the all-1-number we see that

the diameter of Q_n is n.

Finally, we claim that Q_n is Hamiltonian whenever $n \geq 2$. We shall prove this claim by induction on n. For $n = 2$ the graph Q_n is a 4-cycle and hence Hamiltonian. So let $n \geq 3$. We consider Q_n as the union of a perfect matching M on 2^{n-1} edges which lies between two copies of Q_{n-1}. Moreover, every edge in M joins the two vertices corresponding to each other in the two copies of Q_{n-1}.



By induction hypothesis Q_{n-1} has a Hamiltonian cycle C. Let xy be a fixed edge on C. We construct a Hamiltonian cycle of Q_n as follows: Take the path $C - xy$ in both copies of Q_{n-1} and join these two paths into a Hamiltonian cycle by adding the edges of M incident to x and y.

The complete bipartite graph $K_{m,n}$: Since the vertices of $K_{m,n}$ are partitioned into two sets, a set A of cardinality m and a set B of cardinality n we have

$$|V(K_{m,n})| = |A| + |B| = m + n.$$

Moreover, the edge set of $K_{m,n}$ is given by $A \times B$, which gives

$$|E(K_{m,n})| = |A| \cdot |B| = mn.$$

Further note that $K_{m,n}$ is bipartite. Thus the girth of $K_{m,n}$ is at least 4. Because $K_{m,n}$ contains a 4-cycle $a_1 \to b_1 \to a_2 \to b_2 \to a_1$, provided there exist vertices $a_1 \neq a_2 \in A$ and $b_1 \neq b_2 \in B$ we conclude

the girth of $K_{m,n}$ is 4, if $m, n \geq 2$ and ∞, otherwise.

The shortest path between any vertex $a \in A$ and any vertex $b \in B$ consists of only the edge ab. While every shortest path between two distinct vertices $a_1, a_2 \in A$

contains two edges $a_1 \to b \to a_2$ for some $b \in B$. Similarly, every two distinct vertices in B have distance two. Thus

the diameter of $K_{m,n}$ is 2.

Finally, we claim that $K_{m,n}$ is Hamiltonian if and only if $m = n$. Since the graph is bipartite with bipartition classes A and B every cycle C must alternatingly take vertices from A and B; in particular the same number of vertices from either side. Thus if C is a Hamiltonian cycle, then $|A| = |B|$, which means $m = n$. Secondly, we shall construct a Hamiltonian cycle for $K_{m,n}$ in case $m = n$ as follows. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$. Then a Hamiltonian cycle is defined as $a_1 \to b_1 \to a_2 \to b_2 \to \cdots \to a_n \to b_n \to a_1$.

The Petersen graph: Since the vertices of the Petersen graph P are in bijection with the 2-element subsets of the 5-element set, we have

$$|V(P)| = \binom{5}{2} = 10.$$

Moreover, every edge of P corresponds one-to-one to two 2-element sets that are disjoint. Hence, P is 3-regular, which implies

$$|E(P)| = \frac{1}{2} \sum_{v \in V(P)} d(v) = \frac{1}{2} \cdot 10 \cdot 3 = 15.$$

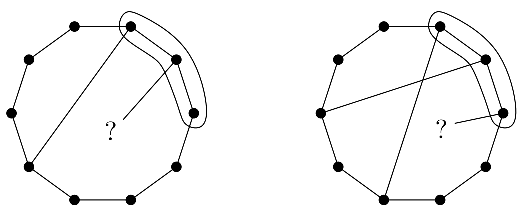
Further note that no three 2-element sets of the 5-element set can be mutually disjoint. Thus the girth of P is at least 4. Moreover, two 2-element sets cannot both be disjoint to two other 2-element sets. Thus the girth of P is actually at least 5. Considering the 5-cycle $\{1,2\} \to \{3,4\} \to \{1,5\} \to \{2,4\} \to \{3,5\} \to \{1,2\}$ we conclude

the girth of P is 5.

Now consider two distinct 2-element sets X and Y. If $X \cap Y = \emptyset$, then the corresponding vertices are adjacent, that is, at distance 1. Otherwise, X and Y have exactly one element in common and the 2-element set that is the complement of $X \cup Y$ corresponds to a neighbor of X and Y. Thus we conclude

the diameter of P is 2.

Finally, we claim that the Petersen graph is not Hamiltonian. Indeed, if P would have a Hamiltonian cycle $C = (v_1, v_2, \ldots, v_{10})$ then the edges not on C must form a perfect matching M as P is 3-regular. Moreover, as the girth of P is 5 every edge in M must connect vertices whose indices differ by at least 4 modulo 10. Moreover, two edges in M that start at vertices that are neighbors on C must not end at vertices that are neighbors on C, since otherwise we would have a 4-cycle. However, these two requirements cannot be simultaneously satisfied by three edges of M starting at three consecutive vertices on C.



□

Problem 10. **5 points**

For a natural number $n \geq 2$ let G be the graph of order $2n+1$ obtained from $K_{n,n}$ by subdividing an edge by a vertex. Show that $\chi'(G) = \Delta(G) + 1 = n + 1$ but $\chi'(G - e) = \Delta(G - e) = n$ for every edge e of G.

Solution.

First we shall find a proper edge-coloring of $K_{n,n}$ with n colors and argue that every proper edge-coloring of $K_{n,n}$ requires n colors.

Note that by Hall's theorem it is possible to find a 1-factor in any regular bipartite graph. So, we can find a 1-factor in $K_{n,n}$, assign color 1 to all its edges, delete this factor and obtain a regular graph in which again a 1-factor exists. We give the edges of the second 1-factor color 2 and proceed in this manner until no edges are left. As a result we properly colored $E(K_{n,n})$ with n colors. On the other hand $\chi'(K_{n,n}) \geq \Delta(K_{n,n}) = n$, thus $\chi'(K_{n,n}) = n$.

Now, let us consider the graph G from the claimed statement. We shall prove that $\chi'(G) \leq n + 1$. Indeed, consider a coloring of $K_{n,n}$ described above. For the subdivided edge, use an old color and a color $n + 1$ on two new resulting edges. Call this coloring c. On the other hand, if $\chi'(G) \leq n$ then there is a color class of size at least

$$\left\lceil \frac{|E(G)|}{n} \right\rceil = \left\lceil \frac{n^2+1}{n} \right\rceil = n + 1 > \frac{2n+1}{2} = \frac{|V(G)|}{2},$$

which is impossible for a matching. Furthermore, note that $\Delta(G) = \Delta(K_{n,n})$ because $n \geq 2$.

Next, consider the graph $G' = G - e$. Since the maximal degree of G' is n the chromatic index is at least n. Thus it remains to show that $\chi'(G') \leq n$. Assume without loss of generality that the subdivided edge xy had color 1, and the new edges xz, zy got colors 1 and $n + 1$, respectively. Now, treat the following cases:

Case 1: e is incident to y. Then we color the remaining edge at y with color 1 and have only n colors in total.

Case 2: e not incident to y and $c(e) = 1$. Then consider the edge f adjacent to e and incident to z; say its color is $i \neq 1$. We recolor f to have color 1. Now, neither y nor z is incident to color i. Thus we can use color i instead of color $n + 1$ on yz, reducing the total number of colors by one.

Case 3: e not incident to y and $c(e) = i \neq 1$. Then consider the subgraph H of G on edges with colors 1 or i. Every vertex in H has degree 2, except for y and z, whose degree is 1. In particular, H is the disjoint union of a y-z-path and some (possibly none) even cycles. Consider the connected component C of H that contains the edge e.

If C contains the edge xy, then we swap the colors 1 and i on the path in $C \setminus e$ that contains xy. As a result, neither y nor z is incident to color 1. Thus we can use color 1 instead of color $n + 1$ on yz, reducing the total number of colors by one.

Otherwise, C is an even cycle. Swapping the colors 1 and i on C the edge e gets color 1 and Case 2 applies.

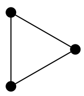
We have shown that $\chi'(G - e) = n$. Clearly, as G contains at least 4 vertices of degree $\Delta(G) = n$ (here we use again that $n \geq 2$), we have $\Delta(G - e) = n$, which concludes the proof. □

Problem 11. **5 points**
For each integer $k > 1$ construct a k-regular graph with no 1-factor. Justify your answer.

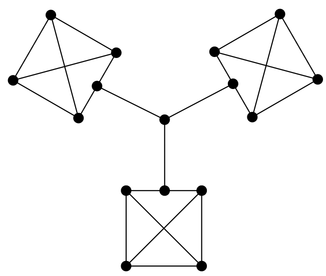
Solution.
Let us distinguish two cases.

Case 1: k is even. Clearly, it suffices to construct a k-regular graph with an odd number of vertices. Such a graph is for instance K_{k+1}, the complete graph on $k+1$ vertices.

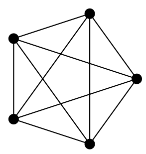
Case 2: k is odd. Here, every k-regular graph has neccessarily an even number of vertices. So we need to work a little harder to obtain a k-regular graph with no 1-factor. Our construction starts with k disjoint copies of K_{k+1}. In each copy we subdivide a single edge with a new vertex of degree 2. Call these vertices $a_1, \ldots, a_k$ and the set of these vertices A. Finally, we add a set B of $k-2$ further vertices denoted by $b_1, \ldots, b_{k-2}$ and all edges of the form $a_i b_j$ $(i = 1, \ldots, k,\ j = 1, \ldots, k-2)$ to the graph. (Here we use the fact that $k > 1$.) The resulting graph we denote by G.



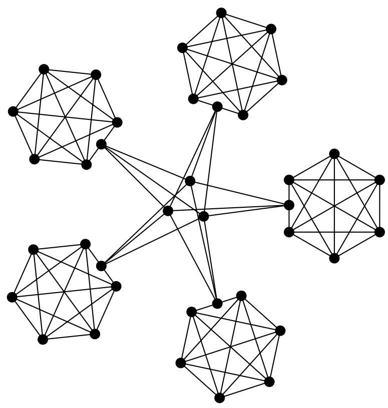
2-regular graph
with no 1-factor



3-regular graph
with no 1-factor



4-regular graph
with no 1-factor



5-regular graph
with no 1-factor

It is straightforward to see that G is k-regular. Indeed, every vertex in one of the K_{k+1}-subgraphs has degree k (even though some edge is subdivided). Moreover, each vertex a_i $(i = 1, \ldots, k)$ is adjacent to exactly two vertices in the corresponding copy of K_{k+1} and the $k-2$ vertices in B, which gives $d(a_i) = k$. Finally, each vertex b_j $(j = 1, \ldots, k-2)$ is adjacent to exactly the vertices in A, hence $d(b_j) = |A| = k$.

Now we claim that G has no 1-factor. Let us assume for the sake of contradiction that M is a 1-factor of G. Consider one of the subdivided copies C_i of K_{k+1}. If the corresponding vertex $a_i \in A$ is matched in M to a vertex of C_i, then the remaining k vertices of C_i must be matched to each other. However, as k is odd, this is impossible. So each $a_i \in A$ is matched in M to a vertex $b_j \in B$. However, this is impossible since $|A| = k > k-2 = |B|$ – a contradiction. □

Problem 12. **5 points**
Let G be a graph on $2n$ vertices with all degrees at least n. Prove that G has a 1-factor.

Solution.
Let $G = (V, E)$ be a graph on $2n$ vertices with minimum degree at least n. We shall show the following stronger claim.

Claim. There is a cycle in G spanning all vertices of G, i.e., G is Hamiltonian.

We will first show that if G contains a path on t vertices, then G also contains a cycle on t vertices. So assume for the sake of contradiction that the longest path in G has t vertices $x_1, \ldots, x_t$, but there is no cycle on t vertices in G. Then all neighbors of x_1 and x_t must lie on the path or else it is not longest. The minimum degree condition implies that both have degree at least $t/2$. But if x_1x_k is an edge in G, then x_tx_k is no edge in G or else we can re-route to get a cycle.

So, each of x_1's $t/2$ neighbors on the path prohibit a potential neighbor of x_t. Yet x_t's neighbors come from indices $1, \ldots, t-1$, so there is not enough space for x_t to have $t/2$ neighbors there, avoiding the prohibited ones. Now if this longest path is not the full $2n$ vertices, then we get a cycle C missing some vertex x. But min-degree n implies that the graph is connected (every vertex lies in a connected component of order at least $n+1$), so there is a shortest path from x to C, and adding this to the cycle gives a longer path than t – a contradiction. □

Open Problem.
Is every Eulerian graph on n vertices the edge-disjoint union of at most $\frac{1}{2}(n-1)$ cycles?