

Problem 17

Theorem 1.1. *In a planar triangulation let n_i be the number of vertices of degree i . Then,*

$$\sum_{i \in \mathbb{N}} (6 - i)n_i = 12$$

Lemma 1.1.1. *Let $G = (V, E)$ and $G' = (V \cup \{v\}, E \cup E')$ be planar triangulations. Then $|E'| = 3$ and all edges in E' are incident to v .*

Proof. Since any planar triangulation on n vertices and e_n edges satisfies $e_n = 3n - 6$, we see inductively that

$$\begin{aligned} e_n &= e_{n-1} + 3 & (n > 3) \\ e_3 &= 3 \end{aligned}$$

Since G' contains $|V| + 1$ vertices, the number of edges in G' exceeds the number of edges in G by $|E'| = 3$.

Next, we will show that the degree of v exceeds or is equal to 3 and thus, all edges of E' have to be incident to v .

By KURATOWSKI, G' is not a topological minor of $K_{3,3}$ or K_5 and any planar triangulation is edge-maximal. By Lemma 4.4.5 (any edge-maximal graph without topological minors $K_{3,3}, K_5$ is 3-connected), G' is 3-connected.

If the degree of v decreased 3, then G' would not be 3-connected (v could be isolated by removing two vertices).

Hence, all three edges of E' are incident to v . □

We will show by induction on the number of vertices n of a planar triangulation G with n_i vertices of degree i ($i \in \mathbb{N}$) that

$$T_G := \sum_{i \in \mathbb{N}} (6 - i)n_i = 12$$

- Base $n = 3$

Then, the graph is a triangle and the condition is satisfied:

$$T_{K_3} = \sum_{i \in \mathbb{N}} (6 - i)n_i = (6 - 2) \cdot 3 = 4 \cdot 3 = 12$$

- Step $n \geq 4$

Any n -vertex planar triangulation $G = (V, E)$ has a subgraph $H = (V', E')$ which is an $(n - 1)$ -vertex planar triangulation.

By Lemma 1.1.1, there is a vertex $v \in V \setminus V'$ of degree 3. Furthermore, the degree of exactly three other vertices $v_i, v_j, v_k \in V$ is increased. Thus, $E \setminus E' = \{\{v, v_i\}, \{v, v_j\}, \{v, v_k\}\}$ and for T_G :

$$\begin{aligned} T_G &= T_H \\ &\quad + (6 - 3) \\ &\quad + (6 - (d(v_i) + 1)) - (6 - d(v_i)) \\ &\quad + (6 - (d(v_j) + 1)) - (6 - d(v_j)) \\ &\quad + (6 - (d(v_k) + 1)) - (6 - d(v_k)) \\ &= T_H + 3 - 1 - 1 - 1 \\ &= T_H \\ &= 12 & \text{(by induction)} \end{aligned}$$



Problem 19

Theorem 2.1. *A plane embedded graph on n vertices that has no triangular face has at most $2n - 4$ edges.*

Lemma 2.1.1. *Each planar triangulation (order ≥ 4) can be modified to a planar graph containing only faces of order 4 by removing one edge for each two faces*

Proof. Proof by induction over the order of a plane triangulation G :

- **Base:** $V(G) = 4$

G contains of 4 faces f_1, f_2, f_3, f_4 . We get the desired graph by removing the shared edge between f_1, f_2 and f_3, f_4 . This is possible because each face is adjacent to one another. The remaining graph is, as desired, created by removing one edge for each 2 faces.

- **Step:** $V(G) = n$

Let $G' = G - \{u\}$ ($u \in V(G)$ with $\deg(u) = 3$ as shown in **Lemma 1.1.1**). Then G' is still a planar triangulation and the amount of faces in G is exactly 2 more than in G' , because by removing the vertex u , 3 edges were removed, thus, 2 triangular faces were removed.

By induction we get the desired graph H' for G' by removing one edge for each 2 faces. Moreover, by inserting u and its 3 adjacent edges in H' at their old position, 2 faces of degree 3 were created (Would there be more than 2 new faces, the face in which u was inserted would have an order of at least 5). Thus, there are only two faces of order 3 remaining. Because all inserted edges are incident to u , these two faces of order 3 are adjacent and can be merged by removing the shared edge.

The resulting graph of order n is still planar (only edges have been removed), has only faces of order 4 and was created of G by removing one edge for each 2 faces.

□

Lemma 2.1.2. *Each planar graph with no triangular face can be modified to a planar graph containing only faces of order 4 and with at least the same amount of edges*

Proof. Let G be such a planar graph with no triangular face. Let G' be the planar triangulation of G . To get G' out of G edges have to be added. Because the smallest face is of order 4, at least 1 edge for each face is needed to reduce the size of all faces, resulting in at least one additional face for each existing face in G , to be more precisely, in exactly one additional face for each inserted edge. So G' has at least twice as much faces as G and exactly one new edge for each new face. As shown in **Lemma 2.1.1**, G' can be modified to a planar graph H containing only faces of order 4 by removing one edge for each two faces. Thus, the amount of faces in H is at least as much as in G , and furthermore the amount of edges is at least as much as in G , because for each two faces exactly one edge has been removed.

All in all, H is a graph containing only faces of order 4 and has at least the same amount of edges as G .

□

Let $G = (V, E_G)$ be a planar graph with no plane triangulation.

As shown in **Lemma 2.1.2**, we get a planar graph $H = (V, E_H)$ (F_H corresponding faces, each face of order 4), containing at least the same amount of edges as G and the same set of vertices V .

By *Euler's Formula* and the fact $|E_H| = \frac{4 * |F_H|}{2}$ (because each face in F_H has order 4 and each edge is exactly counted twice), we get the following:

$$2 = |V| - |E_H| + |F_H| \leftrightarrow 2 = |V| - |E_H| + \frac{E_H}{2} \leftrightarrow |E_H| = 2 * |V| - 4.$$

This inequality allows us to bound the amount of edges as following:

$$|E_G| \leq |E_H| = 2 * n - 4 \quad (n = |V|)$$

□

Problem 20

Theorem 3.1. *For a planar embedded graph G on n vertices such that $G^* \cong G$, the number of edges $e = 2(n - 1)$.*

Proof. The faces of G correspond to the vertices of G^* . Considering their isomorphism, the number of faces f of G equals the number of vertices of G .

G is planar, by *Euler's Characteristic*: $n - e + f = 2 \Rightarrow 2n - e = 2 \Rightarrow e = 2(n - 1)$. □