Solution sheet 6

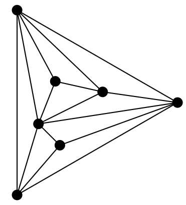
Date: December 5. Discussion of solutions: December 6.

Problem 21. **5 points**

(a) Prove that every planar triangulation on at least four vertices contains a vertex whose neighborhood induces a cycle.

(b) Prove that every n-vertex planar graph has at most $3n - 8$ triangles.

Hint: Show with (a) that there is a vertex of degree 3.



A planar graph with 7 vertices and 13 triangles.

Solution.

(a) Let $G = (V, E)$ be a planar triangulation with $n = |V| \geq 4$. We fix a plane embedding of G. Now call a triangle Δ in G a *filled triangle* if Δ is not an inner face of G. Since $n \geq 4$, the outer triangle of G is a filled triangle.

For a given filled triangle Δ let G_Δ denote the subgraph of G induced by the vertices of Δ and all vertices "inside Δ" in the plane embedding of G. Then G_Δ itself is a plane triangulation on at least four vertices, whose outer triangle is Δ, and whose filled triangles are also filled triangles of G.

Now let Δ_{min} be an inclusion-minimal filled triangle of G, that is, the "inside" of Δ_{min} should have smallest area among all filled triangles of G. Let v be any inner vertex of $G_{\Delta_{\text{min}}}$. By the minimality of Δ_{min} it follows that v is not part of any filled triangle of G.

We claim that the neighborhood $N(v)$ of v induces a cycle in G. Indeed, any edge uw between two non-consecutive neighbors of v would form a filled triangle $u - v - w$ containing v – a contradiction.

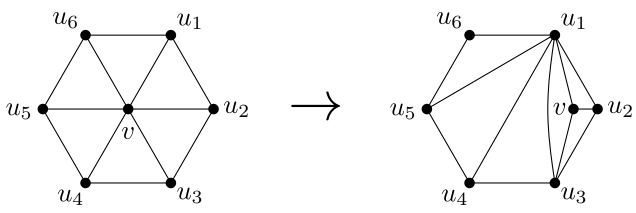
(b) Let $G = (V, E)$ be an n-vertex planar graph with the largest possible number of triangles. First consider the case that $n \leq 3$. If $n \leq 2$ then G has clearly no triangles, and if $n = 3$ then G is a triangle itself. Thus G has indeed no more then $3n - 8 \leq 3 \cdot 3 - 8 = 1$ triangles.

So let G have $n \geq 4$ vertices. Without loss of generality G has the largest possible number of edges, because adding edges to G could only introduce more triangles. Thus G is a planar triangulation on $n \geq 4$ vertices.

Claim. G contains a vertex of degree 3.

By (a) there exists a vertex v in G whose neighborhood induces a cycle. Since G is a triangulation we have $\deg(v) \geq 3$. If $\deg(v) = 3$ we are done. Otherwise, if $\deg(v) \geq 4$, and we claim that we can construct a planar n-vertex graph G' with strictly more triangles than G – a contradiction to the maximality of G.

Indeed $G[N(v) \cup v]$ has exactly $d = \deg(v)$ triangles since $d \geq 4$ and $G[N(v)]$ is a cycle. Let $u_1, \ldots, u_d$ be the cyclic order of neighbors around v. Because $G[N(v)]$ is a cycle, we have that $u_1 u_i \notin E$ for $i = 3, \ldots, d - 1$. (Note that $d - 1 \geq 3$ since $\deg(v) \geq 4$.)



$G[N(v) \cup v]$ $G'[N(v) \cup v]$

Now define G' to be the graph obtained from G by removing the edges vu_i ($i = 4, \ldots, d$) and adding the edges u_1u_i ($i = 3, \ldots, d-1$). Clearly, G' is planar. Moreover, $G'[\{u_1, \ldots, u_d, v\}]$ has $d+1$ triangles. Thus G' has strictly more triangles than G, which proves the claim.

Claim. G has at most $3n - 8$ triangles.

We prove this claim by induction on n. The base cases $n \leq 3$ are already covered above. So let $n \geq 4$. By our claim there is a vertex v of degree 3. Clearly, v is part of exactly 3 triangles. Let $G' = G - v$ be the planar graph obtained from G by removing v. By induction hypothesis (clearly $|V(G')| = n - 1$) G' has at most $3(n-1) - 8$ triangles, which are precisely the triangles of G not containing v. Together with the 3 triangles containing v we conclude that G has at most $3(n-1) - 8 + 3 = 3n - 8$ triangles.

$\square$

Problem 22. **5 points**

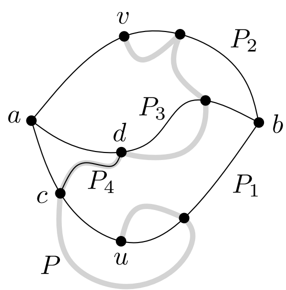
(a) Find the largest number of edges in an n-vertex TK_3-free graph.

(b) Prove that if G is 3-connected then $TK_4 \subseteq G$.

Solution.

(a) A graph is a TK_3 if and only if it is a cycle. Thus, A TK_3-free graph is an acyclic graph, so it has $n - c \leq n - 1$ edges, where c is the number of connected components. A tree on n vertices has $n-1$ edges and is acyclic, so this is the largest number.

(b) Let $a, b \in V(G)$ be two vertices, and P_1, P_2, P_3 be three independent paths between them; these exist by Menger's theorem. At most one of them can be of length 1, so without loss of generality, let u and v be vertices in P_1 and P_2, respectively, with $\{u, v\} \cap \{a, b\} = \emptyset$. Let P be a path linking u and v in $G - a - b$, which exists because G is 3-connected. Let P_4 be a shortest subpath of P such that its endpoints c and d lie on different paths in the set $\{P_1, P_2, P_3\}$. Such a segment exists because P is one candidate. Then the path P_4 is internally disjoint from the paths P_1, P_2, P_3. So we have four vertices a, b, c, d pairwise linked by independent paths; this is a TK_4.



$\square$

Problem 23. **5 points**

Use Kuratowski's theorem to prove that a graph is outerplanar if and only if it has no subdivision of K_4 or $K_{2,3}$.

Solution.
Let $G = (V, E)$ be any graph. Consider the graph $G' = (V \cup \{v^*\}, E \cup \{vv^* \mid v \in V\})$ obtained by adding a new vertex v^* to G and joining it to all vertices in G. We shall prove that G is outerplanar if and only if G' is planar. By Kuratowski's theorem it then remains to show that $TK_4 \subseteq G$ if and only if $TK_5 \subseteq G'$, and that $TK_{2,3} \subseteq G$ if and only if $TK_{3,3} \subseteq G'$.

Claim. G is outerplanar if and only if G' is planar.

If G is outerplanar, then we can draw v^* and its adjacent edges into the outer face, obtaining a planar drawing of G'. Conversely, if G' is planar, then a drawing of it, after removing v^*, is a drawing of G and the face of G where v^* was has all vertices of G on its boundary, so G is outerplanar.

Claim. $TK_4 \subseteq G$ if and only if $TK_5 \subseteq G'$.

Let H be a TK_4 in G. Because v^* is adjacent to every node in G, we can add v^* and the edges $\{vv^* \mid \deg_H(v) = 3\}$ to H to obtain a TK_5 in G'.

Conversely, let H be a TK_5 in G'. If $v^* \notin H$, then H it is already a subgraph of G and contains a TK_4. If $v^* \in H$, then $H - v^*$ is a subgraph of G and contains a subdivision of K_4.

Claim. $TK_{2,3} \subseteq G$ if and only if $TK_{3,3} \subseteq G'$.

Let H be a $TK_{2,3}$ in G. Because v^* is adjacent to every node in G, we can add v^* and the edges $\{vv^* \mid \deg_H(v) = 3\}$ to H to obtain a $TK_{3,3}$ in G'.

Conversely, let H be a $TK_{3,3}$ in G'. If $v^* \notin H$, then H it is already a subgraph of G and contains a $TK_{2,3}$. If $v^* \in H$, then $H - v^*$ is a subgraph of G and contains a subdivision of $K_{2,3}$. □

Problem 24. **5 points**
Without using the 4-Color-Theorem, prove that every outerplanar graph is 3-colorable.

Solution.
We prove this by induction on the number of vertices in G. Note that any subgraph of an outerplanar graph is again outerplanar. As induction base consider the 1-vertex graph, which is clearly 3-colorable.

So let G be outerplanar with a fixed outerplanar embedding and with $|V(G)| \geq 2$. We distinguish several cases.

Case 1. G is disconnected. Then combining any 3-colorings of its components (which exist by induction) yields a 3-coloring of G.

Case 2. G is connected but not 2-connected. Similar to the previous case consider a 3-coloring of every block of G (which exist by induction). By permuting the colors we can assume without loss of generality that any two intersecting blocks agree on the same color of the cut vertex they have in common. We remark that we use here that the block-cut-vertex graph is a tree and that every tree can be build up from a single vertex by adding leaves one at a time. Thus, all 3-colorings combined give a 3-coloring of G.

Case 3. G is 2-connected. Then every face of G is bounded by a cycle. In particular, the outer face is bounded by a cycle, which contains all vertices of G and we denote by C.

A cycle is 3-colorable, so assume there is an edge $e = uv \notin C$. Let P_1, P_2 be the two paths linking u and v on C. Both these paths have lengths at least two. Consider the induced subgraphs $G_i = G[P_i]$, $i = 1, 2$, both of which have fewer vertices than G. If we can show that $E(G) = E(G_1) \cup E(G_2)$, then we can take 3-colorings of both of them, possibly permute the colors to agree on $V(G_1) \cap V(G_2) = \{u, v\}$ and combine them to a 3-coloring of G.

So let $e' = xy \in E(G)$ and assume $e' \notin E(G_1) \cup E(G_2)$, i.e., without loss of generality $x \in P_1 \setminus P_2$ and $y \in P_2 \setminus P_1$. In particular, $x, y \notin \{u, v\}$. Because $e' \notin E(C)$, the interior of w lies in the same region of $\mathbb{R}^2 \setminus (P_1 \cup P_2)$ that contains the interior of $e = uv$. But this implies that e and e' cross, a contradiction to the planarity of G. $\square$

Open Problem.
Prove or disprove that for $n \geq 6$, the largest number of edges in an n-vertex graph with no TK_5 is $3n - 6$.