Solution sheet 7

Date: December 12. Discussion of solutions: December 13.

Problem 25. **5 points**

Let G be a graph whose odd cycles are pairwise intersecting, i.e., every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$ and find such a graph G with $\chi(G) = 5$.

Solution.

Let $G = (V, E)$ be a graph satisfying the properties in the problem statement. If G has no odd cycles, it is bipartite and $\chi(G) \leq 2 < 5$. So let C be a shortest odd cycle in G. This is actually an induced subgraph of G, because if it had a chord, the chord together with one of the two halfs of the cycle would form a smaller odd cycle.

The subgraph $G' = G[V - V(C)]$ does not contain any odd cycles, as such a cycle would not intersect C in G. So G' is bipartite and can be colored with 2 colors. The cycle C can be colored with 3 colors. These colorings can be combined to a valid 5-coloring of G.

Finally, a 5-chromatic graph in which every two odd cycles are pairwise intersecting is given by K_5. □

Problem 26. **5 points**

Prove or disprove each of the following.

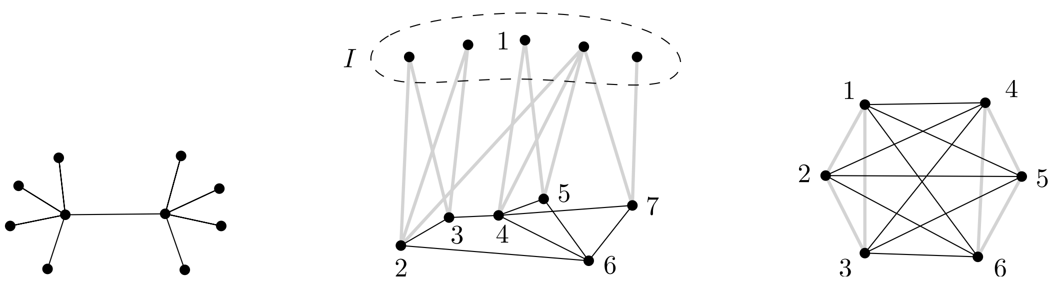
(a) Every graph G with $\chi(G) = k$ has a proper k-coloring in which one color class has size at least $\alpha(G)$.

(b) $\chi(G) \leq |V(G)| - \alpha(G) + 1$.

(c) If $G = F \cup H$, then $\chi(G) \leq \chi(F) + \chi(H)$.

Solution.

(a) **The statement is false.** Consider the graph G shown below. We have $\chi(G) = 2$, $\alpha(G) = 8$, but if we try to color the unique set of 8 independent vertices with the same color, then the remaining two vertices are adjacent, so this cannot lead to a proper 2-coloring.



(a) (b) (c)

(b) **The statement is true.** Let $I \subseteq V(G)$ be a maximum set of independet vertices, so $|I| = \alpha(G)$. Color every vertex of G that is not in I with a different color, and all vertices in I with another additional color. These are $(|V(G)| - |I|) + 1 = |V(G)| - \alpha(G) + 1$ colors and the coloring is valid: The vertices in I are independent, so no edge connects vertices of the same color.

(c) **The statement is false.** Let G be the complete graph on six vertices $\{1,\ldots,6\}$ depicted above, let F be the $K_{3,3}$ contained in G with partite sets $\{1,2,3\}$ and $\{4,5,6\}$ and let H be the union of the two triangles on $\{1,2,3\}$ and $\{4,5,6\}$. We have $G = F \cup H$, but $\chi(G) = 6$, because it is a complete graph, $\chi(F) = 2$, because it is bipartite, and $\chi(H) = 3$, because it is the disjoint union of two complete graphs.

□

Problem 27. **5 points**

Show that for every graph G on n vertices we have

$$\chi(G) + \chi(\overline{G}) \geq 2\sqrt{n}.$$

Solution.

Note that for arbitrary numbers a and b, we have $a+b \geq 2\sqrt{ab}$, because $(\sqrt{a}-\sqrt{b})^2 \geq 0$.

If we have a proper coloring of G using $\chi(G)$ colors from the set C_1 and a proper coloring of $\overline{G}$ using $\chi(\overline{G})$ colors from the set C_2, then we can take the union of G and $\overline{G}$, which is a K_n, and color it with the product set $C_1 \times C_2$ accordingly. This is a valid coloring, as every edge of K_n is either in G or in $\overline{G}$, so one component of the pair of colors at its endpoints differs. This requires $\chi(G)\chi(\overline{G})$ colors. Since $\chi(K_n) = n$ we find $\chi(G)\chi(\overline{G}) \geq n$, and hence

$$\chi(G) + \chi(\overline{G}) \geq 2\sqrt{\chi(G)\chi(\overline{G})} \geq 2\sqrt{n}.$$

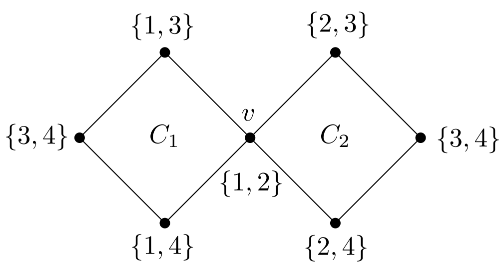
Alternative solution: Let $\chi(G) = k$. Then there is an independent set in G of size at least $\frac{n}{k}$, in particular, $\chi(\overline{G}) \geq \frac{n}{k}$. Thus $\chi(G) + \chi(\overline{G}) \geq k + n/k \geq 2\sqrt{n}$. □

Problem 28. **5 points**

Find the list-chromatic number of $K_{4,4}$. Justify your answer.

Solution.

Because $\chi(K_{4,4}) = 2$, $\mathrm{ch}(K_{4,4}) \geq 2$. Furthermore, $\mathrm{ch}(K_{4,4}) \neq 2$, because there are two a cycles of length 4, sharing exactly one vertex v. Assign v the list $\{1,2\}$, the other vertices of the first cycle C_1 the color lists $\{1,3\}$, $\{3,4\}$ and $\{1,4\}$ in order, and the second cycle C_2 the color lists $\{2,3\}$, $\{3,4\}$ and $\{2,4\}$, also in order. Now we claim that this graph cannot be colored with respect to these lists. Indeed, if v is colored 1, then its neighbors on C_1 are colored 3 and 4, respectively, which implies that the vertex opposite to v on C_1 is not colorable. So v is not colored 1 and symmetrically v is not colored 2.



It remains to show that $\mathrm{ch}(K_{4,4}) \leq 3$. Let $U = \{u_1,\ldots,u_4\}$ and $W = \{w_1,\ldots,w_4\}$ be the two partite sets of $K_{4,4}$, and let L be a list assignment with $|L(v)| = 3$ for all $v \in U \cup W$.

If all vertices in U can be colored with two colors 1 and 2, i.e., if $\{1,2\} \cap L(u_i) \neq \emptyset$ for $i = 1, \ldots, 4$, then we color U with colors 1 or 2 and each vertices $w_i \in W$ with any color in $L(w_i) \setminus \{1,2\} \neq \emptyset$. Since vertices in W are mutually independent we obtain a proper coloring of $K_{4,4}$ and we are done.

So we may assume for the remainder that U can not be colored with only two colors. In particular, that no color appears in the list of three vertices from U. Let us distinguish the following cases.

Case 1. Some color, say 1*, appears in the lists of two vertices in* U*, say* u_1 *and* u_2*.* We color u_1 and u_2 with color 1. By our assumption, $L(u_3) \cap L(u_4) = \emptyset$, as otherwise U could be colored with two colors only. Now we have $|L(u_3)| \cdot |L(u_4)| = 9$ possibilities to color u_3 and u_4 from their lists. Since every vertex $w_i \in W$ can have at most two such combinations in its list, at most 8 of these combinations appear in a list of some $w_i \in W$. Therefore, there it is possible to color u_3 and u_4, say with colors 2 and 3, such that $L(w_i) \setminus \{1,2,3\} \neq \emptyset$ for every $i = 1, \ldots, 4$. This way we can color entire $K_{4,4}$ properly.

Case 2. The lists $L(u_1)$*,* $L(u_2)$*,* $L(u_3)$*,* $L(u_4)$ *are pairwise disjoint.* So for the vertices in U, we have $|L(u_1)| \cdot |L(u_2)| \cdot |L(u_3)| \cdot |L(u_4)| = 81$ different possibilities to color them. For each $w_i \in W$ at most 3 of these possibilities make it impossible to color w_i. So in total at most 12 combinations are "forbidden", which implies that there are plenty of proper colorings for these lists. □

Open Problem.
Prove or disprove that for every graph G we have

$$MK_{\chi(G)} \subseteq G.$$