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Problem 34

In the following we will first proof a lemma and then the theorem.

Lemma 1.0.1. *In a triangle-free graph $G = (V, E)$ with $|V| = n$ if there is a vertex $v \in V$ with $deg(v) = a$, then all the neighbours of v have at most a degree of $n - a$*

Proof. Let $u \in V$ be any neighbour of v. u cannot be connected to nodes adjacent to v or we would have a triangle with u and v. So $deg(u) \leq |V - neighbourhood(v)| = n - deg(v) \leq n - a$. This line of argumentation can be used on all neighbours of v. □

Theorem 1.1. *For every triangle-free graph $G = (V, E)$ with $|V| = n$ the inequality $\sum_{v \in V} deg(v)^2 \leq \frac{n^3}{4}$ holds.*

Proof. Let $v \in V$ be the vertex with the highest degree in G, $deg(v) = a$.
Using the lemma we know that for all $u \in neighbours(v) : deg(u) \leq n - a$. Now we now that we have a vertices with maximum degree $n - a$ and the rest $n - a$ vertices have maximum degree a. This gives us the upper bound for the formula $\sum_{v \in V} deg(v)^2 \leq a * (n - a)^2 + (n - a) * a^2$ with the parameter a. Now we can seek for the maximum of the function to get a precise upper bound $f(a) = a * (n - a)^2 + (n - a) * a^2 = a * n^2 - a^2 * n$. $f'(a) = n^2 - 2 * a * n = n(n - 2a)$. So f has a maximum at $\frac{n}{2}$, so to achieve the maximum for $\sum_{v \in V} deg(v)^2$ it is best for all the vertices to have degree $\frac{n}{2}$. Now we easily get the upper bound $\sum_{v \in V} deg(v)^2 \leq f(a) \leq f(\frac{n}{2}) = \frac{n^3}{4}$. □