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Problem 21

Lemma 1.0.1. *In every planar triangulation G on atleast four vertices, there exists a vertex v which does not lie on the outer bound of G*

Proof. For the sake of contradiction let's assume there is no such vertex v. Then all the vertices lie on the outer bound of G forming a cycle with atleast four vertices. This is a contradiction with G being a planar triangulation, because we can add a edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be atleast one vertex v not on the outer bound of G. □

Theorem 1.1. *Every planar triangulation G on atleast four vertices contains a vertex whose neighbourhood induces a cycle.*

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G. Let $N(v) = p_1, p_2, ..., p_n$ be the neighbourhood of v. $N(v)$ induces a cycle if all p_i are connected to a cycle and if there are no additionaly edges between p_i and p_j with $|i - j| > 1$. We first will proof that all p_i form a cycle, hence the enduced subgraph $N(v)$ has a cycle as a subgraph. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G, or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v,p_i,p_{i+1} and atleast one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence G is no triangulation. Now we will proof that either $N(v)$ has no additional edges and thus is a cycle or that one of the vertices adjacent to v induces a cycle.

- *There are no additional edges connecting two vertices p_i and p_j with $|i - j| > 1$.*
  In this case $N(v)$ is a cycle and we found a vertex whose neighbourhood induces a cycle.
- *There is an edge $p_i p_{i+2}$*
  In this case the neighbourhood induces graph of p_{i+1} is a cycle namely $p_i p_{i+1} v$.
- *There is an edge between to vertices p_i and p_j with $|i - j| > 2$ and without loss of generalitiv $i < v$.*
  This case there is a bounded face in G which is not a triangle. This face is the face bounded by at least the edges $p_i p_{i+1}$, $p_j p_{j-1}$, $p_i p_j$ and atleast one or more edges forming a path from p_{i+1} to p_{j-1}. Hence this case can never occur in a planar triangulation.

In summary we now that we can always find a vertex v whoose neighbourhood induces a cycle or one of the neighbours of v here called p_{i+1} has a neibourhood inducing a cycle. □

Lemma 1.1.1. *Any planar triangulation G hast at most $2n - 4$ faces and $2n - 5$ triangles*

Proof. Every inner face of G is a triangle, hence every face is adjacent to atleast 3 edges. Additionally, every edge is adjacent to exactly two faces, hence $2e \geq 3 * f \Leftrightarrow e \geq \frac{3}{2} f$(e: number of edges, f: number of faces). By Euler's formula we know $2 = n - e + f \Leftrightarrow e - f = n - 2 \Rightarrow \frac{1}{2} f \leq n - 2 \Leftrightarrow f \leq 2n - 4$. Now we know that G has at most $2n - 4$ faces and because the unbounded face cannot be a triangle we know that G has at most $2n - 5$ triangles. □

Theorem 1.2. *Every n-vertex planar graph has at most $3n - 8$ triangles.*

Proof. A planar graph which G is maximal in the number of triangles has to be a triangulation, because if there would be a inner face which is adjacent to more than 3 edges, we could divide the face into triangles thus forming a graph with same number of vertices but more triangles. Now we can apply lemma 1.1.1 and we know that G has at most $2n - 5$ triangles. Now we know that $2n - 5 \leq 3n - 8$ for every $n > 2$, hence we have proven an even stronger theorem. □

Problem 22

Theorem 2.1. *Any TK_3-free graph G on n vertices contains a maximum of $n - 1$ edges.*

Proof. First, K_3 is the triangle C_3. Subdividing any edge of C_i results in C_{i+1}. Moreover, any cycle has a $TC_3 = TK_3$.

Hence, a graph G is TK_3-free if and only if it is acyclic. Further we assume that G is connected (since joining two disjoint acyclic components will not create a cycle but increase the edge count).

From these considerations, the maximum number of edges of an n-vertex, TK_3-free graph equals the maximum number of edges in an n-vertex tree. Any n-vertex tree contains a maximum of $n - 1$ edges. □

Theorem 2.2. *If a graph G is 3-connected then $TK_4 \subseteq G$.*

Proof. By TUTTE (1961), any 3-connected graph has a construction sequence $G_0, G_1, ..., G_n$ whereby $G_0 = K_4$ and $G_n = G$.

For any $i < n$, $G_i = (V_i, E_i)$ can be constructed by contracting an edge $e = \{x, y\}$ of G_{i+1} ($x, y \in V_{i+1}$, $d(x), d(y) \geq 3$).

Since $d(y) \geq 3$ and contracting e results in G_i, we can effectly say that there is a third vertex z in G_i which is also in G_{i+1} for which $\{\{x, y\}, \{y, z\}\}$ is a subdivision of $\{x, z\}$.

Thus, G_{i+1} has a TG_i and inductively, by the transitivity of topological minority, G_{i+1} has a $TG_0 = TK_4$. □