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# Problem 9

**Theorem 1.1.** *A hypercube $Q_n$ is Hamiltonian. It has a girth of 4 for $n \geq 2$ and $\infty$ otherwise. It's diameter is $n$, it's order $2^n$ and it has a size of ?.*

*Proof.* Let $S$ be a set of cardinality $|S| = n$. We construct $Q_n = (V_Q, E_Q)$ by creating a vertex for each subset of $S$ and moreover add edges between those subsets which differ by only one element. In the following, we will use binary representations of the vertices of $Q_n$ since $V_Q = \mathcal{P}(S) \cong (\mathbb{Z}/2\mathbb{Z})^n$ (we can denote a 1 for including an element and a 0 for excluding an element in a subset).

**Order:** Since $V_Q = \mathcal{P}(S)$ and $|\mathcal{P}(S)| = 2^n$, *the order of $Q_n$ is $2^n$.*

**Size:**

**Girth:** We differ between two cases.

- **Case 1:** $n = 1$. Our graph contains exactly one edge and is therefore acyclic. *Hence, the girth is $\infty$ for $n = 1$.*
- **Case 2:** $n \geq 2$ Our graph contains the cycle $(\emptyset, \{a\}, \{a, b\}, \{b\}, \emptyset)$ $(a, b \in S)$ which has length 4.

  A shorter cycle $(A, B, C, A)$ $(A, B, C \in V_Q)$ does not exist due to the property that two adjacent vertices differ by exactly one element. For such a cycle, $B$ and $A$ differed by one element, and hence $A$ and $C$ differed by two or are equal. However, a difference of zero or two elements between two consecutive elements renders any walk invalid. The edge $\{C, A\}$ could not be contained in $Q_n$.

  *From these considerations, for $n \geq 2$, the girth is 4.*

**Diameter:** For any set $A \in V_Q$, we are able to get to any other element $B \in V_Q$ by inserting or removing a maximum of $n$ elements. Thus, a path of length $n$ is sufficient to walk from any $A$ to any $B$. Furthermore, there exist $A$ and $B$ such that a path of length $n$ is the shortest path between them. E.g. $A = \emptyset \in V_Q$, $B = S \in V_Q$. *Thus, the diameter of $Q_n$ is $n$.*

**Hamiltonian:** A Hamiltonian cycle is equivalent to an enumeration of $(\mathbb{Z}/2\mathbb{Z})^n$ in which consecutive elements differ by exactly one element. We provide such an enumeration: the *Gray Code*[a]. *Thus, there exists a Hamiltonian cycle and $Q_n$ is Hamiltonian.* □

[a] For $n = 2$: 00,01,11,10,00. Generally, the $k$'th vertex in the Hamiltonian cycle is $k \otimes \lfloor \frac{k}{2} \rfloor$ whereby $\cdot \otimes \cdot$ denotes the exclusive or.

**Theorem 1.2.** *A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n$. It's girth is 4 for $m, n \neq 1$ and and $\infty$ otherwise. It's diameter is 1 for $m = n = 1$ and 2 otherwise. The graph's order is $m + n$ and it's size is $m \cdot n$.*

*Proof.* Let $V = \{v_1, ..., v_m\}$ and $W = \{w_1, ..., w_n\}$ denote the two partitions of $K_{m,n} = (V_K, E_K)$.

**Order:** The first partition has $m$ elements, the second $n$ elements. *Thus, $K_{m,n}$ has an order of $m+n$.*

**Size:** Each of the $m$ elements of the first partition are connected to each of the $n$ elements in the second partition. *Thus, $K_{m,n}$ has a size of $m \cdot n$.*

**Girth:** If either $m = 1$ or $n = 1$, then all vertices of one partition are indicent to and only to the single vertex of the other partition. Hence, there is no cycle in $K_{1,n}$ or $K_{m,1}$ and *the girth of $K_{m,n}$ is $\infty$ if $n = 1$ or $m = 1$.*

If $m, n \neq 1$, each cycle must have even length since any two consectuive vertices in a path of $K_{n,m}$ are in different partitions. Thus, we require an even amount of edge-crossings to enclose a walk. Any cycle has a length of at least 3, thus the girth of $K_{m,n}$ has to exceed or be equal to 4.
Furthermore, we find such a cycle of length 4 easily since both partitions $V, W$ have at least 2 vertices: $(v_1, w_1, v_2, w_2, v_1)$. *From these considerations, the girth of $K_{m,n}$ must be equal to 4.*

**Diameter:** For $m = n = 1$, there are exactly two vertices in different partitions. They have a distance of 1 and thus, *the diameter of $K_{1,1}$ is 1.*

Since two consectuive vertices in a path of $K_{n,m}$ are in different partitions $V, W$ the distance between two vertices in the same partition has to be at least 2. Moreover, we find a path of distance 2 between $v_1 \in V$ to $v_2 \in V$: $(v_1, w, v_2)$ for any $w \in W$. Thus, the diameter does not deceed 2.
For any two vertices in different partitions $V, W$, they are directly connected by a path of length 1.
*Hence, the diameter of $K_{m,n}$ is 2 if not $m = n = 1$.*

**Hamiltonian:** □

**Theorem 1.3.** *The Petersen graph is not Hamiltonian, it has a girth of* 5*, a diameter of* 2*, an order of* 10 *and a size of* 15*.*

*Proof.* **Order:** The graph has 10 vertices and thus, it's order is 10.

**Size:** The graph has 15 edges and thus, it's size is 15.

**Girth:**

**Diameter:**

**Hamiltonian:**

□

# Problem 11

For each even integer $k > 1$, the complete graph $K_{(n+1)}$ is a $k$-regular graph with no 1-factor. For each odd $k > 1$ we can construct a k-regular graph with no 1-factor in the following way.
In order to garantee that the graph has no 1-factor we can use Tutt's theorem. We construct the graph by starting of with one vertex $v$ connected to k subgraphs $S$ which are not inter-connected. Then we construct

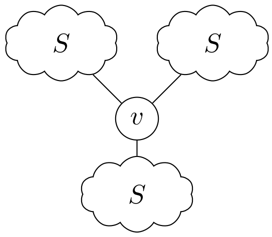


Figure 1: Example with k=3

$S = (V, E)$ in such a way that $|V|$ is odd and that all vertices in $V$ have degree k except of one vertex $u \in V$ with degree $k - 1$. If we then connect $u$ to $v$ we get a k-regular graph. Using Tutt's theorem we know that the resulting graph has no 1-factor. Because if $v$ is removed we get $k$ components $S$ with odd number of vertices and $k$ is greater than 1.

In order to contruct $S$ we first need the following lemma.

**Lemma 2.0.1.** *For any odd integer $k > 1$ it is possible to construct $(k-1)$-regular graph $G = (V, E)$ with $k + 1$ vertices.*

*Proof.* We can obtain $G$ from $K_{k+1}$ by removing by removing all the edges from $K_{k+1}$ contained in a perfect matching. By removing the edges the degree of every vertex decreases exactly by one. $K_{k+1}$ is by definition $k$-regular, hence $G$ is $k - 1$-regular. A perfect matching in $K_{k+1}$ exists, because $k + 1$ is even, and the condition from Tutte's theorem always holds in a complete graph. To find such a matching we can just randomly choose $(u, v)$ edges and remove $u$ and $v$ from $K_{k+1}$. □

*Constructing a connected graph $S = (V, E)$ with $|V| = k + 2$ and the degree sequence $(k, k, ..., k, (k - 1))$.* First we construct a $(k-1)$-regular graph $S' = (V', E')$ with $k + 1$ vertices as described in the lemma. Then we can add one vertex to $S'$ and connect it to all vertices in $V'$ except of one vertex. Thus we get a new graph $S$. Because we only added one vertex $|V| = |V'| + 1 = k + 2$ and the degree of the newly added vertex is $k$. The degree of all the other vertices except of the last one is increased by one. Hence, $S$ hat the degree sequence $(k, k, ..., k, (k - 1))$.

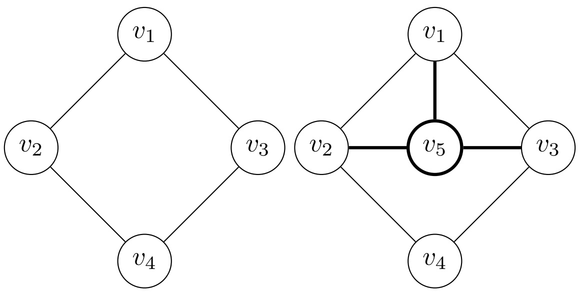


Figure 2: $S'$ and $S$ for k=3

# Problem 12

In the following I will show that any graph $G$ with $2n$ vertices and all degrees at least $n$ has a 1-factor. I will show that if we divide such a $G$ into two parts we can remove edges until we get an bipartite 1-regular graph. Then using the corollary of Hall's theorem we know we can find a perfect matching.

**Theorem 3.1.** *Let $G = (V, E)$ be a graph with $|V| = 2n$ vertices with all degree atleast $n$, then $G$ has a 1-factor.*

*Proof.* Because $|V| = 2n$ we can divide the vertices $V$ into two subsets $A, B \subset V$ with $|A| = |B| = n$ and $A \cap B = \emptyset$.

$\square$