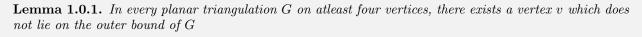
Graph Theory - Sheet 6 - November 29, 2013 J. Batzill (1698622), M. Franzen (1696933), J. Labeit (1656460)

Problem 21



Proof. For the sake of contradiction let's assume there is no such vertex v. Then all the vertices lie on the outer bound of G forming a cycle with at least four vertices. This is a contradiction with G being a planar triangulation, because we can add a edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be at least one vertex v not on the outer bound of G.

Theorem 1.1. Every planar triangulation G on at least four vertices contains a vertex whose neighbourhood induces a cycle.

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G. Let $N(v) = p_1, p_2, ..., p_n$ be the neighbourhood of v. N(v) induces a cycle if all p_i are connected to a cycle. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G, or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v, p_i, p_{i+1} and at least one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence G is no triangulation.

Lemma 1.1.1. Any planar triangulation G hast at most 2n-4 faces and 2n-5 triangles

Proof. Every inner face of G is a triangle, hence every face is adjacent to at least 3 edges. Additionally, every edge is adjacent to exactly two faces, hence $2e \geq 3*f \Leftrightarrow e \geq \frac{3}{2}f$ (e: number of edges, f: number of faces). By Euler's formula we know $2 = n - e + f \Leftrightarrow e - f = n - 2 \Rightarrow \frac{1}{2}f \leq n - 2 \Leftrightarrow f \leq 2n - 4$. Now we know that G has at most 2n - 4 faces and because the unbounded face cannot be a triangle we know that G has at most 2n - 5 triangles.

Theorem 1.2. Every n-vertex planar graph has at most 3n - 8 triangles.

Proof. A planar graph which G is maximal in the number of triangles has to be a triangulation, because if there would be a inner face which is adjacent to more than 3 edges, we could divide the face into triangles thus forming a graph with same number of vertices but more triangles. Now we can apply lemma 1.1.1 and we know that G has at most 2n-5 triangles. Now we know that $2n-5 \le 3n-8$ for every n>2, hence we have proven an even stronger theorem.

Problem 22