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# Problem 17

**Theorem 1.1.** *In a planar triangulation let $n_i$ be the number of vertices of degree $i$. Then,*

$$\sum_{i\in\mathbb{N}}(6-i)n_i = 12$$

.

**Lemma 1.1.1.** *Let $G = (V, E)$ and $G' = (V \cup \{v\}, E \cup E')$ be planar triangulations. Then $|E'| = 3$ and all edges in $E'$ are incident to $v$.*

*Proof.* Since any planar triangulation of $n$ vertices and $e_n$ edges satisifes $e_n = 3n-6$, we see inductively that

$$\begin{aligned} e_n &= e_{n-1} + 3 && (n > 3) \\ e_3 &= 3 \end{aligned}$$

Since $G'$ has exactly one vertex more than $G$ and both are planar triangulations, $|E'| = 3$.

Next, we will show that the degree of $v$ exceeds or is equal to 3 and thus, all edges of $E'$ have to be incident to $v$.

By KURATOWSKI, $G'$ is not a topological minor of $K_{3,3}$ or $K_5$ and any planar triangulation is edge-maximal. By *Lemma 4.4.5 (any edge-maximal graph without topological minors $K_{3,3}, K_5$ is 3-connected)*, $G'$ is 3-connected.

If the degree of $v$ deceeded 3, then $G'$ would not be 3-connected ($v$ could be isolated by removing two vertices) .

Hence, all three edges of $E'$ are incident to $v$. □

We will show by induction on the number of vertices $n$ of a planar triangulation $G$ with $n_i$ vertices of degree $i$ ($i \in \mathbb{N}$) that

$$T_G := \sum_{i\in\mathbb{N}}(6-i)n_i = 12$$

.

- Base $\mathbf{n = 3}$

  Then, the graph is a triangle and the condition is satisifed:

  $$T_{K_3} = \sum_{i\in\mathbb{N}}(6-i)n_i = (6-2)\cdot 3 = 4 \cdot 3 = 12$$

- Step $\mathbf{n \geq 4}$

  Any $n$-vertex planar triangulation $G = (V, E)$ has a subgraph $H = (V', E')$ which is an $(n-1)$-vertex planar triangulation.

  By *Lemma 1.1.1*, there is a vertex $v \in V \setminus V'$ of degree 3. Furthermore, the degree of exactly three other vertices $v_i, v_j, v_k \in V$ is increased. Thus, $E \setminus E' = \{\{v, v_i\}, \{v, v_j\}, \{v, v_k\}\}$ and for $T_G$:

  $$\begin{aligned} T_G =& T_H \\ &+ (6-3) \\ &+ (6-(d(v_i)+1)) - (6-d(v_i))) \\ &+ (6-(d(v_j)+1)) - (6-d(v_j)) \\ &+ (6-(d(v_k)+1)) - (6-d(v_k)) \\ =& T_H + 3 - 1 - 1 - 1 \\ =& T_H \\ =& 12 && \text{(by induction)} \end{aligned}$$

□

# Problem 19

**Theorem 2.1.** *A plane embedded graph on $n$ vertices that has no triangular face has at most $2n-4$ edges.*

**Lemma 2.1.1.** *Each plane triangulation (order $\geq 4$) can be modified to a plane graph containing only faces of order 4 by removing one edge for each two faces*

*Proof.* Proof by induction over the order of a plane triangulation $G$:

- **Base:** $V(G) = 4$
  G contains of 4 faces $f_1, f_2, f_3, f_4$. We get the desired graph by removing the shared edge between $f_1, f_2$ and $f_3, f_4$. This is possible because each face is adjacent to one another. The remaining graph is, as desired, created by removing one edge foreach 2 faces.

- **Step:** $V(G) = n$
  Let $G' = G - \{u\}$ ($u \in V(G)$ with $deg(u) = 3$ as shown in **Lemma 1.1.1**). Then $G'$ is still a plane triangulation and the amount of faces in $G$ is exactly 2 more than in $G'$, because by removing the vertex $u$, 3 edges were removed, thus, 2 triangular faces were removed.
  By induction we get the desired graph $H'$ for $G'$ by removing one edge for each 2 faces. Moreover, by inserting $u$ and its 3 adjacent edges in $H'$ at their old position, 2 faces of degree 3 were created (Would there be more than 2 new faces, the face in which $u$ was inserted would have an order of at least 5). Thus, there are only two faces of order 3 remaining. Because all inserted edges are incident to $u$, these two faces of oder 3 are adjacent and can be merged by removing the shared edge.
  The resulting graph of order $n$ is still plane (only edges have been removed), has only faces of order 4 and was created of $G$ by removing one edge for each 2 faces.

□

**Lemma 2.1.2.** *Each plane graph with no triangular face can be modified to a plane graph containing only faces of order 4 and with at least the same amount of edges*

*Proof.* Let $G$ be such a plane graph with no triangular face. Let $G'$ be the plane triangulation of $G$. To get $G'$ out of $G$ edges have to be added. Because the smallest face is of order 4, at least 1 edge for each face is needed to reduce the size of all faces, resulting in at least one additional face for each existing face in $G$, to be more precisely, in exactly one additional face for each inserted edge. So $G'$ has at least twice as much faces as $G$ and exactly one new edge for each new face. As shown in **Lemma 2.1.1**, $G'$ can be modified to a plane graph $H$ containing only faces of order 4 by removing one edge for each two faces. Thus, the amount of faces in $H$ is at least as much as in $G$, and furthermore the amount of edges is at least as much as in $G$, because for each two faces exactly one edge has been removed.
All in all, $H$ is a graph containing only faces of order 4 and has at least the same amount of edges as $G$. □

Let $G = (V, E_G)$ be a plane graph with no plane triangulation.
As shown in **Lemma 2.1.2**, we get a plane graph $H = (V, E_H)$ ($F_H$ corresponding faces, each face of order 4), containing at least the same amount of edges as $G$ and the same set of vertices $V$.

By $Euler'sFormula$ and the fact $|E_H| = \frac{4*|F_H|}{2}$ (because each face in $F_H$ has order 4 and each edge is exactly counted twice), we get the following:
$2 = |V| - |E_H| + |F_H| \leftrightarrow 2 = |V| - |E_H| + \frac{E_H}{2} \leftrightarrow |E_H| = 2 * |V| - 4$.
This inequality allows us to bound the amount of edges as following:

$|E_G| \leq |E_H| = 2 * n - 4 \quad (n = |V|)$

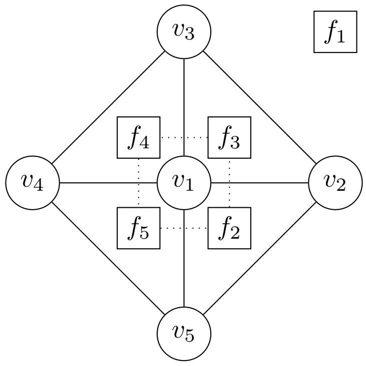
□

# Problem 20

**Theorem 3.1.** *For a planar embedded graph $G$ on $n$ vertices such that $G* \cong G$, the number of edges $e = 2(n-1)$.*

*Proof.* The faces of $G$ corrospond to the vertices of $G^*$. Considering their ismorphism, the number of faces $f$ of $G$ equals the number of vertices of $G$.

$G$ is planar, by *Euler's Characteristic*: $n - e + f = 2 \Rightarrow 2n - e = 2 \Rightarrow e = 2(n-1)$.
How to find such a graph $G = (V, E)$ which is isomorph to it's plane dual $G*$ with $|V| = n \geq 4$. Let $v_2, v_3, ...v_n$ be a simple cycle with $v_1$ connected with an edge to every other vertex. $G$ can be planar embedded by drawing $v_1$ in the middle of the cycle. We can see that $G \cong G*$ holds by finding a bijection identifying every vertex of $G$ with exactly one face of $G$. The vertices $v_2, v_3...v_n$ can be identified with the face below it. The vertex $v_1$ can be identified with the face surrounding $G$. Now it is easy to see that $G \cong G*$ holds.




□