Problem 13

Theorem 1.1. *If a graph has an ear-decomposition, then it is 2-connected.*

Proof. Let $G_1, ..., G_n$ be the ear-decomposition of $G = (V, E)$ existing by definition.
As far as $G_n = G$ it is sufficient to proof that $\forall i \in \{1, .., n\}$ G_i is 2-connected.
This can be done by inducion.

Base$(\mathbf{i = 1})$: G_1 is a cycle, hence it is 2-connected.

Step$(\mathbf{i \geq 2})$: Per defintion $G_i = G_{i-1} + P_i$, P_i path and $P_i \cap G_{i-1}$ contains exactly the two endpoints of P_i.
Moreover we know that G_{i-1} is 2-connected by induction and P_i is connected by definition.
To proof that G_i is 2-connected we have to proof that $H := G_i - \{u\}$ $(u \in V(G_i))$ is connected.
Hence 2 cases have to be considered:

$\mathbf{u \in G_{i-1}}$ Because G_{i-1} is 2-connected by induction H is connected.
Furthermore we know that H contains still one endpoint of P_i or more.
Thus H is a composition of two connected graphs, hence H is connected.

$\mathbf{u \in P_i - G_{i-1}}$ We know that u is no endpoint of P_i, thus $P' := P_i - \{u\}$ is disconnected and is now a forest containing exactly two Trees T_1, T_2.
Nevertheless each tree of P' contains exactly one endpoint of P_i. Hence $G_{i-1} + T_1 + T_2$ is still connected, because T_1, T_2 and G_{i-1} are connected. As far as $G_{i-1} + T_1 + T_2 = G_{i-1} + P_i - \{u\} = G_i - \{u\} = H$, H is connected.

Considering these two cases we know that H is connected, hence G_i is 2-connected.

We finally proofed by induction that $\forall i \in \{1, ..., n\}$ G_i is 2-connected.
Thus $G = G_n$ is 2-connected.

$\square$

Problem 14

For $0 < l < m \leq d$, we will construct a graph $F(d, l, m)$.

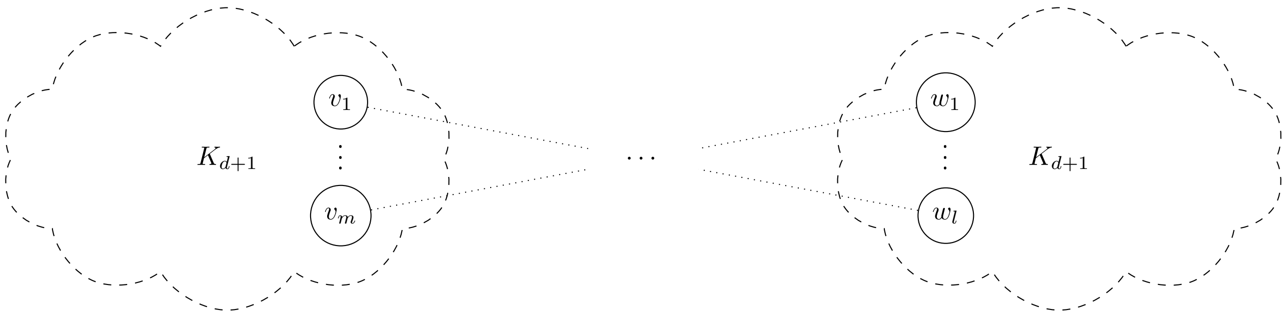

Figure 1: $F(d, l, m)$

First, we construct two complete graphs on $d+1$ vertices. $(V, E) \simeq K_{d+1}$, $(W, E') \simeq K_{d+1}$.

Then, we join m vertices $v_1, ..., v_m \in V$ of the first complete graph and l vertices $w_1, ..., w_l \in W$ of the second such that each v_i has a degree of exactly $d+1$ and each w_j of at least $d+1$ ($i \in [m], j \in [l]$).

Formally, for our constructed graph $F(d, l, m) := (V_F, E_F)$, the vertex set is the union of both complete graphs ($V_F = V \cup W$) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid \delta_{ij} = 1 \ (i, j \in \mathbb{N})\} \tag{1}$$

for a delta function δ_{ij} $(i, j \in \mathbb{N})$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i > l, j = l \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{d}$
No degree of a vertex of the complete graphs has been decreased. *Thus,* $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$.

Moreover, we have increased the degree of exactly $l + m < 2(d+1)$ vertices. Indeed, the complete graph on $d+1$ vertices is d-regular and hence there is at least one vertex of degree d in $F(d, l, m)$. *Thus,* $\delta(F(d, l, m)) \leq d$.

From these considerations, $\delta(F(d, l, m)) = d$.

$\kappa(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{l}$
In $F(d, l, m)$, the two complete graphs are only joined by edges between l vertices of one and m vertices of another complete graph. The graph obviously disconnects by removing those first l vertices. *Thus,* $\kappa(F(d, l, m)) \leq l$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa(K_{d+1}) = d > l$. Hence, if we found a subset of $l' < l$ vertices that disconnects $F(d, l, m)$, it had to consist of the complete graphs' vertices that we have affected in our construction. However, between the complete graphs there are l edges not sharing an endpoint.

Thus, it is neither possible to disconnect one of the complete graph by removing less than l vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than l vertices.

From these considerations, $\kappa(F(d, l, m)) = l$.

$\kappa'(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{m}$
In $F(d, l, m)$, the two complete graphs are only joined by exactly m edges and a removal of those m vertices obviously disconnects $F(d, l, m)$. *Thus,* $\kappa'(F(d, l, m)) \leq m$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa'(K_{d+1}) = d > m$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than m edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than m edges.

From these considerations, $\kappa'(F(d, l, m)) = m$.

Problem 15

I will prove that any block-cut-vertex graph is a tree, by showing by contradiction that any block-cut-vertex graph is acyclic and connected.

Theorem 3.1. *The block-cut-vertex graph $G = (V, E)$ of any connected graph $G' = (V', E')$ is a tree.*

Proof. Let's assume for the sake of contradiction that G has a cycle $C = (b_1 b_2 ... b_1)$. Let's denote the subgraphs $B_1, B_2, ...B_n$ of G' which are the 2-connected components and bridges corresponding to the nodes $b_1, b_2, ...b_n$ of G. Let B_1 and B_2 be as stated above two different subgraphs of G'. Because the corresponding nodes b_1 and b_2 are adjacent in G, B_1 and B_2 have to share a vertex $x \in V(B_1) \cap V(B_2)$. We can use the same argument for each pair B_i, B_{i+1}. Additionally, we know because each component B_j is either 2-connected or a bridge. Thus we can find a circle through all the components $B_1, B_2 ... B_n$ which is 2-connected. this is a contradiction to $B_1, B_2 ... B_n$ being the blocks of an block-cut-vertex graph, because by definition these blocks are either bridges or maximal 2-connected components. $\square$

Problem 16