

## Problem 1

**Theorem 1.1.** Any tree  $T$  has at least  $\Delta(T)$  leaves.

**Lemma 1.1.1.** Any connected subgraph of a tree is a tree as well.

*Proof.* If a graph  $G = (V, E)$  is acyclic, then  $E$  does not contain any cyclic subset and hence there is no cyclic subgraph of  $G$ . From these considerations, any connected subgraph of  $G$  is acyclic and therefore a tree.  $\square$

**Lemma 1.1.2.** Any tree  $T$  with  $\Delta(T) + 1$  vertices has  $\Delta(T)$  leaves ( $\Delta(T) \geq 1$ ).

*Proof.* Let  $v_0 \in V_T$  denote the vertex of  $T = (V_T, E_T)$  with  $d(v_0) = \Delta(T)$ .

For any  $v, w \in V_T \setminus \{v_0\}$  ( $v \neq w$ ):

- $d(v) \geq 1$ :  $v_0$  is adjacent to  $v$  since  $|V_T| = \Delta(T) + 1 \Rightarrow d(v) \geq 1$
- $d(v) \leq 1$ : Any edge  $\{v, w\} \in E_T$  would create a cycle  $(v_0, v, w, v_0)$  which would render  $T$  invalid as a tree.

$\Rightarrow \forall v \in V_T \setminus \{v_0\} : d(v) = 1 \Leftrightarrow v$  is a leaf. Notably,  $|V_T \setminus \{v_0\}| = \Delta(T)$ .  $\square$

**Lemma 1.1.3.** For any tree  $T = (V_T, E_T)$  with  $|V_T| > \Delta(T) + 1 \geq 3$ , there exists a partition  $(S, S')$  of  $T$  with  $\Delta(S) + \Delta(S') = \Delta(T)$ .

*Proof.* Let  $v_0 \in V_T$  denote a vertex with  $d(v_0) = \Delta(T)$ . Since  $|V_T| > \Delta(T) + 1$ ,  $v_0$  has at least one non-leaf adjacent vertex  $v_1 \in V_T$ . Now, let  $(S_0, S_1)$  denote the partition at the edge  $\{v_0, v_1\}$  whereby  $S_0$  contains  $v_0$  and  $(S_1, v_1)$  respectively.

As seen in Lemma 1.1,  $S_0$  and  $S_1$  are trees.

- case  $d(v_1) = 2$ : In this case, a cut at the edge  $\{v_0, v_1\}$  would make

$\square$

*Proof by induction.* Let  $T = (V_T, E_T)$  be a tree.

**Basis:**  $\Delta(T) = 1$

Inherently,  $V_T = \{v_1, \dots, v_m\}$  ( $v_i = v_j \Rightarrow i = j$ ) and  $E_T = \{\{v_1, v_2\}, \dots, \{v_{m-1}, v_m\}\}$  ( $m, i, j \in \mathbb{N}$ ). Any modification would violate our preconditions.  $v_1, v_m$  are the only vertices with degree 1. Therefore, the number of leaves is  $2 \geq \Delta(T)$ .

**Step:** For some  $n \in \mathbb{N}$ : Any  $T = (V_T, E_T)$  with  $\Delta(T) = n$  has at least  $n$  leaves.

Let  $T' = (V_{T'}, E_{T'})$  be a tree with  $\Delta(T') > n$ .  $\square$

## Problem 2

**Theorem 2.1.** *If any removal of an edge increases the number of connected components of a graph  $G$ , then  $G$  is acyclic.*

*Proof.* Let  $S$  be a connected component of  $G$ . If  $S$  contained any cycle  $C = (v_0, \dots, v_i, v_j, \dots, v_0)$ , then the removal of an edge  $\{v_i, v_j\}$  would still leave a complete walkthrough  $(v_j, \dots, v_0, \dots, v_i)$  of  $S$  and therefore maintain the component's connectivity. But - as our preconditions state - the removal of any edge increases the number of connected components (disconnects a component).

Thus, a component of  $G$  does not contain any cycles. Considering that none of the graph's connected components contains a cycle,  $G$  is acyclic as well.  $\square$

**Theorem 2.2.** *If adding any edge introduces a cycle in an acyclic graph  $G = (V, E)$ , then any two vertices in  $G$  are joined by a unique path.*

*Proof.* If adding an edge  $\{v_0, v_1\}$  joining two non-adjacent vertices  $v_0, v_1 \in V$  introduces a cycle  $(v_0, \dots, v_1, v_0)$ , then there had to be *at least* one path from  $v_0$  to  $v_1$ .

Furthermore, if there was *more than one* path joining  $v_0$  and  $v_1$ , then there would have already been a cycle (but  $G$  is acyclic).  $\Rightarrow$  Any vertex had to be joined by a unique path.  $\square$

**Theorem 2.3.** *If any two vertices in a graph are joined by a unique path, then any removal of an edge increases the number of connected components.*

*Proof.* Let  $G = (V, E)$  be a graph in which all vertices are joined by a unique path. Let  $e = \{v_0, v_1\} \in E$  be an edge. Thus, the unique path from  $v_0$  to  $v_1$  runs over (and is exactly)  $e$ . From these considerations, removing  $e$  would make  $v_1$  inaccessible from  $v_0$  and would thereby increase the number of connected components.  $\square$

## Problem 3

**Theorem 3.1.** *Either a graph or its complement is connected.*

*Proof.* For a *connected* graph, we're done.

Let  $G = (V, E)$  be disconnected.

Claim: Any two vertices  $u, v \in V$  are connected in  $\bar{G}$ .

Proof: There are only two cases to distinguish, either  $u$  and  $v$  lie in the same component or in different components.

- **Case 1:**  $u$  and  $v$  are in different components  $V_S$  and  $V_T$  with  $u \in V_S$  and  $v \in V_T$ .  
Then  $G$  does not contain the edge  $e = \{u, v\}$ . Otherwise,  $S$  and  $T$  were interconnected. From these considerations, the graph's complement  $\bar{G}$  does contain  $e$ .
- **Case 2:**  $u$  and  $v$  are in the same component  $V_S$ ,  $u, v \in V_S$ :  
 $G$  is disconnected, hence there is atleast another not empty component  $V_T$  with  $V_T \neq V_S$ . Considering that  $V_T$  is not empty, then there is a vertex  $w \in V_T$  such that the edges  $\{u, w\}$  and  $\{v, w\}$  exist in  $G$  (see Case 1). From these considerations,  $\bar{G}$  also contains the path  $(u, w, v)$ .

Therefore, any two vertices in a *disconnected graph*  $G$  are connected in  $\bar{G}$  either by one or two edges and hereby  $\bar{G}$  is connected.

□

## Problem 4

I will prove the theorem that if  $u$  and  $v$  are the only vertices with odd degree in a graph  $G$ , then there is a path connecting  $u$  and  $v$ , by assuming there is no path connecting  $u$  and  $v$  and leading this assumption to the contradiction that  $G$  is no valid graph.

**Theorem 4.1.** *If  $u$  and  $v$  are the only vertices of odd degree in a graph then there is a  $u$ - $v$ -path.*

*Proof.* Let  $G$  be a graph with edges  $u, v \in V(G)$  and let  $u$  and  $v$  be the only vertices with odd degree in  $G$ . Assuming there is no path connecting  $u$  and  $v$  then  $u$  and  $v$  have to be in different components, because if they were in the same component there would be a path connecting  $u$  and  $v$ . Let  $A$  be the component of  $u$ , then  $u$  is the only vertex of odd degree in  $A$  because as stated above  $v$  is not in  $A$ . Because  $u$  is the only vertex with odd degree in  $A$  the sum over the degree of all vertices in  $A$  is odd. Using the theorem that the sum over the degrees over all vertices in a graph has to be even this leads to the conclusion that  $A$  is no valid graph. This is a contradiction to  $A$  being a valid component of  $G$ , hence the assumption that there is no path connecting  $u$  and  $v$  must be wrong leaving the only conclusion that there is a  $u$ - $v$ -path.  $\square$