

Problem 13

Theorem 1.1. *If a graph has an ear-decomposition, then it is 2-connected.*

Proof. By *Menger's Theorem*, a graph G is k -connected if and only if for any two vertices a, b in G there exist k independent a - b -paths. We find those 2 paths for any ear-composable graph. \square

Problem 14

For $0 < l < m \leq d$, we will construct a graph $F(d, l, m)$.

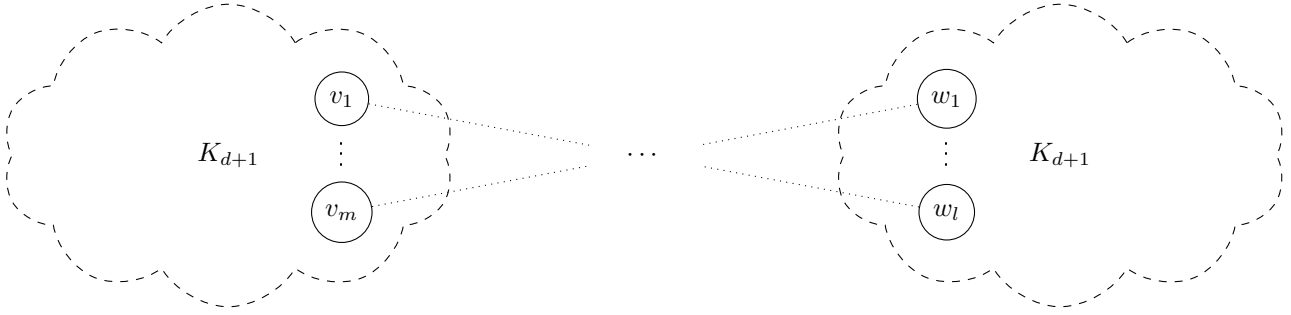


Figure 1: $F(d, l, m)$

First, we construct two complete graphs on $d + 1$ vertices. $(V, E) \simeq K_{d+1}$, $(W, E') \simeq K_{d+1}$.

Then, we join m vertices $v_1, \dots, v_m \in V$ of the first complete graph and l vertices $w_1, \dots, w_l \in W$ of the second such that each v_i has a degree of exactly $d + 1$ and each w_j of at least $d + 1$ ($i \in [m], j \in [l]$).

Formally, for our constructed graph $F(d, l, m) := (V_F, E_F)$, the vertex set is the union of both complete graphs ($V_F = V \cup W$) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid \delta_{ij} = 1 \ (i, j \in \mathbb{N})\} \quad (1)$$

for a delta function δ_{ij} ($i, j \in \mathbb{N}$)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i > l, j = l \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(F(d, l, m)) = d$

No degree of a vertex of the complete graphs has been decreased. *Thus, $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$.*

Moreover, we have increased the degree of exactly $l + m < 2(d + 1)$ vertices. Indeed, the complete graph on $d + 1$ vertices is d -regular and hence there is at least one vertex of degree d in $F(d, l, m)$. *Thus, $\delta(F(d, l, m)) \leq d$.*

From these considerations, $\delta(F(d, l, m)) = d$.

$\kappa(F(d, l, m)) = l$

In $F(d, l, m)$, the two complete graphs are only joined by edges between l vertices of one and m vertices of another complete graph. The graph obviously disconnects by removing those first l vertices. *Thus, $\kappa(F(d, l, m)) \leq l$.*

Moreover, a complete graph on $d + 1$ vertices is internally connected with $\kappa(K_{d+1}) = d > l$. Hence, if we found a subset of $l' < l$ vertices that disconnects $F(d, l, m)$, it had to consist of the complete graphs' vertices that we have affected in our construction. However, between the complete graphs there are l edges not sharing an endpoint.

Thus, it is neither possible to disconnect one of the complete graph by removing less than l vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than l vertices.

From these considerations, $\kappa(F(d, l, m)) = l$.

$\kappa'(F(d, l, m)) = m$

In $F(d, l, m)$, the two complete graphs are only joined by exactly m edges and a removal of those m vertices obviously disconnects $F(d, l, m)$. *Thus, $\kappa'(F(d, l, m)) \leq m$.*

Moreover, a complete graph on $d + 1$ vertices is internally connected with $\kappa'(K_{d+1}) = d > m$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than m edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than m edges.

From these considerations, $\kappa'(F(d, l, m)) = m$.

Problem 15

I will prove that any block-cut-vertex graph is a tree, by showing by contradiction that any block-cut-vertex graph is acyclic and connected.

Theorem 3.1. *The block-cut-vertex graph $G = (V, E)$ of any connected graph $G' = (V', E')$ is a tree.*

Proof. Let's assume for the sake of contradiction that G has a cycle $C = (b_1 b_2 \dots b_1)$. Let's denote the subgraphs B_1, B_2, \dots, B_n of G' which are the 2-connected components and bridges corresponding to the nodes b_1, b_2, \dots, b_n of G . Let B_1 and B_2 be as stated above two different subgraphs of G' . Because the corresponding nodes b_1 and b_2 are adjacent in G , B_1 and B_2 have to share a vertex $x \in V(B_1) \cap V(B_2)$. We can use the same argument for each pair B_i, B_{i+1} . Additionally, we know because each component B_j is either 2-connected or a bridge. Thus we can find a circle through all the components B_1, B_2, \dots, B_n which is 2-connected. this is a contradiction to B_1, B_2, \dots, B_n being the blocks of an block-cut-vertex graph, because by definition these blocks are either bridges or maximal 2-connected components. \square

Problem 16