

Problem 29

Theorem 1.1. *For every $k \in \mathbb{N}$ there exists a tree T_k with $\Gamma(T_k) = k$*

Proof. In the following we will show how to construct T_k by induction. Additionally to $\Gamma(T_k) = k$, every T_k will have a greedy coloring so that the root of T_k is colored with color k .

- *Base*

T_1 is just a single vertex without edges. Obviously $\Gamma(T_1) = 1$ and the color of the root of T_1 is 1.

- *Induction step*

For any $k > 1$, T_k can be constructed by using one new vertex v and T_1, T_2, \dots, T_{k-1} . By connecting the roots of T_1, T_2, \dots, T_{k-1} to v we ensure that there is a greedy coloring in which v has to have the color k . This greedy coloring can be achieved by first coloring T_1, T_2, \dots, T_{k-1} by induction so that the roots have the colors 1, 2, \dots , $k-1$. Now we can color v with the color k because by construction all colors smaller than v are already taken by the nodes adjacent to v . Hence, we constructed T_k with a root node v which has color k and $\Gamma(T_k) = k$.

□

Finally, by using the proven theorem we know that $\min\{k \in \mathbb{N} \mid \Gamma(T) \leq k \text{ for all trees } T\} = \infty$.

Theorem 1.2. *For any Graph G $\Gamma(G) \leq \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$*

Proof. For the sake of contradiction let's assume that there is a graph G with $\Gamma(G) = k$ and $k > \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$. Additionally, let G be colored with a greedy coloring using k colors which obviously has to exist if $\Gamma(G) = k$.

Let $v \in V(G)$ with $c(v) = k$ be one of the vertices with the highest color. Now, $\deg(v) \geq k-1$ for $c(v) = k$ to be possible in a greedy coloring, because v has to have at least $k-1$ adjacent vertices which are colored in colors 1 through $k-1$.

By assumption we know that for all neighbours u of v , $k > \min\{\deg(v), \deg(u)\} + 1$. Now we easily see that $\deg(u) < \deg(v)$ because else $k > \min\{\deg(v), \deg(u)\} + 1 = \deg(v) + 1$ but earlier we saw that $\deg(v) \geq k-1$. Hence, we know that for all neighbours u of v the following inequality holds: $k > \min\{\deg(u), \deg(v)\} + 1 = \deg(u) + 1 \Leftrightarrow \deg(u) < k-1$.

Now for $c(v) = k$ to hold in a proper greedy coloring we need to find a neighbour u of v with $c(u) = k-1$ which was colored before v . Because $\deg(u) > k-1$ u only has $k-2$ neighbours which are not v . Hence if v is not already colored $c(u) < k-1$ so it is impossible to greedy color v with the color k . This finally leads to a contradiction, hence $k > \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$ was a wrong assumption and for every graph $\Gamma(G) > \max_{uv \in E(G)} \min\{\deg(u), \deg(v)\} + 1$ holds. □

Problem 31

Let $G = (V = \{v_1, \dots, v_n\}, E)$ be a graph.

Lemma 2.0.1. $(A(G)^k)[i, j] = a_{i,j}^k$ represents the amount of v_i - v_j -paths of length k ($i, j \in \{1, \dots, n\}$).

proof *Prove by induction over k .*

- Base: $k = 1$

$$A(G)[i, j] = \{e \in E : e = v_i v_j\} = \{p - \text{path} : p = (v_i, v_j)\} \in \{0, 1\}$$

- Step: $k \geq 1$

$$(A(G)^k)[i, j] = (A(G)^{k-1} * A(G))[i, j] = (\sum_{a=1}^n \sum_{b=1}^n \sum_{d=1}^n (A(G)^{k-1})[a, d] * A(G)[d, b])[i, j] = \sum_{d=1}^n (A(G)^{k-1})[i, d] * A(G)[d, j].$$

*By induction $(A(G)^{k-1})[i, d]$ is the amount of v_i - v_d -paths of length $k-1$, and $A(G)[d, j]$ indicates if there is a v_d - v_j -paths of length 1. So $(A(G)^{k-1})[i, d] * A(G)[d, j]$ is the amount of v_i - v_j -paths of length k containing $v_d v_j$ as last edge. To get all paths of length k , all possible incident edges of v_j have to be checked.*

*So all in all, $(A(G)^k)[i, j] = \sum_{d=1}^n (A(G)^{k-1})[i, d] * A(G)[d, j]$ is the amount of all v_i - v_j -paths of length k .*

- (a) $\text{tr}(A(G)) = 0$

Because G doesn't have any self-loops, $A(G)[i, i] = 0 \forall i \in \{1, \dots, n\}$.

$$\text{Therefore: } \text{tr}(A(G)) = \sum_{i=1}^n A[i, i] = \sum_{i=1}^n 0 = 0$$

- (b) $\text{tr}(A(G)^2)$ equals twice the number of edges of G .

$$\begin{aligned} \text{tr}(A(G)^2) &= \sum_{i=1}^n \sum_{k=1}^n A[i, k] * A[k, i] = \sum_{i=1}^n \sum_{k=1}^n A[k, i]^2 \\ &= \sum_{i=1}^n \sum_{k=1}^n A[i, k] = \sum_{i=1}^n d(v_i) = 2 * |E|. \end{aligned}$$

- (c) $\text{tr}(A(G)^3)$ equals six times the number of triangles in G .

As proven in Lemma 2.0.1, $(A(G)^3)[i, i]$ is the amount of all v_i - v_i -paths of length 3.

Because a closed path of length 3 is a triangle, $(A(G)^3)[i, i]$ is the amount of triangles containing v_i .

Moreover, a triangle $t = (v_i, v_j, v_k, v_i)$ can be interpreted as two different paths starting/ending at v_i : $p_1 = v_i v_j v_k v_i$ and $p_2 = v_i v_k v_j v_i$. Thus, for each triangle there are two paths.

All in all, let t be a triangle in G . Then for each vertex in t there are two paths of length 3. This makes $3 * 2 = 6$ paths in $\text{tr}(A(G)^3)$ for each triangle in G .

Because $\text{tr}(A(G)^3)$ is the amount of all closed paths of length 3 ($v \in V(G)$), and a triangle is a closed path of length 3, $\text{tr}(A(G)^3) = \sum_{i=1}^n (2 * |\{t - \text{triangle in } G : t \text{ contains } v_i\}|)$ is exactly six times the amount of all triangles in G .

Problem 32

Theorem 3.1. *The adjacency matrix of any d -regular graph has an eigenvalue of d .*

Proof. Let $G = (V, E)$ be a d -regular graph and let $A(G) = (a_{ij})_{i,j=1,\dots,n}$ ($n \in \mathbb{N}$) denote its adjacency matrix. We show that

$$A(G) \cdot (1, \dots, 1)^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

and thereby that d is an eigenvalue of G . Since G is d -regular, every row sum equals exactly d :

$$\forall i \in [1, \dots, n] : \sum_{j=1}^n a_{ij} = d$$

Trivially,

$$A(G) \cdot (1, \dots, 1)^\top = \left(\sum_{j=1}^n a_{1j}, \dots, \sum_{j=1}^n a_{nj} \right)^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

□

Theorem 3.2. *The adjacency matrix of any bipartite, d -regular graph has an eigenvalue of $-d$.*

Proof. Let $G = (V, E)$ be a bipartite, d -regular graph and let $A(G) = (a_{ij})_{i,j=1,\dots,n}$ ($n \in \mathbb{N}$) denote its adjacency matrix.

For all of G 's vertices v_1, \dots, v_n ($n \in \mathbb{N}$), let $\sigma(v_i)$ denote a selection function that assigns each vertex its partition:

$$\sigma(v) = \begin{cases} -1 & \text{if } v \text{ in partition \#1} \\ 1 & \text{if } v \text{ in partition \#2} \end{cases}$$

Let \mathbf{x} denote

$$A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = (x_1, \dots, x_n)^\top$$

We will prove that $(\sigma(v_1), \dots, \sigma(v_n))^\top$ is an eigenvector for the eigenvalue $-d$ by showing that $\mathbf{x} = (-d \cdot \sigma(v_1), \dots, -d \cdot \sigma(v_n))$.

For any $i \in \{1, \dots, n\}$,

$$x_i := \sum_{j=1}^n a_{ij} \cdot \sigma(v_j)$$

Since v_i is only adjacent to vertices which are not in its partition and $\sigma(v_i) = -\sigma(v_j)$ if i, j are in different partitions.

$$x_i := \sum_{j=1}^n a_{ij} \cdot -\sigma(v_i)$$

Moreover, G is d -regular. Thus,

$$x_i := \sum_{i=1}^d 1 \cdot -\sigma(v_i) = -d \cdot \sigma(v_i)$$

Hence, $A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = -d \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top$ and thereby, $-d$ is an eigenvalue of G . □