Problem Problem 29

Theorem 1.1. *For every $k \in \mathbb{N}$ there exists a tree T_k with $\Gamma(T_k) = k$*

Proof. In the following we will show how to construct T_k by induction. Additionally to $\Gamma(T_k) = k$, every T_k will have a greedy coloring so that the root of T_k is colored with color k.

- *Base*
  T_1 is just a single vertex without edges. Obviously $\Gamma(T_1) = 1$ and the color of the root of T_1 is 1.
- *Induction step*
  For any $k > 1$, T_k can be constructed by using one new vertex v and T_1, T_2, ... T_{k-1}. By connecting the roots of T_1, T_2 ... T_{k-1} to v we ensure that there is a greedy coloring in which v has to have the color k. This greedy coloring can be achieved by first coloring T_1, T_2, ...T_{k+1} by induction so that the roots have the colors 1, 2, ... $k-1$. Now we can color v with the color k because by construction all colors smaller than v are already taken by the nodes adjacent to v. Hence, we constructed T_k with a root node v which has color k and $\Gamma(T_k) = k$.

□

Finally, by using the proven theorem we know that $min\{k \in \mathbb{N} | \Gamma(T) \leq k \ for \ all \ trees \ T\} = \infty$.

Theorem 1.2. *For any Graph G $\Gamma(G) \leq max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$*

Proof. For the sake of contradiction let's assume that there is a graph G with $\Gamma(G) = k$ and $k > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$. Additionally, let G be colored with a greedy coloring using k colors which obviously has to exist if $\Gamma(G) = k$.
Let $v \in V(G)$ $with$ $c(v) = k$ be one of the vertices with the highest color. Now, $deg(v) \geq k - 1$ for $c(v) = k$ to be possible in a greedy coloring, because v has to have atleast $k - 1$ adjacent vertices which are colored in colors 1 through $k - 1$.
By assumption we know that for all neighbours u of v, $k > min\{deg(v), deg(u)\} + 1$. Now we easily see that $deg(u) < deg(v)$ because else $k > min\{deg(v), deg(u)\} + 1 = deg(v) + 1$ but earlier we saw that $deg(v) \geq k - 1$. Hence, we know that for all neighbours u of v the following inequality holds: $k > min\{deg(u), deg(v)\} + 1 = deg(u) + 1 \Leftrightarrow deg(u) < k - 1$.
Now for $c(v) = k$ to hold in a proper greedy coloring we need to find a neighbour u of v with $c(u) = k - 1$ which was colored before v. Because $deg(u) > k - 1$ u only has $k - 2$ neighbours which are not v. Hence if v is not already colored $c(u) < k - 1$ so it is impossible to greedy color v with the color k. This finally leads to a contradiction, hence $k > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$ was a wrong assumption and for every graph $\Gamma(G) > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$ holds. □

Problem Problem 32

Theorem 2.1. *The adjacency matrix of any d-regular graph has an eigenvalue of d.*

Proof. Let $G = (V, E)$ be a d-regular graph and let $A(G) = (a_{ij})_{i,j=1,...,n}$ $(n \in \mathbb{N})$ denote it's adjacency matrix. We show that

$$A(G) \cdot (1, \dots, 1)^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

and thereby that d is an eigenvalue of G. Since G is d-regular, every row sum equals exactly d:

$$\forall i \in [1, \dots, n] : \sum_{j=1}^{n} a_{ij} = d$$

Trivially,

$$A(G) \cdot (1, \dots, 1)^\top = (\sum_{j=1}^{n} a_{1j}, \dots, \sum_{j=1}^{n} a_{nj})^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

□

Theorem 2.2. *The adjacency matrix of any bipartite, d-regular graph has an eigenvalue of $-d$.*

Proof. Let $G = (V, E)$ be a bipartite, d-regular graph and let $A(G) = (a_{ij})_{i,j=1,...,n}$ $(n \in \mathbb{N})$ denote it's adjacency matrix.

For all of G's vertices $v_1, ..., v_n$ $(n \in \mathbb{N})$, let $\sigma(v_i)$ denote a selection function that assigns each vertex it's partition:

$$\sigma(v) = \begin{cases} -1 & \text{if } v \text{ in partition \#1} \\ 1 & \text{if } v \text{ in partition \#2} \end{cases}$$

Let $\mathbf{x}$ denote

$$A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = (x_1, \cdots, x_n)^\top$$

We will prove that $(\sigma(v_1), \dots, \sigma(v_n))^\top$ is an eigenvector for the eigenvalue $-d$ by showing that $\mathbf{x} = (-d \cdot \sigma(v_1), \dots, -d \cdot \sigma(v_n))$.

For any $i \in \{1, \dots, n\}$,

$$x_i := \sum_{j=1}^{n} a_{ij} \cdot \sigma(v_j)$$

Since v_i is only adjacent to vertices which are not in it's partition and $\sigma(v_i) = -\sigma(v_j)$ if i, j are in different partitions.

$$x_i := \sum_{j=1}^{n} a_{ij} \cdot -\sigma(v_i)$$

Moreover, G is d-regular. Thus,

$$x_i := \sum_{i=1}^{d} 1 \cdot -\sigma(v_i) = -d \cdot \sigma(v_i)$$

Hence, $A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = -d \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top$ and thereby, $-d$ is an eigenvalue of G. □