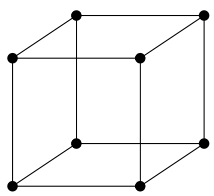
Problem sheet 3

Due date: **November 13, 12:00 am**. Discussion of solutions: November 15.

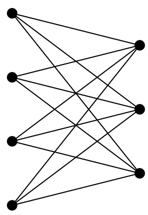
Let $G = (V, E)$ be a graph. Then we define:

- The *diameter of* G is the length of a longest shortest path in G (counted by number of edges). That is, if $\mathrm{dist}(u, v)$ denotes the distance of two vertices u and v in G, then the diameter of G is given by $\max_{u,v\in V} \mathrm{dist}(u, v)$. The diameter of a disconnected graph is defined to be infinity.
- The graph G is called *Hamiltonian* if there exists a cycle in G that contains all vertices of G. Such a cycle is then called a *Hamiltonian cycle*.
- The *girth of* G is the length of a shortest cycle in G. The girth of an acyclic graph is defined to be infinity.
- The *chromatic index of* G, denoted by $\chi'(G)$, is the minimum number of colors needed to color the edges of G so that no two edges of the same color have a common endpoint. Such colorings are also called *proper edge-colorings of* G.

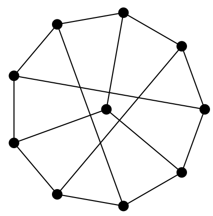
Problem 9. **5 points**
Find order, size, girth and diameter of the following graphs and figure out whether they are Hamiltonian: Q_n for all integer $n \geq 1$, $K_{m,n}$ for all integer $m, n \geq 1$, Petersen's graph. Justify your answer.



hypercube Q_3



complete bipartite graph $K_{4,3}$



Petersen graph

Problem 10. **5 points**
For a natural number $n \geq 2$ let G be the graph of order $2n + 1$ obtained from $K_{n,n}$ by subdividing an edge by a vertex. Show that $\chi'(G) = \Delta(G) + 1 = n + 1$ but $\chi'(G - e) = \Delta(G - e) = n$ for every edge e of G.

Problem 11. **5 points**
For each integer $k > 1$ construct a k-regular graph with no 1-factor. Justify your answer.

Problem 12. **5 points**
Let G be a graph on $2n$ vertices with all degrees at least n. Prove that G has a 1-factor.

Open Problem.
Is every Eulerian graph on n vertices the edge-disjoint union of at most $\frac{1}{2}(n - 1)$ cycles?