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Problem 13

Theorem 1.1. *If a graph has an ear-decomposition, then it is 2-connected.*

Proof. Let $G_1, ..., G_n$ be an ear-decomposition of $G = (V, E)$ (existing by definition).
As far as $G_n = G$, it is sufficient to inductively prove that G_i is 2-connected ($i \in \{1, .., n\}$).

Base $(\mathbf{i = 1})$: G_1 is a cycle and hence, it is 2-connected.

Step $(\mathbf{i \geq 2})$: By defintion $G_i = G_{i-1} + P_i$, P_i is a path and $P_i \cap G_{i-1}$ contains exactly the two endpoints of P_i. By induction, we know that G_{i-1} is 2-connected and by definition, P_i is connected.

To prove that G_i is 2-connected, we have to show that $H := G_i - \{u\}$ ($u \in V(G_i)$) is connected. We differ between two cases:

$\mathbf{u \in G_{i-1}}$ Because G_{i-1} is 2-connected (induction), H is connected.
Furthermore we know that H contains still one or more endpoint of P_i.
Thus, H is a composition of two connected graphs and hence, H is connected.

$\mathbf{u \in P_i - G_{i-1}}$ We know that u is no endpoint of P_i. Thus, $P' := P_i - \{u\}$ is disconnected and a forest containing exactly two trees T_1, T_2.
Nevertheless, each tree of P' contains exactly one endpoint of P_i. Hence, $G_{i-1} + T_1 + T_2$ is still connected because T_1, T_2 and G_{i-1} are connected. As far as $G_{i-1} + T_1 + T_2 = G_{i-1} + P_i - \{u\} = G_i - \{u\} = H$, H is connected.

From these two considerations, we know that H is connected and hence, G_i is 2-connected.

Inductively, we have shown that $\forall i \in \{1, ..., n\}$ G_i is 2-connected.
Thus, $G = G_n$ is 2-connected.

□

Problem 14

For $0 < l < m \leq d$, we will construct a graph $F(d, l, m)$.

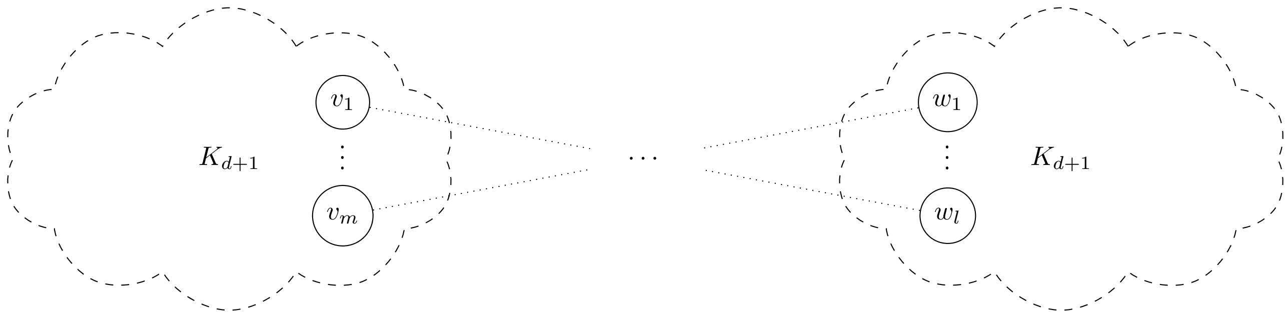

Figure 1: $F(d, l, m)$

First, we construct two complete graphs on $d+1$ vertices. $(V, E) \simeq K_{d+1}$, $(W, E') \simeq K_{d+1}$.

Then, we join m vertices $v_1, ..., v_m \in V$ of the first complete graph and l vertices $w_1, ..., w_l \in W$ of the second such that each v_i has a degree of exactly $d+1$ and each w_j of at least $d+1$ ($i \in [m], j \in [l]$).

Formally, for our constructed graph $F(d, l, m) := (V_F, E_F)$, the vertex set is the union of both complete graphs ($V_F = V \cup W$) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid (i, j) \in [m] \times [l] : (i = j) \vee (i > l, j = l)\} \tag{1}$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{d}$

No degree of a vertex of the complete graphs has been decreased. *Thus,* $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$.

Moreover, we have increased the degree of exactly $l + m < 2(d+1)$ vertices. Indeed, the complete graph on $d+1$ vertices is d-regular and hence there is at least one vertex of degree d in $F(d, l, m)$. *Thus,* $\delta(F(d, l, m)) \leq d$.

From these considerations, $\delta(F(d, l, m)) = d$.

$\kappa(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{l}$

In $F(d, l, m)$, the two complete graphs are only joined by edges between l vertices of one and m vertices of another complete graph. The graph obviously disconnects by removing those first l vertices. *Thus,* $\kappa(F(d, l, m)) \leq l$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa(K_{d+1}) = d > l$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than l vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than l vertices.

From these considerations, $\kappa(F(d, l, m)) = l$.

$\kappa'(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{m}$

In $F(d, l, m)$, the two complete graphs are only joined by exactly m edges and a removal of those m vertices obviously disconnects $F(d, l, m)$. *Thus,* $\kappa'(F(d, l, m)) \leq m$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa'(K_{d+1}) = d > m$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than m edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than m edges.

From these considerations, $\kappa'(F(d, l, m)) = m$.

Problem 15

Theorem 3.1. *The block-cut-vertex graph $G = (V, E)$ of any connected graph $G' = (V', E')$ is a tree.*

Proof. Considering that a tree is an acyclic, connected graph, we prove the aforementioned theorem by contradiction.

First, we will show that G is *acyclic*.

For the sake of contradiction, let's assume for the that G has a cycle $C = (b_1, b_2, ..., b_1)$. We denote G''s subgraphs as $B_1, B_2, ...B_n$. They are the 2-connected components and bridges of G corresponding to the nodes $b_1, b_2, ..., b_n$.

As stated above, B_1 and B_2 are two different subgraphs of G'. Because the corresponding nodes b_1 and b_2 are adjacent in G, B_1 and B_2 have to share a vertex $x \in V(B_1) \cap V(B_2)$. We are able to apply the same argumentation for each pair B_i, B_{i+1}.

Additionally, we know that each component B_j is either 2-connected or a bridge. Thus, we find a circle through all the components $B_1, B_2...B_n$ which is, like every cycle, 2-connected. these blocks are either bridges or maximally 2-connected components by definition (in this case though, we found a larger two connected component).

Thus, G has to be acyclic.

Next, we will prove that G is *connected*.

For the sake of contradiction, let's assume that G is not connected (but G' is connected).

If G is not connected, we have at least two non-connected components in G. Because G is the block-cut-vertex graph of G', each node in G' is represented by at least one component of G (since each vertex in G' is either part of a 2-connected component or is incident to a bridge).

Because G' is connected, there is at least one edge $e = \{u, v\}$ in G' with u and v being in a 2-connected component or bridge which are in G represented as different non-connected components.

Now, there are two cases. Either u and v are connected by an additional path, not using e, then u and v would be 2-connected and share a 2-connected component. Or there is no additional path joining u and v (e is a bridge then).

Either way, u and v would be represented in G by one and the same node which is a direct contradiction with u and v being represented in different components of G.

Thus, G has to be connected if G' is connected. $\square$