# Problem 21

**Lemma 1.0.1.** *In every planar triangulation $G$ on atleast four vertices, there exists a vertex $v$ which does not lie on the outer bound of $G$*

*Proof.* For the sake of contradiction let's assume there is no such vertex $v$. Then all the vertices lie on the outer bound of $G$ forming a cycle with atleast four vertices. This is a contradiction with $G$ being a planar triangulation, because we can add a edge between two not adjacent vertices to $G$ and the result is still planar. Hence, there has to be atleast one vertex $v$ not on the outer bound of $G$. □

**Theorem 1.1.** *Every planar triangulation $G$ on atleast four vertices contains a vertex whose neighbourhood induces a cycle.*

*Proof.* After lemma 1.0.1 there exists a vertex $v$ which does not lie on the outer bound of $G$. Let $N(v) = p_1, p_2, ..., p_n$ be the neighbourhood of $v$. $N(v)$ induces a cycle if all $p_i$ are connected to a cycle and if there are no additionally edges between $p_i$ and $p_j$ with $|i - j| > 1$. We first will proof that all $p_i$ form a cycle, hence the induced subgraph $N(v)$ has a cycle as a subgraph. For the sake of contradiction let's assume this is not the case, hence there is $p_i$ and $p_{i+1}$ which are not connected. This is either a contradiction with $v$ not lying on the outer bound of $G$, or with $G$ being a maximal planar graph. Because if $v_i$ and $v_{i+1}$ are not connected and $v$ is not on the outer bound of $G$ there is a face bounded by $v$,$p_i$,$p_{i+1}$ and atleast one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence $G$ is no triangulation. Now we will proof that either $N(v)$ has no additional edges and thus is a cycle or that one of the vertices adjacent to $v$ induces a cycle.

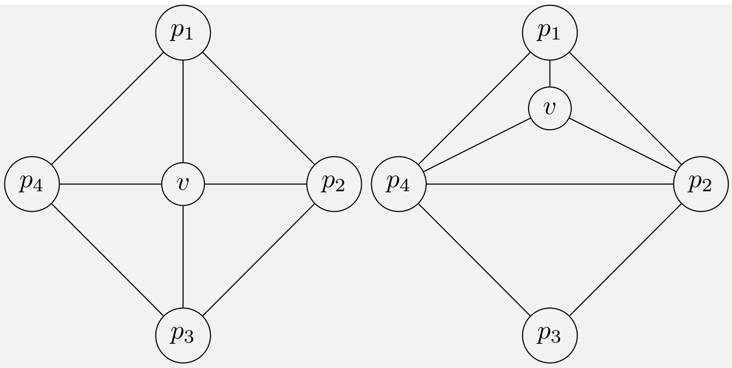
- *There are no additional edges connecting two vertices $p_i$ and $p_j$ with $|i - j| > 1$.*
  In this case $N(v)$ is a cycle and we found a vertex whose neighbourhood induces a cycle.
- *There is an edge $p_i p_{i+2}$*
  In this case the neighbourhood induces graph of $p_{i+1}$ is a cycle namely $p_i p_{i+1} v$.
- *There is an edge between to vertices $p_i$ and $p_j$ with $|i - j| > 2$ and without loss of generality $i < v$.*
  This case there is a bounded face in $G$ which is not a triangle. This face is the face bounded by at least the edges $p_i p_{i+1}$, $p_j p_{j-1}$, $p_i p_j$ and atleast one or more edges forming a path from $p_{i+1}$ to $p_{j-1}$. Hence this case can never occur in a planar triangulation.

In summary we now that we can always find a vertex $v$ whose neighbourhood induces a cycle or one of the neighbours of $v$ here called $p_{i+1}$ has a neighbourhood inducing a cycle. □

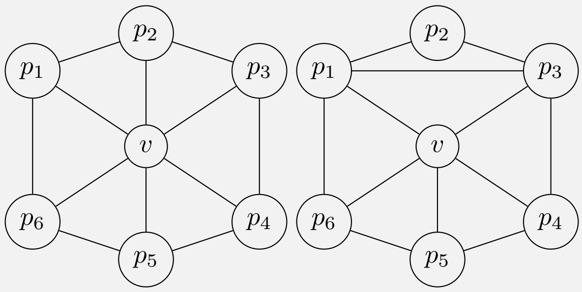
**Lemma 1.1.1.** *Every planar graph $G$ with a maximal number of triangles and atleast 4 vertices has a vertex of degree 3.*

*Proof.* Using theorem 1.1 we can find a vertex $v$ whose neighbourhood induces a cycle $p_1 p_2 ... p_n$. In the following we will show by induction over the degree of $v$ that if $G$ has a maximal number of triangles, $v$ cannot have a degree greater than 3.

**Basis: $v$ has a degree of 4** If $v$ has a degree 4, then the subgraph with the nodes $v, p_1, p_2, p_3, p_4$ only has 4 triangles. We can change the edge set of this subgraph to get a subgraph with the same vertices and the same outer structure but with containing 6 triangles. We can achieve this by removing edge $vp_3$ and adding the edge $p_2 p_4$. Thus we know that if $G$ has a maximal number of triangles $v$ cannot have degree 4.



**Induction step** Let the degree of $v$ be $n > 3$. We now can again change the edges set without changing the outer structure of the induced subgraph $N(v)$ so that $v$ has degree $n - 1$ and the number of triangles in $N(v)$ stays the same. We can achieve this by removing the edge $vp_2$ and adding the edge $p_1p_3$. By induction we can repeat this operation until $n = 4$ and have the base case. Thus we can increase the number of triangles in the subgraph without changing the vertex count or the outer structure. Hence, the degree of $v$ cannot be greater than 3 if $G$ has a maximal number of triangles.



□

**Theorem 1.2.** *Every $n$-vertex planar graph has at most $3n - 8$ triangles.*

*Proof.* Let $G$ be a planar graph with a maximal number of triangles. In the following we will show by induction that a planar graph $G$ with atleast 3 vertices with a maximum number of triangles has exactly $3n - 8$ triangles.

**Base:** $n = 3$ The planar graph with maximum number of triangles is $K3$ which has one triangle. $3n - 8 = 3 * 3 - 8 = 1$ so for the base case the formula is correct.

**Induction Step** Let $G$ be such a graph with $n$ nodes. By using lemma 1.1.1 we can find a vertex $v$ with degree 3 whose neighbourhood induces a cycle. If we remove this node we decrease the number of nodes by one and the number of triangles by 3. By induction we know that the resulting graph has no more than $3 * (n-1) - 8$ triangles. Hence, we know that $G$ has no more than $3 * (n-1) - 8 + 3 = 3 * n - 8$ triangles. □

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# Problem 22

**Theorem 2.1.** *Any $TK_3$-free graph $G$ on $n$ vertices contains a maximum of $n-1$ edges.*

*Proof.* First, $K_3$ is the triangle $C_3$. Subdividing any edge of $C_i$ results in $C_{i+1}$. Moreover, any cycle has a $TC_3 = TK_3$.

Hence, a graph $G$ is $TK_3$-free if and only if it is acyclic. Further we assume that $G$ is connected (since joining two disjoint acyclic components will not create a cycle but increase the edge count).

From these considerations, the maximum number of edges of an $n$-vertex, $TK_3$-free graph equals the maximum number of edges in an $n$-vertex tree. Any $n$-vertex tree contains a maximum of $n-1$ edges. □

**Theorem 2.2.** *If a graph $G$ is 3-connected then $TK_4 \subseteq G$.*

*Proof.* By TUTTE (1961), any 3-connected graph has a construction sequence $G_0, G_1, ..., G_n$ whereby $G_0 = K_4$ and $G_n = G$.

For any $i < n$, $G_i = (V_i, E_i)$ can be constructed by contracting an edge $e = \{x, y\}$ of $G_{i+1}$ ($x, y \in V_{i+1}$, $d(x), d(y) \geq 3$).

Since $d(y) \geq 3$ and contracting $e$ results in $G_i$, we can effectively say that there is a third vertex $z$ in $G_{i+1}$ for which $\{\{x, y\}, \{y, z\}\} \subset E_{i+1}$ is a subdivision of $\{x, z\} \in E_i$.

Thus, $G_{i+1}$ has a $TG_i$ and inductively, by the transitivity of topological minority, $G_{i+1}$ has a $TG_0 = TK_4$. □

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# Problem 24

**Theorem 3.1.** *Every outerplanar graph is 3-colorable.*

*Proof.* Let $G = (V_G, E_G)$ be a outerplanar graph and $G_1, ..., G_k$ the components of $G$. If $\forall i \in \{1, .., k\}$ $G_i$ is 3-colorable, then $G$ itself is 3-colorable. Thus, because $\forall i \in \{1, .., k\}$ $G_i$ is outerplanar, we have to proof, that a connected, outerplanar graph is 3-colorable.

Let $H = (V_H, E_H)$ be a connected, outerplanar graph.
If $|V_H| <= 2$, $H$ can be colored with 3 colors by giving each vertex a different color.
Else, if $|V_H| > 2$, it can be proofed by induction:

- *Base:* $|V_H| = 3$
  Because $H$ contains exactly three vertices, $H$ can be simply colored by giving each vertex a different color. So $H$ is 3-colorable.
- *Step:* $|V_H| > 3$
  Let $E_N$ be the edges of $H$ which are not in the border of the unbounded face. Then two cases have to be considered:
  - $|E_N| = 0$
    In this case, all edges are in the border of the unbounded face. Hence, $H$ is either a circle, thus, is 3-colorable, or $H$ contains a cut vertex. If $H$ contains a cut vertex $v$, $H - \{v\}$ consists of at least two connected components. These components merged with $\{v\}$ and the related edges in $H$, named $C_1, ..., C_n$ $(V = C_1 \cup ... \cup C_n)$, are induced subgraphs of $H$, hence, they are outerplanar. So, by induction, $\forall i \in \{1, ..., n\}$ $C_i$ is 3-colorable. Because colors can be renamed, the color of $v$ can be set and still each $C_i$ is 3-colorable. Moreover, there is no edge between two components of $H$, because all components of $H - \{v\}$ are connected and maximal.
    By these conditions, $H$ is 3-colorable.
  - $|E_N| > 0$
    In this case, there is at least one edge not laying in the border of the unbounded face, hence, laying in another face.
    Let $e = (u, v) \in E_N$. Because $H$ is outerplanar, $e$ cuts $H$ into two connected parts $H_1$ and $H_2$ $(V(H_1) \cap V(H_2) = \{u, v\}$ and $E(H_1) \cap E(H_2) = \{e\})$. Assume there is an edge $e_2$ between $H_1 \setminus \{u, v\}$ and $H_2 \setminus \{u, v\}$. Then either $e_2$ have to cross $e$, or $e_2$ cut w.l.o.g $u$ from the border of the unbounded face. A contradiction in both cases, hence, there is no such edge.
    As before, $H_1$ and $H_2$ are induced subgraphs of $H$, $H_1 \cup H_2 = H$, so both are 3-colorable by induction.
    By these conditions, $H$ is 3-colorable.

All in all, a connected outerplanar graph is 3-colorable.
Hence, because each outerplanar graph consists of connected, outerplanar components, each outerplanar graph is 3-colorable.

$\square$