Problem 21

Lemma 1.0.1. *In every planar triangulation G on atleast four vertices, there exists a vertex v which does not lie on the outer bound of G*

Proof. For the sake of contradiction let's assume there is no such vertex v. Then all the vertices lie on the outer bound of G forming a cycle with atleast four vertices. This is a contradiction with G being a planar triangulation, because we can add a edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be atleast one vertex v not on the outer bound of G. □

Theorem 1.1. *Every planar triangulation G on atleast four vertices contains a vertex whose neighbourhood induces a cycle.*

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G. Let $N(v) = p_1, p_2, ..., p_n$ be the neighbourhood of v. $N(v)$ induces a cycle if all p_i are connected to a cycle. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G, or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v,p_i,p_{i+1} and atleast one additional vertex. This face could be again divided into to smaller faces by adding an edge, hence G is no triangulation. □

Lemma 1.1.1. *Any planar triangulation G hast at most $2n - 4$ faces and $2n - 5$ triangles*

Proof. Every inner face of G is a triangle, hence every face is adjacent to atleast 3 edges. Additionally, every edge is adjacent to exactly two faces, hence $2e \geq 3 * f \Leftrightarrow e \geq \frac{3}{2}f$(e: number of edges, f: number of faces). By Euler's formula we know $2 = n - e + f \Leftrightarrow e - f = n - 2 \Rightarrow \frac{1}{2}f \leq n - 2 \Leftrightarrow f \leq 2n - 4$. Now we know that G has at most $2n - 4$ faces and because the unbounded face cannot be a triangle we know that G has at most $2n - 5$ triangles. □

Theorem 1.2. *Every n-vertex planar graph has at most $3n - 8$ triangles.*

Proof. A planar graph which G is maximal in the number of triangles has to be a triangulation, because if there would be a inner face which is adjacent to more than 3 edges, we could divide the face into triangles thus forming a graph with same number of vertices but more triangles. Now we can apply lemma 1.1.1 and we know that G has at most $2n - 5$ triangles. Now we know that $2n - 5 \leq 3n - 8$ for every $n > 2$, hence we have proven an even stronger theorem. □

Problem 22