Problem 29

Theorem 1.1. *For every $k \in \mathbb{N}$ there exists a tree T_k with $\Gamma(T_k) = k$*

Proof. In the following we will show how to construct T_k by induction. Additionally to $\Gamma(T_k) = k$, every T_k will have a greedy coloring so that the root of T_k is colored with color k.

- *Base*
  T_1 is just a single vertex without edges. Obviously $\Gamma(T_1) = 1$ and the color of the root of T_1 is 1.
- *Induction step*
  For any $k > 1$, T_k can be constructed by using one new vertex v and T_1, T_2, ... T_{k-1}. By connecting the roots of T_1, T_2 ... T_{k-1} to v we ensure that there is a greedy coloring in which v has to have the color k. This greedy coloring can be achieved by first coloring T_1, T_2, ...T_{k+1} by induction so that the roots have the colors 1, 2, ... $k-1$. Now we can color v with the color k because by construction all colors smaller than v are already taken by the nodes adjacent to v. Hence, we constructed T_k with a root node v which has color k and $\Gamma(T_k) = k$.

□

Finally, by using the proven theorem we know that $min\{k \in \mathbb{N} | \Gamma(T) \leq k \ for \ all \ trees \ T\} = \infty$.

Theorem 1.2. *For any Graph G $\Gamma(G) \leq max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$*

Proof. For the sake of contradiction let's assume that there is a graph G with $\Gamma(G) = k$ and $k > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$. Additionally, let G be colored with a greedy coloring using k colors which obviously has to exist if $\Gamma(G) = k$.
Let $v \in V(G) \ with \ c(v) = k$ be one of the vertices with the highest color. Now, $deg(v) \geq k-1$ for $c(v) = k$ to be possible in a greedy coloring, because v has to have atleast $k-1$ adjacent vertices which are colored in colors 1 through $k-1$.
By assumption we know that for all neighbours u of v, $k > min\{deg(v), deg(u)\} + 1$. Now we easily see that $deg(u) < deg(v)$ because else $k > min\{deg(v), deg(u)\} + 1 = deg(v) + 1$ but earlier we saw that $deg(v) \geq k-1$. Hence, we know that for all neighbours u of v the following inequality holds: $k > min\{deg(u), deg(v)\} + 1 = deg(u) + 1 \Leftrightarrow deg(u) < k-1$.
Now for $c(v) = k$ to hold in a proper greedy coloring we need to find a neighbour u of v with $c(u) = k-1$ which was colored before v. Because $deg(u) > k-1$ u only has $k-2$ neighbours which are not v. Hence if v is not already colored $c(u) < k-1$ so it is impossible to greedy color v with the color k. This finally leads to a contradiction, hence $k > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$ was a wrong assumption and for every graph $\Gamma(G) > max_{uv \in E(G)} min\{deg(u), deg(v)\} + 1$ holds. □

Problem 31

Let $G = (V = \{v_1, ..., v_n\}, E)$ be a graph.

Lemma 2.0.1. $(A(G)^k)[i, j]) = a^k_{i,j}$ *represents the amount of* v_i*-*v_j*-paths of lengt hk* $(i, j \in \{1, ..., n\}$.

proof *Prove by induction ofer k.*

- Base: $k = 1$
  $A(G)[i, j] = \{e \in E \ : \ e = v_i v_j\} = \{p \ - \ path \ : \ p = (v_i, v_j)\} \ \in \{0, 1\}$
- Step: $k \geq 1$
  $(A(G)^k)[i, j] = (A(G)^{k-1} * A(G))[i, j] = (\sum_{a=1}^n \sum_{b=1}^n \sum_{d=1}^n (A(G)^{k-1})[a, d] * A(G)[d, b])[i, j] = \sum_{d=1}^n (A(G)^{k-1})[i, d] * A(G)[d, j]$.
  By induction $(A(G)^{k-1})[i, d]$ *is the amount of* v_i*-*v_d*-paths of length* $k - 1$*, and* $A(G)[d, j]$ *is the amount of* v_d*-*v_j*-paths of length* 1*. So* $(A(G)^{k-1})[i, d] * A(G)[d, j]$ *is the amount of* v_i*-*v_j*-paths of length* k *containing* $v_d v_j$*. To get all paths of length k, all possible incident edges of* v_j *have to be checked.*
  So all in all, $(A(G)^k)[i, j] = \sum_{d=1}^n (A(G)^{k-1})[i, d] * A(G)[d, j]$ *is the amount of all* v_i*-*v_j*-paths of length* k.

- (a)$tr(A(G)) = 0$
  Because G doesn't have any self-loops, $A(G)[i, i] = 0 \ \forall i \in \{1, ..., n\}$.
  Therefore: $tr(A(G)) = \sum_{i=1}^n A[i, i] = \sum_{i=1}^n 0 = 0$

- (b)$tr(A(G)^2$ *equals* twice the number of edges of G.
  $tr(A(G)^2) = \sum_{i=1}^n \sum_{k=1}^n A[i, k] * A[k, i] = \sum_{i=1}^n \sum_{k=1}^n A[k, i]^2$
  $= \sum_{i=1}^n \sum_{k=1}^n A[i, k] = \sum_{i=1}^n d(v_n) = 2 * |E|$.
- (c)$tr(A(G)^3)$ *equals* six times the number of triangles in G.
  As proven in *Lemma* 2.0.1, $(A(G)^k)[i, i]$ is the amount of all v_i-v_i-*paths of length* 3.
  Because a closed path of length 3 is a triangle, $(A(G)^k)[i, i]$ is the amount of triangles containing v_i.
  Moreover, a triangle $t = (v_i, v_j, v_k)$ can be interpreted as two different paths starting/ending at v_i: $p_1 = v_i v_j v_k v_i$ and $p_2 = v_i v_k v_j v_i$. Thus, for each triangle there are two paths.
  All in all, let t be a triangle in G. Then for each vertex in t there are two paths of length 3. This makes $3 * 2 = 6$ paths in $tr(A(G)^3)$ for each triangle in G.
  Because $tr(A(G)^3)$ is the amount of all *closed paths of length* 3 $(v \in V(G))$, and a triangle is a closed path of length 3, $tr(A(G)^3) = \sum_{i=1}^n (2 * |\{t \ - \ triangle\ in\ G \ : \ t\ contains\ v_i\}|)$ is exactly six times the amount of all triangles in G.

Problem 32

Theorem 3.1. *The adjacency matrix of any d-regular graph has an eigenvalue of d.*

Proof. Let $G = (V, E)$ be a d-regular graph and let $A(G) = (a_{ij})_{i,j=1,...,n}$ $(n \in \mathbb{N})$ denote it's adjacency matrix. We show that

$$A(G) \cdot (1, \dots, 1)^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

and thereby that d is an eigenvalue of G. Since G is d-regular, every row sum equals exactly d:

$$\forall i \in [1, \dots, n] : \sum_{j=1}^{n} a_{ij} = d$$

Trivially,

$$A(G) \cdot (1, \dots, 1)^\top = (\sum_{j=1}^{n} a_{1j}, \dots, \sum_{j=1}^{n} a_{nj})^\top = (d, \dots, d)^\top = d \cdot (1, \dots, 1)^\top$$

□

Theorem 3.2. *The adjacency matrix of any bipartite, d-regular graph has an eigenvalue of $-d$.*

Proof. Let $G = (V, E)$ be a bipartite, d-regular graph and let $A(G) = (a_{ij})_{i,j=1,...,n}$ $(n \in \mathbb{N})$ denote it's adjacency matrix.

For all of G's vertices $v_1, ..., v_n$ $(n \in \mathbb{N})$, let $\sigma(v_i)$ denote a selection function that assigns each vertex it's partition:

$$\sigma(v) = \begin{cases} -1 & \text{if } v \text{ in partition } \#1 \\ 1 & \text{if } v \text{ in partition } \#2 \end{cases}$$

Let $\mathbf{x}$ denote

$$A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = (x_1, \cdots, x_n)^\top$$

We will prove that $(\sigma(v_1), \dots, \sigma(v_n))^\top$ is an eigenvector for the eigenvalue $-d$ by showing that $\mathbf{x} = (-d \cdot \sigma(v_1), \dots, -d \cdot \sigma(v_n))$.

For any $i \in \{1, \dots, n\}$,

$$x_i := \sum_{j=1}^{n} a_{ij} \cdot \sigma(v_j)$$

Since v_i is only adjacent to vertices which are not in it's partition and $\sigma(v_i) = -\sigma(v_j)$ if i, j are in different partitions.

$$x_i := \sum_{j=1}^{n} a_{ij} \cdot -\sigma(v_i)$$

Moreover, G is d-regular. Thus,

$$x_i := \sum_{i=1}^{d} 1 \cdot -\sigma(v_i) = -d \cdot \sigma(v_i)$$

Hence, $A(G) \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top = \mathbf{x} = -d \cdot (\sigma(v_1), \dots, \sigma(v_n))^\top$ and thereby, $-d$ is an eigenvalue of G. □