

Solution sheet 1

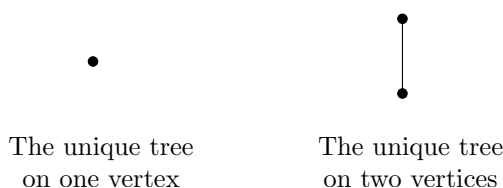
Date: October 31.

Discussion of solutions: none.

Problem 1.**5 points**Show that any tree T has at least $\Delta(T)$ leaves.**Solution. (Variant 1: Induction)**

Let $T = (V, E)$ be any tree and $\Delta(T)$ be the maximum degree of T . We shall show by induction on the number $|V|$ of vertices of T that T has at least $\Delta(T)$ leaves.

The claim is true when T is the empty tree or the unique tree of order 1, which both have no leaf and maximum degree zero. It is also true for the unique tree of order 2, which has two leaves and maximum degree one. Note that in this case T has actually $\Delta(T) + 1$ leaves.



Now assume that $|V| > 2$, and hence $\Delta(T) \geq 2$, and that the claim holds for all trees of order strictly less than $|V|$. Let v be a leaf of T and u be its neighbor. (The existence of a leaf has been proven in the lecture.) Consider the graph $T' = T - v$ of order $|V| - 1$. It has been shown in the lecture that T' is also a tree. Thus, we can apply induction to T' and conclude that T' has at least $\Delta(T')$ leaves. Clearly, the maximum degree of T' is either $\Delta(T)$ or $\Delta(T) - 1$. We distinguish these two cases.

Case 1: If $\Delta(T') = \Delta(T)$, then all leaves of T' are leaves of T , with possibly the exception of u . Together with v we obtain that there are at least $\Delta(T') - 1 + 1 = \Delta(T') = \Delta(T)$ leaves.

Case 2: Now if $\Delta(T') = \Delta(T) - 1$, then this is only possible if u is the only vertex with degree $\Delta(T)$ in T . If $\Delta(T') \geq 2$, then u is no leaf in T' . Hence the leaves of T are precisely the leaves of T' plus v . In particular, there are at least $\Delta(T') + 1 = \Delta(T) - 1 + 1 = \Delta(T)$ leaves in T . On the other hand if $\Delta(T') = 1$, then from the induction base we know that T' has $\Delta(T') + 1 = 2$ leaves. Again, by replacing the leaf u in T' with the leaf v in T we obtain $\Delta(T') + 1 = \Delta(T)$ leaves in T .

We have shown that in all cases a tree T of maximum degree $\Delta(T)$ has at least $\Delta(T)$ leaves, which concludes the proof. \square

Solution. (Variant 2: Counting)

Let $T = (V, E)$ be any tree and $\Delta(T)$ be the maximum degree of T . Moreover, assume without loss of generality that $|V| \geq 1$. (The claim clearly holds for the empty tree.) Let $L \subseteq V$ be the set of leaves in T and $N = V \setminus L$ the set of non-leaves in T . We shall show that $|L| \geq \Delta(T)$, i.e., that T has at least $\Delta(T)$ leaves, by counting the degrees of vertices in L and N separately.

Let u be a vertex with maximum degree $\Delta(T)$. In the lecture it was proven that the sum of all vertex degrees is twice the number of edges. It was moreover shown that any non-empty tree on $|V|$ vertices has precisely $|V| - 1$ edges. Hence we have

$$\begin{aligned}
2 \cdot (|V| - 1) &= \sum_{v \in V(T)} d(v) \\
&= \sum_{v \in L} d(v) + \sum_{v \in N \setminus \{u\}} d(v) + d(u) \\
&\geq \sum_{v \in L} 1 + \sum_{v \in N \setminus \{u\}} 2 + \Delta(T) \\
&= |L| \cdot 1 + (|V| - |L| - 1) \cdot 2 + \Delta(T) \\
&= 2 \cdot (|V| - 1) - |L| + \Delta(T).
\end{aligned}$$

From this it follows $|L| \geq \Delta(T)$, as desired. \square

Problem 2.

5 points

We define the following properties for any graph G .

- (P1) G is acyclic.
- (P2) The removal of any edge in G increases the number of connected components.
- (P3) Adding an edge between any two non-adjacent vertices in G introduces a cycle.
- (P4) Any two vertices are joined by a unique path in G .

Show each of the following implications directly.

- (i) $(P2) \implies (P1)$
- (ii) $(P1) \text{ and } (P3) \implies (P4)$
- (iii) $(P4) \implies (P2)$

Solution.

Let $G = (V, E)$ be any graph. We shall show each of the three claimed implications directly, that is, we avoid to mention any of the four properties (P1)–(P4) not involved in the claimed implication.

- (i) We shall prove the implication $(P2) \implies (P1)$ by contraposition. That is, we assume that G fails to have property (P1) and shall show that then G also fails to have property (P2).

So, because G is *not* acyclic, it contains a cycle C . Let uw be any edge on C . We consider the graph $G' = (V, E - uw)$ and claim that G' has as many connected components as G (and not more), which means that G does not have property (P2).

Let x, y be two vertices in V . We shall show that there exists an x - y -path in G' provided that there exists an x - y -path in G . Let P be a path in G from x to y . If P does not contain the edge uw then P is also a path from x to y in G' , as desired. On the other hand if P contains the edge uw , we define a walk W from x to y in G' by replacing the edge uw in P with the u - w -path $C - uw$ in G' . Here we use the fact that C is a cycle. In the lecture it was proven that if there is a walk from x to y then there is also a path from x to y , i.e., x and y are still in the same connected component of G' .

- (ii) We shall prove the implication (P1) and (P3) \implies (P4) again by contraposition. That is, we assume that G fails to have property (P4) and shall show that then G fails to have property (P1) or fails to have property (P3).

So, because G fails to have property (P4) there exists two vertices u, w in G that are *not* joined by a unique path in G . Hence there exist either no u - w -path in G or at least two distinct u - w -paths in G . We distinguish these two cases.

Case 1: If there is no path between u and w in G , then we consider the graph $G' = (V, E \cup uw)$. That is, we add an edge between the two non-adjacent vertices u and w . If this would introduce a cycle C then $C - uw$ would be a u - w -path in G , which does not exist by the assumption in this case. Hence, adding uw does *not* introduce a cycle, which means that G fails to have property (P3), as desired.

Case 2: If there are at least two distinct paths P_1, P_2 between u and w in G , then we consider the closed walk W which is the concatenation of P_1 and the reversal of P_2 . This is indeed a closed walk since P_1 and P_2 have the same endpoints. Moreover, because $P_1 \neq P_2$ by assumption in this case, there is at least one non-repeated edge in W . It was shown in the lecture that under these assumptions G contains a cycle. In particular G fails to have property (P1).

Assuming that G contains two vertices that are not joined by a unique path, we have shown that then G fails to have property (P3) (Case 1) or fails to have property (P1) (Case 2). This concludes the proof.

- (iii) We shall prove the implication (P4) \implies (P2) directly. We assume that G has property (P4), consider any edge $uw \in E$ and show that the graph $G' = (V, E - uw)$ has more connected components than G .

First note that because every two vertices in G are joined by some path, we have that G is connected, i.e., has exactly one connected component. Now let uw be any edge in G . Clearly, this edge forms a path between u and w in G and by assumption this is the only such path. In particular, every path between u and w contains the edge uw . In other words there is no path between u and w in $G' = (V, E - uw)$, that is G' is disconnected. I.e., G' has at least two connected components, which is strictly more than the number of connected components in G . Since the edge uw was chosen arbitrarily G has property (P2), which concludes the proof.

□

Problem 3.

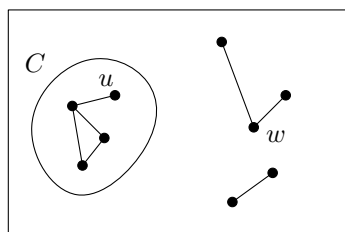
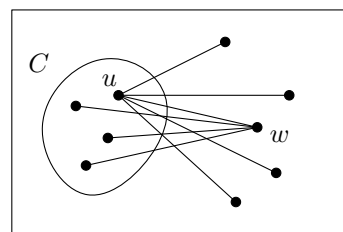
5 points

Prove that either a graph or its complement is connected.

Solution.

Let $G = (V, E)$ be any non-empty graph. We assume that the graph G is not connected and shall argue that the complement $\overline{G} = (V, \binom{V}{2} \setminus E)$ of G is connected, i.e., that there exists a connected subgraph of \overline{G} spanning the entire vertex set V .

Since G is assumed to be disconnected, we find two vertices $u, w \in V$ and a connected component C of G such that $u \in C$ and $w \notin C$. Now in \overline{G} all vertices outside of C are adjacent to u and all vertices in C are adjacent to w . And in particular $uw \in E(\overline{G})$. So all vertices lie in a single connected component of \overline{G} , which is therefore connected.

the graph G a connected spanning subgraph of \overline{G}

□

Problem 4.**5 points**

Prove or disprove that if u and v are the only vertices of odd degree in G then there is a u - v -path in G .

Solution.

Let $G = (V, E)$ be any graph and assume that $u \neq v$ are the only vertices of G of odd degree in G . We shall show that there indeed is a u - v -path in G . We assume for the sake of contradiction that there is no u - v -path in G .

By the definition of connectedness it follows that u and v are not in the same connected component of G , say u lies in the connected component C with $v \notin C$. Since by assumption u and v are the only vertices of odd degree, all vertices in C different from u have even degree. In particular, C contains exactly one vertex of odd degree. As proven in the lecture, the sum of all degrees of vertices in C is twice the number of edges in C , and hence even. On the other hand this sum has exactly one odd summand, which implies that the sum is odd – a contradiction. □