Problem 9

Theorem 1.1. *A hypercube Q_n is Hamiltonian. It has a girth of 4 for $n \geq 2$ and ∞ otherwise. It's diameter is n, it's order 2^n and it has a size of $2^{n-1} \cdot n$.*

Proof. Let S be a set of cardinality $|S| = n$. We construct $Q_n = (V_Q, E_Q)$ by creating a vertex for each subset of S and moreover add edges between those subsets which differ by only one element. In the following, we may use binary representations of the vertices of Q_n since $V_Q = \mathcal{P}(S) \cong (\mathbb{Z}/2\mathbb{Z})^n$ (we can denote a 1 for including an element and a 0 for excluding an element in a subset).

Order: Since $V_Q = \mathcal{P}(S)$ and $|\mathcal{P}(S)| = 2^n$, *the order of Q_n is 2^n.*

Size: Each of the 2^n vertices is adjacent to n other vertices since we can insert / remove each of the n elements of S. For undirected edges, we have $\frac{2^n \cdot n}{2} = 2^{n-1} \cdot n$ edges. *Thus, the size of Q_n is $2^{n-1} \cdot n$.*

Girth: We differ between two cases.

- **Case 1:** $n = 1$. Our graph contains exactly one edge and is therefore acyclic. *Hence, the girth is ∞ for $n = 1$.*
- **Case 2:** $n \geq 2$ Our graph contains the cycle $(\emptyset, \{a\}, \{a, b\}, \{b\}, \emptyset)$ $(a, b \in S)$ which has length 4.

  A shorter cycle $(A,\ B,\ C,\ A)$ $(A, B, C \in V_Q)$ does not exist due to the property that two adjacent vertices differ by exactly one element. For such a cycle, B and A differed by one element, and hence A and C differed by two or are equal. However, a difference of zero or two elements between two consecutive elements renders any walk invalid. The edge $\{C, A\}$ could not be contained in Q_n.

  From these considerations, for $n \geq 2$, the girth is 4.

Diameter: For any set $A \in V_Q$, we are able to get to any other element $B \in V_Q$ by inserting or removing a maximum of n elements. Thus, a path of length n is sufficient to walk from any A to any B. Furthermore, there exist A and B such that a path of length n is the shortest path between them. E.g. $A = \emptyset \in V_Q$, $B = S \in V_Q$. *Thus, the diameter of Q_n is n.*

Hamiltonian: A Hamiltonian cycle is equivalent to an enumeration of $(\mathbb{Z}/2\mathbb{Z})^n$ in which consecutive elements differ by exactly one element. We provide such an enumeration: the *Gray Code*[a]. *Thus, there exists a Hamiltonian cycle and Q_n is Hamiltonian.* □

[a] For $n = 2$: 00,01,11,10,00. Generally, the k'th vertex in the Hamiltonian cycle is $k \otimes \lfloor \frac{k}{2} \rfloor$ whereby $\cdot \otimes \cdot$ denotes the exclusive or.

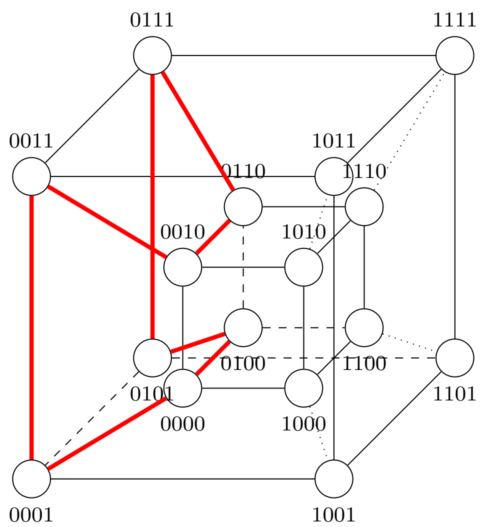

Figure 1: The hypercube Q_4 and the Hamiltonian cycle of Q_3 as a subgraph of Q_4.

Theorem 1.2. *A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n$. It's girth is 4 for $m, n \neq 1$ and and ∞ otherwise. It's diameter is 1 for $m = n = 1$ and 2 otherwise. The graph's order is $m + n$ and it's size is $m \cdot n$.*

Proof. Let $V = \{v_1, ..., v_m\}$ and $W = \{w_1, ..., w_n\}$ denote the two partitions of $K_{m,n} = (V_K, E_K)$.

Order: The first partition has m elements, the second n elements. *Thus, $K_{m,n}$ has an order of $m+n$.*

Size: Each of the m elements of the first partition are connected to each of the n elements in the second partition. *Thus, $K_{m,n}$ has a size of $m \cdot n$.*

Girth: If either $m = 1$ or $n = 1$, then all vertices of one partition are indicent to and only to the single vertex of the other partition. Hence, there is no cycle in $K_{1,n}$ or $K_{m,1}$ and *the girth of $K_{m,n}$ is ∞ if $n = 1$ or $m = 1$.*

If $m, n \neq 1$, each cycle must have even length since any two consectuive vertices in a path of $K_{n,m}$ are in different partitions. Thus, we require an even amount of edge-crossings to enclose a walk. Any cycle has a length of at least 3, thus the girth of $K_{m,n}$ has to exceed or be equal to 4.
Furthermore, we find such a cycle of length 4 easily since both partitions V, W have at least 2 vertices: $(v_1, w_1, v_2, w_2, v_1)$. *From these considerations, the girth of $K_{m,n}$ must be equal to 4.*

Diameter: For $m = n = 1$, there are exactly two vertices in different partitions. They have a distance of 1 and thus, *the diameter of $K_{1,1}$ is 1.*

Since two consectuive vertices in a path of $K_{n,m}$ are in different partitions V, W the distance between two vertices in the same partition has to be at least 2. Moreover, we find a path of distance 2 between $v_1 \in V$ to $v_2 \in V$: (v_1, w, v_2) for any $w \in W$. Thus, the diameter does not deceed 2.
For any two vertices in different partitions V, W, they are directly connected by a path of length 1.
Hence, the diameter of $K_{m,n}$ is 2 if not $m = n = 1$.

Hamiltonian: For $m = n$, we find always find a Hamiltonian cycle: $(v_1, w_1, v_2, w_2, ..., v_n, w_n, v_1)$.
However, for $m \neq n$, there can not exist a Hamiltonian cycle.
We assume such a Hamiltonian cycle $c = (u_1, u_2, ..., u_{n+m}, u_1)$ existed for $m \neq n$.

- **Case 1:** $m + n$ odd. Then, u_{n+m} and u_1 were in the same partition which rendered the edge $\{u_{n+m}, u_1\}$ invalid. *Hence, $K_{m,n}$ is not Hamiltonian for an odd $n + m$.*
- **Case 2:** $m + n$ is even. Then, there was an $n_0 < n + m$ such that $(u_1, ..., u_{n_0})$ is the shortest sub walk that covers one partition but not both. Again, we inspect two cases.
  - $\mathbf{n_0 = m + n - 1}$. Then, both partitions have the same number of vertices which is contradictory to our precondition that $m \neq n$.
  - $\mathbf{n_0 < m + n - 1}$. Then, we are trapped in one partition for we are not able to cover two vertices of the same partition consecutively.

Hence, $K_{m,n}$ $(m \neq n)$ is not Hamiltonian for an even $m + n$.

All in all, $K_{m,n}$ is Hamiltonian if and only if $m = n$. □

Theorem 1.3. *The Petersen graph is not Hamiltonian, it has a girth of* 5*, a diameter of* 2*, an order of* 10 *and a size of* 15*.*

Proof. **Order:** The graph has 10 vertices and thus, it's order is 10.

Size: The graph has 15 edges and thus, it's size is 15.

Girth:

Diameter:

Hamiltonian:

□

Problem 11

For each even integer $k > 1$, the complete graph $K_{(n+1)}$ is a k-regular graph without a 1-factor. For each odd $k > 1$, we are able to construct a k-regular graph without a 1-factor in the following way.

In order to guarantee that the graph has no 1-factor, we can use Tutte's theorem. We construct the graph by starting with a single vertex $v \in V$ connected to k subgraphs S which are not inter-connected. Then, we

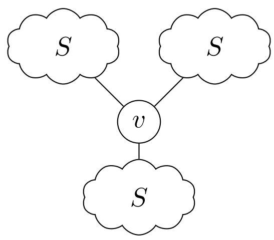

Figure 2: Example with k=3

construct $S = (V, E)$ in such a way that $|V|$ is odd and exactly one vertex $u \in V$ has degree $k - 1$ while all other vertices have degree k.

If we then connect u to v, we obtain a k-regular graph. If we removed v we would have k components S with an odd number of vertices. Furthermore, $k > 1$ and thus, by Tutte's theorem, we know that the resulting graph has no 1-factor.

In order to contruct S we first need the following lemma.

Lemma 2.0.1. *For any odd integer $k > 1$ it is possible to construct a $(k-1)$-regular graph $G = (V, E)$ with $k + 1$ vertices.*

Proof. We can obtain G from K_{k+1} by removing all edges from K_{k+1} which are contained in a perfect matching.

By removing those edges, the degree of every vertex decreases exactly by one. K_{k+1} is by definition k-regular and hence G is $k - 1$-regular.

A perfect matching in K_{k+1} exists, because $k + 1$ is even, and the conditions of Tutte's theorem are always satisfied in a complete graph. Moreover, we find such a matching by randomly choosing edges $\{u, v\}$ and removing u and v from K_{k+1}. $\square$

Constructing a connected graph $S = (V, E)$ with $|V| = k + 2$ and the degree sequence $(k, k, ..., k, (k-1))$. First we construct a $(k-1)$-regular graph $S' = (V', E')$ with $k + 1$ vertices as described in the aforementioned way. Then, we can add one vertex to S' and connect it to all except one vertices in V'. Thus, we have a new graph S. Because we have added only one vertex $|V| = |V'| + 1 = k + 2$ and the degree of the newly added vertex is k, the degree of all the other vertices except of the last one is increased by one.

Hence, S hat the degree sequence $(k, k, ..., k, (k-1))$ and we have constructed a k-regular graph without a perfect matching.

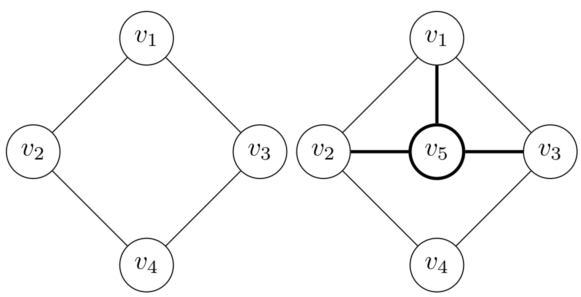

Figure 3: S' and S for k=3

Problem 12

Theorem 3.1. *Any graph G with $2n$ vertices and $\delta(G) \geq n$ has a 1-factor.*

Proof. Let $G = (V, E)$ be a graph with $|V| = 2n$ and $d(v) \geq n \, \forall v \in V$.

In the following, we will prove for nontrivial G that the number of odd components in a graph $G - S$ deceeds the number of vertices in S. By *Tutte's Matching Theorem*, G then has a 1-factor.

$\mathbf{n = 1}$ Then, G is a simple graph with two vertices that are connected by one edge. This is of course a perfect matching of G.

$\mathbf{n \geq 2}$ Let $S \subseteq V$ be a set of vertices, $G' = (V', E') := G - S$ and $k := |S|$.
As G' is created by removing all vertices of S and their incident edges from G, we obtain the following properties:

- $\forall v \in V : d(v) \geq n - k$
- For any component C of G', $|V(C)| \geq n - k + 1$
- the order of G is $2n - k$

In the following cases, we prove that $\lambda := \#\textit{odd components} \leq k$.

$\mathbf{k = 0}$:

Because G consists of one even component, $\lambda = 0 \leq k$

$\mathbf{k = 1}$:

After removing any vertex of G, the degree of a vertex in G' is reduced by one or less. Hence, $\forall v \in V : d(v) >= n - 1$. This implies that the size of any component in G' is at least n. As far as $|V'| = 2n - 1$ there can only exist one component in G' with order $2n - 1$ (which is odd).

All in all, we have shown that $\lambda = 1 \leq k$.

$\mathbf{2 \leq k \leq n}$:

As the minimum size of a component in G' is $n - k + 1$ and $|V'| = 2n - k$, we can bound the amount of components by the following term: $\frac{2n-k}{n-k+1}$.

We now have to prove that $\frac{2n-k}{n-k+1} \leq k \iff 2n - k \leq k * n - k^2 + k \iff 0 \leq (k-2)n - k^2 + 2k =: f(k)$.

To prove this inequality, we have to determine the minimum value of f in the defined boundaries.

- $f'(k) = n - 2k + 2 \stackrel{!}{=} 0 \iff k = \frac{n}{2} + 1$
- $f''(k) = -2$

So f has a maximum but no local minimum. To find the minimum value within the given range, we have to check the borders:

$f(2) = 0 = f(n)$. Thus, $\min(f) = 0 \geq f(k)$ which proves the inequality.

All in all, we have shown that the number of compomonents deceeds or is equal to k. This implies that $\lambda \leq k$.

$\mathbf{n \leq k \leq 2n}$:

Because removing one vertex can not increase the number of components by more than one and $V' = 2n - k \leq n \leq k$, there can not be more than k odd components.

Hence, $\lambda \leq k$.

Finally, we have shown that the number of resulting components in G' is bounded by the order of S. In other words: $\forall S \subseteq V(G) : \#\textit{odd components of } G - S \leq |S|$.

Thus, we have shown that all conditions for *Tutte's Matching Theorem* are satisfied. Hence, G has a pefect matching aka 1-factor.

□