J. Batzill (1698622), M. Franzen (1696933), J. Labeit (1656460)

Problem 21

Lemma 1.0.1. *In every planar triangulation G on atleast four vertices, there existes a vertex v which does not lie on the outer bound of G*

Proof. For the sake of contradiction let's assume there is no such vertex v. Then all the vertices lie on the outer bound of G forming a cycle with atleast four vertices. This is a contradiction with G beeing a planar triangulation, because we can add a edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be atleast one vertex v not on the outer bound of G. □

Theorem 1.1. *Every planar triangulation G on atleast four vertices contains a vertex whose neighborhood induces a cycle.*

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G. Let $N(v) = p_1, p_2, ..., p_n$ be the neighborhood of v. $N(v)$ induces a cycle if all p_i are connected to a cycle. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G, or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v,p_i,p_{i+1} and atleast one additional vertex. This face could be again devided into to smaller faces by adding an edge, hence G is no triangulation. □

Theorem 1.2. *Every n-vertex planar graph has at mose $3n - 8$ triangles.*

Proof. □

Problem 22