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Problem 9

Theorem 1.1. *A hypercube Q_n is Hamiltonian. It has a girth of 4 for $n \geq 2$ and ∞ otherwise. It's diameter is n, it's order 2^n and it has a size of $2^{n-1} \cdot n$.*

Proof. Let S be a set of cardinality $|S| = n$. We construct $Q_n = (V_Q, E_Q)$ by creating a vertex for each subset of S and moreover add edges between those subsets which differ by only one element. In the following, we may use binary representations of the vertices of Q_n since $V_Q = \mathcal{P}(S) \cong (\mathbb{Z}/2\mathbb{Z})^n$ (we can denote a 1 for including an element and a 0 for excluding an element in a subset).

Order: Since $V_Q = \mathcal{P}(S)$ and $|\mathcal{P}(S)| = 2^n$, *the order of Q_n is 2^n.*

Size: Each of the 2^n vertices is adjacent to n other vertices since we can insert / remove each of the n elements of S. For undirected edges, we have $\frac{2^n \cdot n}{2} = 2^{n-1} \cdot n$ edges. *Thus, the size of Q_n is $2^{n-1} \cdot n$.*

Girth: We differ between two cases.

- **Case 1:** $n = 1$. Our graph contains exactly one edge and is therefore acyclic. *Hence, the girth is ∞ for $n = 1$.*
- **Case 2:** $n \geq 2$ Our graph contains the cycle $(\emptyset, \{a\}, \{a,b\}, \{b\}, \emptyset)$ $(a, b \in S)$ which has length 4.

  A shorter cycle (A, B, C, A) $(A, B, C \in V_Q)$ does not exist due to the property that two adjacent vertices differ by exactly one element. For such a cycle, B and A differed by one element, and hence A and C differed by two or are equal. However, a difference of zero or two elements between two consecutive elements renders any walk invalid. The edge $\{C, A\}$ could not be contained in Q_n.

  From these considerations, for $n \geq 2$, the girth is 4.

Diameter: For any set $A \in V_Q$, we are able to get to any other element $B \in V_Q$ by inserting or removing a maximum of n elements. Thus, a path of length n is sufficient to walk from any A to any B. Furthermore, there exist A and B such that a path of length n is the shortest path between them. E.g. $A = \emptyset \in V_Q$, $B = S \in V_Q$. *Thus, the diameter of Q_n is n.*

Hamiltonian: A Hamiltonian cycle is equivalent to an enumeration of $(\mathbb{Z}/2\mathbb{Z})^n$ in which consecutive elements differ by exactly one element. We provide such an enumeration: the *Gray Code*[a]. *Thus, there exists a Hamiltonian cycle and Q_n is Hamiltonian.* □

[a] For $n = 2$: 00,01,11,10,00. Generally, the k'th vertex in the Hamiltonian cycle is $k \otimes \lfloor \frac{k}{2} \rfloor$ whereby $\cdot \otimes \cdot$ denotes the exclusive or.

Theorem 1.2. *A complete bipartite graph $K_{m,n}$ is Hamiltonian iff $m = n$. It's girth is 4 for $m, n \neq 1$ and and ∞ otherwise. It's diameter is 1 for $m = n = 1$ and 2 otherwise. The graph's order is $m + n$ and it's size is $m \cdot n$.*

Proof. Let $V = \{v_1, ..., v_m\}$ and $W = \{w_1, ..., w_n\}$ denote the two partitions of $K_{m,n} = (V_K, E_K)$.

Order: The first partition has m elements, the second n elements. *Thus, $K_{m,n}$ has an order of $m+n$.*

Size: Each of the m elements of the first partition are connected to each of the n elements in the second partition. *Thus, $K_{m,n}$ has a size of $m \cdot n$.*

Girth: If either $m = 1$ or $n = 1$, then all vertices of one partition are indicent to and only to the single vertex of the other partition. Hence, there is no cycle in $K_{1,n}$ or $K_{m,1}$ and *the girth of $K_{m,n}$ is ∞ if $n = 1$ or $m = 1$.*

If $m, n \neq 1$, each cycle must have even length since any two consectuive vertices in a path of $K_{n,m}$ are in different partitions. Thus, we require an even amount of edge-crossings to enclose a walk. Any cycle has a length of at least 3, thus the girth of $K_{m,n}$ has to exceed or be equal to 4.
Furthermore, we find such a cycle of length 4 easily since both partitions V, W have at least 2 vertices: $(v_1, w_1, v_2, w_2, v_1)$. *From these considerations, the girth of $K_{m,n}$ must be equal to 4.*

Diameter: For $m = n = 1$, there are exactly two vertices in different partitions. They have a distance of 1 and thus, *the diameter of $K_{1,1}$ is 1.*

Since two consectuive vertices in a path of $K_{n,m}$ are in different partitions V, W the distance between two vertices in the same partition has to be at least 2. Moreover, we find a path of distance 2 between $v_1 \in V$ to $v_2 \in V$: (v_1, w, v_2) for any $w \in W$. Thus, the diameter does not deceed 2.
For any two vertices in different partitions V, W, they are directly connected by a path of length 1.
Hence, the diameter of $K_{m,n}$ is 2 if not $m = n = 1$.

Hamiltonian: $\square$

Theorem 1.3. *The Petersen graph is not Hamiltonian, it has a girth of* 5*, a diameter of* 2*, an order of* 10 *and a size of* 15*.*

Proof. **Order:** The graph has 10 vertices and thus, it's order is 10.

Size: The graph has 15 edges and thus, it's size is 15.

Girth:

Diameter:

Hamiltonian:

□

Problem 11

For each even integer $k > 1$, the complete graph $K_{(n+1)}$ is a k-regular graph with no 1-factor. For each odd $k > 1$ we can construct a k-regular graph with no 1-factor in the following way.
In order to garantee that the graph has no 1-factor we can use Tutt's theorem. We construct the graph by starting of with one vertex v connected to k subgraphs S which are not inter-connected. Then we construct

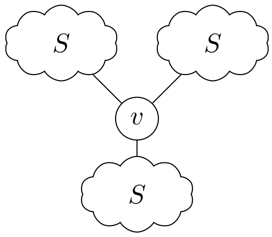

Figure 1: Example with k=3

$S = (V, E)$ in such a way that $|V|$ is odd and that all vertices in V have degree k except of one vertex $u \in V$ with degree $k - 1$. If we then connect u to v we get a k-regular graph. Using Tutt's theorem we know that the resulting graph has no 1-factor. Because if v is removed we get k components S with odd number of vertices and k is greater than 1.

In order to contruct S we first need the following lemma.

Lemma 2.0.1. *For any odd integer $k > 1$ it is possible to construct $(k-1)$-regular graph $G = (V, E)$ with $k + 1$ vertices.*

Proof. We can obtain G from K_{k+1} by removing by removing all the edges from K_{k+1} contained in a perfect matching. By removing the edges the degree of every vertex decreases exactly by one. K_{k+1} is by definition k-regular, hence G is $k-1$-regular. A perfect matching in K_{k+1} exists, because $k+1$ is even, and the condition from Tutte's theorem always holds in a complete graph. To find such a matching we can just randomly choose (u, v) edges and remove u and v from K_{k+1}. □

Constructing a connected graph $S = (V, E)$ with $|V| = k + 2$ and the degree sequence $(k, k, ..., k, (k-1))$. First we construct a $(k-1)$-regular graph $S' = (V', E')$ with $k+1$ vertices as described in the lemma. Then we can add one vertex to S' and connect it to all vertices in V' except of one vertex. Thus we get a new graph S. Because we only added one vertex $|V| = |V'| + 1 = k + 2$ and the degree of the newly added vertex is k. The degree of all the other vertices except of the last one is increased by one. Hence, S hat the degree sequence $(k, k, ..., k, (k-1))$.

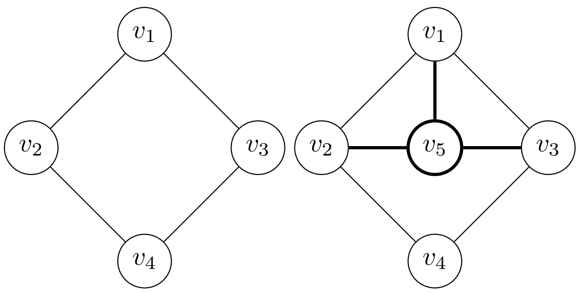

Figure 2: S' and S for k=3

Problem 12

Theorem 3.1. *Any graph G with $2n$ vertices and $\delta(G) \geq n$ has a 1-factor.*

Proof. Let $G = (V, E)$ be a graph with $|V| = 2n$ and $d(v) \geq n \, \forall v \in V$.

In the following, we will prove for nontrivial G that the number of odd components in a graph $G - S$ deceeds the number of vertices in S. By *Tutte's Matching Theorem*, G then has a 1-factor.

n = 1 Then, G is a simple graph with two vertices that are connected by one edge. This is of course a perfect matching of G.

n ≥ 2 Let $S \subseteq V$ be a set of vertices, $G' = (V', E') := G - S$ and $k := |S|$.
As G' is created by removing all vertices of S and their incident edges from G, we obtain the following properties:

- $\forall v \in V : d(v) \geq n - k$
- For any component C of G', $|V(C)| \geq n - k + 1$
- the order of G is $2n - k$

In the following cases, we prove that $\lambda := \#\textit{odd components} \leq k$.

k = 0 :

Because G consists of one even component, $\lambda = 0 \leq k$

k = 1 :

After removing any vertex of G, the degree of a vertex in G' is reduced by one or less. Hence, $\forall v \in V : d(v) >= n - 1$. This implies that the size of any component in G' is at least n. As far as $|V'| = 2n - 1$ there can only exist one component in G' with order $2n - 1$ (which is odd).

All in all, we have shown that $\lambda = 1 \leq k$.

2 ≤ k ≤ n :

As the minimum size of a component in G' is $n - k + 1$ and $|V'| = 2n - k$, we can bound the amount of components by the following term: $\frac{2n-k}{n-k+1}$.

We now have to prove that $\frac{2n-k}{n-k+1} \leq k \iff 2n - k \leq k * n - k^2 + k \iff 0 \leq (k-2)n - k^2 + 2k =: f(k)$.

To prove this inequality, we have to determine the minimum value of f in the defined boundaries.

* $f'(k) = n - 2k + 2 \overset{!}{=} 0 \iff k = \frac{n}{2} + 1$
* $f''(k) = -2$

So f has a maximum but no local minimum. To find the minimum value within the given range, we have to check the borders:

$f(2) = 0 = f(n)$. Thus, $\min(f) = 0 \geq f(k)$ which proves the inequality.

All in all, we have shown that the number of compomonents deceeds or is equal to k. This implies that $\lambda \leq k$.

n ≤ k ≤ 2n :

Because removing one vertex can not increase the number of components by more than one and $V' = 2n - k \leq n \leq k$, there can not be more than k odd components.

Hence, $\lambda \leq k$.

Finally, we have shown that the number of resulting components in G' is bounded by the order of S. In other words: $\forall S \subseteq V(G) : \#\textit{odd components of } G - S \leq |S|$.

Thus, we have shown that all conditions for *Tutte's Matching Theorem* are satisfied. Hence, G has a pefect matching aka 1-factor.

□