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Problem 17

Theorem 1.1. *In a planar triangulation let n_i be the number of vertices of degree i. Then,*

$$\sum_{i \in \mathbb{N}} (6-i)n_i = 12$$

.

Lemma 1.1.1. *Let $G = (V, E)$ and $G' = (V \cup \{v\}, E \cup E')$ be planar triangulations. Then $|E'| = 3$ and all edges in E' are indicent to v.*

Proof. Since any planar triangulation of n vertices and e_n edges satisifes $e_n = 3n-6$, we see inductively that

$$\begin{aligned} e_n &= e_{n-1} + 3 && (n > 3) \\ e_3 &= 3 \end{aligned}$$

Since G' has exactly one vertex more than G and both are planar triangulations, $|E'| = 3$.

Next, we will show that the degree of v exceeds or is equal to 3 and thus, all edges of E' have to be indicent to v.

By KURATOWSKI, G' is not a topological minor of $K_{3,3}$ or K_5 and any planar triangulation is edge-maximal. By *Lemma 4.4.5 (any edge-maximal graph without topological minors $K_{3,3}, K_5$ is 3-connected)*, G' is 3-connected.

If the degree of v deceeded 3, then G' would not be 3-connected (it could be isolated by removing two vertices) .

Hence, all three edges of E' are indicent to v. □

We will show by induction on the number of vertices n of a planar triangulation G with n_i vertices of degree i ($i \in \mathbb{N}$) that

$$T_G := \sum_{i \in \mathbb{N}} (6-i)n_i = 12$$

.

- Base $\mathbf{n = 3}$

  Then, the graph is a triangle and the condition is satisifed:

  $$T_{K_3} = \sum_{i \in \mathbb{N}} (6-i)n_i = (6-2) \cdot 3 = 4 \cdot 3 = 12$$

- Step $\mathbf{n \geq 4}$

  Any n-vertex planar triangulation $G = (V, E)$ has a subgraph $H = (V', E')$ which is a $(n-1)$-vertex planar triangulation.

  By *Lemma 1.1.1*, there is a vertex $v \in V \setminus V'$ of degree 3. Furthermore, the degree of exactly three other vertices $v_i, v_j, v_k \in V$ is increased. Thus $E \setminus E' = \{\{v, v_i\}, \{v, v_j\}, \{v, v_k\}\}$ and for T_G:

  $$\begin{aligned} T_G =& T_H \\ &+ (6-3) \\ &+ (6 - (d(v_i) + 1)) - (6 - d(v_i))) \\ &+ (6 - (d(v_j) + 1)) - (6 - d(v_j)) \\ &+ (6 - (d(v_k) + 1)) - (6 - d(v_k)) \\ =& T_H + 3 - 1 - 1 - 1 \\ =& T_H \\ =& 12 && \text{(by induction)} \end{aligned}$$

□

Problem 18

Problem 19

Problem 20