

Problem 21

Lemma 1.0.1. *In every planar triangulation G on at least four vertices, there exists a vertex v which does not lie on the outer bound of G*

Proof. For the sake of contradiction let's assume there is no such vertex v . Then all the vertices lie on the outer bound of G forming a cycle with at least four vertices. This is a contradiction with G being a planar triangulation, because we can add an edge between two not adjacent vertices to G and the result is still planar. Hence, there has to be at least one vertex v not on the outer bound of G . \square

Theorem 1.1. *Every planar triangulation G on at least four vertices contains a vertex whose neighbourhood induces a cycle.*

Proof. After lemma 1.0.1 there exists a vertex v which does not lie on the outer bound of G . Let $N(v) = p_1, p_2, \dots, p_n$ be the neighbourhood of v . $N(v)$ induces a cycle if all p_i are connected to a cycle and if there are no additional edges between p_i and p_j with $|i - j| > 1$. We first will prove that all p_i form a cycle, hence the induced subgraph $N(v)$ has a cycle as a subgraph. For the sake of contradiction let's assume this is not the case, hence there is p_i and p_{i+1} which are not connected. This is either a contradiction with v not lying on the outer bound of G , or with G being a maximal planar graph. Because if v_i and v_{i+1} are not connected and v is not on the outer bound of G there is a face bounded by v, p_i, p_{i+1} and at least one additional vertex. This face could be again divided into smaller faces by adding an edge, hence G is not a triangulation. Now we will prove that either $N(v)$ has no additional edges and thus is a cycle or that one of the vertices adjacent to v induces a cycle.

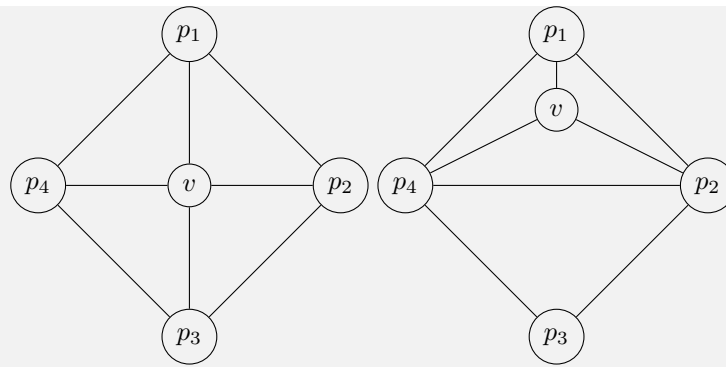
- *There are no additional edges connecting two vertices p_i and p_j with $|i - j| > 1$.*
In this case $N(v)$ is a cycle and we found a vertex whose neighbourhood induces a cycle.
- *There is an edge $p_i p_{i+2}$*
In this case the neighbourhood induces a graph of p_{i+1} is a cycle namely $p_i p_{i+1} v$.
- *There is an edge between two vertices p_i and p_j with $|i - j| > 2$ and without loss of generality $i < j$.*
This case there is a bounded face in G which is not a triangle. This face is the face bounded by at least the edges $p_i p_{i+1}$, $p_j p_{j-1}$, $p_i p_j$ and at least one or more edges forming a path from p_{i+1} to p_{j-1} . Hence this case can never occur in a planar triangulation.

In summary we now know that we can always find a vertex v whose neighbourhood induces a cycle or one of the neighbours of v here called p_{i+1} has a neighbourhood inducing a cycle. \square

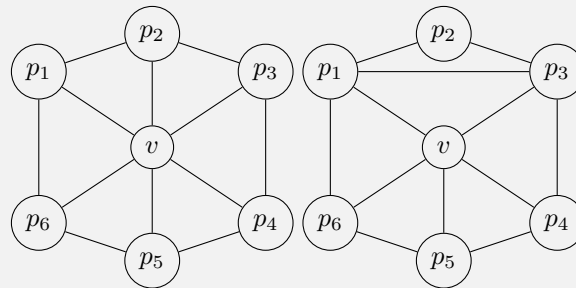
Lemma 1.1.1. *Every planar graph G with a maximal number of triangles and at least 4 vertices has a vertex of degree 3.*

Proof. Using theorem 1.1 we can find a vertex v whose neighbourhood induces a cycle $p_1 p_2 \dots p_n$. In the following we will show by induction over the degree of v that if G has a maximal number of triangles, v cannot have a degree greater than 3.

Basis: v has a degree of 4 If v has a degree 4, then the subgraph with the nodes v, p_1, p_2, p_3, p_4 only has 4 triangles. We can change the edge set of this subgraph to get a subgraph with the same vertices and the same outer structure but with containing 6 triangles. We can achieve this by removing edge vp_3 and adding the edge $p_2 p_4$. Thus we know that if G has a maximal number of triangles v cannot have degree 4.



Induction step Let the degree of v be $n > 3$. We now can again change the edges set without changing the outer structure of the induced subgraph $N(v)$ so that v has degree $n - 1$ and the number of triangles in $N(v)$ stays the same. We can achieve this by removing the edge vp_2 and adding the edge p_1p_3 . By induction we can repeat this operation until $n = 4$ and have the base case. Thus we can increase the number of triangles in the subgraph without changing the vertex count or the outer structure. Hence, the degree of v cannot be greater than 3 if G has a maximal number of triangles.



□

Theorem 1.2. Every n -vertex planar graph has at most $3n - 8$ triangles.

Proof. Let G be a planar graph with a maximal number of triangles. In the following we will show by induction that a planar graph G with at least 3 vertices with a maximum number of triangles has exactly $3n - 8$ triangles.

Base: $n = 3$ The planar graph with maximum number of triangles is K_3 which has one triangle. $3n - 8 = 3 * 3 - 8 = 1$ so for the base case the formula is correct.

Induction Step Let G be such a graph with n nodes. By using lemma 1.1.1 we can find a vertex v with degree 3 whose neighbourhood induces a cycle. If we remove this node we decrease the number of nodes by one and the number of triangles by 3. By induction we know that the resulting graph has no more than $3 * (n - 1) - 8$ triangles. Hence, we know that G has no more than $3 * (n - 1) - 8 + 3 = 3 * n - 8$ triangles. □

Problem 22

Theorem 2.1. *Any TK_3 -free graph G on n vertices contains a maximum of $n - 1$ edges.*

Proof. First, K_3 is the triangle C_3 . Subdividing any edge of C_i results in C_{i+1} . Moreover, any cycle has a $TC_3 = TK_3$.

Hence, a graph G is TK_3 -free if and only if it is acyclic. Further we assume that G is connected (since joining two disjoint acyclic components will not create a cycle but increase the edge count).

From these considerations, the maximum number of edges of an n -vertex, TK_3 -free graph equals the maximum number of edges in an n -vertex tree. Any n -vertex tree contains a maximum of $n - 1$ edges. \square

Theorem 2.2. *If a graph G is 3-connected then $TK_4 \subseteq G$.*

Proof. By TUTTE (1961), any 3-connected graph has a construction sequence G_0, G_1, \dots, G_n whereby $G_0 = K_4$ and $G_n = G$.

For any $i < n$, $G_i = (V_i, E_i)$ can be constructed by contracting an edge $e = \{x, y\}$ of G_{i+1} ($x, y \in V_{i+1}$, $d(x), d(y) \geq 3$).

Since $d(y) \geq 3$ and contracting e results in G_i , we can effectively say that there is a third vertex z in G_{i+1} for which $\{\{x, y\}, \{y, z\}\} \subset E_{i+1}$ is a subdivision of $\{x, z\} \in E_i$.

Thus, G_{i+1} has a TG_i and inductively, by the transitivity of topological minority, G_{i+1} has a $TG_0 = TK_4$. \square

1,...

Problem 24

Theorem 3.1. *Every outerplanar graph is 3-colorable.*

Proof. Let $G = (V_G, E_G)$ be an outerplanar graph and G_1, \dots, G_k the components of G . If $\forall i \in \{1, \dots, k\}$ G_i is 3-colorable, then G itself is 3-colorable. Thus, because $\forall i \in \{1, \dots, k\}$ G_i is outerplanar, we have to prove, that a connected, outerplanar graph is 3-colorable.

Let $H = (V_H, E_H)$ be a connected, outerplanar graph.

If $|V_H| \leq 2$, H can be colored with 3 colors by giving each vertex a different color.

Else, if $|V_H| > 2$, it can be proved by induction:

- *Base:* $|V_H| = 3$

Because H contains exactly three vertices, H can be simply colored by giving each vertex a different color. So H is 3-colorable.

- *Step:* $|V_H| > 3$

Let E_N be the edges of H which are not in the border of the unbounded face. Then two cases have to be considered:

- $|E_N| = 0$

In this case, all edges are in the border of the unbounded face. Hence, H is either a circle, thus, is 3-colorable, or H contains a cut vertex. If H contains a cut vertex v , $H - \{v\}$ consists of at least two connected components. These components merged with $\{v\}$ and the related edges in H , named C_1, \dots, C_n ($V = C_1 \cup \dots \cup C_n$), are induced subgraphs of H , hence, they are outerplanar. So, by induction, $\forall i \in \{1, \dots, n\}$ C_i is 3-colorable. Because colors can be renamed, the color of v can be set and still each C_i is 3-colorable. Moreover, there is no edge between two components of H , because all components of $H - \{v\}$ are connected and maximal.

By these conditions, H is 3-colorable.

- $|E_N| > 0$

In this case, there is at least one edge not laying in the border of the unbounded face, hence, laying in another face.

Let $e = (u, v) \in E_N$. Because H is outerplanar, e cuts H into two connected parts H_1 and H_2 ($V(H_1) \cap V(H_2) = \{u, v\}$ and $E(H_1) \cap E(H_2) = \{e\}$). Assume there is an edge e_2 between $H_1 \setminus \{u, v\}$ and $H_2 \setminus \{u, v\}$. Then either e_2 have to cross e , or e_2 cut w.l.o.g u from the border of the unbounded face. A contradiction in both cases, hence, there is no such edge.

As before, H_1 and H_2 are induced subgraphs of H , $H_1 \cup H_2 = H$, so both are 3-colorable by induction.

By these conditions, H is 3-colorable.

All in all, a connected outerplanar graph is 3-colorable.

Hence, because each outerplanar graph consists of connected, outerplanar components, each outerplanar graph is 3-colorable.

□