Problem sheet 9

Due date: **January 8, 12:00 am**. Discussion of solutions: January 10.

(Please prepare solutions for **at most three** problems.)

Problem 33. **5 points**

Let G be a graph with n vertices and minimum degree $\delta(G) \geq \lfloor \frac{r-2}{r-1} \cdot n \rfloor + 1$. Show that there is a copy of K_r in G.

Problem 34. **5 points**

Let $G = (V, E)$ be a triangle-free graph on n vertices. Prove that

$$\sum_{v \in V} \deg(v)^2 \leq \frac{n^3}{4}.$$

Problem 35. **5 points**

Let $x_1, \ldots, x_n$ be points on the plane such that $||x_i - x_j|| \leq 1$ for all i, j. Prove that at most $\frac{n^2}{3}$ pairs of points are at distance greater than $\frac{1}{\sqrt{2}}$.

Hint: Show that among any four points there are two whose distance is at most $\frac{1}{\sqrt{2}}$.

Problem 36. **5 points**

Prove that the maximum number of edges in an n-vertex graph without an even cycle is $\lfloor \frac{3(n-1)}{2} \rfloor$. Compare it with the maximum number of edges in an n-vertex graph without an odd cycle.

Bonus Problem. **5 bonus points**

Santa Claus wants to decorate his house with the newest chain of lights. As a first feature, the light bulbs are no longer arranged consecutively in a chain rather than in an arbitrarily complicated net. The second feature is supposed to improve the way lights are switched on and off. When a light bulb is touched its status switches automatically from on to off or vice versa, and for convenience the status of each light bulb directly adjacent to it is swapped as well.

However, before ordering a set of such net of lights, Santa worries whether he will ever be able to have all lights turned on at the same time. But his personal adjunct calms him, saying that no matter how the actual net looks like, there is always a sequence of light bulbs that if touched switches all lights from off to on or vice versa.

Is the adjunct right? Or are Santa's doubts justified?

Open Problem.

Determine the largest number of triangles in an n-vertex graph without a copy of K_4.