Solution sheet 10

Date: January 10. Discussion of solutions: January 17.

Hint: You may use the fact that $\sum_{v \in V} \binom{\deg(v)}{2} \geq \frac{m}{n}(2m - n)$ *for any* n*-vertex,* m*-edge graph* $G = (V, E)$*. This inequality will be proven in the problem class.*

Problem 37. **5 points**

Show that every graph G without C_6 has a subgraph H with $|E(H)| \geq |E(G)|/2$, which contains no C_4.

Solution.

Let G be any graph without a copy of C_6. We shall prove the existence of a subgraph H of G, which has at least half the number of edges of G and contains no copy of C_4 with the following claim.

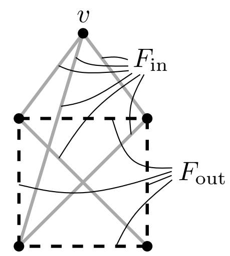
Claim. If a graph G contains a C_4 and no C_6 then there exist disjoint edge-sets F_{in} and F_{out} of G such that $|F_{\text{in}}| \geq |F_{\text{out}}|$ and every C_4 that contains an edge of F_{in} also contains an edge of F_{out}.

One crucial observation is that no two copies of C_4 in G share exactly one edge, as otherwise we obtain a copy of C_6. To prove the claim we now distinguish the following cases.

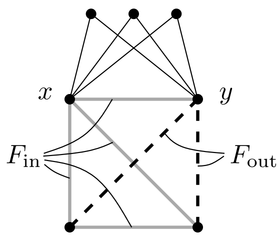
Case 1: G contains a copy K of K_4. Each of the six edges in K lies on some C_4 in K. Hence, every C_4 containing some edge from K must have all four edges in K or two consecutive edges in K. We further distinguish two cases.

Case 1.1: Some vertex v is adjacent to at least three vertices of K. Since G does not contain C_6 every C_4 that contains two edges of K must contain v. Moreover, every edge between v and K is contained in a copy of K_4 in $K \cup v$. Hence every C_4 that contains such an edge is completely contained in $K \cup v$. Now we choose as F_{out} the edges of a 4-cycle in K and as F_{in} the edges in $(K \cup v) - F_{\text{out}}$. These edge-sets enjoy the claimed properties.

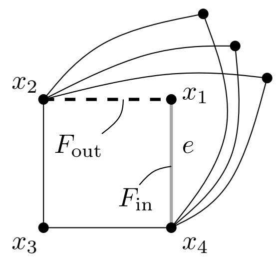
Case 1.2: No vertex is adjacent to three or more vertices of K. It follows that all copies of C_4's that contain two edges of K leave and enter K at the same pair x, y of vertices, as otherwise we obtain a copy of C_6. It is now easy to see that choosing as F_{out} the two edges incident to x in $K \setminus xy$ and as F_{in} the edge-set $E(K) - F_{\text{out}}$ satisfies the properties of the claim.



Case 1.1



Case 1.2



Case 2

Case 2: G contains no copy of K_4. Let $C = x_1, x_2, x_3, x_4$ be any copy of C_4 in G. Consider the edge $e = x_2x_3$. As G does not contain C_6 every copy of C_4 containing e different from C contains one more edge from C. As G does not contain K_4 such a copy contains either the edge x_1x_2 or the edge x_3x_4, say there is a copy $C' \neq C$ containing e and x_1x_2. As G does not contain C_6 there is no copy of C_4 different from C containing e and x_3x_4. Hence the edge-sets $F_{\text{in}} = \{e\}$ and $F_{\text{out}} = \{x_1x_2\}$ satisfy the properties of the claim.

With the claim proven, we can now easily prove our statement by induction on the number of C_4's in G. If (induction base) G has no copy of C_4, then we simply take $H = G$ and are done. In case G contains at least one C_4 (induction step) we consider the edge-sets F_{in} and F_{out} from the claim. We apply induction to $G' = G - (F_{\text{in}} \cup F_{\text{out}})$ and obtain a subgraph H' without C_6 and with $|E(H')| \geq |E(G')|/2$. But then $H = H' \cup F_{\text{in}}$ is a subgraph of G without C_6 and since $|F_{\text{in}}| \geq |F_{\text{out}}|$ we have $|E(H)| \geq |E(G)|/2$, as desired. $\square$

Problem 38. **5 points**
Show that any graph on n vertices and at least $\lfloor \frac{n^2}{4} \rfloor + 1$ edges contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

Solution.
Let G be any graph on n vertices and exactly $\lfloor \frac{n^2}{4} \rfloor + 1$ edges. We shall prove that G contains at least $\lfloor \frac{n}{2} \rfloor$ triangles by induction on n.

Induction base: For $n = 1, 2$ the claim holds trivially, since there is no such graph.

Induction step: Assume that $n \geq 3$. First, note that $\lfloor \frac{n^2}{4} \rfloor = t(n, 2)$, so by Turán's theorem, G contains at least one triangle. Now let's consider the following cases:

Case 1: n is odd, i.e. $n = 2k + 1$ for some k.
We have $|E(G)| = \lfloor \frac{n^2}{4} \rfloor + 1 = \lfloor k^2 + k + \frac{1}{4} \rfloor + 1 = k^2 + k + 1$. Then there is a vertex v of degree at most k (otherwise, we would have at least $\frac{(2k+1)(k+1)}{2} > k^2 + k + 1$ edges). Removing this vertex creates a graph $G - v$ with $n - 1$ vertices and at least $k^2 + 1 = \frac{(n-1)^2}{4} + 1$ edges, which, by the induction hypothesis, contains at least $\lfloor \frac{n-1}{2} \rfloor = k = \lfloor k + \frac{1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor$ triangles.

Case 2: n is even, i.e. $n = 2k$ for some k.
We have $|E(G)| = \lfloor \frac{n^2}{4} \rfloor + 1 = k^2 + 1$.

Case 2.1: G contains a vertex v of $\deg(v) \leq k - 1$.
By removing this vertex, we create a graph $G - v$ with $n - 1$ vertices and at least $k^2 + 2 - k = \lfloor \frac{(n-1)^2}{4} \rfloor + 2$ edges. Thus, by Turán's theorem, $G - v$ contains at least one triangle.

Take any edge e of this triangle. By removing it, we get the graph $(G - v) - e$ with $n - 1$ vertices and at least $\lfloor \frac{(n-1)^2}{4} \rfloor + 1$ edges, so by the induction hypothesis, $(G - v) - e$ contains at least $\lfloor \frac{n-1}{2} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$ triangles. Thus by adding e again, $G - v$ (and thereby G) contains at least $\lfloor \frac{n}{2} \rfloor$ triangles.

Case 2.2: All vertices in G have degree at least k.
Then there are at most two vertices of degree at least $k + 1$, as otherwise

$$|E(G)| = \frac{1}{2} \sum_{v \in V(G)} \deg(v) > \frac{1}{2}(2k \cdot k + 2) = \lfloor \frac{n^2}{4} \rfloor + 1,$$

which would be too many edges. Now by Turán's theorem, G contains at least one triangle, and by the preliminary considerations, we know that at least one vertex v in this triangle has degree k. By removing it, we get the graph $G - v$ with $n - 1$ vertices and $k^2 - k + 1 = \lfloor \frac{(n-1)^2}{4} \rfloor + 1$ edges, so by the induction hypothesis, $G - v$ contains $\lfloor \frac{n-1}{k} \rfloor = \lfloor \frac{n}{2} \rfloor - 1$ triangles, and the original graph G additionally contains the triangle containing v, i.e. $\lfloor \frac{n}{2} \rfloor$ triangles.

We remark that the bound $\lfloor n/2 \rfloor$ is best possible. To show that, we just take the bipartite graph between sets of size $\lfloor n/2 \rfloor$ and $\lceil n/2 \rceil$ and add one extra edge in the set of size $\lceil n/2 \rceil$. $\square$

Problem 39. **5 points**
Let $G = (V, E)$ be a graph on n vertices and m edges. For $i = 0, 1, 2, 3$ let t_i denote the number of vertex triples of G inducing exactly i edges.

(a) Prove that $t_0 + t_3 = \binom{n}{3} - (n-2)m + \sum_{v \in V} \binom{\deg(v)}{2}$.

(b) Conclude with (a) that $t_3 \geq \frac{m}{3n}(4m - n^2)$.

Solution.

(a) We shall prove the equality $t_0 + t_3 = \binom{n}{3} - (n-2)m + \sum_{v \in V} \binom{\deg(v)}{2}$ by three simple equations, one for each term in the right side.

- Considering all vertex triples of G we see
$$\binom{n}{3} = t_0 + t_1 + t_2 + t_3. \tag{1}$$
- Considering all pairs of an edge e in G and a vertex v not incident to e we see
$$m(n-2) = t_1 + 2t_2 + 3t_3, \tag{2}$$
since every triple inducing i edges occurs exactly i times this way.
- Considering all pairs of a vertex v in G and a pairs of neighbors of v we see
$$\sum_{v \in V} \binom{\deg(v)}{2} = 3t_3 + t_2. \tag{3}$$

Combining (1), (2) and (3) we get
$$t_0+t_3 = (t_0+t_1+t_2+t_3)-(t_1+2t_2+3t_3)+(3t_3+t_2) = \binom{n}{3}-(n-2)m+\sum_{v \in V} \binom{\deg(v)}{2}.$$

(b) The claimed inequality follows from two crucial observations and the inequality provided in the hint. First, we apply the equation in (a) to the complement $\overline{G}$ of G and obtain
$$t_3 + t_0 = \binom{n}{3} - (n-2)\overline{m} + \sum_{v \in V} \binom{n - 1 - \deg(v)}{2},$$
where $\overline{m}$ denotes the number of edges in $\overline{G}$.

Second, we apply (3) to $\overline{G}$ and obtain

$$t_0 \leq \frac{1}{3} \sum_{v \in V} \binom{n-1-\deg(v)}{2}.$$

Together we obtain

$$\begin{aligned} t_3 &\geq \binom{n}{3} - (n-2)\overline{m} + \frac{2}{3} \sum_{v \in V} \binom{n-1-\deg(v)}{2} \\ &\geq \binom{n}{3} - (n-2)\overline{m} + \frac{2}{3} \frac{\overline{m}}{n}(2\overline{m} - n) \\ &= \frac{m}{3n}(4m - n^2), \end{aligned}$$

where the second inequality uses the hint and the last equality uses $\overline{m} = \binom{n}{2} - m$.

□

Problem 40. **5 points**

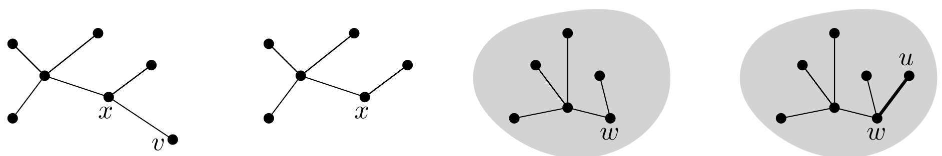
Prove that any graph G with $\delta(G) \geq k$ contains all trees on k edges as a subgraph.

Solution.

Let G be any graph with $\delta(G) \geq k$ and T be any tree on k edges. We shall prove that T is a subgraph of G by induction on k.

Induction base $k = 0$. There is nothing to show.

Induction step $k \geq 1$. Let v be a leaf of T. Then $T' = T - v$ is a tree on $k-1$ edges, which by induction hypothesis is a subgraph of G. Let x be the neighbor of v in T and w be the corresponding vertex in the copy of T' in G. Since $\deg(w) \geq k$ and $|V(T') \setminus w| = k - 1$, there is a neighbor u of w in G that is not in the copy of T' in G. Thus we have found a copy of T in G.



T $\quad$ $T' = T - v$ $\quad$ copy of T' in G $\quad$ copy of T in G

□

Open Problem.

Prove or disprove that for all trees T with k edges $\mathrm{ex}(n, T) \leq \frac{n(k-1)}{2}$.