Problem 9

Theorem 1.1. *A hypercube Q_n is Hamiltonian. It has a girth of 4, a diameter of n, an order of 2^n and a size of ?.*

Proof. □

Theorem 1.2. *A bipartite complete graph $K_{m,n}$ is Hamiltonian iff $m = n$. It's girth is 4 for $m, n \geq 2$ and ∞ otherwise. It's diameter is 2. The graph's order is $m + n$ and it's size is $m \cdot n$.*

Proof. □

Theorem 1.3. *The Petersen graph is Hamiltonian, it has a girth of 5, a diameter of 2, an order of 10 and a size of 15.*

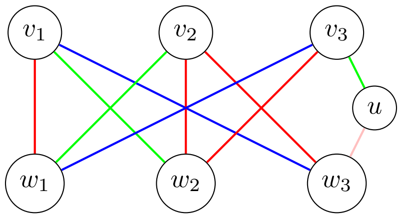
Proof. □

Problem 10

Theorem 2.1. *For a natural number $n \geq 2$ let $G = (V_G, E_G)$ be the graph of order $2n+1$ obtained from $K_{n,n}$ by subdividing an edge by a vertex. Then, $\chi'(X) = \Delta(G) + 1 = n+1$ but $\chi'(G-e) = \Delta(G-e)$ for any edge e of G.*

Proof. Let $V = \{v_1, ..., v_n\}$ and $W = \{w_1, ..., w_n\}$ denote the two partitions of $K_{n,n}$ and $\{c_k \mid k \in \mathbb{N}\}$ be a set of colors. Furthermore, let $u \in V_G$ be the vertex subdividing an edge between these two partitions.

First, we will prove that $\chi'(X) = \Delta(G) + 1 = n+1$. We will argue, that any partial coloring of G will inevitably result in $n+1$ colors.



For a partially colored G, let $v \in V_G$ be a vertex of G.

- **Case 1:** $v \neq u$: Since v has n adjacent vertices, we need n colors to color the edges incident to v. Of course,

$\square$

Problem 11

For each even integer $k > 1$, the complete graph $K_{(n+1)}$ is a k-regular graph with no 1-factor. For each odd $k > 1$ we can construct a k-regular graph with no 1-factor in the following way.
In order to garantee that the graph has no 1-factor we can use Tutt's theorem. We construct the graph by starting of with one vertex v connected to k subgraphs S which are not inter-connected. Then we construct

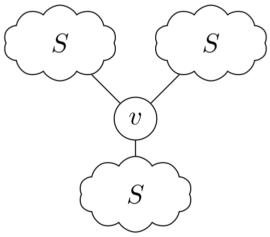


Figure 1: Example with k=3

$S = (V, E)$ in such a way that $|V|$ is odd and that all vertices in V have degree k except of one vertex $u \in V$ with degree $k-1$. If we then connect u to v we get a k-regular graph. Using Tutt's theorem we know that the resulting graph has no 1-factor. Because if v is removed we get k components S with odd number of vertices and k is greater than 1.

In order to contruct S we first need the following lemma.

Lemma 3.0.1. *For any odd integer $k > 1$ it is possible to construct $(k-1)$-regular graph $G = (V, E)$ with $k+1$ vertices.*

Proof. We can obtain G from K_{k+1} by removing by removing all the edges from K_{k+1} contained in a perfect matching. By removing the edges the degree of every vertex decreases exactly by one. K_{k+1} is by definition k-regular, hence G is $k-1$-regular. A perfect matching in K_{k+1} exists, because $k+1$ is even, and the condition from Tutte's theorem always holds in a complete graph. To find such a matching we can just randomly choose (u, v) edges and remove u and v from K_{k+1}. □

Constructing a connected graph $S = (V, E)$ with $|V| = k+2$ and the degree sequence $(k, k, ..., k, (k-1))$. First we construct a $(k-1)$-regular graph $S' = (V', E')$ with $k+1$ vertices as described in the lemma. Then we can add one vertex to S' and connect it to all vertices in V' except of one vertex. Thus we get a new graph S. Because we only added one vertex $|V| = |V'| + 1 = k+2$ and the degree of the newly added vertex is k. The degree of all the other vertices except of the last one is increased by one. Hence, S hat the degree sequence $(k, k, ..., k, (k-1))$.

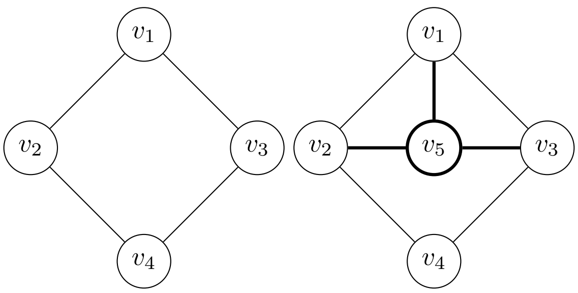


Figure 2: S' and S for k=3

Problem 12

In the following I will show that any graph G with $2n$ vertices and all degrees at least n has a 1-factor. I will show that if we divide such a G into two parts we can remove edges until we get an bipartite 1-regular graph. Then using the corollary of Hall's theorem we know we can find a perfect matching.

Theorem 4.1. *Let $G = (V, E)$ be a graph with $|V| = 2n$ vertices with all degree atleast n, then G has a 1-factor.*

Proof. Because $|V| = 2n$ we can divide the vertices V into two subsets $A, B \subset V$ with $|A| = |B| = n$ and $A \cap B = \emptyset$.

□