Solution sheet 11

Date: January 17. Discussion of solutions: January 24.

Problem 41. **5 points**

Show that an ϵ-regular partition of a graph G is also an ϵ'-regular partition of its complement $\overline{G}$ for some ϵ'.

Solution.

This follows immediately from the definition. □

Problem 42. **5 points**

Let G be any graph of chromatic number t. Show that $\mathrm{ex}(n, G) \geq \mathrm{ex}(n, K_t)$ and that equality holds only if there is an edge e such that $\chi(G - e) < t$.

Solution.

Let G be a graph with $\chi(G) = t$. Considering the Turán graph $T(n, t-1)$ we see $\chi(T(n, t-1)) = t - 1$ and hence it does not contain G as a subgraph. Thus we have $\mathrm{ex}(n, G) \geq \mathrm{ex}(n, K_t)$.

Now assume that G does not fulfil the additional assertion, i.e., that removing *any* e edge from G still leaves one with a graph of chromatic number t. Again consider the Turán graph $T(n, r-1)$ and add a single edge e in one of the independent sets. We will show that this graph, which necessarily contains a copy of K_t, does not contain a copy of G.

Assume for the sake of contradiction that we do have a copy of G. As $\chi(G) = t$, any copy of G must contain the edge e. So for any copy we get two vertices in the same maximal independent set of $T(n, t-1)$ and, by construction, every other vertex must lie in distinct independent sets. But then the edge e is the only obstruction to making the graph $(t-1)$-chromatic, so deleting the edge e from G will yield a graph of chromatic number $t - 1$ – a contradiction to our assumptions on G. □

Problem 43. **5 points**

Show that for any graph $G = (V, E)$, $|V| = n$, there exists a set S, $|S| \leq \lfloor n^2/4 \rfloor$, and for each vertex $v \in V$ a subset $S_v \subseteq S$ such that $S_v \cap S_w \neq \emptyset$ if and only if $vw \in E$.

Hint: Try to cover the edges of G with cliques.

Solution.

Let $G = (V, E)$ be any n-vertex graph. We shall find a set S of edges and triangles in G such that every edge e of G is covered by S, that is, $e \in S$ or e is part of a triangle in S. Then we can associate to each vertex v the set $S_v \subseteq S$ of all edges and triangles in S that contain v. Then we have $S_v \cap S_w \neq \emptyset$ if and only if v and w are contained in the same edge or the same triangle in S. So if $vw \notin E$ then $S_v \cap S_w = \emptyset$ and otherwise $S_v \cap S_w \neq \emptyset$, since S covers all edges of G.

Thus the statement follows from the following claim.

Claim. The edges of G can be covered by at most $\lfloor n^2/4 \rfloor$ edges and triangles of G.

We prove the claim by induction on n.

Induction base $n \leq 2$. For $n = 1$ there is nothing to show. For $n = 2$ we have $\lfloor n^2/4 \rfloor = 1 = \binom{n}{2}$, so we can take $S = E$.

Induction step $n \geq 3$. If $|E| \leq \lfloor n^2/4 \rfloor$ then it is enough to take $S = E$ and we are done. So assume that $|E| \geq \lfloor n^2/4 \rfloor + 1$. Then by Turáns theorem there is some triangle in G. Let u and v any two vertices contained in a triangle of G. We cover all edges incident to u and v with a set S' of triangles and edge by chosing at most one edge or triangle for each of the $n-2$ vertices in $V \setminus \{u, v\}$.

Let $w \in V \setminus \{u, v\}$. If w is adjacent to at most one of u, v, then let S' contain the edge (if existent) between w and $\{u, v\}$. Otherwise, if w is adjacent to both of u, v, then let S' contain the triangle $\{u, v, w\}$. Clearly, S' covers all edges of G incident to u, v, and $|S'| \leq n-2$.

By induction on $G' = G - \{u, v\}$ we obtain a set S'', $|S''| \leq \lfloor (n-2)^2/4 \rfloor$, of edges and triangles of G covering all edges of G'. Then $S = S' \cup S''$ cover entire E and we have

$$|S| = |S'| + |S''| \leq n - 2 + \lfloor (n-2)^2/4 \rfloor \leq \lfloor n^2/4 \rfloor,$$

which concludes the proof. □

Problem 44. **5 points**

Determine the smallest number n such that the following holds. Whenever one colors the edges of K_n with two colors, red and blue, one creates a red triangle or a blue K_4.

Solution.

We shall prove that the number we seek is $n = 9$, i.e., that every red-blue coloring of the edges of K_9 induces a red triangle or a blue K_4 and that there is a red-blue coloring of the edges of K_8 that does neither induce a red triangle nor a blue K_4.

Let n be any number and the edges of K_n be colored arbitrarily with red and blue.

Claim. If some vertex v is incident to four red edges, then there is a red K_3 or blue K_4.

Indeed, let $v_1, \ldots, v_4$ be the endpoints of the four red edges at v. If one of the edges among $\{v_1, \ldots, v_4\}$ is red we obtain together with v a red K_3. Otherwise, if all these edges are blue, then $\{v_1, \ldots, v_4\}$ induce a blue K_4.

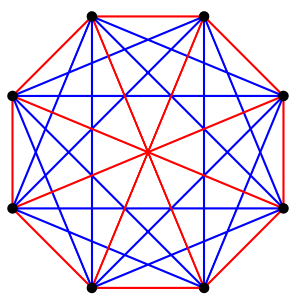
Claim. If some vertex v is incident to six blue edges, then there is a red K_3 or blue K_4.

Indeed, let $v_1, \ldots, v_6$ be the endpoints of the six blue edges at v. From the lecture we know that $\{v_1, \ldots, v_6\}$ induce a red triangle or a blue triangle. In the former case we have a red K_3 and in the latter case we obtain together with v a blue K_4.

Claim. If $n = 9$ then there is a red K_3 or blue K_4.

Assume for the sake of contradiction that there is neither a red K_3 nor a blue K_4. Since K_9 is 8-regular it follows from the above claims that every vertex is incident to exactly three red edges and five blue edges. However, then the subgraph of K_9 on all red edges would be 3-regular, which is impossible since the number of vertices is odd – a contradiction.

Claim. If $n = 8$ then there is a red-blue coloring inducing neither a red K_3 nor blue K_4.



□

Open Problem.

Prove or disprove that if G is a triangle-free graph on n vertices, then there is a set of at most $n^2/25$ edges in G whose deletion destroys all odd cycles in G.