Solution sheet 2

Date: November 7. Discussion of solutions: November 8.

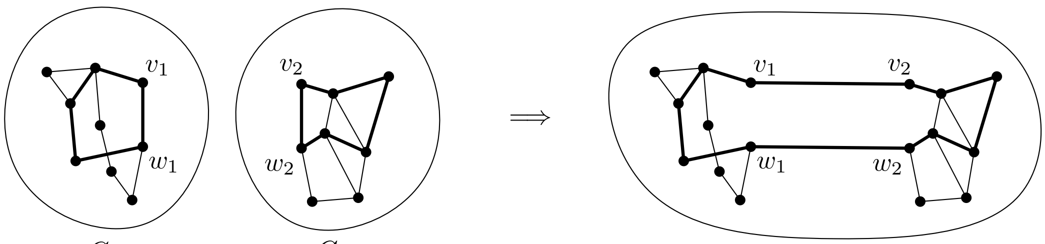
Problem 5. **5 points**

Let G be a non-empty graph with minimum degree at least two. Show that there is a connected graph having the same degree sequence as G.

Solution.

Let $G = (V, E)$ be any non-empty graph. We show the claim by induction on the number of connected components of G. If this is one, the graph is already connected, so assume we have two components C_1 and C_2 of G. By assumption of the claim, there are no vertices of degree one. So C_i cannot be a tree, $i = 1, 2$. (As every tree has a leaf, as shown in the lecture.) For $i = 1, 2$, because C_i is connected, it must contain a cycle and an edge $e_i = v_iw_i \in C_i$ on the cycle.

The graphs $C_i - e_i$ ($i = 1, 2$) are still connected, so $C' := (C_1 - e_1) \cup (C_2 - e_2) + v_1v_2 + w_1w_2$ is connected. Since we only touched edges incident to $\{v_1, v_2, w_1, w_2\}$ and for each of these vertices, we have removed one edge and added one edge, the degree of each vertex is preserved. The graph $(G - C_1 - C_2) \cup C'$ thus has fewer components than G but same degree sequence as G. Applying induction to G' we obtain a connected graph with the same degree sequence as G' (and hence as G), as desired.



□

Problem 6. **5 points**

Let T be a non-empty tree with an even number of vertices. Show that T has exactly one spanning subgraph in which every vertex has odd degree.

Solution. (Variant 1)

Let T be a non-empty tree with an even number of vertices. We prove the claim by induction on number of vertices of T. First note that the desired spanning subgraph must contain all edges incident to leaves.

Now if $|V| = 2$ (recall that T has an even number of vertices.) then the desired subgraph is the graph T itself.

So assume that $|V| \geq 4$. Consider a vertex x in T such that all but at most one of its neighbors are leaves in T. (For example the second-to-last vertex on any maximum path in T is such a vertex.) Let $x_1, \ldots, x_k$ be the leaves at x. Let x' be the neighbor of x such that $x' \neq x_i$, $i = 1, \ldots, k$. If such x' does not exist, it means that T is a star with odd number of edges and we are done by taking T itself as the desired subgraph. Next we distinguish two cases.

Case 1: If k is odd then our spanning subgraph must contain the edge xx_i for all $i = 1, \dots, k$ and must *not* contain the edge xx'. Let $T' = T - \{x, x_1, \dots, x_k\}$. The graph T' is a tree (proved in the lecture) with even order and it has smaller number of vertices than T. Applying induction to T' we obtain a unique spanning subgraph G' with all vertices of odd order. Together with a star on vertices $x, x_1, \dots, x_k$ we have a unique spanning subgraph of T with all vertices of odd order.

Case 2: Let k be even. Again, xx_i must be in the desired spanning subgraph. Let $T' = T - \{x_1, \dots, x_k\}$. The graph T' is a tree of even order and x is a leaf of T'. Thus, if G' is a unique spanning subgraph of T' found by induction, then xx' is its edge. Therefore $G' \cup \{xx_1, \dots, xx_k\}$ is a desired spanning subgraph of T. □

Solution. (Variant 2)
Let T be a non-empty tree with an even number of vertices. We shall find a spanning subgraph G of T in which every vertex has odd degree as follows. The vertex set of G is the same as of T, $E(G) := \{e \in E(T) \mid Te \text{ has two odd components}\}$. First we shall prove that the degree of each vertex in G is odd, second we shall prove that any spanning subgraph G' of T with each vertex having odd degree satisfies the property that $G' = G$.

Consider any vertex x in G. Assume for the sake of contradiction that $d(x)$ is even. Then, deleting all edges of G incident to x from the tree T results in odd number of odd components. Therefore, the total number of vertices in T is odd – a contradiction. Thus every vertex in G has odd degree.

On the other hand, consider a subgraph G' of T with all vertices of odd degree. Let e be an edge of G, i.e., $T - e$ has two odd components, T_1, T_2. Assume for the sake of contradiction that $e \notin E(G')$. Then $G'[V(T_1)]$ has odd number of vertices of odd degree – a contradiction. Thus, if $e \in E(G)$ then $e \in E(G')$.

Now assume for the sake of contradiction that $E(G') \setminus E(G)$ contains an edge e. Consider the two components T_1, T_2 of Te. Both of them have even number of vertices. If $e \in E(G')$ then in $G'[V(T_1)]$ we have all vertices except for an endpoint of e of odd degree, i.e., we have odd number of vertices of odd degree – a contradiction. Thus, if $e \in E(T) \setminus E(G)$ then $e \notin E(G')$.

Therefore, we have that $e \in E(G)$ if and only if $e \in E(G')$. □

Problem 7. **5 points**
For any graph G let $\pi(G)$ denote the minimum number of walks in G so that every edge of G appears once in exactly one walk and does not appear in other walks.

Find an expression/formula for $\pi(G)$.

Solution.
Let $G = (V, E)$ be any graph and $\pi(G)$ be defined as above. We define $s(G)$ to be the number of vertices in G of odd degree and $t(G)$ to be the number of non-empty Eulerian components of G. We claim the following.

Claim. For every graph G we have $\pi(G) = s(G)/2 + t(G)$.

We shall prove the claim in two steps. First, we identify a set of $s(G)/2 + t(G)$ walks in G such that every edge appears exactly once in these walks, which proves $\pi(G) \leq s(G)/2 + t(G)$. And second, we prove $\pi(G) \geq s(G)/2 + t(G)$ by counting the endpoints of an arbitrary set of walks in which every edge appears exactly once in these walks.

I) We shall show that $\pi(G) \leq s(G)/2 + t(G)$. More precisely, we shall find a set S of walks, $|S| = s(G)/2 + t(G)$, such that every edge of G appears exactly once in a walk in S. We define these walks for each connected component of G separately. So let C be an arbitrary non-empty connected component of G. We distinguish two cases.

If C is Eulerian then we add any Eulerian tour (seen as a closed walk) into the set S. By definition every edge in C appears exactly once in this walk.

On the other hand, if C is not Eulerian, then it was proven in the lecture that C must contain at least one vertex of odd degree. Indeed, as the sum of degrees of vertices in G is even, there is an even number of vertices in C that have odd degree in G. Let U be the set of those vertices. We have $|U| = 2k$ for some natural number $k \geq 1$. We partition the vertices in U into k pairs of vertices, denoted by $\{u_i, w_i\}$ for $i = 1, \ldots, k$. We define a new graph G' by taking only the component C of G and for each $i = 1, \ldots, k$ introducing a new vertex v_i with edges only to u_i and w_i.

It is easily seen that every vertex in G' has an even degree. Indeed, the degree of u_i and w_i in G' is exactly one more than its degree in G, the degree of a vertex in $C \setminus U$ is the same in G and G', and the degree of v_i is two, $i = 1, \ldots, k$. Thus (as proven in the lecture) there exists a Eulerian tour of G'. Removing all subsequences $u_i - v_i - w_i$ for $i = 1, \ldots, k$, we obtain a set of k walks in G such that every edge of G appears exactly once in these walks.

Altogether we have identified one walk per Eulerian component of G ($t(G)$ in total) and $s(C)/2$ walks per non-Eulerian component C of G ($s(G)/2$ in total). Moreover, every edge appears exactly once in these walks, proving $\pi(G) \leq s(G)/2 + t(G)$.

II) We shall show that $\pi(G) \geq s(G)/2 + t(G)$. To this end consider any set S of walks in G such that every edge of G appears exactly once in these walks. Let C be an arbitrary connected component of G.

If C is Eulerian and non-empty, then clearly at least one walk has an endpoint in C. And if C is not Eulerian then every vertex of C of odd degree is the endpoint of at least one walk in S. Together this shows $\pi(G) \geq s(G)/2 + t(G)$.

With I) and II) we have shown $\pi(G) = s(G)/2 + t(G)$, which concludes the proof. □

Problem 8. **5 points**

A *permutation matrix* is a matrix of zeros and ones such that each row and each column contains exactly one 1.

Show that a square matrix A with non-negative integer entries is a sum of k permutation matrices if and only if the sum of elements in each row and in each column of A is k.

Solution.

Let A be any $n \times n$-matrix. First assume that $P_1, \ldots, P_k$ are k permutation matrices, such that $A = P_1 + \cdots + P_k$, i.e., A is the sum of k permutation matrices. Then clearly the sum of elements in a row of A is the sum of the corresponding row sums of $P_1, \ldots, P_k$. Since each summand is 1 (as each P_i is a permutation matrix) the row sum for A is exactly k.

Next, we show that reverse direction, that is, we assume that the sum of elements in each row and in each column of A is k and shall show that A is the sum of k permutation matrices.

We do induction on k. If $k = 1$, then A is a permutation matrix and we are done. So let $k > 1$. We construct a bipartite graph G with partite sets $\{u_1, \ldots, u_n\}$ and $\{v_1, \ldots, v_n\}$

such that $u_i v_j \in E(G)$ if and only if $A[i, j] \neq 0$. We shall show that the conditions of Halls theorem are satisfied for G.

Indeed, let $S = \{u_{i_1}, \ldots, u_{i_l}\}$, then $|N(S)|$ is exactly the number of columns $\{c_1, \ldots, c_q\}$ in A such that some row i_j has a non-zero entry in that column. Each row has total sum of elements k in it. Thus total sum of elements in rows $i_1, \ldots, i_l$ is kl. On the other hand, if the number of columns passing through non-zero positions of these rows is less than l, then, since the total sum of element in these columns is at least kl, there is a column with at least $kl/q > kl/l = k$ sum of elements – a contradiction.

Thus Halls condition is satisfied and there is a perfect matching in G, say it consists of the edges $u_1 v_{i_1}, \ldots, u_n v_{i_n}$. Now, the matrix P such that $P[j, i_j] = 1$, $j = 1, \ldots, n$ is a permutation matrix and $A - P$ is a matrix satisfying the conditions of the problem with $k - 1$. Thus applying induction to $A - P$ we obtain $k - 1$ permutation matrices whose sum equals $A - P$, which implies that A is the sum of these $k - 1$ permutation matrices and the k-th matrix P, as desired. □