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Problem 13

Theorem 1.1. *If a graph has an ear-decomposition, then it is* 2*-connected.*

Proof. By *Menger's Theorem*, a graph G is k-connected if and only if for any two vertices a, b in G there exist k independent a-b-paths. We find those 2 paths for any ear-composable graph. $\square$

Problem 14

For $0 < l < m \leq d$, we will construct a graph $F(d, l, m)$.

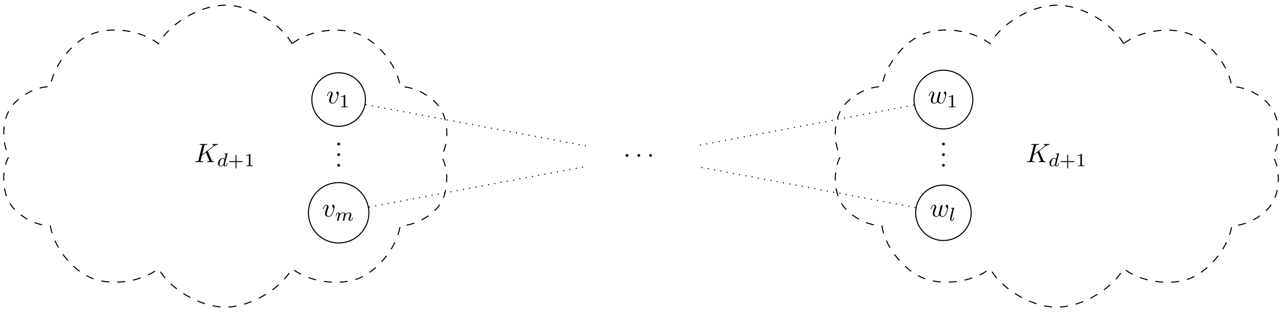

Figure 1: $F(d, l, m)$

First, we construct two complete graphs on $d+1$ vertices. $(V, E) \simeq K_{d+1}$, $(W, E') \simeq K_{d+1}$.

Then, we join m vertices $v_1, ..., v_m \in V$ of the first complete graph and l vertices $w_1, ..., w_l \in W$ of the second such that each v_i has a degree of exactly $d+1$ and each w_j of at least $d+1$ ($i \in [m], j \in [l]$).

Formally, for our constructed graph $F(d, l, m) := (V_F, E_F)$, the vertex set is the union of both complete graphs ($V_F = V \cup W$) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid \delta_{ij} = 1 \ (i, j \in \mathbb{N})\} \tag{1}$$

for a delta function δ_{ij} $(i, j \in \mathbb{N})$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i > l, j = l \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{d}$
No degree of a vertex of the complete graphs has been decreased. *Thus,* $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$.

Moreover, we have increased the degree of exactly $l + m < 2(d+1)$ vertices. Indeed, the complete graph on $d+1$ vertices is d-regular and hence there is at least one vertex of degree d in $F(d, l, m)$. *Thus,* $\delta(F(d, l, m)) \leq d$.

From these considerations, $\delta(F(d, l, m)) = d$.

$\kappa(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{l}$
In $F(d, l, m)$, the two complete graphs are only joined by edges between l vertices of one and m vertices of another complete graph. The graph obviously disconnects by removing those first l vertices. *Thus,* $\kappa(F(d, l, m)) \leq l$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa(K_{d+1}) = d > l$. Hence, if we found a subset of $l' < l$ vertices that disconnects $F(d, l, m)$, it had to consist of the complete graphs' vertices that we have affected in our construction. However, between the complete graphs there are l edges not sharing an endpoint.

Thus, it is neither possible to disconnect one of the complete graph by removing less than l vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than l vertices.

From these considerations, $\kappa(F(d, l, m)) = l$.

$\kappa'(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{m}$
In $F(d, l, m)$, the two complete graphs are only joined by exactly m edges and a removal of those m vertices obviously disconnects $F(d, l, m)$. *Thus,* $\kappa'(F(d, l, m)) \leq m$.

Moreover, a complete graph on $d+1$ vertices is internally connected with $\kappa'(K_{d+1}) = d > m$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than m edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than m edges.

From these considerations, $\kappa'(F(d, l, m)) = m$.

Problem 15

I will prove that any block-cut-vertex graph is a tree, by showing by contradiction that any block-cut-vertex graph is acyclic and connected.

Theorem 3.1. *The block-cut-vertex graph $G = (V, E)$ of any connected graph $G' = (V', E')$ is a tree.*

Proof. Acyclic
Let's assume for the sake of contradiction that G has a cycle $C = (b_1 b_2 ... b_1)$. Let's denote the subgraphs $B_1, B_2, ...B_n$ of G' which are the 2-connected components and bridges corresponding to the nodes $b_1, b_2, ...b_n$ of G. Let B_1 and B_2 be as stated above two different subgraphs of G'. Because the corresponding nodes b_1 and b_2 are adjacent in G, B_1 and B_2 have to share a vertex $x \in V(B_1) \cap V(B_2)$. We can use the same argument for each pair B_i, B_{i+1}. Additionally, we know that each component B_j is either 2-connected or a bridge. Thus we can find a circle through all the components $B_1, B_2 ... B_n$ which is, like very cycle, 2-connected. this is a contradiction to $B_1, B_2 ... B_n$ being the blocks of an block-cut-vertex graph, because by definition these blocks are either bridges or maximal 2-connected components. In this case though, we found a larger two connected component. Thus G has to be acyclic.

Connected
Let's assume for the sake of contradiction that G is not connected (but G' is connected). If G is not connected we have atleast two not connected components in G. Because G is the block-cut-vertex graph of G' each node in G' is represented by atleast one component of G because each vertex in G' is either part of a 2-connected component or is incident to a bridge. Additionally because G' is connected, there is atleast one edge $e = (uv) \in G'$ with with u and v being in a 2-connected component or bridge which are in G represented in different not connected components. Now there are two cases:
Either u and v are connected by an additionally path, not using e, then u and v would be 2-connected and would lie in the same 2-connected component. Or there is no additional path connecting u to v, hence e is a bridge. Either way, u and v would be represented in G by one and the same node which is a direct contradiction with u and v being represented in different components in G. Thus G has to be connected if G' is connected. □

Problem 16