

## Problem 17

**Theorem 1.1.** *In a planar triangulation let  $n_i$  be the number of vertices of degree  $i$ . Then,*

$$\sum_{i \in \mathbb{N}} (6 - i)n_i = 12$$

**Lemma 1.1.1.** *Let  $G = (V, E)$  and  $G' = (V \cup \{v\}, E \cup E')$  be planar triangulations. Then  $|E'| = 3$  and all edges in  $E'$  are incident to  $v$ .*

*Proof.* Since any planar triangulation of  $n$  vertices and  $e_n$  edges satisfies  $e_n = 3n - 6$ , we see inductively that

$$\begin{aligned} e_n &= e_{n-1} + 3 & (n > 3) \\ e_3 &= 3 \end{aligned}$$

Since  $G'$  has exactly one vertex more than  $G$  and both are planar triangulations,  $|E'| = 3$ .

Next, we will show that the degree of  $v$  exceeds or is equal to 3 and thus, all edges of  $E'$  have to be incident to  $v$ .

By KURATOWSKI,  $G'$  is not a topological minor of  $K_{3,3}$  or  $K_5$  and any planar triangulation is edge-maximal. By Lemma 4.4.5 (any edge-maximal graph without topological minors  $K_{3,3}, K_5$  is 3-connected),  $G'$  is 3-connected.

If the degree of  $v$  decreased 3, then  $G'$  would not be 3-connected ( $v$  could be isolated by removing two vertices).

Hence, all three edges of  $E'$  are incident to  $v$ . □

We will show by induction on the number of vertices  $n$  of a planar triangulation  $G$  with  $n_i$  vertices of degree  $i$  ( $i \in \mathbb{N}$ ) that

$$T_G := \sum_{i \in \mathbb{N}} (6 - i)n_i = 12$$

- Base  $n = 3$

Then, the graph is a triangle and the condition is satisfied:

$$T_{K_3} = \sum_{i \in \mathbb{N}} (6 - i)n_i = (6 - 2) \cdot 3 = 4 \cdot 3 = 12$$

- Step  $n \geq 4$

Any  $n$ -vertex planar triangulation  $G = (V, E)$  has a subgraph  $H = (V', E')$  which is an  $(n - 1)$ -vertex planar triangulation.

By Lemma 1.1.1, there is a vertex  $v \in V \setminus V'$  of degree 3. Furthermore, the degree of exactly three other vertices  $v_i, v_j, v_k \in V$  is increased. Thus,  $E \setminus E' = \{\{v, v_i\}, \{v, v_j\}, \{v, v_k\}\}$  and for  $T_G$ :

$$\begin{aligned} T_G &= T_H \\ &\quad + (6 - 3) \\ &\quad + (6 - (d(v_i) + 1)) - (6 - d(v_i)) \\ &\quad + (6 - (d(v_j) + 1)) - (6 - d(v_j)) \\ &\quad + (6 - (d(v_k) + 1)) - (6 - d(v_k)) \\ &= T_H + 3 - 1 - 1 - 1 \\ &= T_H \\ &= 12 & \text{(by induction)} \end{aligned}$$

□

## Problem 19

**Theorem 2.1.** *A plane embedded graph on  $n$  vertices that has no triangular face has at most  $2n - 4$  edges.*

**Lemma 2.1.1.** *Each plane triangulation (order  $\geq 4$ ) can be modified to a plane graph containing only faces of order 4 by removing one edge for each two faces*

*Proof.* Proof by induction over the order of a plane triangulation  $G$ :

- **Base:**  $V(G) = 4$

$G$  contains of 4 faces  $f_1, f_2, f_3, f_4$ . We get the desired graph by removing the shared edge between  $f_1, f_2$  and  $f_3, f_4$ . This is possible because each face is adjacent to one another. The remaining graph is, as desired, created by removing one edge foreach 2 faces.

- **Step:**  $V(G) = n$

Let  $G' = G - \{u\}$  ( $u \in V(G)$  with  $\deg(u) = 3$  as shown in **Lemma 1.1.1**). Then  $G'$  is still a plane triangulation and the amount of faces in  $G$  is exactly 2 more than in  $G'$ , because by removing the vertex  $u$ , 3 edges were removed, thus, 2 triangular faces were removed.

By induction we get the desired graph  $H'$  for  $G'$  by removing one edge for each 2 faces. Moreover, by inserting  $u$  and its 3 adjacent edges in  $H'$  at their old position, 2 faces of degree 3 were created (Would there be more than 2 new faces, the face in which  $u$  was inserted would have an order of at least 5). Thus, there are only two faces of order 3 remaining. Because all inserted edges are incident to  $u$ , these two faces of oder 3 are adjacent and can be merged by removing the shared edge.

The resulting graph of order  $n$  is still plane (only edges have been removed), has only faces of order 4 and was created of  $G$  by removing one edge for each 2 faces.

□

**Lemma 2.1.2.** *Each plane graph with no triangular face can be modified to a plane graph containing only faces of order 4 and with at least the same amount of edges*

*Proof.* Let  $G$  be such a plane graph with no triangular face. Let  $G'$  be the plane triangulation of  $G$ . To get  $G'$  out of  $G$  edges have to be added. Because the smallest face is of order 4, at least 1 edge for each face is needed to reduce the size of all faces, resulting in at least one additional face for each existing face in  $G$ , to be more precisely, in exactly one additional face for each inserted edge. So  $G'$  has at least twice as much faces as  $G$  and exactly one new edge for each new face. As shown in **Lemma 2.1.1**,  $G'$  can be modified to a plane graph  $H$  containing only faces of order 4 by removing one edge for each two faces. Thus, the amount of faces in  $H$  is at least as much as in  $G$ , and furthermore the amount of edges is at least as much as in  $G$ , because for each two faces exactly one edge has been removed.

All in all,  $H$  is a graph containing only faces of order 4 and has at least the same amount of edges as  $G$ .

□

Let  $G = (V, E_G)$  be a plane graph with no plane triangulation.

As shown in **Lemma 2.1.2**, we get a plane graph  $H = (V, E_H)$  ( $F_H$  corresponding faces, each face of order 4), containing at least the same amount of edges as  $G$  and the same set of vertices  $V$ .

By *Euler's Formula* and the fact  $|E_H| = \frac{4 \cdot |F_H|}{2}$  (because each face in  $F_H$  has order 4 and each edge is exactly counted twice), we get the following:

$$2 = |V| - |E_H| + |F_H| \leftrightarrow 2 = |V| - |E_H| + \frac{|E_H|}{2} \leftrightarrow |E_H| = 2 \cdot |V| - 4.$$

This inequality allows us to bound the amount of edges as following:

$$|E_G| \leq |E_H| = 2 \cdot n - 4 \quad (n = |V|)$$

□

## Problem 20

**Theorem 3.1.** *For a planar embedded graph  $G$  on  $n$  vertices such that  $G^* \cong G$ , the number of edges  $e = 2(n - 1)$ .*

*Proof.* The faces of  $G$  correspond to the vertices of  $G^*$ . Considering their isomorphism, the number of faces  $f$  of  $G$  equals the number of vertices of  $G$ .

$G$  is planar, by *Euler's Characteristic*:  $n - e + f = 2 \Rightarrow 2n - e = 2 \Rightarrow e = 2(n - 1)$ . □