

Problem 13

Theorem 1.1. *If a graph has an ear-decomposition, then it is 2-connected.*

Proof. Let G_1, \dots, G_n be the ear-decomposition of $G = (V, E)$ existing by definition. As far as $G_n = G$ it is sufficient to proof that $\forall i \in \{1, \dots, n\}$ G_i is 2-connected. This can be done by induction.

Base($i = 1$) : G_1 is a cycle, hence it is 2-connected.

Step($i \geq 2$) : Per definition $G_i = G_{i-1} + P_i$, P_i path and $P_i \cap G_{i-1}$ contains exactly the two endpoints of P_i .

Moreover we know that G_{i-1} is 2-connected by induction and P_i is connected by definition.

To proof that G_i is 2-connected we have to proof that $H := G_i - \{u\}$ ($u \in V(G_i)$) is connected.

Hence 2 cases have to be considered:

$u \in G_{i-1}$ Because G_{i-1} is 2-connected by induction H is connected.

Furthermore we know that H contains still one endpoint of P_i or more.

Thus H is a composition of two connected graphs, hence H is connected.

$u \in P_i - G_{i-1}$ We know that u is no endpoint of P_i , thus $P' := P_i - \{u\}$ is disconnected and is now a forest containing exactly two Trees T_1, T_2 .

Nevertheless each tree of P' contains exactly one endpoint of P_i . Hence $G_{i-1} + T_1 + T_2$ is still connected, because T_1, T_2 and G_{i-1} are connected. As far as $G_{i-1} + T_1 + T_2 = G_{i-1} + P_i - \{u\} = G_i - \{u\} = H$, H is connected.

Considering these two cases we know that H is connected, hence G_i is 2-connected.

We finally proofed by induction that $\forall i \in \{1, \dots, n\}$ G_i is 2-connected.

Thus $G = G_n$ is 2-connected.

□

Problem 14

For $0 < l < m \leq d$, we will construct a graph $F(d, l, m)$.

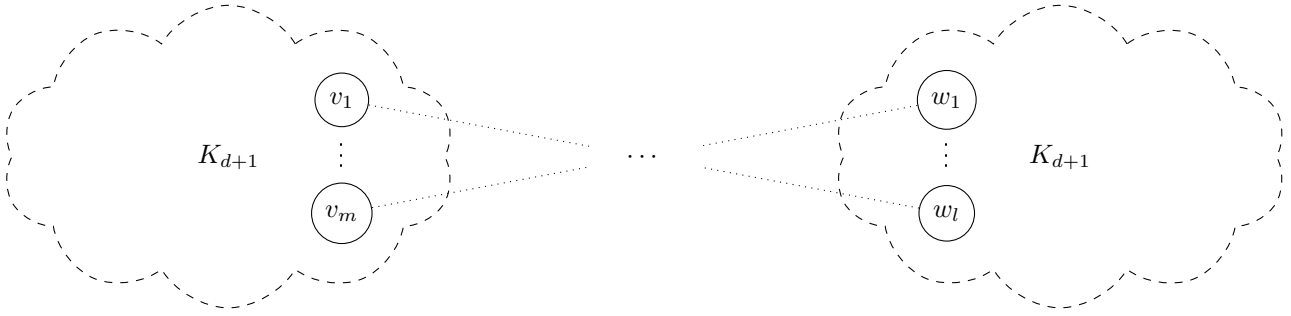


Figure 1: $F(d, l, m)$

First, we construct two complete graphs on $d + 1$ vertices. $(V, E) \simeq K_{d+1}$, $(W, E') \simeq K_{d+1}$.

Then, we join m vertices $v_1, \dots, v_m \in V$ of the first complete graph and l vertices $w_1, \dots, w_l \in W$ of the second such that each v_i has a degree of exactly $d + 1$ and each w_j of at least $d + 1$ ($i \in [m], j \in [l]$).

Formally, for our constructed graph $F(d, l, m) := (V_F, E_F)$, the vertex set is the union of both complete graphs ($V_F = V \cup W$) and it's edge set is defined by

$$E_F = E \cup E' \cup \{\{v_i, w_j\} \mid \delta_{ij} = 1 \ (i, j \in \mathbb{N})\} \quad (1)$$

for a delta function δ_{ij} ($i, j \in \mathbb{N}$)

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \text{ or } i > l, j = l \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

We will show that

- $\delta(F(d, l, m)) = d$
- $\kappa(F(d, l, m)) = l$
- $\kappa'(F(d, l, m)) = m$

$\delta(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{d}$

No degree of a vertex of the complete graphs has been decreased. Thus, $\delta(F(d, l, m)) \geq \delta(K_{d+1}) = d$.

Moreover, we have increased the degree of exactly $l + m < 2(d + 1)$ vertices. Indeed, the complete graph on $d + 1$ vertices is d -regular and hence there is at least one vertex of degree d in $F(d, l, m)$. Thus, $\delta(F(d, l, m)) \leq d$.

From these considerations, $\delta(F(d, l, m)) = d$.

$\kappa(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{l}$

In $F(d, l, m)$, the two complete graphs are only joined by edges between l vertices of one and m vertices of another complete graph. The graph obviously disconnects by removing those first l vertices. Thus, $\kappa(F(d, l, m)) \leq l$.

Moreover, a complete graph on $d + 1$ vertices is internally connected with $\kappa(K_{d+1}) = d > l$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than l vertices nor is it possible to remove the inter-connection between the two complete graphs by removing less than l vertices.

From these considerations, $\kappa(F(d, l, m)) = l$.

$\kappa'(\mathbf{F}(\mathbf{d}, \mathbf{l}, \mathbf{m})) = \mathbf{m}$

In $F(d, l, m)$, the two complete graphs are only joined by exactly m edges and a removal of those m vertices obviously disconnects $F(d, l, m)$. Thus, $\kappa'(F(d, l, m)) \leq m$.

Moreover, a complete graph on $d + 1$ vertices is internally connected with $\kappa'(K_{d+1}) = d > m$.

Thus, it is neither possible to disconnect one of the complete graph by removing less than m edges nor is it possible to remove the inter-connection between the two complete graphs by removing less than m edges.

From these considerations, $\kappa'(F(d, l, m)) = m$.

Problem 15

I will prove that any block-cut-vertex graph is a tree, by showing by contradiction that any block-cut-vertex graph is acyclic and connected.

Theorem 3.1. *The block-cut-vertex graph $G = (V, E)$ of any connected graph $G' = (V', E')$ is a tree.*

Proof. Acyclic

Let's assume for the sake of contradiction that G has a cycle $C = (b_1 b_2 \dots b_1)$. Let's denote the subgraphs B_1, B_2, \dots, B_n of G' which are the 2-connected components and bridges corresponding to the nodes b_1, b_2, \dots, b_n of G . Let B_1 and B_2 be as stated above two different subgraphs of G' . Because the corresponding nodes b_1 and b_2 are adjacent in G , B_1 and B_2 have to share a vertex $x \in V(B_1) \cap V(B_2)$. We can use the same argument for each pair B_i, B_{i+1} . Additionally, we know that each component B_j is either 2-connected or a bridge. Thus we can find a circle through all the components B_1, B_2, \dots, B_n which is, like every cycle, 2-connected. This is a contradiction to B_1, B_2, \dots, B_n being the blocks of an block-cut-vertex graph, because by definition these blocks are either bridges or maximal 2-connected components. In this case though, we found a larger two connected component. Thus G has to be acyclic.

Connected

Let's assume for the sake of contradiction that G is not connected (but G' is connected). If G is not connected we have at least two not connected components in G . Because G is the block-cut-vertex graph of G' each node in G' is represented by at least one component of G because each vertex in G' is either part of a 2-connected component or is incident to a bridge. Additionally because G' is connected, there is at least one edge $e = (uv) \in G'$ with u and v being in a 2-connected component or bridge which are in G represented in different not connected components. Now there are two cases:

Either u and v are connected by an additional path, not using e , then u and v would be 2-connected and would lie in the same 2-connected component. Or there is no additional path connecting u to v , hence e is a bridge. Either way, u and v would be represented in G by one and the same node which is a direct contradiction with u and v being represented in different components in G . Thus G has to be connected if G' is connected. \square

Problem 16