Problem 37

Lemma 1.0.1. *For every edge $e = (u, v)$ in a graph G without C_6 there are at most 2 C_4 in G which contain e.*

Proof. □

By using the lemma we can simply construct H by removing edges from G.

Theorem 1.1. *Every graph G without C_6 has a subgraph H with $|E(H)| \geq \frac{|E(G)|}{2}$, which contains C_4.*

Proof. Let G be such a graph. We know by using the lemma that evere edge is at most part of 2 C_4s. Let's denote k with the number of C_4s in G. $k \leq \frac{2}{4} * |E(G)| = \frac{|E(G)|}{2}$, because every edge is at most in 2 C_4s and $|C_4| = 4$. Now we can construct H by removing an edge from every C_4 in G. It is easy to see that by removing an edge no new cycles are created, hence H contains no C_4 and $|E(H)| = |E(G)| - k \geq |E(G)| - \frac{|E(G)|}{2} \leq \frac{|E(G)|}{2}$. □