Problem 17

Theorem 1.1. *In a planar triangulation let n_i be the number of vertices of degree i. Then,*

$$\sum_{i\in\mathbb{N}}(6-i)n_i = 12$$

.

Lemma 1.1.1. *Let $G = (V, E)$ and $G' = (V \cup \{v\}, E \cup E')$ be planar triangulations. Then $|E'| = 3$ and all edges in E' are indicent to v.*

Proof. Since any planar triangulation of n vertices and e_n edges satisifes $e_n = 3n-6$, we see inductively that

$$\begin{aligned} e_n &= e_{n-1} + 3 && (n > 3) \\ e_3 &= 3 \end{aligned}$$

Since G' has exactly one vertex more than G and both are planar triangulations, $|E'| = 3$.

Next, we will show that the degree of v exceeds or is equal to 3 and thus, all edges of E' have to be indicent to v.

By KURATOWSKI, G' is not a topological minor of $K_{3,3}$ or K_5 and any planar triangulation is edge-maximal. By *Lemma 4.4.5 (any edge-maximal graph without topological minors $K_{3,3}, K_5$ is 3-connected)*, G' is 3-connected.

If the degree of v deceeded 3, then G' would not be 3-connected (it could be isolated by removing two vertices) .

Hence, all three edges of E' are indicent to v. □

We will show by induction on the number of vertices n of a planar triangulation G with n_i vertices of degree i ($i \in \mathbb{N}$) that

$$T_G := \sum_{i\in\mathbb{N}}(6-i)n_i = 12$$

.

- Base $\mathbf{n = 3}$

  Then, the graph is a triangle and the condition is satisifed:

$$T_{K_3} = \sum_{i\in\mathbb{N}}(6-i)n_i = (6-2)\cdot 3 = 4\cdot 3 = 12$$

- Step $\mathbf{n \geq 4}$

  Any n-vertex planar triangulation $G = (V, E)$ has a subgraph $H = (V', E')$ which is a $(n-1)$-vertex planar triangulation.

  By *Lemma 1.1.1*, there is a vertex $v \in V \setminus V'$ of degree 3. Furthermore, the degree of exactly three other vertices $v_i, v_j, v_k \in V$ is increased. Thus $E \setminus E' = \{\{v, v_i\}, \{v, v_j\}, \{v, v_k\}\}$ and for T_G:

$$\begin{aligned} T_G =& T_H \\ &+ (6-3) \\ &+ (6-(d(v_i)+1)) - (6-d(v_i))) \\ &+ (6-(d(v_j)+1)) - (6-d(v_j)) \\ &+ (6-(d(v_k)+1)) - (6-d(v_k)) \\ =& T_H + 3 - 1 - 1 - 1 \\ =& T_H \\ =& 12 && \text{(by induction)} \end{aligned}$$

□

Problem 18

Problem 19

Theorem 3.1. *Each plane triangulation (order >= 4) can be modified to a plane graph containing only faces of order 4 by removing one edge for each two faces*

Proof. Proof by induction over the order of a plane triangulation G:

- Base: $V(G) = 4$
  G contains of 4 faces f_1, f_2, f_3, f_4. We get the desired graph by removing the shared edge between f_1, f_2 and f_3, f_4. This is possible because each face is adjacent to one another. The remaining graph is as desired, created by removing one edge foreach 2 faces.

- Step: $V(G) = n$
  Let $G' = G - \{u\}$ $(u \in V(G))$. Then G' is still a plane triangulation and the amount of faces in G is exactly 2 more than in G', because by removing the vertex u 3 edges were removed, thus 2 triangular faces were removed.
  By induction we get the desired graph H' for G' by removing one edge for each 2 faces. Because the order of the unbounded face in H' is 4, exactly one edge was removed of its border. Hence, by inserting u and its 3 adjacent edges, in H', there is one triangular face less than before. Thus, there are only two faces of order 3 remaining. Because all inserted edges are incident to u, these two faces of oder 3 are adjacent and can be merged by removing the shared edge.
  The resulting graph of order n is still plane, has only faces of order 4 and was created of G by removing one edge for each 2 faces.

□

Theorem 3.2. *Each plane graph with no triangular face can be modified to a plane graph containing only faces of order 4 and at least the same amount of edges*

Proof. Let G be such a plane graph with no triangular face. Let G' be the plane triangulation of G.
To get G' out of G edges have to be added. Because the smallest face is of order 4, at least 1 edge is needed to reduce the size of one face, resulting in at least one additional face, to be more precisely, in one additional face for each inserted edge. So G' has at least twice as much faces as G and one new edge for each new face. As shown in **Theorem 3.1**, G' can be modified to a plane graph H containing only faces of order 4 by removing one edge for each two faces. Thus, the amount of faces in H is at least as much as in G, and furthermore the amount of edges is at least as much as in G, because for each face removed exactly one edge has been removed.
All in all, H is a graph containing only faces of order 4 and has at least the same amount of edges as G. □

Let $G = (V, E_G)$ be a plane graph with no planar triangulation.
As shown in **Theorem 3.2**, we get a graph $H = (V, E_H)$ (F_H corresponding faces) containing at least the same amount of edges as G and the same set of vertices V.

By $Euler'sFormula$ and the fact $|E_H| = \frac{4*|F_H|}{2}$ (because each face in F_H has order 4 and each edge is exactly counted twice), we get the following:
$2 = |V| - |E_H| + |F_H| \leftrightarrow 2 = |V| - |E_H| + \frac{E_H}{2} \leftrightarrow |E_H| = 2 * |V| - 4.$
This inequality allows us to bound the amount of edges as following:

$|E_G| \leq |E_H| = 2 * n - 4 \quad (n = |V|)$

Problem 20