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Novel similarity measures in spherical fuzzy environment and their applications

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ABSTRACT

Spherical fuzzy sets (SFSs) have gained great attention from researchers in various fields. The spherical fuzzy set is characterized by three membership functions expressing the degrees of membership, non-membership and the indeterminacy to provide a larger preference domain. It was proposed as a generalization of picture fuzzy sets and Pythagorean fuzzy sets in order to deal with uncertainty and vagueness information. The similarity measure is one of the essential and advantageous tools to determine the degree of similarity between items. Several studies on similarity measures have been developed due to the importance of similarity measure and application in decision making, data mining, medical diagnosis, and pattern recognition in the literature. The contribution of this study is to present some novel spherical fuzzy similarity measures. We develop the Jaccard, exponential, and square root cosine similarity measures under spherical fuzzy environment. Each of these similarity measures is analyzed with respect to decision-makers' optimistic or pessimistic point of views. Then, we apply these similarity measures to medical diagnose and green supplier selection problems. These similarity measures can be computed easily and they can express the dependability similarity relation apparently.

1. Introduction

Fuzzy sets have gained attention from researchers in various applications since it has emerged in 1965. Zadeh (1965) developed fuzzy sets with notions of inclusion, union, intersection, complement, relation, convexity, and linguistic hedges as a class of objects with a continuum of grades of membership. The ordinary fuzzy sets have been extended to many new types of fuzzy sets. The other extensions of ordinary fuzzy sets are as follows: Type-2 fuzzy sets (Zadeh, 1975), Interval-valued fuzzy sets (Sambuc, 1975; Zadeh, 1975; Jahn, 1975; Grattan-Guinness, 1976), Intuitionistic fuzzy sets (Atanassov, 1986, 2016), fuzzy multisets (Yager, 1986), neutrosophic sets (Smarandache, 1998), nonstationary fuzzy sets (Garibaldi and Ozen, 2007), hesitant fuzzy sets (Torra, 2010), Pythagorean fuzzy sets (Yager, 2013), picture fuzzy sets (Cuong and Kreinovich, 2013), q-rung orthopair fuzzy sets (Yager, 2016) and spherical fuzzy sets (Kahraman and Kutlu Gündoğdu, 2018; Kutlu Gündoğdu and Kahraman, 2019d).

Kahraman and Kutlu Gündoğdu (2018) introduced spherical fuzzy sets as an extension of picture fuzzy sets and Pythagorean fuzzy sets. The basic idea behind SFSs is to allow evaluators to generalize other extensions of fuzzy sets by defining a membership function on a spherical surface and independently assign the parameters of that function with a larger domain. Sequentially, Ashraf and Abdullah (2019) proposed spherical fuzzy sets with some operational rules, and aggregation operations based on Archimedean t-norm and t-conorms (Ashraf et al., 2019a). A linguistic spherical fuzzy set was presented by Jin et al. (2019a) and they also presented linguistic spherical fuzzy weighted averaging and geometric operators to multi-attribute group decision-making problems. Kutlu Gündoğdu (2019) summarized the previously introduced spherical fuzzy sets and spherical fuzzy TOPSIS method employed for the site selection of photovoltaic power. Kutlu Gundogdu and Kahraman (2019a) also introduced novel interval-valued spherical fuzzy sets, and applied it to develop the extension of TOPSIS under spherical fuzzy environment for solving 3D printer selection. Ashraf et al. (2019b) proposed Spherical fuzzy Dombi weighted averaging and geometric aggregation operators and discussed several properties of these aggregation operators for the decision-making problems. In another research, Ashraf et al. (2019c) presented spherical fuzzy distance-weighted averaging operators. Jin et al. (2019b) proposed some novel logarithmic operations of spherical fuzzy sets with spherical fuzzy entropy. Donyatalab et al. (2019) and Farrokhizadeh et al. (2020) developed Harmonic mean and Bonferroni mean aggregation

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operators under spherical fuzzy environment and their proposed operators were used for the decision-making problems. Zeng et al. (2019) adopted a new approach of covering-based spherical fuzzy rough set (CSFRS) models to hybrid spherical fuzzy sets with notions of covering the rough set and presented TOPSIS approach through CSFRS models. Kutlu Gündoğdu and Kahraman (2019b,c) proposed the novel method namely the spherical fuzzy analytic hierarchy process (SF-AHP) and their proposed method was applied to industrial robot selection and renewable energy selection problems. Liu et al. (2020) developed the linguistic spherical fuzzy numbers (Lt-SFNs) to suggest the public's knowledge of language valuation and they also developed TODIM method and a MABAC method based on Lt-SFNs.

Similarity measures and distances are widely used to determine the relationship between two individuals in many domains composed of decision-making, pattern recognition, medical engineering, and network comparison (Chiclana et al., 2013). A similarity measure indicates how similar two objects are. The existing similarity measures comprise Dice's measure, Jaccard's measure, the cosine formula, and the correlation coefficient of Pearson (Egghe, 2010). The Dice, Jaccard, and cosine measures are types of vector similarity measures. With the development of fuzzy sets, "classical" similarity measures have been extended under various fuzzy environments (Zhang et al., 2014). Some new similarity measures in the framework of spherical fuzzy sets and T-spherical fuzzy sets including cosine similarity measures, gray similarity measures, and set theoretic similarity measures were developed by Ullah et al. (2018) and these similarity measures were applied to a building material recognition problem. Rafiq et al. (2019) also investigated the novel cosine similarity measures under spherical fuzzy environment by considering the membership, hesitancy, non-membership and refusal degrees. Wei et al. (2019) defined ten different similarity measures between spherical fuzzy sets (SFSs) based on the cosine and cotangent functions by considering the membership, non-membership, indeterminacy and refusal degrees. They also developed the weighted similarity measures of cosine and cotangent functions. They applied these similarity functions to pattern recognition and medical diagnosis.

The aim of this study is to develop novel similarity measures for recently introduced spherical fuzzy sets. In this paper, it is observed that the existing similarity measures developed in the literature have some limitations and cannot be applied to the problems where information is provided in spherical fuzzy environment. Some similarity measures in spherical fuzzy environment were developed in the literature but, the proposed similarity measures in this article have some differences from those similarity measures from the optimistic and pessimistic point of view. Some of these measures have optimistic view and give greater similarity degree, while some of them have pessimistic nature and give smaller similarity degree. So, depending on which similarity measure we use, the results can be different. Decision-makers with different points of view can use the proper similarity measure for their aims. To solve this issue, some novel similarity measuring tools including Jaccard similarity measures, exponential similarity measure based on Hamming and Euclidean distance and square root cosine similarity measure are proposed for spherical fuzzy sets. It is also proposed that existing similarity measures become special cases of the developed similarity measures showing the novelty and diversity of the proposed similarity measures. By using these new similarity measures, medical diagnosis and green supplier selection problems are solved successfully and the results are discussed.

The rest of this manuscript is organized as follows. Section 2 briefly introduces the basic preliminaries of the novel concept of "spherical fuzzy set". In Section 3, some existing similarity measures for SFSs in the literature are summarized. In Section 4, the proposed similarity measures such as Jaccard, exponential and square root cosine similarity measures of spherical fuzzy sets are presented. Section 5 includes some descriptive analyses related to applications of these similarity measures. Finally, Section 6 presents the conclusions and future directions.

2. Preliminaries of Spherical Fuzzy Sets (SFSs)

Spherical fuzzy sets are the latest extension of picture fuzzy sets and Pythagorean fuzzy sets. Spherical Fuzzy Sets (SFSs) give a larger preference domain for evaluators as well as each decision-maker may also determine his/her hesitancy information independently under spherical fuzzy environment.

Definition 1. A Single valued Spherical Fuzzy Set (SFSs) $\tilde{A}_S$ of the universe of discourse X is given by Kutlu Gündoğdu and Kahraman (2019d):

$$\tilde{A}_s = \left\{ x, \mu_{\tilde{A}_s}(x), \vartheta_{\tilde{A}_s}(x), I_{\tilde{A}_s}(x) \middle| x \in X \right\} \tag{1}$$

where $\mu_{\tilde{A}_s}(x), \vartheta_{\tilde{A}_s}(x), I_{\tilde{A}_s}(x) : X \to [0,1]$ are the degree of membership, non-membership and indeterminacy of x to $\tilde{A}_S$, respectively, and

$$0 \le \mu^2_{\tilde{A}_s}(x) + \vartheta^2_{\tilde{A}_s}(x) + I^2_{\tilde{A}_s}(x) \le 1 \tag{2}$$

Then, $1 - \sqrt{\mu^2_{\tilde{A}_s}(x) + \vartheta^2_{\tilde{A}_s}(x) + I^2_{\tilde{A}_s}(x)}$ is defined as refusal degree of x in X.

Definition 2. Assume that $\tilde{A}_s$ and $\tilde{B}_s$ are any two Spherical Fuzzy sets. So the basic operations of SFSs can be defined as follows (Kutlu Gündoğdu and Kahraman, 2019d):

$$\tilde{A}_s \oplus \tilde{B}_s = \left\{ \sqrt{\mu^2_{\tilde{A}_s} + \mu^2_{\tilde{B}_s} - \mu^2_{\tilde{A}_s}\mu^2_{\tilde{B}_s}}, \vartheta_{\tilde{A}_s}\vartheta_{\tilde{B}_s}, \sqrt{(1-\mu^2_{\tilde{B}_s})I^2_{\tilde{A}_s} + (1-\mu^2_{\tilde{A}_s})I^2_{\tilde{B}_s} - I^2_{\tilde{A}_s}I^2_{\tilde{B}_s}} \right\} \tag{3}$$

$$\tilde{A}_s \otimes \tilde{B}_s = \left\{ \mu_{\tilde{A}_s}\mu_{\tilde{B}_s}, \sqrt{\vartheta^2_{\tilde{A}_s} + \vartheta^2_{\tilde{B}_s} - \vartheta^2_{\tilde{A}_s}\vartheta^2_{\tilde{B}_s}}, \sqrt{(1-\vartheta^2_{\tilde{B}_s})I^2_{\tilde{A}_s} + (1-\vartheta^2_{\tilde{A}_s})I^2_{\tilde{B}_s} - I^2_{\tilde{A}_s}I^2_{\tilde{B}_s}} \right\} \tag{4}$$

$$k\tilde{A}_s = \left\{ \sqrt{1-(1-\mu^2_{\tilde{A}_s})^k}, \vartheta^k_{\tilde{A}_s}, \sqrt{(1-\mu^2_{\tilde{A}_s})^k - (1-\mu^2_{\tilde{A}_s} - I^2_{\tilde{A}_s})^k} \right\}; k > 0 \tag{5}$$

$$\tilde{A}^k_s = \left\{ \mu^k_{\tilde{A}_s}, \sqrt{1-(1-\vartheta^2_{\tilde{A}_s})^k}, \sqrt{(1-\vartheta^2_{\tilde{A}_s})^k - (1-\vartheta^2_{\tilde{A}_s} - I^2_{\tilde{A}_s})^k} \right\}; k > 0 \tag{6}$$

Definition 3. For the SFSs $\tilde{A}_s = (\mu_{A_s}, \vartheta_{A_s}, I_{A_s})$ and $\tilde{B}_s = (\mu_{B_s}, \vartheta_{B_s}, I_{B_s})$, the followings are valid, provided that k, k_1 and $k_2 \ge 0$ (Kutlu Gündoğdu and Kahraman, 2019d).

I. $\tilde{A}_s \oplus \tilde{B}_s = \tilde{B}_s \oplus \tilde{A}_s$ (7)

II. $\tilde{A}_s \otimes \tilde{B}_s = \tilde{B}_s \otimes \tilde{A}_s$ (8)

III. $k(\tilde{A}_s \oplus \tilde{B}_s) = k\tilde{A}_s \oplus k\tilde{B}_{s_s}$ (9)

IV. $k_1\tilde{A}_s \oplus k_2\tilde{A}_s = (k_1 + k_2)\tilde{A}_s$ (10)

V. $(\tilde{A}_s \otimes \tilde{B}_s)^k = \tilde{A}^k_s \otimes \tilde{B}^k_s$ (11)

VI. $\tilde{A}^{k_1}_s \otimes \tilde{A}^{k_2}_s = \tilde{A}^{k_1+k_2}_s$ (12)

Definition 4. Suppose that $\tilde{A}_s$ and $\tilde{B}_s$ are two Spherical fuzzy sets. Then to compare these SFSs, the score function (SC) and accuracy function (AC) are defined as follows (Kutlu Gündoğdu and Kahraman, 2019d):

$$SC\left(\tilde{A}_s\right) = (\mu_{\tilde{A}_s} - \frac{I_{\tilde{A}_s}}{2})^2 - (\vartheta_{\tilde{A}_s} - \frac{I_{\tilde{A}_s}}{2})^2 \tag{13}$$

$$AC\left(\tilde{A}_s\right) = \mu^2_{\tilde{A}_s} + \vartheta^2_{\tilde{A}_s} + I^2_{\tilde{A}_s} \tag{14}$$

After calculating the results of score and accuracy functions, the comparison rules are as follows:
If $SC\left(\tilde{A}_s\right) > SC\left(\tilde{B}_s\right)$, then $\tilde{A}_s > \tilde{B}_s$;
If $SC\left(\tilde{A}_s\right) = SC\left(\tilde{B}_s\right)$ and $AC\left(\tilde{A}_s\right) > AC\left(\tilde{B}_s\right)$, then $\tilde{A}_s > \tilde{B}_s$;
If $SC\left(\tilde{A}_s\right) = SC\left(\tilde{B}_s\right)$, $AC\left(\tilde{A}_s\right) = AC\left(\tilde{B}_s\right)$, then $\tilde{A}_s > \tilde{B}_s$;
If $SC\left(\tilde{A}_s\right) = SC\left(\tilde{B}_s\right)$, $AC\left(\tilde{A}_s\right) = AC\left(\tilde{B}_s\right)$, then $\tilde{A}_s = \tilde{B}_s$;

Definition 5. Spherical Fuzzy Weighted Arithmetic Mean $(SFWAM)$ with respect to, $w = (w_1, w_2, \dots, w_n)$; $w_i \in [0,1]$; $\sum_{i=1}^n w_i = 1$, SFWAM is defined as (Kutlu Gündoğdu and Kahraman, 2019d):

$$SFWAM_w\left(\tilde{A}_{S1}, \tilde{A}_{S2}, \dots, \tilde{A}_{Sn}\right) = w_1\tilde{A}_{S1} + w_2\tilde{A}_{S2} + \dots + w_n\tilde{A}_{Sn}$$

$$=\left\{\sqrt{1-\prod_{i=1}^{n}\left(1-\mu_{A_s}^2\right)^{w_i}},\prod_{i=1}^{n}\vartheta_{A_s}^{w_i},\sqrt{\prod_{i=1}^{n}(1-\mu_{A_s}^2)^{w_i}-\prod_{i=1}^{n}(1-\mu_{A_s}^2-I_{A_s}^2)^{w_i}}\right\} \tag{15}$$

Definition 6. Spherical Fuzzy Weighted Geometric Mean ($SFWGM$) with respect to, $w=(w_1,w_2,\ldots,w_n)$; $w_i\in[0,1]$; $\sum_{i=1}^{n}w_i=1$, SFWGM is defined as (Kutlu Gündoğdu and Kahraman, 2019d):

$$SFWGM_w\left(\tilde{A}_{S1},\tilde{A}_{S2},\ldots,\tilde{A}_{Sn}\right)=\tilde{A}_{S1}^{w_i}+\tilde{A}_{S2}^{w_i}+\cdots+\tilde{A}_{Sn}^{w_i}$$
$$=\left\{\prod_{i=1}^{n}\mu_{A_s}^{w_i},\sqrt{1-\prod_{i=1}^{n}\left(1-\vartheta_{A_s}^2\right)^{w_i}},\sqrt{\prod_{i=1}^{n}(1-\vartheta_{A_s}^2)^{w_i}-\prod_{i=1}^{n}(1-\vartheta_{A_s}^2-I_{A_s}^2)^{w_i}}\right\} \tag{16}$$

After presenting main mathematical operations and aggregation operators of spherical fuzzy sets, we will briefly indicate some existing similarity measures of spherical fuzzy sets and afterward introduce new developed similarity measures.

3. Existing similarity measures for SFSs

In this section, we recall some existing similarity measures that have been developed for spherical fuzzy sets to illustrate and compare our proposed similarity measure with the existing similarity measures. Some cosine and cotangent similarity measures are proposed by Wei et al. (2019). The cosine similarity measure (SFC) between two SFSs $\tilde{A}$ and $\tilde{B}$ is given in Eq. (17).

$$SFC^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\mu_A^2(x_i)\mu_B^2(x_i)+\vartheta_A^2(x_i)\vartheta_B^2(x_i)+I_A^2(x_i)I_B^2(x_i)\right)}{\sqrt{\left(\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right)}\sqrt{\left(\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right)}} \tag{17}$$

The cosine similarity measure with refusal degree is as in Eq. (18) given in Box I. The weighted cosine similarity measure without and with refusal degree is shown as Eqs. (19) and (20) in Box II. The cotangent similarity measures (SFCT) which are proposed by Wei et al. (2019) are as follows.

The cotangent similarity measure without refusal degree is given in Eq. (21).

$$SFCT^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}cot\left[\frac{\pi}{4}+\frac{\pi}{8}(|\mu_A^2(x_i)-\mu_B^2(x_i)|+|\vartheta_A^2(x_i)-\vartheta_B^2(x_i)|+|I_A^2(x_i)-I_B^2(x_i)|)\right] \tag{21}$$

The cotangent similarity measure with refusal degree is given in Eq. (22).

$$SFCT^2(A,B)=\frac{1}{n}\sum_{i=1}^{n}cot\left[\begin{array}{l}\frac{\pi}{4}+\frac{\pi}{8}(|\mu_A^2(x_i)-\mu_B^2(x_i)|+|\vartheta_A^2(x_i)-\vartheta_B^2(x_i)|\\+|I_A^2(x_i)-I_B^2(x_i)|++|R_A^2(x_i)-R_B^2(x_i)|)\end{array}\right] \tag{22}$$

The weighted cotangent similarity measures (WSFCT) without and with refusal degree are given in Eq. (23) and Eq. (24), respectively.

$$WSFCT^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}w_i cot\left[\frac{\pi}{4}+\frac{\pi}{8}(|\mu_A^2(x_i)-\mu_B^2(x_i)|+|\vartheta_A^2(x_i)-\vartheta_B^2(x_i)|+|I_A^2(x_i)-I_B^2(x_i)|)\right] \tag{23}$$

$$WSFCT^2(A,B)=\frac{1}{n}\sum_{i=1}^{n}w_i cot\left[\begin{array}{l}\frac{\pi}{4}+\frac{\pi}{8}(|\mu_A^2(x_i)-\mu_B^2(x_i)|+|\vartheta_A^2(x_i)-\vartheta_B^2(x_i)|\\+|I_A^2(x_i)-I_B^2(x_i)|++|R_A^2(x_i)-R_B^2(x_i)|)\end{array}\right] \tag{24}$$

4. Novel similarity measures for spherical Fuzzy sets

In this section, we propose some novel similarity measures and weighted and generalized similarity measures between SFSs along with the concepts of the Jaccard, exponential, square root cosine functions.

4.1. Jaccard similarity measure

Jaccard's index (Jaccard, 1908), stands out as one of the most useful and widely used similarity indices for binary data. The Jaccard index, also known as intersection over union and the Jaccard similarity coefficient is a statistic used for measuring the similarity and diversity of sample sets.

Let $\tilde{X}$ and $\tilde{Y}$ be two sample sets. The Jaccard index $J(X,Y)$ is defined as the size of the intersection between $\tilde{X}$ and $\tilde{Y}$ divided by the size of the union between $\tilde{X}$ and $\tilde{Y}$. Let X and Y be any two binary objects, each with n attributes, in which each attribute of the two objects can take either 0 or 1, then the binary Jaccard coefficient $J(X,Y)$ is a measure of percentage of overlap between sets. In other words, the Jaccard coefficient $J(X,Y)$ is defined as the ratio of the number of shared attributes of X and Y to the numbers possessed by X or Y as defined in Eq. (25).

$$J(X,Y)=\frac{|X\cap Y|}{|X\cup Y|}=\frac{|X\cap Y|}{|X|+|Y|-|X\cap Y|} \tag{25}$$

Definition 7. Let $X=(x_1,x_2,\ldots,x_n)$ and $Y=(y_1,y_2,\ldots,y_n)$ be the two vectors of length n, where all of the n points are positive, then the Jaccard similarity measure between these two vectors is defined as (Hwang and Yang, 2014):

$$J(X,Y)=\frac{X.Y}{\|X\|_2^2+\|Y\|_2^2-X.Y}=\frac{\sum_{i=1}^{n}x_iy_i}{\sum_{i=1}^{n}x_i^2+\sum_{i=1}^{n}y_i^2-\sum_{i=1}^{n}x_iy_i} \tag{26}$$

where $X.Y=\sum_{i=1}^{n}x_iy_i$ is the inner product of the two vectors X and Y, and $\|X\|_2^2=\sum_{i=1}^{n}x_i^2$, $\|Y\|_2^2=\sum_{i=1}^{n}y_i^2$ are the Euclidean norms (L_2) of X and Y.

The Jaccard similarity measure satisfies the following properties:

1. $0\le J(X,Y)\le 1$;
2. $J(X,Y)=J(Y,X)$;
3. $X=Y$ if and only if $J(X,Y)=1$.

4.1.1. Some similarity measures based on Jaccard coefficient for SFSs

In this section, we propose a new similarity measure between SFSs $\tilde{A}$ and $\tilde{B}$ using the Jaccard similarity coefficient. The Jaccard similarity measure or the Jaccard similarity *coefficient* compares members for two sets to see which members are shared and which are distinct. It is a measure of similarity for the two sets of data, with a range from 0 to 1. The similarity measure, closer to 1, the more similar the two sets. The Jaccard similarity measure, which can calculate the degree of similarity and proximity between any two schemes, have been applied in many practical multiple attribute decision-making (MADM) problems. To select the best alternatives in decision-making problems, the Jaccard similarity measure utilized to obtain the similarity degree between each alternative and the ideal alternative. In this section, some Jaccard similarity measures for spherical fuzzy sets will be presented.

Definition 8. Suppose that there are two sets of SFSs $\tilde{A}_s=\left\{x_i\left\langle\mu_A(x_i),\vartheta_A(x_i),I_A(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$ and $\tilde{B}_s=\left\{x_i\left\langle\mu_B(x_i),\vartheta_B(x_i),I_B(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$ be two SFSs, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership and Indeterminacy degrees, respectively. Then, using Eq. (26) the Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ ($JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$) is defined as follow:

$JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$

$$SFC^2(A,B)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\right)}{\sqrt{\left(\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)+R_A^4(x_i)\right)}\sqrt{\left(\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)+R_B^4(x_i)\right)}} \quad (18)$$

Box I.

$$WSFC^1(A,B)=\frac{1}{n}\sum_{i=1}^{n}w_i\frac{\left(\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right)}{\sqrt{\left(\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right)}\sqrt{\left(\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right)}} \quad (19)$$

$$WSFC^2(A,B)=\frac{1}{n}\sum_{i=1}^{n}w_i\frac{\left(\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\right)}{\sqrt{\left(\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)+R_A^4(x_i)\right)}\sqrt{\left(\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)+R_B^4(x_i)\right)}} \quad (20)$$

Box II.

$$=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}{\begin{array}{c}\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]\end{array}}\right] \quad (27)$$

Theorem 1. *If $\tilde{A}_s=\left\{x_i\left\langle\mu_A(x_i),\vartheta_A(x_i),I_A(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$ and $\tilde{B}_s=\left\{x_i\left\langle\mu_B(x_i),\vartheta_B(x_i),I_B(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$ be two SFSs, then the Jaccard similarity measure $JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ satisfies the following properties:*

$P1.\ 0\le JS_{SFS}(\tilde{A}_s,\tilde{B}_s)\le 1;$

$P2.\ JS_{SFS}(\tilde{A}_s,\tilde{B}_s)=JS_{SFS}(\tilde{B}_s,\tilde{A}_s);$

$P3.\ \tilde{A}_s=\tilde{B}_s\ if\ and\ only\ if\ JS_{SFS}(\tilde{A}_s,\tilde{B}_s)=1\rightarrow\mu_A(x_i)=\mu_B(x_i),$
$\vartheta_A(x_i)=\vartheta_B(x_i)\ and\ I_A(x_i)=I_B(x_i);$

Proof.
P1. Let $\tilde{A}_s$ and $\tilde{B}_s$ be two SFSs. Then $JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ according to Eq. (3) is defined in Box III: We know that $JS_{SFS}(\tilde{A}_s,\tilde{B}_s)\ge 0$. So, according to inequality $a^2+b^2\ge 2ab$:

$$\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]\ge 2\left[\mu_A^2(x_i)+\vartheta_A^2(x_i)+I_A^2(x_i)\right]\times\left[\mu_B^2(x_i)+\vartheta_B^2(x_i)+I_B^2(x_i)\right]$$

Therefore, $0\le JS_{SFS}(\tilde{A}_s,\tilde{B}_s)\le 1$. From Eq. (3), the summation of n terms is also $0\le JS_{SFS}(\tilde{A}_s,\tilde{B}_s)\le 1$.
P2. Without need for any proof, it is obvious that the proposition is true.
P3. We know that if $\tilde{A}_s=\tilde{B}_s$, then $\mu_A(x_i)=\mu_B(x_i),\vartheta_A(x_i)=\vartheta_B(x_i)$ and $I_A(x_i)=I_B(x_i)$. So, there is $JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ defined in Box IV.

Definition 9. Suppose that there are two sets of SFSs $\tilde{A}_s=\left\{x_i\left\langle\mu_A(x_i),\vartheta_A(x_i),I_A(x_i),R_A(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$ and $\tilde{B}_s=\left\{x_i\left\langle\mu_B(x_i),\vartheta_B(x_i),I_B(x_i),R_A(x_i)\right\rangle|x_i\in X\right\};\forall i=1,2,\ldots,n$, so that $\mu(x_i)$, $\vartheta(x_i)$, $I(x_i)$ and $R(x_i)=1-\sqrt{\mu^2_{\tilde{A}_s}(x)+\vartheta^2_{\tilde{A}_s}(x)+I^2_{\tilde{A}_s}(x)}$ are membership, non-membership, indeterminacy and refusal degrees, respectively. We further propose the Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ ($JS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$) considering refusal degree as shown as Eq. (28) in Box V:

The Jaccard similarity measure considering refusal degree $JS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 1.

Due to the space constraint and the similarity of proving these properties with the properties proof of previous section, we do not prove these properties for the upcoming similarity measures.

If we consider the weights of x_i with weight vector $w_i=(w_1,w_2,\ldots,w_n)$ which $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$, then the weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ ($WJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$) is defined as in Eq. (29).

$$WJS_{SFS}(\tilde{A}_s,\tilde{B}_s)=\sum_{i=1}^{n}\left[w_i\frac{\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}{\begin{array}{c}\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]\end{array}}\right] \quad (29)$$

The weighted Jaccard similarity measure $WJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 1.

In the weighted Jaccard similarity measure $WJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$, if the refusal degree is considered as well as membership, non-membership and indeterminacy degrees, we can get the weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ ($WJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$) which takes into account refusal degree as given as Eq. (30) in Box VI. The $WJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 1.

If we consider the parameter Γ, satisfying $0\le\Gamma\le 1$, the Generalized Weighted Jaccard Similarity Measure $GWJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ for two SFSs $\tilde{A}$ and $\tilde{B}$ considering weight vector $w_i=(w_1,w_2,\ldots,w_n)$ which $w_i\in[0,1]$ and $\sum_{i=1}^n w_i=1$ can be obtained. The $GWJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ is given as in Eq. (31).

$$GWJS_{SFS}(\tilde{A}_s,\tilde{B}_s)=\sum_{i=1}^{n}\left[w_i\frac{\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}{\begin{array}{c}\Gamma\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+(1-\Gamma)\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]\end{array}}\right] \quad (31)$$

The $GWJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 1.

The Generalized Weighted Jaccard Similarity Measure for SFSs considering refusal degree, $GWJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$ can be calculated by considering the refusal degree of $\tilde{A}_s$ and $\tilde{B}_s$. The $GWJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$ is defined as in Eq. (32) in Box VII. where $0\le\Gamma\le 1$.

The $GWJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 1.

$$JS_{SFS}(\tilde{A}_s,\tilde{B}_s)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}\right]$$

Box III.

$$\begin{aligned}JS_{SFS}(\tilde{A}_s,\tilde{B}_s)&=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)\right]-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)\right]}\right]\\&=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^2(x_i)\,\mu_A^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_A^2(x_i)+I_A^2(x_i)\,I_A^2(x_i)\right]}{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]+\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]-\left[\mu_A^2(x_i)\,\mu_A^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_A^2(x_i)+I_A^2(x_i)\,I_A^2(x_i)\right]}\right]\\&=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]}{2\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]-\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]}\right]=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]}{\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)\right]}\right]=1.\end{aligned}$$

Box IV.

$$JS_{SFS}^{R}(\tilde{A}_s,\tilde{B}_s)=\frac{1}{n}\sum_{i=1}^{n}\left[\frac{\begin{bmatrix}\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)\\+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\end{bmatrix}}{\begin{array}{c}\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)+R_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)+R^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\right]\end{array}}\right] \tag{28}$$

Box V.

$$WJS_{SFS}^{R}(\tilde{A}_s,\tilde{B}_s)=\sum_{i=1}^{n}\left[w_i\frac{\begin{bmatrix}\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)\\+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\end{bmatrix}}{\begin{array}{c}\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)+R_A^4(x_i)\right]+\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)+R^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\right]\end{array}}\right] \tag{30}$$

Box VI.

$$GWJS_{SFS}^{R}(\tilde{A}_s,\tilde{B}_s)=\sum_{i=1}^{n}\left[w_i\frac{\begin{bmatrix}\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)\\+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\end{bmatrix}}{\begin{array}{c}\Gamma\left[\mu_A^4(x_i)+\vartheta_A^4(x_i)+I_A^4(x_i)+R_A^4(x_i)\right]+(1-\Gamma)\left[\mu_B^4(x_i)+\vartheta_B^4(x_i)+I_B^4(x_i)+R^4(x_i)\right]\\-\left[\mu_A^2(x_i)\,\mu_B^2(x_i)+\vartheta_A^2(x_i)\,\vartheta_B^2(x_i)+I_A^2(x_i)\,I_B^2(x_i)+R_A^2(x_i)\,R_B^2(x_i)\right]\end{array}}\right] \tag{32}$$

Box VII.

4.2. Exponential similarity measure

Shepard (1987) proposed a universal law that distance and perceived similarities are related to an exponential function. The similarity measure based on the exponential function is given in Eq. (33).

$$S(A,B)=e^{-d(A,B)} \tag{33}$$

In this section, we propose some new similarity measures between SFSs $\tilde{A}$ and $\tilde{B}$ using the exponential function, based on Euclidean distance and Hamming distance between two spherical sets $\tilde{A}$ and $\tilde{B}$.

Definition 10. Let $x\in X$ be a universe set and $\tilde{A}$ and $\tilde{B}$ be two sample spherical fuzzy sets. Then normalized Hamming distance $Hd_{SFS}(\tilde{A},\tilde{B})$ is given in Eq. (34) (Ashraf et al., 2019d).

$$Hd_{SFS}(\tilde{A},\tilde{B})=\frac{1}{n}\sum_{i=1}^{n}\left(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|\right) \tag{34}$$

Definition 11. Let $x\in X$ be a universe set and $\tilde{A}$ and $\tilde{B}$ be two sample spherical fuzzy sets. Then normalized Euclidean distance $Ed_{SFS}(\tilde{A},\tilde{B})$ is given in Eq. (35) (Ashraf et al., 2019d). The concept of Euclidean distance is illustrated in Fig. 1.

$$Ed_{SFS}(\tilde{A},\tilde{B})=\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[\left(\mu_A(x_i)-\mu_B(x_i)\right)^2+\left(\vartheta_A(x_i)-\vartheta_B(x_i)\right)^2+\left(I_A(x_i)-I_B(x_i)\right)^2\right]} \tag{35}$$

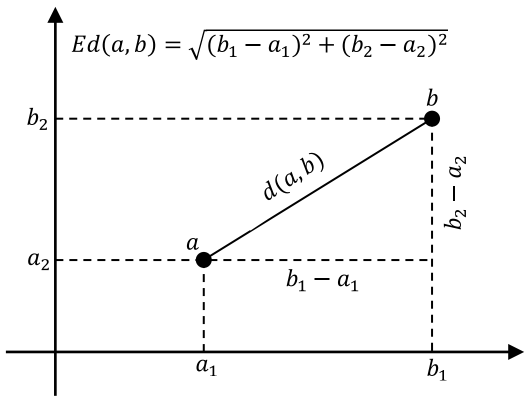

Fig. 1. Graphical concept of Euclidean distance.

All these distance measures have to obey certain properties, called the distance axioms. The four distance axioms are:
1. *Equal self-similarity*: $d(A,A) = d(B,B)$ for all points A and B. Therefore, $S(A,A) = S(B,B)$ for all stimuli A and B.
2. *Minimality*: $d(A,B) > d(A,A)$ for all points $A \neq B$. Therefore, $S(A,B) < S(A,A)$ for all stimuli $A \neq B$.
3. *Symmetry*: $d(A,B) = d(B,A)$ for all points A and B. Therefore, $S(A,B) = S(B,A)$ for all stimuli A and B.
4. *Triangle Inequality*: $d(A,B)+d(B,C) \geq d(A,C)$ for all points A, B, and C. Therefore, the dissimilarities among any set of three stimuli should satisfy this same condition. The triangle inequality also implies that if stimuli A and B are similar and stimuli B and C are similar, then stimuli A and C must also be similar.

4.2.1. Exponential similarity measures based on Hamming distance

Definition 12. Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ be two SFSs, so that $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership and Indeterminacy degrees, respectively. Then, using Eqs. (33) and (34), the exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance ($ES_H(\tilde{A}_s,\tilde{B}_s)$) is defined as Eq. (36):

$$ES_H(\tilde{A}_s,\tilde{B}_s) = e^{-\left[\frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\vartheta_A(x_i)-\vartheta_B(x_i)|+|I_A(x_i)-I_B(x_i)|)\right]} \tag{36}$$

Theorem 2. *If $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ be two SFSs, then, inspired by distance properties which mentioned in Definition 11, the exponential similarity measure based on Hamming distance $ES_H(\tilde{A}_s,\tilde{B}_s)$ satisfies the following properties:*

$P1.\ 0 \leq ES_H(\tilde{A}_s,\tilde{B}_s) \leq 1;$

$P2.\ ES_H(\tilde{A}_s,\tilde{B}_s) = ES_H(\tilde{B}_s,\tilde{A}_s);$

$P3.\ ES_H(\tilde{A}_s,\tilde{B}_s) = 1\ if\ \tilde{A}_s = \tilde{B}_s \rightarrow \mu_A(x_i) = \mu_B(x_i),$
$\vartheta_A(x_i) = \vartheta_B(x_i)\ and\ I_A(x_i) = I_B(x_i);$

Proof.
P1. By definition of SFSs, we have $0 \leq \mu_A(x_i), \vartheta_A(x_i), I_A(x_i) \leq 1$ and $0 \leq \mu_B(x_i), \vartheta_B(x_i), I_B(x_i) \leq 1$. Also, we have $0 \leq \mu_A^2(x_i) + \vartheta_A^2(x_i) + I_A^2(x_i) \leq 1$ and $0 \leq \mu_B^2(x_i) + \vartheta_B^2(x_i) + I_B^2(x_i) \leq 1$. Thus,

$$\frac{1}{n}\sum_{i=1}^{n}(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|) \geq 0 =$$
$$-[\frac{1}{n}\sum_{i=1}^{n}(\left|\mu_A(x_i)-\mu_B(x_i)\right|+\left|\vartheta_A(x_i)-\vartheta_B(x_i)\right|+\left|I_A(x_i)-I_B(x_i)\right|)] \leq 0$$

Therefore, the power of e is always zero or a negative number. Also, we know that $e^0 = 1$. Thus, $0 \leq e^{0 \geq Power \geq negative\ number} \leq 1$. Thus, we can conclude that,

$$0 \leq e^{-\left[\frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\vartheta_A(x_i)-\vartheta_B(x_i)|+|I_A(x_i)-I_B(x_i)|)\right]} \leq 1.$$

Thus, P1 holds.
P2. It is obtained from Symmetry properties of distance measure.
P3. We know that if $\tilde{A}_s = \tilde{B}_s$, then $\mu_A(x_i) = \mu_B(x_i), \vartheta_A(x_i) = \vartheta_B(x_i)$ and $I_A(x_i) = I_B(x_i)$. So, there is

$$ES_H(\tilde{A}_s,\tilde{B}_s) =$$
$$e^{-\left[\frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_A(x_i)|+|\vartheta_A(x_i)-\vartheta_A(x_i)|+|I_A(x_i)-I_A(x_i)|)\right]} = e^0 = 1$$

Definition 13. Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i), R_A(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i), R_A(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$, whose $\mu(x_i)$, $\vartheta(x_i)$, $I(x_i)$ and $R(x_i) = 1 - \sqrt{\mu_{\tilde{A}_s}^2(x) + \vartheta_{\tilde{A}_s}^2(x) + I_{\tilde{A}_s}^2(x)}$ are membership, non-membership, indeterminacy and refusal degrees, respectively. Then, using Eqs. (33) and (34), the exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree based on Hamming distance ($ES_H^R(\tilde{A}_s,\tilde{B}_s)$) is defined as Eq. (37).

$$ES_H^R(\tilde{A}_s,\tilde{B}_s) = e^{-\left[\frac{1}{n}\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\vartheta_A(x_i)-\vartheta_B(x_i)|+|I_A(x_i)-I_B(x_i)|+|R_A(x_i)-R_B(x_i)|)\right]} \tag{37}$$

Theorem 3. *The exponential similarity measure considering refusal degree based on Hamming distance $ES_H^R(\tilde{A}_s,\tilde{B}_s)$ satisfies the following properties:*

$P1.\ 0 \leq ES_H^R(\tilde{A}_s,\tilde{B}_s) \leq 1;$

$P2.\ ES_H^R(\tilde{A}_s,\tilde{B}_s) = ES_H^R(\tilde{B}_s,\tilde{A}_s);$

$P3.\ ES_H^R(\tilde{A}_s,\tilde{B}_s) = 1\ if\ \tilde{A}_s = \tilde{B}_s \rightarrow \mu_A(x_i) = \mu_B(x_i),$
$\vartheta_A(x_i) = \vartheta_B(x_i)\ and\ I_A(x_i) = I_B(x_i);$

Due to the space constraint and the similarity of proving these properties with the properties proof of previous section, we do not prove these properties for the upcoming similarity measures.

If we consider the weights of x_i with weight vector $w_i = (w_1, w_2, \ldots, w_n)$ which $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, then, using Eqs. (33) and (34), the weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance ($WES_H(\tilde{A}_s,\tilde{B}_s)$) is defined as in Eq. (38).

$$WES_H(\tilde{A}_s,\tilde{B}_s) = \sum_{i=1}^{n} w_i e^{-[\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\vartheta_A(x_i)-\vartheta_B(x_i)|+|I_A(x_i)-I_B(x_i)|)]} \tag{38}$$

The weighted exponential similarity measure also has to satisfy the properties mentioned in Theorem 2.

In the weighted exponential similarity measure, if the refusal degree of two SFSs $\tilde{A}$ and $\tilde{B}$ is considered beside the membership, non-membership and indeterminacy degrees, we can obtain the weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance considering refusal degree. The $WES_H^R(\tilde{A}_s,\tilde{B}_s)$ is given in Eq. (39).

$$WES_H^R(\tilde{A}_s,\tilde{B}_s) = \sum_{i=1}^{n} w_i e^{-[\sum_{i=1}^{n}(|\mu_A(x_i)-\mu_B(x_i)|+|\vartheta_A(x_i)-\vartheta_B(x_i)|+|I_A(x_i)-I_B(x_i)|+|R_A(x_i)-R_B(x_i)|)]} \tag{39}$$

The $WES_H^R(\tilde{A}_s,\tilde{B}_s)$ also has to satisfy the properties mentioned in Theorem 2.

4.2.2. Exponential similarity measure based on Euclidean distance

Definition 14. Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i)\rangle | x_i \in X\}; \forall i = 1,2,\ldots,n$ and $\tilde{B}_s = \{x_i \langle \mu_B$

$(x_i), \vartheta_B(x_i), I_B(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$, whose $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership and indeterminacy degrees, respectively. Then, using Eqs. (33) and (34), the exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance ($ES_E(\tilde{A}_s, \tilde{B}_s)$) is proposed as in Eq. (40):

$$ES_E(\tilde{A}_s, \tilde{B}_s) = e^{-\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))^2+(\vartheta_A(x_i)-\vartheta_B(x_i))^2+(I_A(x_i)-I_B(x_i))^2\right]}\right]} \tag{40}$$

Theorem 4. *If $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$ be two SFSs, then, inspired by distance properties which mentioned in Definition 11, the exponential similarity measure based on Euclidean distance $ES_E(\tilde{A}_s, \tilde{B}_s)$ satisfies the following properties:*

$P1.\ 0 \le ES_E(\tilde{A}_s, \tilde{B}_s) \le 1;$

$P2.\ ES_E(\tilde{A}_s, \tilde{B}_s) = ES_E(\tilde{B}_s, \tilde{A}_s);$

$P3.\ ES_E(\tilde{A}_s, \tilde{B}_s) = 1\ if\ \tilde{A}_s = \tilde{B}_s \to \mu_A(x_i) = \mu_B(x_i),$
$\vartheta_A(x_i) = \vartheta_B(x_i)\ and\ I_A(x_i) = I_B(x_i);$

Definition 15. Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i), R_A(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i), R_A(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$, whose $\mu(x_i)$, $\vartheta(x_i)$, $I(x_i)$ and $R(x_i) = 1 - \sqrt{\mu^2_{\tilde{A}_s}(x) + \vartheta^2_{\tilde{A}_s}(x) + I^2_{\tilde{A}_s}(x)}$ are membership, non-membership, indeterminacy and refusal degrees, respectively. Then, using Eqs. (33) and (35), the exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree based on Euclidean distance ($ES^R_E(\tilde{A}_s, \tilde{B}_s)$) is defined as follow:

$$ES^R_E(\tilde{A}_s, \tilde{B}_s) = e^{-\left[\sqrt{\frac{1}{n}\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))^2+(\vartheta_A(x_i)-\vartheta_B(x_i))^2+(I_A(x_i)-I_B(x_i))^2+(R_A(x_i)-R_B(x_i))^2\right]}\right]} \tag{41}$$

Theorem 5. *The exponential similarity measure considering refusal degree based on Euclidean distance $ES^R_E(\tilde{A}_s, \tilde{B}_s)$ satisfies the following properties:*

$P1.\ 0 \le ES^R_E(\tilde{A}_s, \tilde{B}_s) \le 1;$

$P2.\ ES^R_E(\tilde{A}_s, \tilde{B}_s) = ES^R_E(\tilde{B}_s, \tilde{A}_s);$

$P3.\ ES^R_E(\tilde{A}_s, \tilde{B}_s) = 1\ if\ \tilde{A}_s = \tilde{B}_s \to \mu_A(x_i) = \mu_B(x_i),$
$\vartheta_A(x_i) = \vartheta_B(x_i)\ and\ I_A(x_i) = I_B(x_i);$

Due to the space constraint and the similarity of proving these properties with the properties proof of previous section, we do not prove these properties for the upcoming similarity measures.

The Weighted Exponential Similarity Measure (WES) based on Euclidean distance can be calculated by considering the weights of x_i with the weight vector $w_i = (w_1, w_2, \ldots, w_n)$ which $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Thus, using Eqs. (33) and (35), the weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance ($WES_E(\tilde{A}_s, \tilde{B}_s)$) is obtained as Eq. (42).

$$WES_E(\tilde{A}_s, \tilde{B}_s) = \sum_{i=1}^{n} w_i e^{-\left[\sqrt{\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))^2+(\vartheta_A(x_i)-\vartheta_B(x_i))^2+(I_A(x_i)-I_B(x_i))^2\right]}\right]} \tag{42}$$

All the properties mentioned in Theorem 5, have to satisfied for $WES_E(\tilde{A}_s, \tilde{B}_s)$.

We can extend the weighted exponential similarity measure by considering the refusal degree based on Euclidean distance. Therefore, the weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance, considering refusal degree ($WES^R_E(\tilde{A}_s, \tilde{B}_s)$) is proposed as Eq. (43).

$$WES^R_E(\tilde{A}_s, \tilde{B}_s) = \sum_{i=1}^{n} w_i e^{-\left[\sqrt{\sum_{i=1}^{n}\left[(\mu_A(x_i)-\mu_B(x_i))^2+(\vartheta_A(x_i)-\vartheta_B(x_i))^2+(I_A(x_i)-I_B(x_i))^2+(R_A(x_i)-R_B(x_i))^2\right]}\right]} \tag{43}$$

All properties mentioned in Definition 11, are valid for $WES^R_E(\tilde{A}_s, \tilde{B}_s)$.

4.3. Square root cosine similarity based on Hellinger distance

The Square Root Cosine Similarity Measure Eq. (44) based on Hellinger distance Eq. (45), is a new similarity measure which is proposed by Zhu et al. (2012).

$$SqrtCos(A, B) = \frac{\sum_{i=1}^{n}\sqrt{a_i b_i}}{\sqrt[4]{\sum_{i=1}^{n} a_i^2}\sqrt[4]{\sum_{i=1}^{n} b_i^2}} \tag{44}$$

$$Hd(a, b) = \sqrt{\sum_{i=1}^{m}(\sqrt{a_i} - \sqrt{b_i})^2} = \sqrt{2 - 2\sum_{i=1}^{m}\sqrt{a_i b_i}} \tag{45}$$

Assuming each document is normalized to 1 in $L1$ norm: $\sum_{i=1}^{m} a_i^2 = 1$, the Hellinger distance leads to the square root cosine similarity measure Eq. (44). Here, we are going to introduce some similarity measures for SFSs based on the square root cosine similarity measure.

Definition 16. Suppose that there are two sets of SFSs $\tilde{A}_s = \{x_i \langle \mu_A(x_i), \vartheta_A(x_i), I_A(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$ and $\tilde{B}_s = \{x_i \langle \mu_B(x_i), \vartheta_B(x_i), I_B(x_i)\rangle | x_i \in X\}; \forall i = 1, 2, \ldots, n$ be two SFSs whose $\mu(x_i)$, $\vartheta(x_i)$ and $I(x_i)$ are membership, non-membership and indeterminacy degrees, respectively. Then the Square Root Cosine Similarity measure (SqrtCS) of $\tilde{A}_s$ and $\tilde{B}_s$ is given by Eq. (46):

$$SqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s) = \frac{1}{n}\sum_{i=1}^{n}\frac{\left(\sqrt{\mu_A^2(x_i)\mu_B^2(x_i) + \vartheta_A^2(x_i)\vartheta_B^2(x_i) + I_A^2(x_i)I_B^2(x_i)}\right)}{\sqrt[4]{\left(\mu_A^4(x_i) + \vartheta_A^4(x_i) + I_A^4(x_i)\right)}\sqrt[4]{\left(\mu_B^4(x_i) + \vartheta_B^4(x_i) + I_B^4(x_i)\right)}} \tag{46}$$

Theorem 6. *The $SqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s)$ satisfies the following properties:*
P1. $0 \le SqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s) \le 1;$
P2. $SqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s) = SqrtCS_{SFS}(\tilde{B}_s, \tilde{A}_s);$
P3. $SqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s) = 1\ if\ \tilde{A}_q = \tilde{B}_q \to \mu_A(x_i) = \mu_B(x_i)\ and\ \vartheta_A(x_i) = \vartheta_B(x_i);$

By considering the refusal degree of SFSs ($R(x_i) = 1 - \sqrt{\mu^2_{\tilde{A}_s}(x) + \vartheta^2_{\tilde{A}_s}(x) + I^2_{\tilde{A}_s}(x)}$), the weights of x_i with weight vector $w_i = (w_1, w_2, \ldots, w_n)$ which $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$ and the parameter Γ, so that $0 \le \Gamma \le 1$, the square root cosine similarity considering refusal degree $SqrtCS^R_{SFS}(\tilde{A}_s, \tilde{B}_s)$, the weighted square root cosine similarity with $WSqrtCS^R_{SFS}(\tilde{A}_s, \tilde{B}_s)$ and without $WSqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s)$ refusal degree, the Generalized weighted square root cosine similarity measure with $GWSqrtCS^R_{SFS}(\tilde{A}_s, \tilde{B}_s)$ and without $GWSqrtCS_{SFS}(\tilde{A}_s, \tilde{B}_s)$ refusal degree can be calculated as given in Eqs. (47)–(49) in Box VIII.

After introducing the new similarity measures, we will now show the applicability of these similarity functions using two different case studies in medical diagnosis and green supplier selection fields.

5. Application of the proposed similarity measures in medical diagnosis and green supplier selection

Several studies on similarity measures have been developed due to the importance of similarity measures and their applications in decision making, data mining, clustering, medical diagnosis, and pattern recognition in the literature. In this section, we are going to illustrate the application of the proposed similarity measures in this research in medical diagnosis and multiple criteria decision-making problems. In case of medical diagnosis, we apply the similarity measures to detect

$$SqrtCS^{R}_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)=\frac{1}{n}\sum_{i=1}^{n}\frac{\left(\sqrt{\mu_A^2\left(x_i\right)\mu_B^2\left(x_i\right)+\vartheta_A^2\left(x_i\right)\vartheta_B^2\left(x_i\right)+I_A^2\left(x_i\right)I_B^2\left(x_i\right)+R_A^2\left(x_i\right)R_B^2\left(x_i\right)}\right)}{\sqrt[4]{\left(\mu_A^4\left(x_i\right)+\vartheta_A^4\left(x_i\right)+I_A^4\left(x_i\right)+R_A^4\left(x_i\right)\right)}\sqrt[4]{\left(\mu_B^4\left(x_i\right)+\vartheta_B^4\left(x_i\right)+I_B^4\left(x_i\right)+R_B^4\left(x_i\right)\right)}} \tag{47}$$

$$WSqrtCS^{R}_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)=\sum_{i=1}^{n}w_i\frac{\left(\sqrt{\mu_A^2\left(x_i\right)\mu_B^2\left(x_i\right)+\vartheta_A^2\left(x_i\right)\vartheta_B^2\left(x_i\right)+I_A^2\left(x_i\right)I_B^2\left(x_i\right)+R_A^2\left(x_i\right)R_B^2\left(x_i\right)}\right)}{\sqrt[4]{\left(\mu_A^4\left(x_i\right)+\vartheta_A^4\left(x_i\right)+I_A^4\left(x_i\right)+R_A^4\left(x_i\right)\right)}\sqrt[4]{\left(\mu_B^4\left(x_i\right)+\vartheta_B^4\left(x_i\right)+I_B^4\left(x_i\right)+R_B^4\left(x_i\right)\right)}} \tag{48}$$

$$WSqrtCS_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)=\sum_{i=1}^{n}w_i\frac{\left(\sqrt{\mu_A^2\left(x_i\right)\mu_B^2\left(x_i\right)+\vartheta_A^2\left(x_i\right)\vartheta_B^2\left(x_i\right)+I_A^2\left(x_i\right)I_B^2\left(x_i\right)}\right)}{\sqrt[4]{\left(\mu_A^4\left(x_i\right)+\vartheta_A^4\left(x_i\right)+I_A^4\left(x_i\right)\right)}\sqrt[4]{\left(\mu_B^4\left(x_i\right)+\vartheta_B^4\left(x_i\right)+I_B^4\left(x_i\right)\right)}} \tag{49}$$

Box VIII.

whether a patient is suffering from a certain disease or not. In case of MCDM problem, the proposed similarity measures are applied to green supplier selection problem. It is noteworthy that in the medical diagnosis problem, we use non-weighted form of similarity measures and in the green supplier selection problem we apply the weighted form of similarity measures.

5.1. Application-1: A case study in medical diagnosis

In this section, we present a medical diagnosis problem where similarity measures can be applied. We demonstrate that this new proposed technique of similarity measures between two SFSs can be applied to detect whether a patient is suffering from Coronavirus Disease (COVID-19) or not.

Coronavirus Disease (COVID-19) has emerged as a global problem. The disease was first identified in 2019 in Wuhan, the capital of China's Hubei province, and has since spread globally. Since spread of the disease to March 2020, more than one million people have been infected by COVID-19. The disease is spread between people during close contact and via respiratory droplets from coughs and sneezes.

Common symptoms include fever, cough, and shortness of breath. Other symptoms may include muscle pain, sputum production, diarrhea, sore throat, loss of smell, and abdominal pain. Now, we try to determine which patient having certain visible symptoms is suffering from dengue Coronavirus Disease (COVID-19) with largest possibility. For this, we first construct the symptom-diagnose matrix for the patients including medical indicators and state of medical indicators. Medical diagnosis data were generated through three different physicians that visit COVID-19 patients, who are working in one of the big hospitals of Turkey in Istanbul. The whole evaluation process was realized based on their knowledge. The obtained results were presented to these experts and they were quite conforming to their expectations. It was assumed that all physicians have equal weights $\left(\frac{1}{n}\right)$. In our first case study, there are three decision-makers and so the weight of judgments of each decision-maker is 0.3334. The physicians represented their diagnosis using SFSs which are stated through linguistic terms which is given in Table 1. Then, we defined a reference indicator (RI) that indicates that a patient is most possibly has symptoms of the COVID-19. We calculated the similarity measures between medical indicators of each patient and reference indicator (RI). The closer the similarity measure to 1, the possibility of a person suffering from COVID-19 is more.

Let us assume that $MI=\left\{fever\left(MI_1\right),cough\left(MI_2\right),shortness\ of\ breath(MI_3)\right\}$ is the collection of medical indicators. Further, let us assume that $S=\left\{No\left(s_1\right),Mild\left(s_2\right),Severe(s_3)\right\}$ is used to describe the state of medical indicators. Using these collections, the pairwise of medical indicators and state of medical indicators (M_j), so that $M_j=(MI_j,S_j)$. Therefore, the profiles of the patients by observing their medical indicators may be represented by using appropriate linguistic labels given in Table 1. Tables 2–4 show the states of patients taken from different decision-makers.

Table 1
Linguistic terms of physicians' diagnosis.

Linguistic terms	SFSs equivalents numbers (μ,ϑ,I)
Very Low Possibility (VLP)	(0.1, 0.9, 0.0)
Low Possibility (LP)	(0.3, 0.7, 0.2)
Equally Possibility (EP)	(0.5, 0.5, 0.5)
More Possibility (MP)	(0.7, 0.3, 0.2)
Very More Possibility (VMP)	(0.9, 0.1, 0.0)

Suppose that there are ten patients P1, P2, P3, P4, P5, P6, P7, P8, P9 and P10 in a hospital with symptoms fever, cough, and shortness of breath.

The steps of the solution methodology are given as follows.

Step1 Construct the spherical fuzzy symptom-diagnose matrix based on evaluations of three decision makers (physicians). The symptom-diagnose matrices are presented in Tables 2–4.

Step 2 Aggregate the symptom-diagnose matrices using $SFWAM$ (Eq. (15)) into a single symptom-diagnose matrix as shown in Table 5. To clarify the calculations, we give an example for the first patient in terms of first medical indicator and state 1. Assessments such as (0.1, 0.9, 0), (0.3, 0.7, 0.2), and (0.3, 0.7, 0.2) are given by three physicians for membership, non-membership and indeterminacy degrees, respectively. It is assumed that all physicians have equal weight of $\frac{1}{3}$. By using the equation defined in Box IX, we can get the aggregated single value which is (0.25, 0.76, 0.17) for this term.

Step 3 Calculate the similarity degree of each patient with reference indicator (RI) using different similarity measures in Eqs. (27), (36), (40) and (46) by using the reference indicators (RI) of each M_j given in Table 6. The similarity degrees are given in Table 7.

Step 4 Check the obtained similarity measures. Patients with the higher similarity measure (closer to 1) are considered as suspicious and susceptible patients.

The results of Table 7 demonstrate the final similarity degree of each patient. The similarity degrees change from one similarity measure to another. Some of these measures have an optimistic view while some of them are pessimistic. The decision-makers can use these numbers in their decisions based on their diagnoses and preferences. For example, each patient with similarity degree greater than 0.6, is suspicious and susceptible for COVID 19 with respect to a physician's point of view.

Therefore, based on $JS_{SFS}(\tilde{A},\tilde{B})$, $ES_E(\tilde{A},\tilde{B})$ and $ES_H(\tilde{A},\tilde{B})$, patient 4 (P4) and based on $SqrtCS_{SFS}(\tilde{A},\tilde{B})$, patients 1, 3, 4, 7, 8 and 10 (P1, P3, P4, P7, P8, P10) are patients having the largest possibilities to be infected. The comparisons between the proposed similarity measures and the existing similarity measure in the literature

Table 2
Symptom Diagnose matrix based on physician 1.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)
P_2	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_3	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)
P_4	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_5	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_6	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)
P_7	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)
P_8	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_9	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_{10}	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)

Table 3
Symptom Diagnose matrix based on physician 2.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)
P_2	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_3	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)
P_4	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_5	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)
P_6	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)
P_7	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)
P_8	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_9	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_{10}	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)

Table 4
Symptom Diagnose matrix based on physician 3.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)
P_2	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)
P_3	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)
P_4	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)
P_5	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)
P_6	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)
P_7	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)
P_8	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)
P_9	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)
P_{10}	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)

Table 5
Aggregated symptom-diagnose matrix.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
P_1	(0.25, 0.76, 0.17)	(0.80, 0.21, 0.14)	(0.45, 0.56, 0.45)	(0.19, 0.83, 0.12)	(0.25, 0.76, 0.17)	(0.90, 0.10, 0.0)	(0.70, 0.30, 0.20)	(0.45, 0.56, 0.45)	(0.10, 0.90, 0.0)
P_2	(0.58, 0.42, 0.41)	(0.38, 0.63, 0.36)	(0.10, 0.90, 0.0)	(0.90, 0.10, 0.0)	(0.45, 0.56, 0.45)	(0.10, 0.90, 0.0)	(0.19, 0.83, 0.12)	(0.25, 0.76, 0.17)	(0.90, 0.10, 0.0)
P_3	(0.70, 0.30, 0.20)	(0.30, 0.70, 0.20)	(0.10, 0.90, 0.0)	(0.25, 0.76, 0.16)	(0.30, 0.70, 0.20)	(0.90, 0.10, 0.0)	(0.90, 0.10, 0.0)	(0.30, 0.70, 0.20)	(0.19, 0.83, 0.12)
P_4	(0.19, 0.83, 0.12)	(0.38, 0.63, 0.36)	(0.90, 0.10, 0.0)	(0.10, 0.90, 0.0)	(0.65, 0.36, 0.32)	(0.90, 0.10, 0.0)	(0.10, 0.90, 0.0)	(0.25, 0.76, 0.17)	(0.90, 0.10, 0.0)
P_5	(0.80, 0.21, 0.14)	(0.30, 0.70, 0.20)	(0.10, 0.90, 0.0)	(0.50, 0.50, 0.50)	(0.65, 0.36, 0.32)	(0.10, 0.90, 0.0)	(0.25, 0.76, 0.17)	(0.45, 0.56, 0.45)	(0.90, 0.10, 0.0)
P_6	(0.80, 0.21, 0.14)	(0.35, 0.68, 0.35)	(0.10, 0.90, 0.0)	(0.90, 0.10, 0.0)	(0.25, 0.76, 0.17)	(0.10, 0.90, 0.0)	(0.90, 0.10, 0.0)	(0.25, 0.56, 0.17)	(0.10, 0.90, 0.0)
P_7	(0.65, 0.36, 0.32)	(0.25, 0.76, 0.17)	(0.10, 0.90, 0.0)	(0.19, 0.83, 0.12)	(0.30, 0.70, 0.20)	(0.90, 0.10, 0.0)	(0.38, 0.63, 0.36)	(0.86, 0.14, 0.08)	(0.19, 0.83, 0.12)
P_8	(0.19, 0.83, 0.12)	(0.70, 0.30, 0.20)	(0.30, 0.70, 0.20)	(0.19, 0.83, 0.12)	(0.90, 0.10, 0.0)	(0.30, 0.70, 0.20)	(0.19, 0.83, 0.12)	(0.30, 0.70, 0.20)	(0.90, 0.10, 0.0)
P_9	(0.86, 0.14, 0.08)	(0.19, 0.83, 0.12)	(0.10, 0.90, 0.0)	(0.86, 0.14, 0.08)	(0.30, 0.70, 0.20)	(0.10, 0.90, 0.0)	(0.19, 0.83, 0.12)	(0.25, 0.76, 0.17)	(0.90, 0.10, 0.0)
P_{10}	(0.19, 0.83, 0.12)	(0.38, 0.63, 0.36)	(0.86, 0.14, 0.08)	(0.19, 0.83, 0.12)	(0.86, 0.14, 0.08)	(0.10, 0.90, 0.0)	(0.25, 0.76, 0.17)	(0.10, 0.90, 0.0)	(0.86, 0.14, 0.08)

Table 6
Reference indicators (RI) of each $\mathbf{M_j}$.

	(MI_1, s_1)	(MI_1, s_2)	(MI_1, s_3)	(MI_2, s_1)	(MI_2, s_2)	(MI_2, s_3)	(MI_3, s_1)	(MI_3, s_2)	(MI_3, s_3)
RI	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.9, 0.1, 0)	(0.1, 0.9, 0)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)

$$SFWAM\left(P1, MI_1 s_1\right) = \sqrt{1-\left(1-0.1^2\right)^{\frac{1}{3}}\left(1-0.3^2\right)^{\frac{1}{3}}\left(1-0.3^2\right)^{\frac{1}{3}}}, (0.9)^{\frac{1}{3}}(0.7)^{\frac{1}{3}}(0.7)^{\frac{1}{3}},$$
$$\sqrt{\left(1-0.1^2\right)^{\frac{1}{3}}\left(1-0.3^2\right)^{\frac{1}{3}}\left(1-0.3^2\right)^{\frac{1}{3}}-\left(1-0.1^2-0.0^2\right)^{\frac{1}{3}}\left(1-0.3^2-0.2^2\right)^{\frac{1}{3}}\left(1-0.3^2-0.2^2\right)^{\frac{1}{3}}} = (0.25, 0.76, 0.17)$$

Box IX.

Table 7
Similarity degree of each patient obtained from different similarity measures.

	Existing similarity	Proposed similarity measures			
	$SFC^1(\tilde{A},\tilde{B})$	$JS_{SFS}(\tilde{A},\tilde{B})$	$ES_H(\tilde{A},\tilde{B})$	$ES_E(\tilde{A},\tilde{B})$	$SqrtCS_{SFS}(\tilde{A},\tilde{B})$
P_1	**0.69**	0.61	0.48	0.56	**0.79**
P_2	0.43	0.35	0.35	0.45	0.58
P_3	0.49	0.36	0.38	0.47	**0.58**
P_4	**0.85**	**0.74**	**0.68**	**0.68**	**0.87**
P_5	0.42	0.34	0.35	0.46	0.59
P_6	0.29	0.16	0.27	0.39	0.36
P_7	0.52	0.51	0.42	0.52	**0.71**
P_8	**0.83**	0.59	0.51	0.55	**0.74**
P_9	0.42	0.35	0.35	0.44	0.53
P_{10}	**0.65**	0.59	0.47	0.53	**0.72**

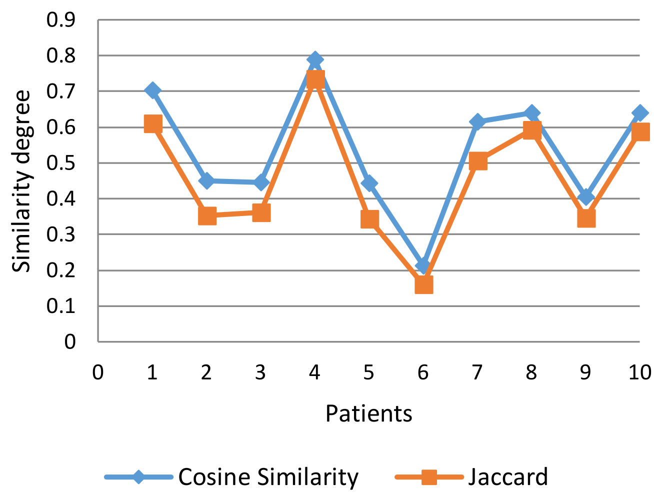

Fig. 2. Comparison between Jaccard and cosine similarities.

are shown in Figs. 2–5. These tables compare the similarity measures based on optimistic and pessimistic point of view. For example, in Tables 3–5, the similarity degrees of proposed measures for each patient are smaller than existing similarity measure, means that the proposed similarity measures are pessimistic than existing one. In Fig. 6, we make a graphical comparison among the results of the proposed similarities on a Radar chart. Radar chart provides an overview of the results and all results can be analyzed and interpreted together. Radar diagram shows that all similarity degrees obtained from different similarity measures are equal or greater than 0.6 in terms of P4. Vertical axes show the similarity degree of each patient and horizontal degrees show the patient number. These figures give a comparison between proposed similarity measures and existing one in case of pessimistic and optimistic point of view. For example, in Figs. 2–4, the similarity degrees of the proposed measures for each patient are smaller than existing similarity measure, which means that the proposed similarity measures are pessimistic than existing one, whereas the proposed square root-cosine similarity in Fig. 5 is larger than cosine similarity, which means that the proposed similarity measure is for optimistic decision-makers.

5.2. Application-2: A case study of green supplier selection

One of the major applications of similarity measures is in multiple-criteria decision making problems. In this section, we try to solve a green supplier selection problem and rank the suppliers using proposed similarity measures. To do this, we define a supplier selection problem based on Banaeian et al. (2018) with some changes.

Appropriate supplier selection is a major aspect of manufacturing industries to meet production demand. Green supplier selection has been one of the most critical factors for environmental protection and

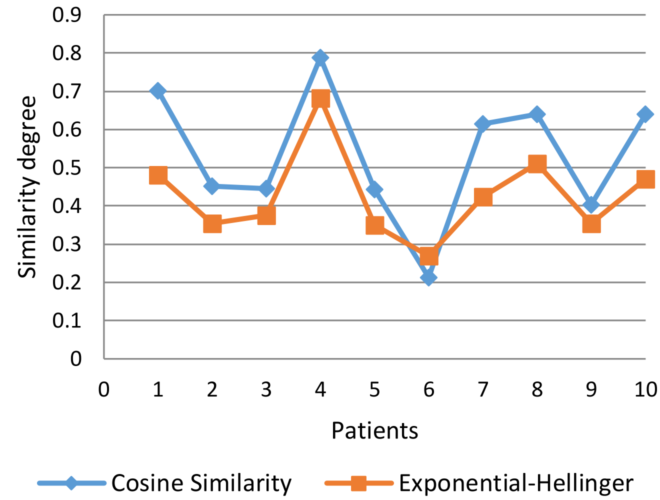

Fig. 3. Comparison between Exp-H and cosine Similarities.

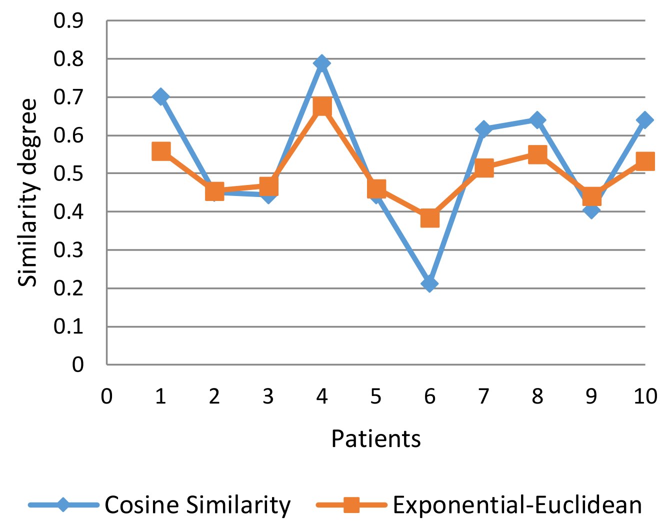

Fig. 4. Comparison between Exp-E and cosine similarities.

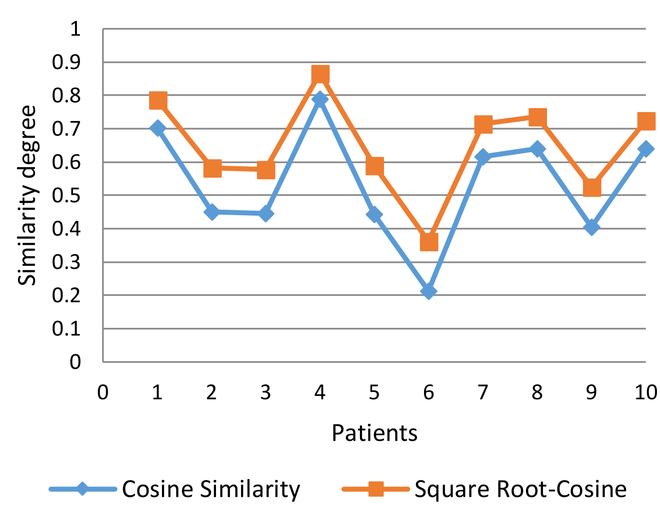

Fig. 5. Comparison between Square root cosine and cosine similarities.

for sustainable development as well. Our case study is for a company that is one of the Iranian manufacturers of eatable vegetable oils and

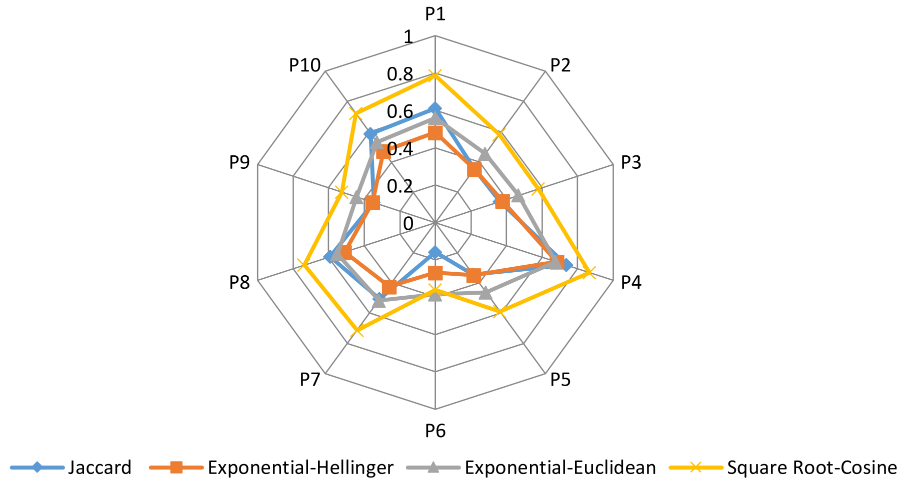

Fig. 6. Radar plot for the results of the medical diagnose.

Table 8
Linguistic variables for supplier evaluations.

Linguistic terms	SFSs equivalents numbers (μ, ϑ, I)
Very Low Importance (VLI)	(0.1, 0.9, 0.0)
Low Importance (LI)	(0.3, 0.7, 0.2)
Slightly Low Importance (SLI)	(0.4, 0.6, 0.3)
Equally Importance (EI)	(0.5, 0.5, 0.5)
Slightly More Importance (SMI)	(0.6, 0.4, 0.3)
More Importance (MI)	(0.7, 0.3, 0.2)
Much More Importance (MMI)	(0.9, 0.1, 0.0)

detergents. Raw oil is the main component in all company products and olive oil, palm oil, sunflower oil and soybean oil are the main raw oils used in this company. The aim is to construct a supplier evaluation and selection process for each raw oil. The available suppliers were identified through the initial screening process which includes four olive oil suppliers (A1, A2, A3, A4), three palm oil suppliers (B1, B2, B3), three sunflower oil (D1, D2, D3) and three soybean oil suppliers (E1, E2, E3). The green supplier selection criteria were identified from the existing literature. These criteria are service level (C1), quality (C2), price (C3), delivery time (C4) and environmental management system (EMS) (C5). The suppliers were evaluated by three different industrial experts against the determined criteria in linguistic terms (Table 8), which were then converted into spherical fuzzy numbers. The evaluated decision matrices are given in Tables 9–12.

In the next step, the similarity degree between each supplier (alternative) and Reference Index (RI) using proposed similarity measures should be calculated. The RI values for each criterion are given in Table 13. Any alternative that has a closer similarity degree to 1, is selected as the best supplier. In this problem, we are going to apply the weighted form of each similarity measure using Eqs. (29), (38), (42) and (49). The compromised distribution of criteria weights satisfying that their sum must be equal to 1 for each criterion is given in Table 14. These weights are given based on the importance of each alternative against each criterion. Using Spherical Fuzzy Weighted

Table 9
Evaluation matrix for olive oil suppliers.

Decision makers	Suppliers	C1	C2	C3	C4	C5
DM1	A1	(0.5, 0.5, 0.5)	(0.4, 0.6, 0.3)	(0.4, 0.6, 0.3)	(0.9, 0.1, 0)	(0.6, 0.4, 0.3)
	A2	(0.7, 0.3, 0.2)	((0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.6, 0.4, .3)	(0.3, 0.7, 0.2)
	A3	(0.6, 0.4, 0.3)	(0.9, 0.1, 0)	(0.6, 0.4, 0.3)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)
	A4	(0.3, 0.7, 0.2)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)	(0.4, 0.6, 0.3)	(0.1, 0.9, 0)
DM2	A1	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)
	A2	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)
	A3	(0.9, 0.1, 0)	(0.4, 0.6, 0.3)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.6, 0.4, 0.3)
	A4	(0.4, 0.6, 0.3)	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)
DM3	A1	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.3, 0.7, 0.2)
	A2	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)
	A3	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)
	A4	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)

Table 10
Evaluation matrix for palm oil suppliers.

Decision makers	Suppliers	C1	C2	C3	C4	C5
DM1	B1	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)
	B2	(0.3, 0.7, 0.2)	(0.4, 0.6, 0.3)	(0.4, 0.6, 0.3)	(0.3, 0.7, 0.2)	(0.7, 0.3, 0.2)
	B3	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.6, 0.4, 0.3)	(0.4, 0.6, 0.3)
DM2	B1	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.4, 0.6, 0.3)
	B2	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.5, 0.5, 0.5)
	B3	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)
DM3	B1	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)
	B2	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.4, 0.6, 0.3)	(0.4, 0.6, 0.3)	(0.6, 0.4, 0.3)
	B3	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)

Table 11
Evaluation matrix for sunflower oil suppliers.

Decision makers	Suppliers	C1	C2	C3	C4	C5
DM1	D1	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.4, 0.6, 0.3)
	D2	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.7, 0.3, 0.2)
	D3	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)
DM2	D1	(0.7, 0.3, 0.2)	(0.4, 0.6, 0.3)	(0.7, 0.3, 0.2)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)
	D2	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.5, 0.5, 0.5)	(0.1, 0.9, 0)
	D3	(0.3, 0.7, 0.2)	(0.4, 0.6, 0.3)	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)	(0.6, 0.4, 0.3)
DM3	D1	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)
	D2	(0.6, 0.4, 0.3)	(0.4, 0.6, 0.3)	(0.1, 0.9, 0)	(0.1, 0.9, 0)	(0.3, 0.7, 0.2)
	D3	(0.7, 0.3, 0.2)	(0.3, 0.7, 0.2)	(0.6, 0.4, 0.3)	(0.6, 0.4, 0.3)	(0.9, 0.1, 0)

Table 12
Evaluation matrix for soybean oil suppliers.

Decision makers	Suppliers	C1	C2	C3	C4	C5
DM1	E1	(0.4, 0.6, 0.3)	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.9, 0.1, 0)
	E2	(0.5, 0.5, 0.5)	(0.6, 0.4, 0.3)	(0.4, 0.6, 0.3)	(0.4, 0.6, 0.3)	(0.7, 0.3, 0.2)
	E3	(0.6, 0.4, 0.3)	(0.3, 0.7, 0.2)	(0.6, 0.4, 0.3)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)
DM2	E1	(0.5, 0.5, 0.5)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)
	E2	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.3, 0.7, 0.2)	(0.6, 0.4, 0.3)	(0.3, 0.7, 0.2)
	E3	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.6, 0.4, 0.3)	(0.3, 0.7, 0.2)	(0.5, 0.5, 0.5)
DM3	E1	(0.4, 0.6, 0.3)	(0.3, 0.7, 0.2)	(0.9, 0.1, 0)	(0.4, 0.6, 0.3)	(0.6, 0.4, 0.3)
	E2	(0.7, 0.3, 0.2)	(0.7, 0.3, 0.2)	(0.6, 0.4, 0.3)	(0.5, 0.5, 0.5)	(0.3, 0.7, 0.2)
	E3	(0.9, 0.1, 0)	(0.3, 0.7, 0.2)	(0.6, 0.4, 0.3)	(0.9, 0.1, 0)	(0.5, 0.5, 0.5)

Table 13
Reference indicator (RI) of each criterion.

	C1	C2	C3	C4	C5
RI	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.9, 0.1, 0)	(0.9, 0.1, 0)

Table 14
Weight of each alternative against each criterion.

	C1	C2	C3	C4	C5
A1	0.05	0.15	0.15	0.35	0.3
A2	0.4	0.1	0.2	0.2	0.1
A3	0.15	0.2	0.3	0.3	0.05
A4	0.05	0.05	0.3	0.35	0.25
B1	0.35	0.3	0.05	0.1	0.2
B2	0.15	0.1	0.3	0.1	0.35
B3	0.15	0.1	0.3	0.1	0.35
D1	0.3	0.15	0.25	0.15	0.15
D2	0.15	0.2	0.3	0.3	0.05
D3	0.1	0.3	0.1	0.15	0.35
E1	0.1	0.1	0.4	0.2	0.2
E2	0.05	0.05	0.3	0.35	0.25
E3	0.35	0.3	0.05	0.1	0.2

Arithmetic Mean aggregation operator $SFWAM$ (Eq. (15)), all evaluation matrices are aggregated into a single aggregated matrix as given in Table 15.

Table 16 demonstrates the final results of similarity measures. According to these results, for Olive oil, supplier A3 is the best selection. For palm oil, sunflower oil and soybean oil, the best suppliers are B1, D3 and E1, respectively. Although the similarity degrees obtained from the proposed similarity measures are different but, all of them give same result (same best supplier) for each raw oil alternative.

For qualitative comparisons, as it is mentioned before, each of these similarity measures has their own properties. Some of them generally are strict and pessimistic ($ES_H(\tilde{A},\tilde{B})$ and $ES_E(\tilde{A},\tilde{B})$) while others are optimistic ($JS_{SFS}(\tilde{A},\tilde{B})$ and $SqrtCS_{SFS}(\tilde{A},\tilde{B})$). Therefore, based on these properties, the results of similarity measures can be different and decision-makers can use these similarity measures depending on their point of view.

Fig. 7 shows the ranking of suppliers with respect to the similarity degree of each proposed similarity measure. Using this plot, we can compare the suppliers with respect to different similarity measures. For instance, the plot shows that supplier A3 has the largest square root similarity among all suppliers and the overall ranking with respect to square root similarity is A3 > B1 > E1 = A2 > A1 > B3 = D3 > D1 > A4 = E3 > E2 > B2 = D2. All of the other rankings with respect to other similarities are covered by this ranking, which means that square root similarity produces the larger similarities than the other similarity measures.

To compare the obtained results of the proposed similarity measures with other MCDM methods in the literature, we made a comparison analysis with Banaeian et al.'s (2018) work. In this research, they investigated and ranked these alternatives by using fuzzy TOPSIS and fuzzy VIKOR methods. In Table 17, the comparison results between the proposed similarity measures and the mentioned methods are given.

As can be seen from the results of Table 17, the best supplier in all raw oils is the same in the existing methods and the proposed similarity measures. There are some differences in the second, third and fourth ranks due to the inherent characteristics of the different similarity measures.

6. Conclusion

In this paper, some novel similarity measures of spherical fuzzy sets have been developed. The Jaccard, Exponential, and Square root cosine similarity measures in the spherical fuzzy environment have been proposed. We have applied these similarity measures to medical diagnose and green supplier selection problems successfully. The new proposed techniques of similarity measure between two SFSs have been applied to detect whether a patient is suffering from Coronavirus Disease (COVID-19) or not, based on the symptoms such as fever, cough, and shortness of breath that a patient may have. This medical diagnosis problem has been solved under imprecise and vagueness information by using proposed spherical fuzzy similarity measures. As a second application, the proposed similarity measures have also been applied to construct a supplier evaluation and selection process to select the best supplier for each raw oil. Comparative analyses have

Table 15
Aggregated evaluation matrix.

Kind of raw oil	Supplier	C1	C2	C3	C4	C5
Olive oil	A1	(0.41, 0.59, 0.38)	(0.56, 0.45, 0.35)	(0.54, 0.48, 0.26)	(0.74, 0.27, 0.28)	(0.72, 0. 3, 0.17)
	A2	(0.76, 0.25, 0.25)	(0.54, 0.47, 0.34)	(0.61, 0.39, 0.35)	(0.61, 0.39, 0.35)	(0.54, 0.47, 0.34)
	A3	(0.72, 0. 3, 0.17)	(0.75, 0.26, 0.17)	(0.78, 0.23, 0.17)	(0.8, 0.21, 0.14)	(0.67, 0.33, 0.23)
	A4	(0.41, 0.59, 0.38)	(0.41, 0.63, 0.23)	(0.54, 0.47, 0.34)	(0.59, 0.42, 0.27)	(0.47, 0.56, 0.36)
Palm oil	B1	(0.8, 0.21, 0.14)	(0.76, 0.25, 0.2)	(0.64, 0.36, 0.27)	(0.67, 0.33, 0.23)	(0.69, 0.35, 0.16)
	B2	(0.44, 0.58, 0.25)	(0.41, 0.59, 0.38)	(0.46, 0.55, 0.28)	(0.34, 0.66, 0.24)	(0.61, 0.39, 0.35)
	B3	(0.57, 0.43, 0.38)	(0.5, 0.5, 0.5)	(0.65, 0.36, 0.32)	(0.51, 0.49, 0.38)	(0.71, 0.31, 0.28)
Sunflower oil	D1	(0.8, 0.21, 0.14)	(0.56, 0.45, 0.35)	(0.78, 0.23, 0.17)	(0.54, 0.46, 0.44)	(0.38, 0.65, 0.37)
	D2	(0.55, 0.46, 0.3)	(0.51, 0.49, 0.38)	(0.47, 0.56, 0.36)	(0.47, 0.56, 0.36)	(0.48, 0.57, 0.18)
	D3	(0.5, 0.53, 0.2)	(0.69, 0.35, 0.16)	(0.51, 0.49, 0.38)	(0.54, 0.46, 0.44)	(0.72, 0. 3, 0.17)
Soybean oil	E1	(0.44, 0.56, 0.39)	(0.54, 0.47, 0.34)	(0.8, 0.21, 0.14)	(0.63, 0.38, 0.23)	(0.76, 0.25, 0.2)
	E2	(0.76, 0.25, 0.25)	(0.58, 0.44, 0.24)	(0.46, 0.55, 0.28)	(0.51, 0.49, 0.38)	(0.5, 0.53, 0.2)
	E3	(0.76, 0.25, 0.2)	(0.38, 0.63, 0.36)	(0. 6, 0.4, 0.3)	(0.74, 0.28, 0.14)	(0.54, 0.46, 0.44)

Table 16
Similarity degree of each supplier obtained from different similarity measures.

Kind of raw oil	Supplier	Existing similarity $SFC^1(\tilde{A},\tilde{B})$	$JS_{SFS}(\tilde{A},\tilde{B})$	$ES_H(\tilde{A},\tilde{B})$	$ES_E(\tilde{A},\tilde{B})$	$SqrtCS_{SFS}(\tilde{A},\tilde{B})$
Olive oil	A1	0.88	0.69	0.48	0.65	0.94
	A2	0.89	0.67	0.46	0.63	0.95
	A3	**0.96**	**0.89**	**0.64**	**0.77**	**1.00**
	A4	0.77	0.41	0.35	0.54	0.84
Palm oil	B1	**0.95**	**0.85**	**0.61**	**0.75**	**0.99**
	B2	0.73	0.35	0.33	0.52	0.78
	B3	0.86	0.62	0.42	0.60	0.93
Sunflower oil	D1	0.85	0.64	0.47	0.64	0.89
	D2	0.72	0.32	0.30	0.50	0.78
	D3	**0.87**	**0.67**	**0.51**	**0.66**	**0.93**
Soybean oil	E1	**0.90**	**0.75**	**0.56**	**0.70**	**0.95**
	E2	0.75	0.37	0.34	0.53	0.82
	E3	0.80	0.56	0.43	0.60	0.84

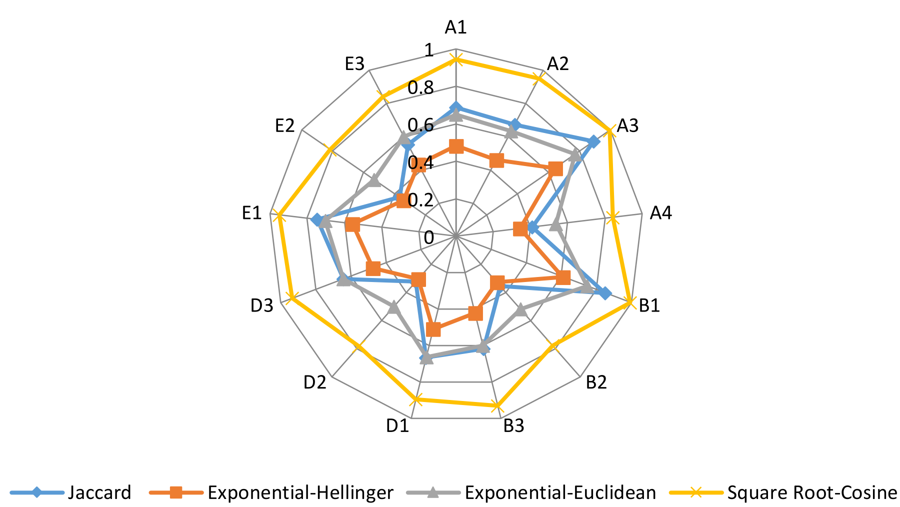

Fig. 7. Radar plot for the results of the green supplier selection.

been performed between existing similarity measures and the proposed similarity measures by using some graphs.

It is noteworthy that all the proposed similarity measures are applicable in the decision-making problems. However, their selection depends on the decision makers' risk attitude. A decision-maker with an optimistic point of view may prefer Jaccard or square root similarity measures while exponential similarity measures could be preferred by decision-makers with a pessimistic point of view.

Similarity calculation is commonly used in the tasks of data mining such as data classification and prediction, data clustering, association rules, sequential patterns, and so on. We propose our developed fuzzy similarity measures to be employed in data mining tasks. For future studies, the proposed spherical fuzzy Jaccard, exponential and square root cosine similarity measures may be also applied to complex group decision-making problems, linguistic summarization risk analysis, pattern recognition, and image processing.

Table 17
Comparison analysis results.

Kind of raw oil	Supplier	Ranking using existing methods		Ranking using proposed similarity measures			
		TOPSIS	VIKOR	$JS_{SFS}(\tilde{A},\tilde{B})$	$ES_H(\tilde{A},\tilde{B})$	$ES_E(\tilde{A},\tilde{B})$	$SqrtCS_{SFS}(\tilde{A},\tilde{B})$
Olive oil	A1	3	3	3	2	2	3
	A2	2	2	2	3	3	2
	A3	**1**	**1**	**1**	**1**	**1**	**1**
	A4	4	4	4	4	4	4
Palm oil	B1	**1**	**1**	**1**	**1**	**1**	**1**
	B2	3	3	3	3	3	3
	B3	2	2	2	2	2	2
Sunflower oil	D1	2	2	2	2	2	2
	D2	3	3	3	3	3	3
	D3	**1**	**1**	**1**	**1**	**1**	**1**
Soybean oil	E1	**1**	**1**	**1**	**1**	**1**	**1**
	E2	3	3	2	3	3	2
	E3	2	2	3	2	2	3

CRediT authorship contribution statement

Seyed Amin Seyfi Shishavan: Conceptualization, Methodology, Original draft preparation. **Fatma Kutlu Gündoğdu:** Investigation, Writing, Reviewing. **Elmira Farrokhizadeh:** Visualization, Investigation. **Yaser Donyatalab:** Methodology, Validation. **Cengiz Kahraman:** Reviewing and editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix

Some important abbreviations used in the article.

Abbreviation	Explanation
$JS_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$
$JS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree
$WJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$
$WJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree
$GWJS_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Generalized weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$
$GWJS^R_{SFS}(\tilde{A}_s,\tilde{B}_s)$	Generalized weighted Jaccard similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree
$ES_H(\tilde{A}_s,\tilde{B}_s)$	Exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance
$ES^R_H(\tilde{A}_s,\tilde{B}_s)$	Exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance considering refusal degree
$WES_H(\tilde{A}_s,\tilde{B}_s)$	Weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance
$WES^R_H(\tilde{A}_s,\tilde{B}_s)$	Weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Hamming distance considering refusal degree
$ES_E(\tilde{A}_s,\tilde{B}_s)$	Exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance
$ES^R_E(\tilde{A}_s,\tilde{B}_s)$	Exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance considering refusal degree
$WES_E(\tilde{A}_s,\tilde{B}_s)$	Weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance
$WES^R_E(\tilde{A}_s,\tilde{B}_s)$	Weighted exponential similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ based on Euclidean distance considering refusal degree
$SqrtCS_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)$	Square Root Cosine Similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$
$SqrtCS^R_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)$	Square Root Cosine Similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree
$WSqrtCS_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)$	Weighted Square Root Cosine Similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$
$WSqrtCS^R_{SFS}\left(\tilde{A}_s,\tilde{B}_s\right)$	Weighted Square Root Cosine Similarity measure of $\tilde{A}_s$ and $\tilde{B}_s$ considering refusal degree

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