Hybrid MPC of an Inverted Pendulum

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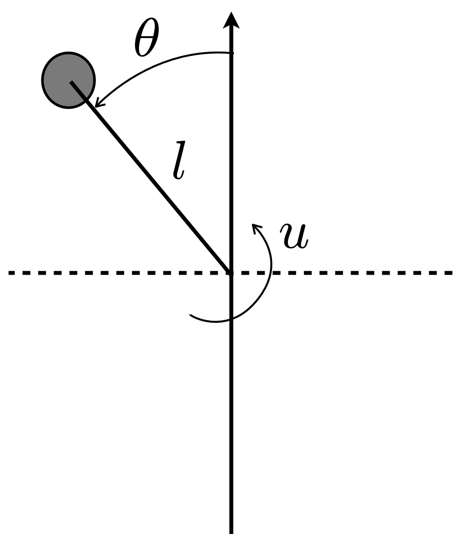


Figure 1: Pendulum model

Consider the inverted pendulum depicted in Figure 1, consisting of a mass $M = 1$ kg rotating by an angle θ at a fixed distance $l = 0.5$ m from the central joint and subject to gravity g. The pendulum is actuated by an input torque u and subject to a viscous friction torque $\beta\dot{\theta}$, $\beta = 0.5$ Nms. The goal is to move and keep the pendulum at the upper equilibrium position, with the constraint on modulus of the input torque u that is either 0 or between 1 and 10 Nm.

1 Modeling

1.1 Simulation model

The nonlinear continuous-time dynamics of the pendulum are

$$l^2 M\ddot{\theta} = Mgl\sin\theta - \beta\dot{\theta} + u \tag{1}$$

1.2 Hybrid discrete-time model

1.2.1 Trigonometric function approximation

Model (1) contains the nonlinearity $\sin\theta$, that we can approximate as the piecewise linear function

$$\sin\theta \approx s \triangleq \begin{cases} -\alpha\theta - \gamma & \text{if} \quad \theta \leq -\frac{\pi}{2} \\ \alpha\theta & \text{if} \quad |\theta| \leq \frac{\pi}{2} \\ -\alpha\theta + \gamma & \text{if} \quad \theta \geq \frac{\pi}{2} \end{cases} \tag{2}$$

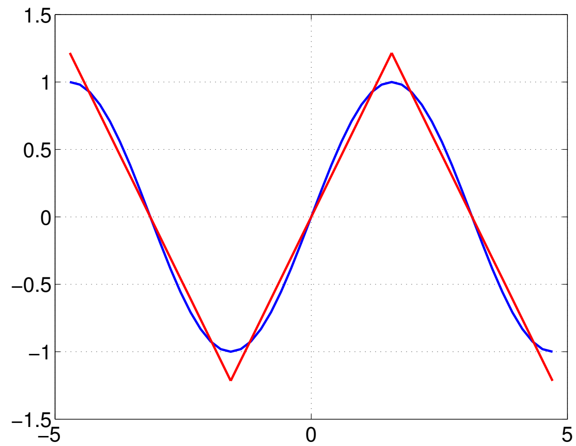


Figure 2: PWL approximation of the sin function plotted on the interval $[-\frac{3}{2}\pi, \frac{3}{2}\pi]$

The optimal value of α is obtained by fitting the linear function $\alpha\theta$ to $\sin\theta$ on the interval $[0, \frac{\pi}{2}]$, by optimizing

$$\begin{aligned}\min_{\alpha} \quad & \int_0^{\frac{\pi}{2}} (\alpha\theta - \sin(\theta))^2 d\theta \\ = \quad & \frac{\theta}{2} - \frac{1}{2}\cos\theta\sin\theta - 2\alpha\sin\theta + \frac{1}{3}\alpha^2\theta^3 + 2\alpha\theta\cos\theta\Big|_0^{\frac{\pi}{2}} \\ = \quad & \frac{1}{24}\pi^3\alpha^2 - 2\alpha + \frac{\pi}{4}\end{aligned}$$

Zeroing the derivative $\frac{\pi^3}{12}\alpha - 2$ of the integral with respect to α brings

$$\alpha = \frac{24}{\pi^3}$$

The coefficient γ is imposed by requiring $s = 0$ for $\theta = \pi$, that is $-\alpha\pi + \gamma = 0$, or $\gamma = \frac{24}{\pi^2}$, which also guarantees $s = 0$ for $\theta = -\pi$.

To represent (2) we introduce the event variables $\delta_3, \delta_4 \in \{0, 1\}$

$$\begin{aligned}[\delta_3 = 1] \quad & \leftrightarrow \quad [\theta \leq -\tfrac{\pi}{2}] \\ [\delta_4 = 1] \quad & \leftrightarrow \quad [\theta \geq \tfrac{\pi}{2}]\end{aligned} \tag{3}$$

and set

$$s = \alpha\theta + s_3 + s_4 \tag{4}$$

where

$$\begin{aligned}s_3 \quad &= \quad \begin{cases} -2\alpha\theta - \gamma & \text{if } \delta_3 = 1 \\ 0 & \text{otherwise} \end{cases} \\ s_4 \quad &= \quad \begin{cases} -2\alpha\theta + \gamma & \text{if } \delta_4 = 1 \\ 0 & \text{otherwise} \end{cases}\end{aligned} \tag{5}$$

To simplify the hybrid model, we impose the logic constraint

$$[\delta_4 = 1] \rightarrow [\delta_3 = 0] \tag{6}$$

Note that the symmetric constraint $[\delta_3 = 1] \rightarrow [\delta_4 = 0]$ would be redundant (both would be translated into the linear inequality $\delta_3 + \delta_4 \leq 1$).

1.2.2 Input constraints

Set $\tau_{\min} \triangleq 1$ Nm and $\tau_{\max} = 10$ Nm to be the minimum and maximum absolute values of the input torque u, respectively,

$$u \in [-\tau_{\max}, -\tau_{\min}] \cup [\tau_{\min}, \tau_{\max}] \tag{7}$$

In order to model the constraint (7), we introduce the auxiliary variable τ_A

$$\tau_A = \begin{cases} u & \text{if } -\tau_{\min} \leq u \leq \tau_{\min} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

and consider $u - \tau_A$ as the torque acting on the pendulum, with

$$u \in [-\tau_{\max}, \tau_{\max}] \tag{9}$$

As the magnitude of the input u will be penalized in the MPC cost function and it has no effect on the dynamics for values in $[-\tau_{\min}, \tau_{\min}]$, the numerical solver will not choose values for u in such a range. To represent (8) we introduce the event variables $\delta_1, \delta_2 \in \{0, 1\}$

$$\begin{aligned} [\delta_1 = 1] &\quad \leftrightarrow \quad [u \leq \tau_{\min}] \\ [\delta_2 = 1] &\quad \leftrightarrow \quad [u \geq -\tau_{\min}] \end{aligned} \tag{10}$$

and set

$$\tau_A = \begin{cases} u & \text{if } [\delta_1 = 1] \wedge [\delta_2 = 1] \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

so that $u - \tau_A$ is zero in $[-\tau_{\min}, \tau_{\min}]$ and equal to u in $[-\tau_{\max}, -\tau_{\min}] \cup [\tau_{\min}, \tau_{\max}]$. To simplify the hybrid model, we impose the logic constraint

$$[\delta_1 = 0] \rightarrow [\delta_2 = 1] \tag{12}$$

1.2.3 Discrete-time dynamics

Under the piecewise linear approximation (2), the dynamics of the pendulum (1) can be rewritten in state-space form

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A_c \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + B_c \begin{bmatrix} s_3 + s_4 \\ u - \tau_A \end{bmatrix} \tag{13a}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \tag{13b}$$

$$A_c \triangleq \begin{bmatrix} 0 & 1 \\ \frac{g}{l}\alpha & -\frac{\beta}{l^2 M} \end{bmatrix}, \quad B_c \triangleq \begin{bmatrix} 0 & 0 \\ \frac{g}{l} & \frac{1}{l^2 M} \end{bmatrix}$$

where $x \triangleq \left[\begin{smallmatrix} \theta \\ \dot{\theta} \end{smallmatrix}\right]$, $y \triangleq \theta$. A reasonably good discrete-time approximation of model (13) with sampling time $T_s = 50$ ms is obtained by setting

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \end{bmatrix} = A \begin{bmatrix} x_1(k) \\ x_2(k) \end{bmatrix} + B \begin{bmatrix} s_3(k) + s_4(k) \\ u(k) - \tau_A(k) \end{bmatrix} \tag{14a}$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(k) \tag{14b}$$

$$A \triangleq e^{T_s A}, \quad B \triangleq \int_0^{T_s} e^{tA} B_c dt$$

An alternative formulation than (14) would be to substitute (5) in (13), get a three-mode continuous-time piecewise affine system, then convert each of its affine dynamics to discrete-time.

1.2.4 Discrete hybrid automaton

The overall discrete-time hybrid model of the pendulum is obtained by collecting (14), (3), (5), (6), (10), (11),(12) subject to constraint (9).

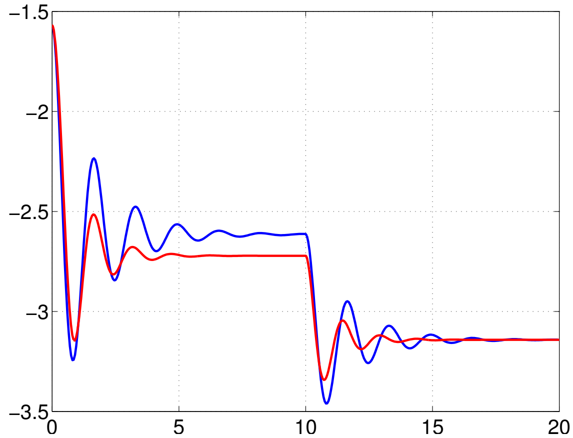


Figure 3: Open-loop simulation: nonlinear model (1) (red) against discrete-time hybrid model approximation (blue)

1.2.5 Open-loop simulation

We validate the hybrid model against the nonlinear model (1) for the initial condition $\theta(0) = -\frac{\pi}{2}$, $\dot{\theta}(0) = 0$ and signal input $u(t) = u_0$, $u_0 = 2$ N for $0 \leq t \leq 10$ s and zero after $t \geq 10$ s. The comparison of the trajectories is reported in Figure 3. Note the offset error in steady-state when the 2 N torque signal is applied, due to the PWL approximation error: in this case $\sin(\theta) = \frac{u_0}{Mgl}$ provides $\theta \approx -2.72$ rad, while the hybrid model provides approximately -2.61 rad. Note that both models converge to $-\pi$ rad when the input torque is released.

2 MPC design

We design a hybrid MPC controller to swing the pendulum up starting from initial condition $\theta = \pi$ rad, $\dot{\theta} = 0$. The objective of the controller is to make the output $y = \theta$ track the reference signal $r = 0$, based on the cost function

$$\sum_{k=0}^{N-1} \|Q(y_k - r(t))\|_\infty + \|Ru_k\|_\infty$$

with prediction horizon $N = 5$, $Q = 1$, $R = 0.01$, and subject to constraint (9). The results of a nominal closed-loop simulation (that is, the nominal discrete-time hybrid model in closed loop with the MPC controller) is shown in Figure 4(a). The results of the closed-loop simulation in which the nonlinear model (1) is commanded by the hybrid MPC controller is shown in Figure 4(b).

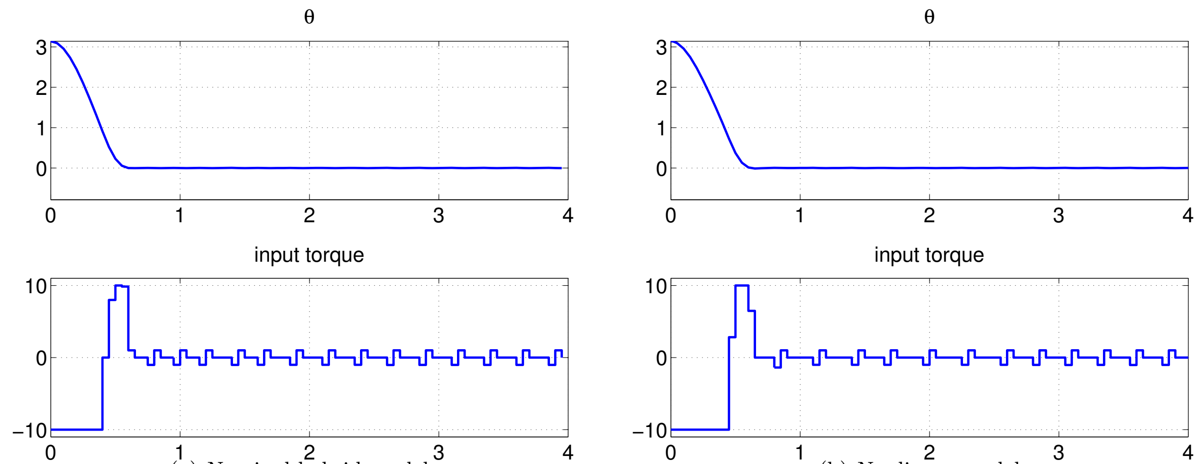
A HYSDEL model

```
/* Hybrid model of a pendulum

   (C) 2012 by A. Bemporad, April 2012 */

SYSTEM hyb_pendulum {

INTERFACE {
    STATE {
        REAL th    [-2*pi,2*pi];
        REAL thdot [-20,20];     }
```



(a) Nominal hybrid model (b) Nonlinear model

Figure 4: Closed-loop simulation results

```
    INPUT {
        REAL u [-11,11];         }
    OUTPUT{
        REAL y;                  }
    PARAMETER {
        REAL tau_min,alpha,gamma;
        REAL a11,a12,a21,a22,b11,b12,b21,b22; }
}

IMPLEMENTATION {
    AUX {
        REAL tauA,s3,s4;
        BOOL d1,d2,d3,d4;        }
    AD {
        d1 = u<=tau_min;
        d2 = u>=-tau_min;
        d3 = th <= -0.5*pi;
        d4 = th >= 0.5*pi;       }
    DA  {
        tauA = {IF d1 & d2 THEN u ELSE 0};
        s3 = {IF d3 THEN -2*alpha*th-gamma ELSE 0};
        s4 = {IF d4 THEN -2*alpha*th+gamma ELSE 0}; }

    CONTINUOUS {
        th    = a11*th+a12*thdot+b11*(s3+s4)+b12*(u-tauA);
        thdot = a21*th+a22*thdot+b21*(s3+s4)+b22*(u-tauA);  }

    OUTPUT {
        y = th;  }
    MUST {
        d4->~d3;
        ~d1->d2; }
  }
}
```