ASSIGNMENT 1

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Question : To solve following PDE with hat function and MATLAB.

-u″ + u = f with boundary conditions u(0) = 0 and u′(1) = 0 in the interval (0,1)

- We have p = q = 1 and let us take f = 1 so the weak formulation is to find u such that u(0) = 0 and

$$\int_0^1 u'v' + uv \, dx$$

for all v with v(0) = 0

- For finding solution of given PDE we have to find Q in KQ = F. Let us first find load matrix F element by element.

$$\psi_i = \frac{x - x_{i-1}}{x_i - x_{i-1}} \qquad x_{i-1} \le x \le x_i$$

$$\psi_i = \frac{x_{i+1} - x}{x_{i+1} - x_i} \qquad x_i \le x \le x_{i+1}$$

$$\psi_i = 0 \qquad \textit{otherwise}$$

$$f_1^i = \int_{x_{i-1}}^{x_i} \psi_{i-1} dx = \int_{x_{i-1}}^{x_i} \frac{x_i - x}{x_i - x_{i-1}} dx = \frac{x_i - x}{2}$$

$$f_2^i = \int_{x_{i-1}}^{x_i} \psi_i dx = \int_{x_{i-1}}^{x_i} \frac{x - x_{i-1}}{x_i - x_{i-1}} dx = \frac{x_i - x}{2}$$

where f_i is element load vector. Following up we get:

$$f_0 = \int_0^1 \psi_0 \mathrm{d}x = \int_{x_0}^{x_1} \psi_0 \mathrm{d}x = f_1^1$$

$$f_1 = \int_0^1 \psi_1 \mathrm{d}x = \int_{x_0}^{x_1} \psi_1 \mathrm{d}x + \int_{x_1}^{x_2} \psi_1 \mathrm{d}x = f_2^1 + f_1^2$$

and so on. At end we get:

$$F = \begin{pmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 + f_1^3 \\ \vdots \\ f_2^N - 1 + f_1^N \\ f_2^N \end{pmatrix}$$

Hence we now have our load matrix F.

- Now let us find element stiffness matrix K.

$$K_{1,1}^i = \int_{x_{i-1}}^{x_i} \psi'_{\mathrm{i}-1}\psi'_{i-1} + \psi_{\mathrm{i}-1}\psi_{i-1}\mathrm{d}x$$

$$K_{1,2}^i = \int_{x_{i-1}}^{x_i} \psi'_{\mathrm{i}}\psi'_{i-1} + \psi_{\mathrm{i}}\psi_{i-1}\mathrm{d}x$$

$$K_{2,1}^i = \int_{x_{i-1}}^{x_i} \psi'_{\mathrm{i}-1}\psi'_{i} + \psi_{\mathrm{i}-1}\psi_{i}\mathrm{d}x$$

$$K_{2,2}^i = \int_{x_{i-1}}^{x_i} \psi'_{\mathrm{i}}\psi'_{i} + \psi_{\mathrm{i}}\psi_{i}\mathrm{d}x$$

which gives us

$$K^i = \begin{pmatrix} \frac{1}{x_i - x_{i-1}} & \frac{-1}{x_i - x_{i-1}} \\ \frac{-1}{x_i - x_{i-1}} & \frac{1}{x_i - x_{i-1}} \end{pmatrix} + \begin{pmatrix} x_i - x_{i-1} & x_i - x_{i-1} \\ x_i - x_{i-1} & x_i - x_{i-1} \end{pmatrix}$$

also in general:

$$K_{j,j} = \int_{x_{j-1}}^{x_i} (\psi'_j)^2 + (\psi_j)^2 \mathrm{d}x = K_{2,2}^j + K_{1,1}^{j+1}$$

$$K_{j,j-1} = \int_{x_{j-1}}^{x_i} \psi'_{\mathrm{j}-1}\psi'_j + \psi_{\mathrm{j}-1}\psi_j \mathrm{d}x = K_{2,1}^j$$

Hence we get our matrix K:

$$K = \begin{pmatrix} K^1_{1,1} & K^1_{1,2} & & & \\ K^1_{2,1} & K^1_{2,2}+K^2_{1,1} & K^2_{1,2} & & \\ & & \ddots & & \\ & & & K^{N-1}_{2,2}+K^N_{1,1} & K^N_{1,2} \\ & & & K^N_{2,1} & a^N_{2,2} \end{pmatrix}$$

- Now we will use boundary conditions. Since we need $u_h(0) = 0$ and our test function must satisfy $v_h(0) = 0$ we will remove first row and first column form matrix K and first row of load matrix F.
- Since we have all the things required to get soultion U. We will solve $KQ = f$ using matlab. The code and grphs produced:

```
% solve −u   + u =1 on (0,1) with u(0)=0 and u (1)=0 using FEM
N=4; % number of elements
x=linspace(0,1,N+1); % set up mesh
h=1/N; % h=constant mesh size
K=zeros(N+1,N+1); % set up empty stiffness matrix
F=zeros(N+1,1); % set up global load vector
for i=1:N % loop over elements
Ktemp = [1 −1;−1 1]/h + [1 1;1 1]h; % compute entries in element stiffness matrix
Ftemp = [h/2;h/2]; % compute entries in element load vector
K(i:i+1,i:i+1)=K(i:i+1,i:i+1)+Ktemp; % add to global stiffness matrix
f(i:i+1)=f(i:i+1)+Ftemp; % add to global load vector
end
K(1,:)=[]; % apply Dirichlet boundary condition by
K(:,1)=[]; % removing first row and column of stiffness matrix
f(1)=[]; % and first row of load vector
U=K\F; % compute solution to matrix problem
U=[0;U]; % add value at x=0 to solution vector
plot(x,U,'rx−') % plot solution
disp(U)
```

- For $N = 4$ we get U as:

$$U = \begin{pmatrix} 0 \\ 0.0956 \\ 0.1500 \\ 0.1778 \\ 0.1863 \end{pmatrix}$$

- For $N = 5$ we get U as:

$$U = \begin{pmatrix} 0 \\ 0.0798 \\ 0.1312 \\ 0.1628 \\ 0.1799 \\ 0.1853 \end{pmatrix}$$

- For $N = 6$ we get U as:

$$U = \begin{pmatrix} 0 \\ 0.0685 \\ 0.1162 \\ 0.1486 \\ 0.1694 \\ 0.1810 \\ 0.1847 \end{pmatrix}$$