1

Consider the initial-boundary value problem for Burgers' equation

$$\begin{aligned} u_t + u * u_x &= \sigma u_{xx}, \quad 0 < x < 1, \ t > 0, \\ u(x,0) &= \phi(x), \\ u(0,t) &= g_L(t), \\ u(1,t) &= g_R(t), \end{aligned}$$

where σ is a positive constant. Burgers proposed this equation to model certain aspects of turbulent flows. it is also used as a proving ground for numerical methods because when σ is small the solution exhibits a shock like structure that is difficult to calculate.

1.1 Theoretical

Consider the two possibilities for discritizing the non linear term:

$$(uu_x)_j^n \approx U_j^n \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

and

$$(uu_x)_j^n = \frac{[(u^2)_x]_j^n}{2} \approx \frac{(U_{j+1}^n)^2 - (U_{j-1}^n)^2}{4\Delta x}.$$

Both formulas are $O(\Delta x^2)$ approximations of uu_x. Rather than decide which one is best, let's use the linear combination

$$(uu_x) \approx \frac{Q(U_j^n)}{2\Delta x}$$

where

$$Q(U_j^n) = p\frac{(U_{j+1}^n)^2 - (U_{j-1}^n)^2}{2} + (1-p)U_j^n(U_{j+1}^n - U_{j-1}^n)$$

with p selected on $[0,1]$.
Suppose that we discretize Burgers' equation using the Crank-Nicolson scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{Q((U_j^{n+1} + U_j^n)/2)}{2\Delta x} = \sigma \frac{\delta^2((U_j^{n+1} + U_j^n)/2)}{\Delta x^2}$$

where, as usual δ^2 denotes the second-centered difference.
Develop a procedure for solving the Crank-Nicolson system using the tridiagonal algorithm and Newton's iteration. Give formulas for evaluating any Jacobians.

1.2 Computational

Write a program to solve the Crank-Nicolson system developed in Part 1. Choose the initial and Dirichlet boundary data so that the exact solution is

$$u(x,t) = 1 - \tanh\frac{x-t}{2\sigma}.$$

Solve the problem with $\sigma = 0.01$ for $t \leq 1$ using $p = 0, 2/3$, and 1. Select $\Delta x = 0.05, 0.02$, and 0.01 and compute with $\frac{\Delta t}{\Delta x} = 0.5$ and 1. Plot the solutions as function of x at $t = 0, 0.5$, and 1. Tabulate or plot the error in the discrete L^2 norm at $t = 1$ as a function of Δx for each p and $\frac{\Delta t}{\Delta x}$. Discuss the results.