Numerical simulation using FEM and Lax-Wendroff Method

Monisha Muralidharan M

What is the exact(analytical) solution $u^{ex}(x,t)$ of this problem?

We consider,

$$u(x,t) = Ae^{i(kx-\omega t)} \tag{1}$$

Differentiating the eqn wrt time(t), we have

$$u_t(x,t) = -A\omega e^{i(kx-\omega t)} = -\omega u(x,t)$$

Differentiating the eqn wrt space(x), we have

$$u_x(x,t) = Ake^{i(kx-\omega t)} = ku(x,t)$$

Substituting the above values in Transport eqn:

$$u_t + cu_x = 0$$

We get

$$-\omega + ck = 0$$

Which implies that

$$\omega = ck \tag{2}$$

Substituting eqn 2 in eqn 1, we have,

$$u(x,t) = Ae^{i(kx-ckt)}$$

$$u(x,t) = Ae^{ik(x-ct)}$$

We consider y=x-ct, Then we have,

$$u(x,t) = Ae^{ik(y)}$$

At t=0,

$$u(x,0) = u(y,0) = u_0(y,0) = Ae^{ik(x-0c)} = Ae^{ik(y)}$$

From the above we can conclude that,

$$u(x,t) = u_0(y) = u_0(x-ct) \tag{3}$$

Equation 3 is the analytical solution for the transport equation.

Choosing Finite Elements with Lax-Wendroff method implement in Matlab the following questions

We choose the following parameters for the numerical simulation.

$$c = 341m/s$$

$$T = 0.001$$

Number of points in time= [1 2 3 4 5 6 7 8 10 15 20 250 30 35 40 50 60 70 80 90 100]
Number of points in 1D space= [70 100 110 120 130 140 150 200 230 250 270 300 330 350 370 400 450 500 700]

Give arguments to testify that the quality of your implementation is correct (using the exact solution derived in the previous question)

We use MATLAB to plot the exact solution for the wave in transit. The exact solution is in line with the numerical simulation as all the curves tend to be around the exact solution with an error lesser than 1.
The function we've chosen here is a sinc function. The function is not smooth at L=25 and L=75 due to abrupt introduction of zeros. This is reflected in the curves where a spike can be seen arounf those points. We've however chosen a propagation time that limits these spikes.

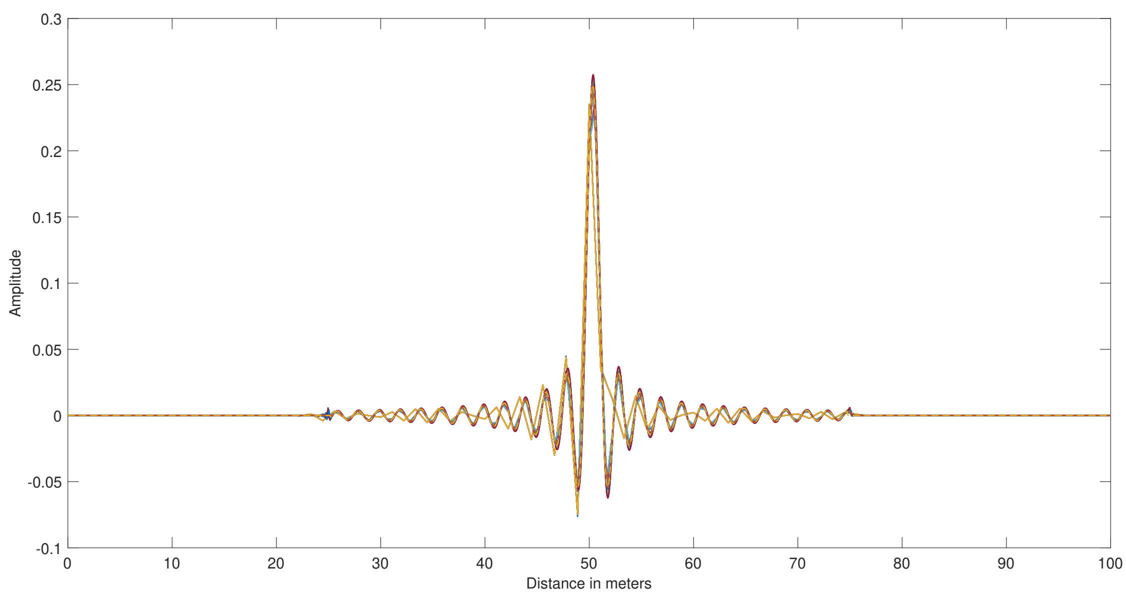


Figure 1: All curves converge towards the exact solution

Compute an error map for different Δx and Δt. Try to describe qualitatively the behavior, as a function of $\Delta x, \Delta t$, and $\alpha = c\Delta t/\Delta x$

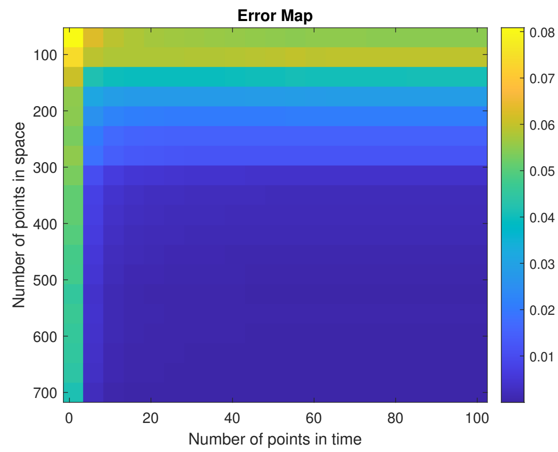


Figure 2: Error Map for FEM-Lax Wendroff method

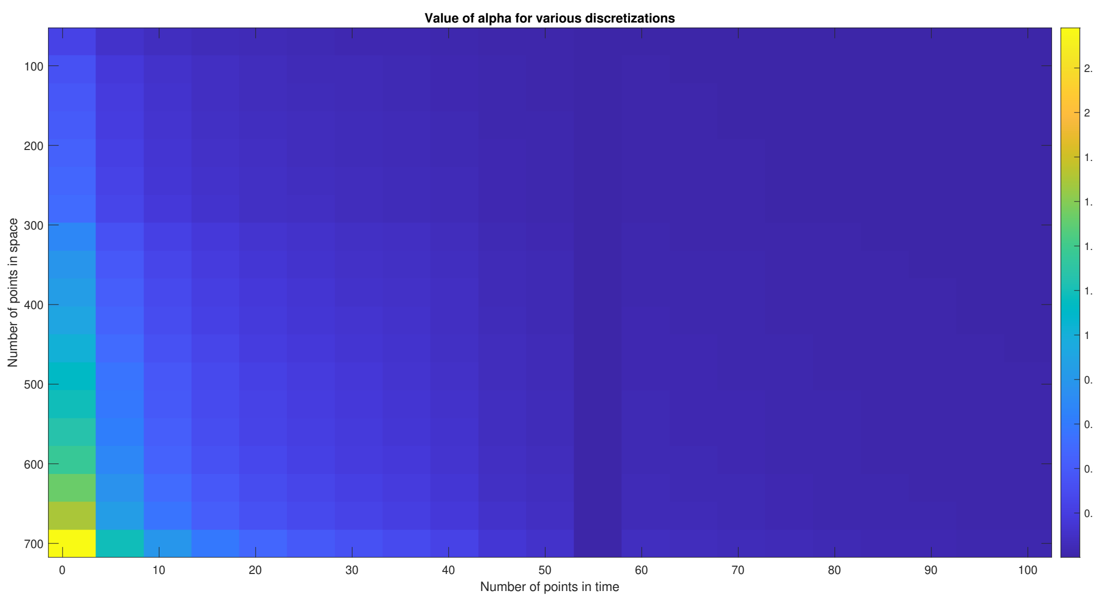


Figure 3: Alpha values for different discretizations

We see that the error is based on the discretization of time and space. The higher the number of points, the better the approximation. We also have the value of α which is the Courant, Friedrichs, Lewy parameter(CFL). The numerical simulation required that this parameter be as small as possible. This can be achieved by reducing the speed of the wave or increasing the number of points to discretize time. A map of the value for various discretizations is computed to understand the affect.

Compare the cases when $c > 0$ and $c < 0$

For $c > 0$, the wave travels forwards(Forward propagation). The graphs in the previous question pertains to forward propagation.

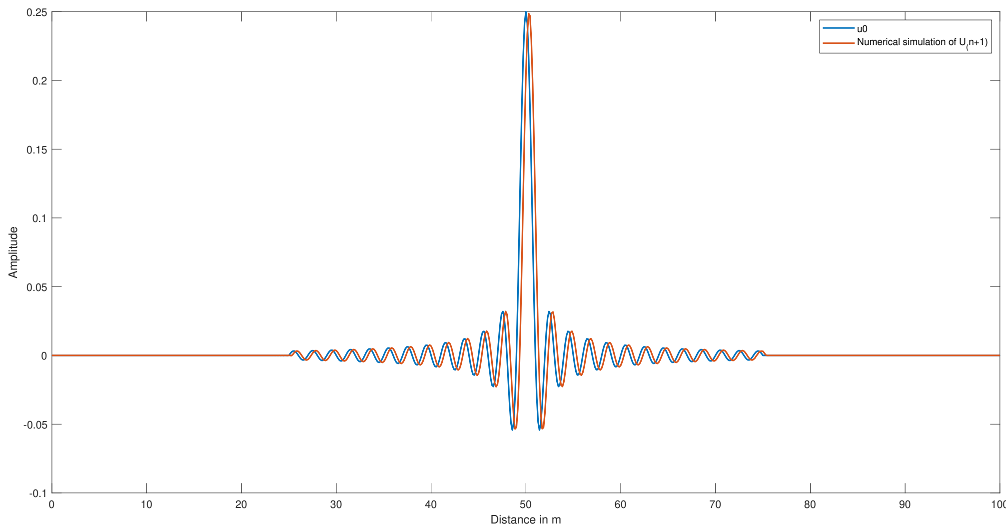


Figure 4: Forward propagation when $c > 0$, here c=340

For $c < 0$, the wave travels backwards(Backward propagation).

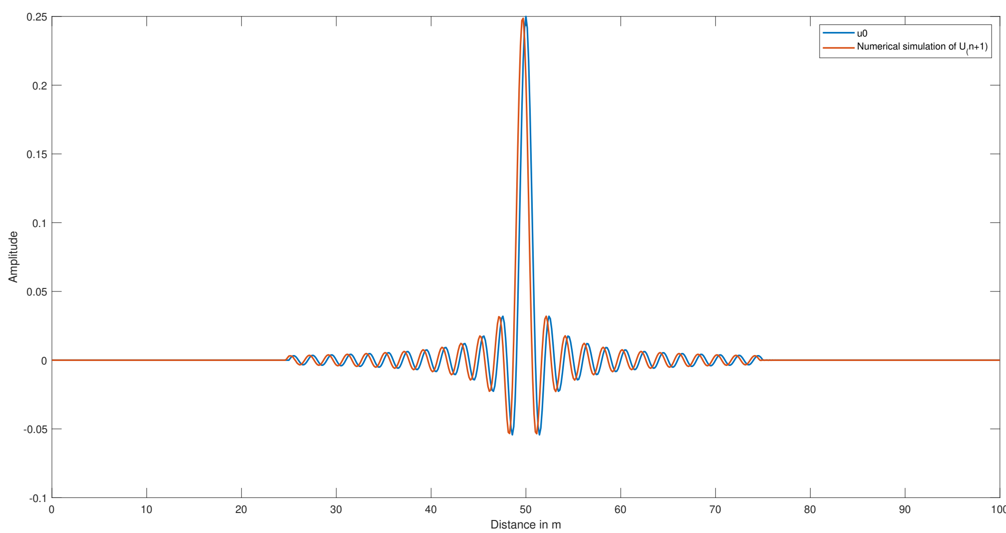


Figure 5: Backward propagation when $c < 0$, here c=-340

The error map looks similar to forward propagation.
The alpha values are also similarly mapped. They have the same absolute values and opposite signs.

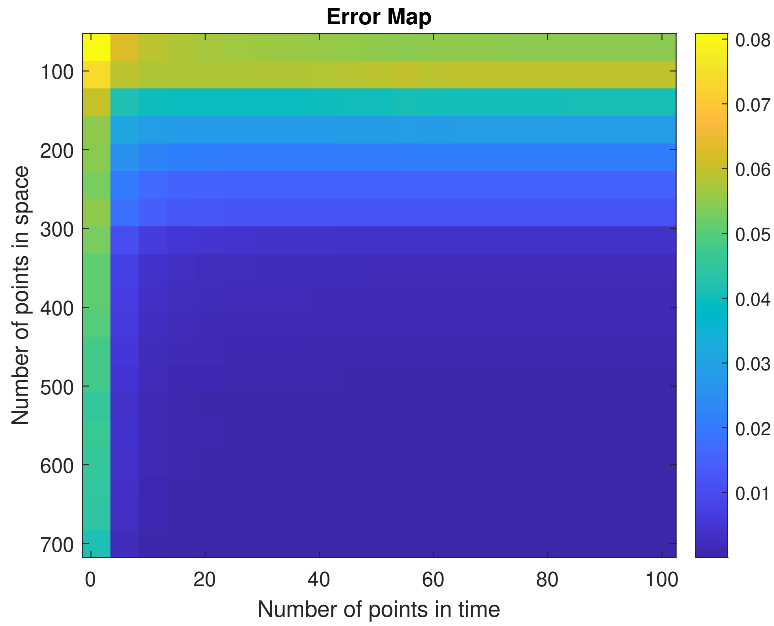


Figure 6: Error Map for FEM-Lax Wendroff method with $c < 0$

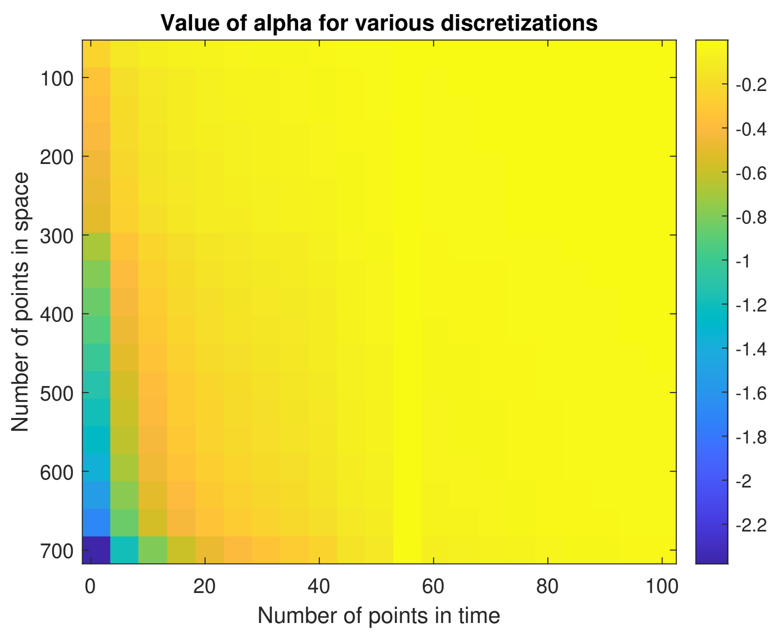


Figure 7: Alpha values for different discretizations when $c < 0$

Compute and compare solutions for different values of λ What is the relation between λ and the wavelength ?

Here, λ is the eigen value of the transformation/amplification matrix $G(\alpha, K)$ We can compute U_{n+1} using,

$$U_{n+1} = (G(\alpha, K))^n U_0$$

The amplification matrix depends on the type of scheme we use for the numerical simulation. For Lax Wendroff, we have,

$$(G(\alpha, K))^n = I - \alpha(I - \alpha K/2)K$$

The eigenvalues provide us the resonant frequencies of the system.

$$\omega = \lambda^{1/2} = 2\pi f$$

By this relationship, the wavelength is inversely proportional to the eigenvalue. The greater the eigenvalue, the lower the wavelength.

$$wavelength = c/f = 2\pi c/\lambda^{1/2}$$

Plot the wavefield as a surface in space time $[L, 1] * [0, T]$

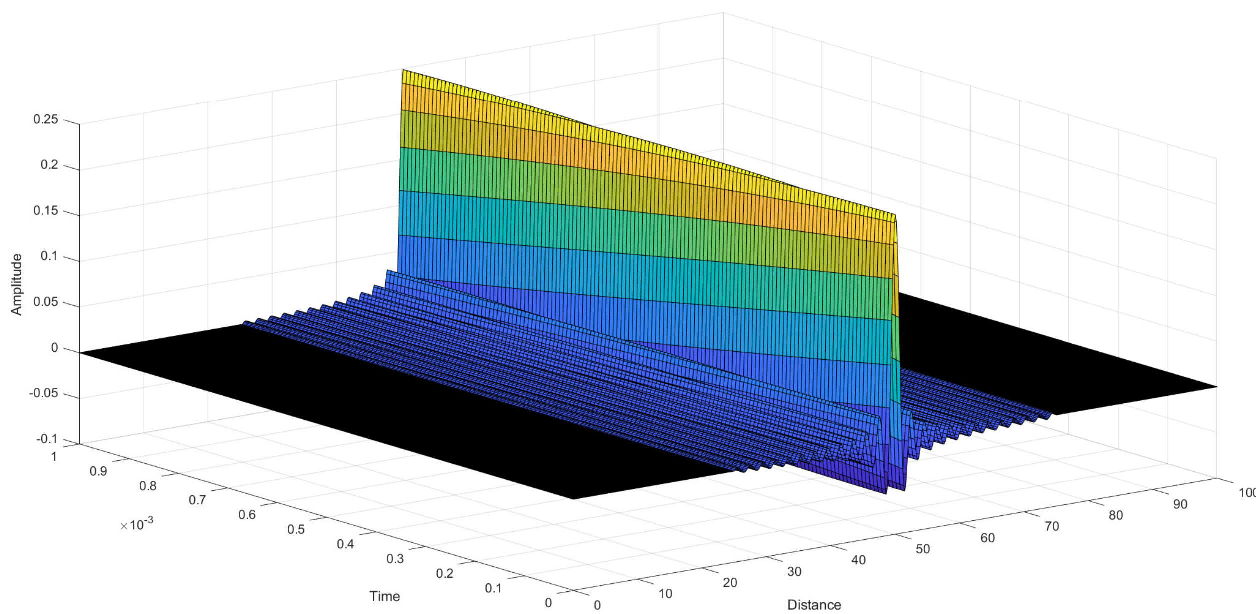
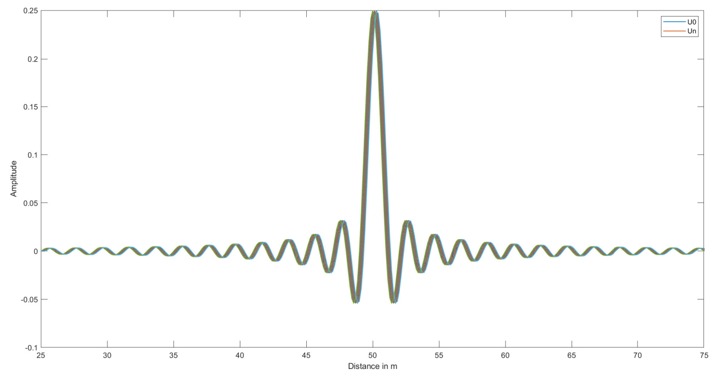


Figure 8: Wave field as a surface in space time

Plot the wavefield as a plot along $[L, L]$ moving with time.



Wavefield moving with time. Please follow link for GIF, `https://drive.google.com/file/d/1t8pe6LAuqGtL6b4j4_tI4XAw_EIc5yUE/view?usp=drive_link`