Numerical simulation using FVM and Lax-Wendroff Method

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What is the exact(analytical) solution $u^{ex}(x,t)$ of this problem?

We consider,

$$u(x,t) = Ae^{i(kx-\omega t)} \tag{1}$$

Differentiating the eqn wrt time(t), we have

$$u_t(x,t) = -A\omega e^{i(kx-\omega t)} = -\omega u(x,t)$$

Differentiating the eqn wrt space(x), we have

$$u_x(x,t) = Ake^{i(kx-\omega t)} = ku(x,t)$$

Substituting the above values in Hopf eqn:

$$u_t + cuu_x = 0$$

We get

$$-\omega + cku = 0$$

Which implies that

$$\omega = cku \tag{2}$$

Substituting eqn 2 in eqn 1, we have,

$$u(x,t) = Ae^{i(kx-ckut)}$$

$$u(x,t) = Ae^{ik(x-cut)}$$

We consider y=x-cut, Then we have,

$$u(x,t) = Ae^{ik(y)}$$

At t=0,

$$u(x,0) = u(y,0) = u_0(y,0) = Ae^{ik(x-0cu)} = Ae^{ik(y)}$$

From the above we can conclude that,

$$u(x,t) = u_0(y) = u_0(x-cut) \tag{3}$$

Alternatively we can use the characteristic method to find the analytical solution.

$$u_t + cuu_x = 0$$

Considering the initial condition,

$$u(x,0) = u_0(x)$$

and that,

$$t \to x(t)$$

$$\frac{dx}{dt}(t) = u(t, x(t))$$

If h(t)=u(t,x(t)) is the solution along the characteristic line, then,

$$h'(t) = u_t(t, x(t)) + x'(t)u_x(t, x(t)) = 0$$

From this we understand that the solution along the characteristic line is a constant. We can identify the initial point along this characteristic line as,

$$u(0, x(0, \xi)) = u_0(\xi)$$

We infer that $x(0, \xi) = \xi$ Hence,

$$\frac{dx}{dt}(t) = u_0(\xi) \Rightarrow x(t) = u_0(\xi)t + \xi$$

$$u(t, x(t, \xi)) = u_0(\xi) = u_0(x(t, \xi) - u_0(\xi)t) = u_0(x(t, \xi) - u(t, x(t, \xi))t)$$

$$u(t, x) = u_0(x - u(t, x)t)$$

Considering the speed of wave, we have,

$$u(t, x) = u_0(x - cu(t, x)t) \tag{4}$$

The above is the solution for Hopf equation using characteristic method.

When c¿0, we have a forward wave propagation that eventually gives rise to shock waves. The part of the wave with the higher amplitude travels faster thereby creating a sudden rise giving rise to shock wave formation.

When c¡0, we have backward propagation of wave and the wave collapses slowly and disperses(reduces in amplitude and certain parts lag in time). If the wave is sharp, it begins to smoothen.

Finite volume with Lax-Wendroff method is chosen to be implemented in Matlab

We choose the following parameters for the numerical simulation.

$$c = 7m/s$$

$$T = 1s$$

Number of points in time= 601
Number of points in 1D space= 500
Domain Length L= 2* 15 m

The exact solution is implemented in the program.
The solution calls for interpolation through Euler's method which follows the solution obtained earlier.

For $c < 0$

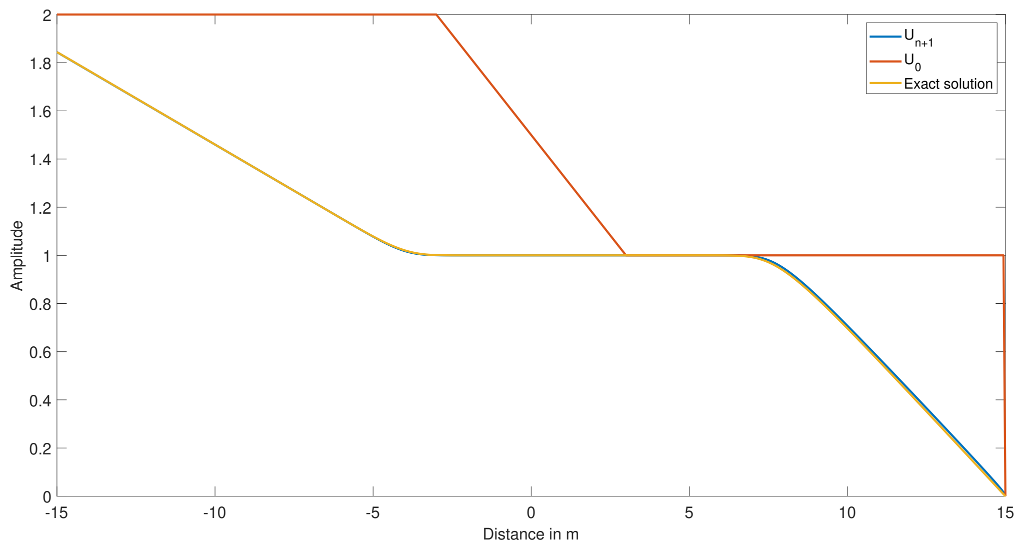

Figure 1: Backward propagation when $c < 0$ and T=1 s

From the above figure we see that the exact solution and the numerical solution are similar. The error obtained is 1.19e-06 which is quite negligible. We see that the wave is spreading through the domain and dispersing slowly. There is rarefaction occuring. The slope in the middle of the initial wave after propagation has become smooth. The portion of the wave with a higher amplitude has travelled farther than the portion of the wave with the least amplitude.

For $c > 0$

The error obtained with forward propagation is slightly higher than that of backward propagation with 1.82e-05 which is again slight.
There is shock formation towards the end of the domain which can be noticed by the steepness of the wave.
In a propagating medium, there would be compression of the matter in the medium.
The exact solution and the numerical solution are coinciding well. It is also imperative to mention that the small step size helped improve the numerical solution. With smaller step sizes there were ripples towards the steep part of the wave which produced higher errors.
Maintaining a low CFL number helped maintain the accuracy of the numerical simulation.

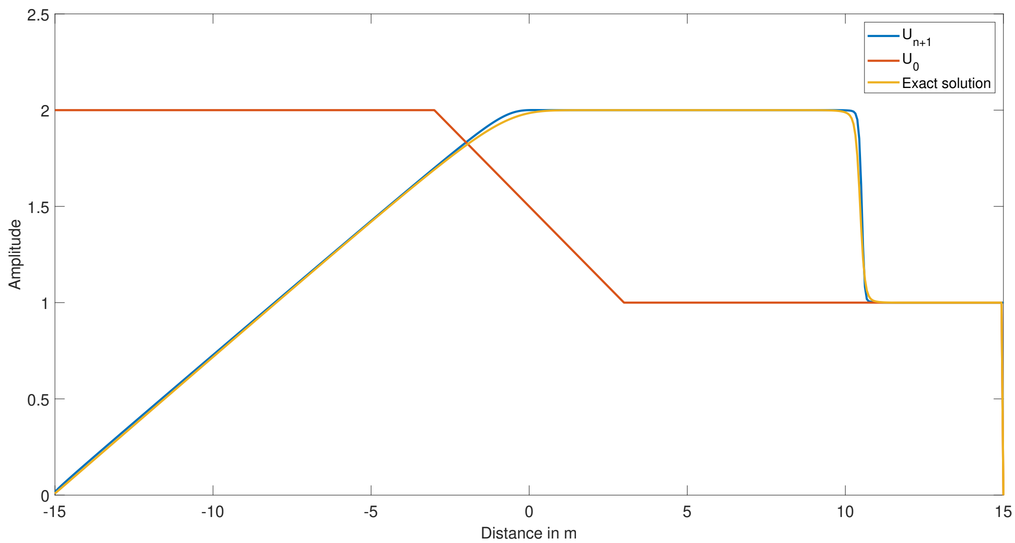

Figure 2: Shock wave formation with forward propagation when $c > 0$ and T=1 s