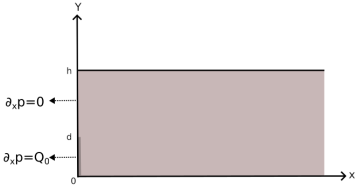
Piston Problem

Time harmonic $e^{-i\omega t}$



A semi-infinite waveguide along x bounded at x=0 has a height h in the y direction. A piston is located at x=0 and occupies y=[0,d] with the following boundary conditions,

$\partial_x p = Q_0 \in x = 0, y = [0, d]$,
$\partial_x p = 0 \in x = 0, y = [d, h]$,
$\partial_y p = 0 \in x > 0, y = 0, h$.

The normal derivative of the pressure on the walls of the waveguide is zero.
The normal derivative of the pressure on the piston is Q_0. Solving the Helmholtz equation $(\nabla + k^2)p = 0$ for the pressure field p in the waveguide, and considering only the outgoing wave condition

$$p(x, y) = \sum_{n=0}^{\infty} A_n e^{ik_n x} g_n(y)$$

. Where, $g_n(y) = \sqrt{\frac{\epsilon}{h}} \cos(\frac{n\pi}{h} y)$, $\epsilon = 2 - \delta_{0n}$, $k_n = \sqrt{k^2 - (\frac{n\pi}{h})^2}$.
Find A_n and reconstruct the pressure field.

Considering only the forward going wave,

$$p(x, y) = \sum_{n=0}^{\infty} A_n e^{ik_n x} g_n(y),$$

The pressure derivative at x=0 is given by,

$$\partial_x p(0, y) = \sum_{n=0}^{\infty} ik_n A_n g_n(y),$$

Using the orthogonality of $g_n(y)$, we get

$$\int_0^h \partial_x p(0, y) g_m(y) dy = \int_0^h \sum_{n=0}^{\infty} ik_n A_n g_n(y) g_m(y) dy,$$

$$\int_0^h \sum_{n=0}^{\infty} ik_n A_n g_n(y) g_m(y) dy = \int_0^d Q_0 g_m(y) dy + \int_d^h 0 dy,$$

$$ik_m A_m = \int_0^d Q_0 g_m(y) dy,$$

$$A_m = \frac{-iQ_0}{k_m}\int_0^d g_m(y)dy,$$

$$A_m = \frac{-iQ_0}{k_m}\int_0^d \sqrt{\frac{\epsilon}{h}}\cos(\frac{m\pi}{h}y)dy,$$

$$A_m = \frac{-iQ_0}{k_m}\sqrt{\frac{\epsilon}{h}}\frac{h}{m\pi}\sin(\frac{m\pi}{h}d).$$

For m>0

$$A_m = \frac{-iQ_0 d}{k_m}\sqrt{\frac{2}{h}}\text{sinc}(\frac{m\pi}{h}d).$$

For m=0

$$A_0 = \frac{-iQ_0 d}{k_0}\sqrt{\frac{1}{h}}.$$

$$\mathbf{A} = \begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_m \end{bmatrix}$$

Where, $A_m = \frac{-iQ_0 d}{k_m}\sqrt{\frac{2-\delta_{0m}}{h}}\text{sinc}(\frac{m\pi d}{h}).$

The pressure field in the waveguide is given by,

$$p(x, y) = \sum_{n=0}^{\infty} A_n e^{ik_n x} g_n(y),$$

$$p(x, y) = \sum_{n=0}^{\infty} \frac{-iQ_0 d}{k_n} \sqrt{\frac{2 - \delta_{0n}}{h}} \operatorname{sinc}(\frac{n\pi}{h} d) e^{ik_n x} \sqrt{\frac{2 - \delta_{0n}}{h}} \cos(\frac{n\pi}{h} y),$$

$$p(x, y) = \frac{-iQ_0 d}{h} \sum_{n=0}^{\infty} \frac{2 - \delta_{0n}}{k_n} \operatorname{sinc}(\frac{n\pi}{h} d) \cos(\frac{n\pi}{h} y) e^{ik_n x}.$$