AN EXTENSION OF A DEPTH INEQUALITY OF AUSLANDER

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ABSTRACT. In this paper, we consider a depth inequality of Auslander which holds for finitely generated Tor-rigid modules over commutative Noetherian local rings. We raise the question of whether such a depth inequality can be extended for n-Tor-rigid modules, and obtain an affirmative answer for 2-Tor-rigid modules that are generically free. Furthermore, in the appendix, we use Dao's eta function and determine new classes of Tor-rigid modules over hypersurfaces that are quotient of unramified regular local rings.

1. INTRODUCTION

Throughout, R denotes a commutative Noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue field k, and all R-modules are assumed to be finitely generated.

In this paper we are concerned with the following theorem of Auslander [2], where $\operatorname{depth}_R(\mathfrak{a}, M)$ denotes the $\mathfrak{a}$-depth of M; see 2.3 and 2.7 for definitions and details.

Theorem 1.1. *(Auslander* [2]*) Let R be a local ring, $\mathfrak{a}$ be an ideal of R, and let M be a nonzero Tor-rigid R-module. Then it follows that* $\operatorname{depth}_R(\mathfrak{a}, M) \leq \operatorname{depth}_R(\mathfrak{a}, R)$.

Our purpose is to investigate to what extent one can generalize Auslander's inequality stated in Theorem 1.1. Prior to stating our main result, we discuss some history and motivation concerning the conclusion, as well as the hypotheses, of Theorem 1.1.

The depth inequality in Theorem 1.1 is a consequence of a result of Auslander [2, 4.3] which states that, if M is a Tor-rigid module over a local ring R, then each non zero-divisor on M is also a non zero-divisor on R, that is, the set of all associated primes of M contains that of R. As the celebrated work of Auslander [2] and Lichtenbaum [37] shows that modules over regular local rings are Tor-rigid, the conclusion of Theorem 1.1 holds over each regular local ring; see 2.7(i). Auslander considered the question whether the same conclusion holds for modules of finite projective dimension and asked if each module of finite projective dimension must be Tor-rigid; this yielded a conjecture known as the Auslander's zero divisor conjecture which claims that, for modules M of finite projective dimension, each non zero-divisor on M is also a non zero-divisor on the ring considered. This conjecture, due to the new intersection theorem established by Roberts, is now a theorem; see [44, 6.2.3, 13.4.1] for the details. The query whether or not modules of finite projective dimension are Tor-rigid also came known as the rigidity conjecture; this was formulated by Peskine and Szpiro [43] who made significant contributions and established the conjecture for torsion modules of projective dimension two; see also [12, 3.1]. The rigidity conjecture did not fare long: Heitmann [27] constructed a torsion-free module of projective dimension two that is not Tor-rigid. However, the rigidity conjecture remains open over complete intersections of codimension at least two, even over those that are one-dimensional domains; see [12, 3.2, 3.3] and [16, 4.2]. There are other questions studied in the literature which are related to Theorem 1.1 including the superheight conjecture; see, for example, [6].

Tor-rigidity, a subject of investigation in commutative algebra, is a delicate assumption in Theorem 1.1. In general, over non-regular local rings, it is very difficult to check if a given module is Tor-rigid.

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The work of Lichtenbaum [37, Theorem 3], along with that of Huneke and Wiegand [34, 1.9], imply that modules of finite projective dimension over hypersurfaces (that are quotient of unramified regular rings) are Tor-rigid; see 2.7(ii). A noteworthy development in this direction has been the utilization of the theta function in the study of Tor-rigidity; the theta function was introduced by Hochster [32] and subsequently used by Dao [15] to generalize the aforementioned fact on Tor-rigidity over hypersurfaces. A consequence of Dao's result is that modules over even dimensional simple hypersurface singularities satisfy the depth inequality stated in Theorem 1.1; see A.2(i). As our results rely upon Tor-rigidity, in the appendix, we discuss the theta function and also consider a generalization of it to determine new classes of n-Tor-rigid modules over complete intersections that are quotient of unramified regular local rings.

The depth inequality stated in Theorem 1.1 can fail in general, even over complete intersection rings as we see next:

Example 1.2. Let $R=\mathbb{C}[[x_1,\dots,x_n,y_1,\dots,y_n]]/(x_1y_1,\dots,x_ny_n)$, $M=R/(y_1,\dots,y_n)$, and let $\mathfrak{a}$ be the ideal of R generated by $x_1,\dots,x_n$. Then it follows that R is a complete intersection of codimension and dimension n. Moreover, $\operatorname{depth}_R(\mathfrak{a},R)=0<n=\operatorname{depth}_R(\mathfrak{a},M)$ so that the depth inequality stated in Theorem 1.1 fails. Note that M is not Tor-rigid: for example, there is an R-module N such that $\operatorname{Tor}_n^R(M,N)=0\neq\operatorname{Tor}_{n+1}^R(M,N)$; see [35, 4.1]. $\square$

Observe that, for the R-module M in Example 1.2, it follows that $\operatorname{depth}_R(\mathfrak{a},M)\leq\operatorname{depth}_R(\mathfrak{a},R)+n$. Also, $M\cong\Omega_R^nN$ for some R-module N, where N is $(n+1)$-Tor-rigid because modules over complete intersection rings of codimension c are $(c+1)$-Tor-rigid [42, 1.6]. Motivated by these facts, we raise the following question:

Question 1.3. Let R be a local ring, M be a nonzero R-module, and let $\mathfrak{a}$ be an ideal of R. Assume $M\cong\Omega_R^nN$ for some $n\geq 0$ and some R-module N which is $(n+1)$-Tor-rigid. Then does it follow that $\operatorname{depth}_R(\mathfrak{a},M)\leq\operatorname{depth}_R(\mathfrak{a},R)+n$? $\square$

Note that, due to Theorem 1.1, Question 1.3 is true in case $n=0$; see also 2.7. The question is also true if R is a complete intersection ring of codimension c and n equals c; see 3.2. The main purpose of this paper is to study Question 1.3 for the case where $n=1$. For that case we are able to obtain an affirmative answer to the question under mild conditions. More precisely, we prove:

Theorem 1.4. *Let R be a local ring, $\mathfrak{a}$ be an ideal of R, and let M be a nonzero R-module such that $M\cong\Omega_RN$ for some R-module N which is 2-Tor-rigid and generically free (e.g., R is reduced). If* $\operatorname{depth}_R(\mathfrak{a},R)\geq 1$, *then it follows that* $\operatorname{depth}_R(\mathfrak{a},M)\leq\operatorname{depth}_R(\mathfrak{a},R)+1$. $\square$

The special case of Theorem 1.1 and Theorem 1.4, where the maximal ideal is considered, is also worth discussing. Note, if a local ring R is not Cohen-Macaulay, then Theorem 1.1 implies that each maximal Cohen-Macaulay R-module is not Tor-rigid. Therefore, Theorem 1.1 is also related to another important conjecture known as the Small Cohen-Macaulay modules conjecture [31]; this conjecture predicts that each complete local ring admits a maximal Cohen-Macaulay module. For example, there are two-dimensional non-complete local domains R such that $\operatorname{depth}_R(M)\leq 1=\operatorname{depth}(R)$ for each R-module M; see [31, page 11].

Theorem 1.1 produces several classes of modules that are not Tor-rigid over local rings of dimension at most three; see [47, 4.8]. For example, by results of Hochster [30, 5.4, 5.6, 5.9], there are three-dimensional non-Cohen-Macaulay local rings that admit maximal Cohen-Macaulay modules; these modules are not Tor-rigid by Theorem 1.1. In Appendix B we give a similar example concerning Theorem 1.4 where the ring considered is four-dimensional. These examples should indicate that the problem of extending Theorem 1.1 is subtle over many rings, even over those that appear in nature.

Let us note that Theorem 1.4 follows as a consequence of our main result, namely Theorem 3.3; see Corollary 3.4. Let us also note that Theorem 3.3 exploits the notion of Tor-rigidity developed by Auslander, and establishes a depth inequality that is more general from the one stated in Theorem 1.4.

The key ingredient for the proof of Theorem 3.3, and hence for the proof of Theorem 1.4, is Proposition 2.8 which yields the existence of a certain short exact sequence involving the syzygy modules. We should point out that Proposition 2.8 corroborates a result of Herzog and Popescu [29, 2.1] and of Takahashi [45, 2.2], and it is proved at the end of section 4; see also Corollary 4.4.

As our results rely upon Tor-rigidity, in Appendix B, we use Dao's eta function and show that modules that are eventually periodic of odd period are c-Tor-rigid over complete intersection rings (that are quotient of unramified regular local rings) of codimension c.

2. Preliminaries

In this section we record several preliminary definitions and results that are used in the paper.

2.1. Let R be a ring and let M and N be R-modules. If $M\oplus F\cong N\oplus G$ for some free R-modules F and G, then M and N are said to be stably isomorphic. As it does not affect our arguments, we do not separate isomorphic and stably isomorphic modules.

2.2. Let R be a ring and let M be an R-module. Given an integer $n\geq 1$, we denote by $\Omega^n_R M$ the nth *syzygy* of M, namely, the image of the n-th differential map in a minimal free resolution of M. As a convention, we set $\Omega^0_R M=M$ and $\Omega^1_R M=\Omega_R M$.

The *transpose* $\mathsf{Tr}M$ of M is the cokernel of $f^*=\mathrm{Hom}_R(f,R)$, where $F_1\xrightarrow{f}F_0\to M\to 0$ is a part of a minimal free resolution of M; see [3, 12.3].

Note that the transpose and the syzygy of M are uniquely determined up to isomorphism, since so is a minimal free resolution of M.

2.3. Let R be a ring, M be an R-module, and let $\mathfrak{a}$ be an ideal of R. If $\mathfrak{a}M\neq M$, then the $\mathfrak{a}$-*depth* of M (or the *grade* of $\mathfrak{a}$ on M), denoted by $\mathrm{depth}_R(\mathfrak{a},M)$, is defined to be the common length of maximal M-regular sequences in $\mathfrak{a}$; see [7, 1.2.6]. In case $\mathfrak{a}M=M$, then we set $\mathrm{depth}_R(\mathfrak{a},M)=\infty$ (in particular, we have $\mathrm{depth}_R(0)=\infty$). Although we write $\mathrm{depth}_R(\mathfrak{a},R)$ throughout the paper, we note that $\mathrm{depth}_R(\mathfrak{a},R)$ is nothing but the height of the ideal $\mathfrak{a}$ in case R is a Cohen-Macaulay ring. Furthermore, we set $\mathrm{depth}_R(M)=\mathrm{depth}_R(\mathfrak{m},M)$.

The following basic facts play an important role in the proofs of Proposition 2.10 and Theorem 3.3.

(i) $\mathrm{depth}_R(\mathfrak{a},M)=\inf\{\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})\mid \mathfrak{p}\in \mathrm{V}(\mathfrak{a})\}$; see [7, 1.2.10(a)].

(ii) $\mathrm{depth}_R(\mathfrak{a},R)=\inf\{i\in\mathbb{Z}:\mathrm{Ext}^i_R(R/\mathfrak{a},R)\neq 0\}$; see [7, 1.2.10(e)].

(iii) If $\underline{x}\subseteq\mathfrak{a}$ is a regular sequence of length n on M, then $\mathrm{depth}_R(\mathfrak{a},M/\underline{x}M)=\mathrm{depth}_R(\mathfrak{a},M)-n$; see [7, 1.2.10(d)]. □

2.4. Let R be a ring, M be an R-module, and let $n\geq 1$ be an integer. Then M is said to satisfy $(\widetilde{S_n})$ if $\mathrm{depth}_{R_\mathfrak{p}}(M_\mathfrak{p})\geq\min\{n,\mathrm{depth}(R_\mathfrak{p})\}$ for all $\mathfrak{p}\in\mathrm{Supp}_R(M)$. Note that, if R is Cohen-Macaulay, then M satisfies $(\widetilde{S_n})$ if and only if M satisfies *Serre's condition* (S_n); see, for example, [23, page 3]. □

We make use of the following properties in the proof of Proposition 2.8 and Corollary 3.10. Note that, if $n\geq 0$, then $\widetilde{X}^n(R)$ denotes the set of all prime ideals $\mathfrak{p}$ of R such that $\mathrm{depth}(R_\mathfrak{p})\leq n$.

2.5. Let R be a ring, M be a nonzero R-module, and let $n\geq 1$ be an integer.

(i) If $\mathrm{Ext}^i_R(M,R)=0$ for all $i=1,\dots,n$, then it follows that $\Omega^n_R\mathsf{Tr}\Omega^n_R M\cong\mathsf{Tr}M$ and so $\mathsf{Tr}M$ is an nth syzygy module; see [3, 2.17].

(ii) If M is an nth syzygy module, then M satisfies $(\widetilde{S_n})$ so that each R-regular sequence of length at most n is also M-regular; see [3, 4.25] and [39, Prop. 2].

(iii) If M is locally free on $\widetilde{X}^{n-1}(R)$ and M satisfies $(\widetilde{S_n})$, then it follows that $M=\Omega^n_R N$ for some R-module N, where $\mathrm{Ext}^i_R(N,R)=0$ for all $i=1,\dots,n$; see [3, 2.17 and 4.25].

2.6. Let R be a ring and let M be an R-module. The *complexity* $\mathrm{cx}_R(M)$ of M is the smallest integer $r \geq 0$ such that the nth Betti number of M is bounded by a polynomial in n of degree $r-1$ for all $n \geq 0$; see [4, 3.1].

It follows that $\mathrm{cx}_R(M) = 0$ if and only if $\mathrm{pd}_R(M) < \infty$, and $\mathrm{cx}_R(M) \leq 1$ if and only if M has bounded Betti numbers. Moreover, if R is a complete intersection, then $\mathrm{cx}_R(M)$ cannot exceed the codimension of R; see, for example, [5, 5.6].

2.7. Let R be a ring, M be an R-module, and let $n \geq 1$ be an integer. Then M is said to be *n-Tor-rigid* provided that the following condition holds: if $\mathrm{Tor}_i^R(M,N) = 0$ for all $i = t+1, \dots, t+n$ for some R-module N and some integer $t \geq 0$, then it follows that $\mathrm{Tor}_i^R(M,N) = 0$ for all $i \geq t+1$. The $n = 1$ case of this definition is known as the *Tor-rigidity* [2]: M is said to be Tor-rigid if it is 1-Tor-rigid.

Tor-rigidity is a subtle property, but examples of such modules are abundant in the literature. Here we record a few examples and refer the reader to [18] for further details and examples.

(i) ([2, 2.2] and [37, Cor. 1]) If R is regular, then each R-module is Tor-rigid.
(ii) ([33, 2.4] and [37, Thm. 3]) If R is a hypersurface, that is a quotient of an unramified regular local ring, then each R-module that has finite length, or has finite projective dimension, is Tor-rigid.
(iii) ([42, 1.6]) If R is a complete intersection of codimension c, then each R-module is $(c+1)$-Tor-rigid. Therefore, if $c = 1$, then each R-module is 2-Tor-rigid.
(iv) Let R be a complete intersection ring of positive codimension c such that $\widehat{R} = S/(\underline{x})$ for some unramified regular local ring $(S, \mathfrak{n})$ and some S-regular sequence $\underline{x} \subseteq \mathfrak{n}^2$ of length c. Each R-module that has complexity strictly less than c is c-Tor-rigid. Therefore, if $c = 2$, then each R-module that has bounded Betti numbers is 2-Tor-rigid; see [14, 6.8].
(v) ([8, Thm. 5(ii)]) If I is a Burch ideal of R, i.e., if $\mathfrak{m}I \neq \mathfrak{m}(I : \mathfrak{m})$, then R/I is 2-Tor-rigid.
(vi) ([36, page 316]) If M is nonzero such that $\mathrm{depth}_R(M) \geq 1$, then $\mathfrak{m}M$ is 2-Tor-rigid. $\square$

The key ingredient of our argument is the following result; it allows us to tackle the problem on hand by using the Tor-rigidity property; see 2.7.

Proposition 2.8. *Let R be a local ring, N a nonzero R-module, and let $M = \Omega_R^n N$ for some $n \geq 1$. Assume there is an R-regular sequence $\underline{x} = x_1, \dots, x_n$ of length n such that $\underline{x} \cdot \mathrm{Ext}_R^1(N, \Omega_R N) = 0$. Then there is a short exact sequence of R-modules*

$$0 \longrightarrow F \longrightarrow \bigoplus_{i=0}^{n} \left(\Omega_R^{i+n-1} N\right)^{\oplus \binom{n}{i}} \longrightarrow \Omega_R^{n-1}(M/\underline{x}M) \longrightarrow 0, \tag{2.8.1}$$

where F is free. $\square$

The proof of Proposition 2.8 is quite involved, and hence it is deferred to Section 4. Here we record an important consequence of the proposition which is used later in the sequel.

Corollary 2.9. *Let R be a local ring, N a nonzero R-module, and let $M = \Omega_R^n N$ for some $n \geq 1$. Assume the following conditions hold:*

(i) N is $(n+1)$-Tor-rigid.
(ii) $\underline{x} \cdot \mathrm{Ext}_R^1(N, \Omega_R N) = 0$ for some R-regular sequence $\underline{x}$ of length n.

Then it follows that $\Omega_R^{n-1}(M/\underline{x}M)$ is Tor-rigid.

Proof. Note, since N is $(n+1)$-Tor-rigid, it follows that $\bigoplus_{i=0}^{n} \left(\Omega_R^{i+n-1} N\right)^{\oplus \binom{n}{i}}$ is Tor-rigid; see 2.7. Therefore, we conclude by (2.8.1) that $\Omega_R^{n-1}(M/\underline{x}M)$ is Tor-rigid. $\square$

Proposition 2.10. *Let R be a local ring, M and N be R-modules, $\mathfrak{a}$ be a proper ideal of R, and let $n \geq 1$. Assume the following conditions hold:*

(i) M satisfies $(\widetilde{S}_n)$.
(ii) $\operatorname{depth}_R(\mathfrak{a},R) \geq n$.
(iii) N is locally free on $\widetilde{X}^{n-1}(R)$.

Then there is a sequence $\underline{x} \subseteq \mathfrak{a}$ of length n such that $\underline{x} \cdot \operatorname{Ext}^1_R(N,\Omega_R N)=0$, and $\underline{x}$ is both R and M-regular.

Proof. We have, by assumption, that $\operatorname{depth}_R(\mathfrak{a},R)=\inf\{\operatorname{depth}(R_{\mathfrak{p}}) \mid \mathfrak{p} \in \mathrm{V}(\mathfrak{a})\} \geq n$; see 2.3(i). Hence, for each $\mathfrak{q} \in \mathrm{V}(\mathfrak{a})$, it follows that $\operatorname{depth}(R_{\mathfrak{q}}) \geq n$.

Set $\mathfrak{b}=\operatorname{Ann}_R(\operatorname{Ext}^1_R(N,\Omega_R N))$. If $\mathfrak{q} \in \mathrm{V}(\mathfrak{b})$, then we have $\operatorname{depth}(R_{\mathfrak{q}}) \geq n$: otherwise, $\mathfrak{q} \in \widetilde{X}^{n-1}(R)$ and hence $\operatorname{Ext}^1_R(N,\Omega_R N)_{\mathfrak{q}}=0$ since N is locally free on $\widetilde{X}^{n-1}(R)$. Therefore, if $\mathfrak{q} \in \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$, then it follows that $\operatorname{depth}(R_{\mathfrak{q}}) \geq n$. Furthermore, if $\mathfrak{q} \in \mathrm{V}(\mathfrak{a}) \cup \mathrm{V}(\mathfrak{b})$, then we have $\operatorname{depth}_{R_{\mathfrak{q}}}(M_{\mathfrak{q}}) \geq n$ since M satisfies $(\widetilde{S}_n)$ and $\operatorname{depth}(R_{\mathfrak{q}}) \geq n$. Consequently, we use 2.3(i) and [7, 1.2.10(c)], and obtain:

$$\operatorname{depth}_R(\mathfrak{a}\cap\mathfrak{b},M\oplus R)=\inf\{\operatorname{depth}_R(\mathfrak{a},M\oplus R),\operatorname{depth}_R(\mathfrak{b},M\oplus R)\} \geq n. \tag{2.10.1}$$

Now, by using (2.10.1), we can choose a sequence $\underline{x} \subseteq \mathfrak{a}\cap\mathfrak{b} \subseteq \mathfrak{a}$ of length n, as claimed. □

The next result is known for the case where $r=0$; see, for example, [10, 3.4].

Lemma 2.11. Let R be a local ring, A and B be R-modules with $A \neq 0$, and let $m \geq 1$, $r \geq 0$ be integers. Assume $\operatorname{Tr}\Omega^m_R B$ is an rth syzygy module. Assume further $\Omega^r_R A$ is Tor-rigid. If $\operatorname{Ext}^m_R(B,A)=0$, then it follows that $\operatorname{Ext}^m_R(B,R)=0$.

Proof. Assume $\operatorname{Ext}^m_R(B,A)=0$, and consider the four term exact sequence that follows from [3, 2.8(b)]:

$$\operatorname{Tor}_2^R(\operatorname{Tr}\Omega^m_R B,A) \to \operatorname{Ext}^m_R(B,R)\otimes_R A \to \operatorname{Ext}^m_R(B,A) \to \operatorname{Tor}_1^R(\operatorname{Tr}\Omega^m_R B,A) \to 0. \tag{2.11.1}$$

Note that, as $\operatorname{Ext}^m_R(B,A)$ vanishes, so does $\operatorname{Tor}_1^R(\operatorname{Tr}\Omega^m_R B,A)$ by (2.11.1). Also, due to the hypothesis, it follows that $\operatorname{Tr}\Omega^m_R B \cong \Omega^r_R X$ for some R-module X. So, since $\operatorname{Tor}_1^R(\operatorname{Tr}\Omega^m_R B,A) \cong \operatorname{Tor}_1^R(X,\Omega^r_R A)$ and $\Omega^r_R A$ is Tor-rigid, we conclude that $\operatorname{Tor}_2^R(\operatorname{Tr}\Omega^m_R B,A)=0$. Hence, as $A \neq 0$, (2.11.1) implies the claim. □

3. Main result and its corollaries

In this section we prove the main result of this paper, namely Theorem 3.3. Prior to that, we note that Question 1.3 is true in case the ring in question is a complete intersection of codimension c and the integer n considered equals $c-1$; this fact has been explained to us by Shunsuke Takagi.

3.1. Let R be a ring such that $R=S/(\underline{x})$ for some local ring $(S,\mathfrak{n})$ and some S-regular sequence $\underline{x} \subseteq \mathfrak{n}$ of length c. Assume the depth inequality $\operatorname{depth}_S(\mathfrak{b},N) \leq \operatorname{depth}_S(\mathfrak{b},S)$ holds for each ideal $\mathfrak{b}$ of S and for each S-module N. Let M be an R-module and let $\mathfrak{a}$ be an ideal of R. Then $\mathfrak{a}=\mathfrak{b}/(\underline{x})$ for some ideal $\mathfrak{b}$ of S. Now, it follows $\operatorname{depth}_R(\mathfrak{a},M)=\operatorname{depth}_S(\mathfrak{b},M) \leq \operatorname{depth}_S(\mathfrak{b},S)=\operatorname{depth}_S(\mathfrak{b},R)+c=\operatorname{depth}_R(\mathfrak{a},R)+c$. □

Recall that each module is Tor-rigid over a regular local ring; see 2.7(i). Therefore, we obtain:

3.2. Let R be a complete intersection ring of codimension c, M be an R-module, and let $\mathfrak{a}$ be an ideal of R. Then it follows from Theorem 1.1 and 3.1 that $\operatorname{depth}_R(\mathfrak{a},M) \leq \operatorname{depth}_R(\mathfrak{a},R)+c$. Note that this depth inequality is sharp; see Example 1.2. □

Next we state and prove Theorem 3.3. We should note that the case where $n=0$ of the theorem is nothing but Theorem 1.1. In other words, Theorem 3.3 yields an extension of Theorem 1.1.

Theorem 3.3. *Let R be a local ring, N be an R-module, and let $\mathfrak{a}$ be an ideal of R. Set $M=\Omega^n_R N$ for some integer $n \geq 0$ and $m=\operatorname{depth}_R(\mathfrak{a},R)$. Assume the following conditions hold:*

(i) $M \neq 0$ and $m \geq n$.
(ii) N is $(n+1)$-Tor-rigid.

If $n \geq 1$, we further assume:

(iii) N is locally free on $\widetilde{X}^{n-1}(R)$.

(iv) $\mathsf{Tr}\,\Omega_R^m(R/\mathfrak{a})$ *is an* $(n-1)$*st syzygy module.*

Then it follows that $\operatorname{depth}_R(\mathfrak{a},M) \leq m+n$.

Proof. Note that there is nothing to prove if $\mathfrak{a}=0$, or $\mathfrak{a}=R$, or $\operatorname{depth}_R(\mathfrak{a},M) \leq n$; see 2.3. Note also that the case where $n=0$ follows from Theorem 1.1. Hence we may assume $\mathfrak{a}$ is a proper ideal and $\operatorname{depth}_R(\mathfrak{a},M) > n \geq 1$.

As M is an nth syzygy module, we see that M satisfies $(\widetilde{S}_n)$; see 2.5(ii). Therefore, since N is locally free on $\widetilde{X}^{n-1}(R)$ and $\operatorname{depth}_R(\mathfrak{a},R) \geq n$, it follows from Proposition 2.10 that there exists a sequence $\underline{x} \subseteq \mathfrak{a}$ of length n which is both R and M-regular and $\underline{x}\cdot \operatorname{Ext}_R^1(N,\Omega_R(N))=0$. Now, as N is $(n+1)$-Tor-rigid, Corollary 2.9 shows that $\Omega_R^{n-1}(M/\underline{x}M)$ is Tor-rigid.

Let $h=\operatorname{depth}_R(\mathfrak{a},M/\underline{x}M)$ and suppose $h>m$. Then it follows that $\operatorname{Ext}_R^m(R/\mathfrak{a},M/\underline{x}M)=0$; see 2.3(ii). Now, letting $A=M/\underline{x}M$, $B=R/\mathfrak{a}$ and $r=n-1$, we conclude from Lemma 2.11 that $\operatorname{Ext}_R^m(R/\mathfrak{a},R)=0$. This yields a contradiction since $m=\operatorname{depth}_R(\mathfrak{a},R)$; see 2.3(ii). Therefore, we have that $h \leq m$. This establishes the required inequality since $h=\operatorname{depth}_R(\mathfrak{a},M)-n$; see 2.3(iii). □

Next we proceed to obtain several consequences of Theorem 3.3. First we separate the case where $n=1$, which is nothing but Theorem 1.4 advertised in the introduction:

Corollary 3.4. *Let R be a local ring, and let* $\mathfrak{a}$ *be an ideal of R such that* $\operatorname{depth}_R(\mathfrak{a},R) \geq 1$. *Set* $M=\Omega_R N$ *for some R-module N, where N is* 2*-Tor-rigid and generically free. If* $M \neq 0$*, then it follows that* $\operatorname{depth}_R(\mathfrak{a},M) \leq \operatorname{depth}_R(\mathfrak{a},R)+1$. □

Corollary 3.5. *Let R be a local complete intersection ring of codimension c such that* $\widehat{R}=S/(\underline{x})$ *for some unramified regular ring* $(S,\mathfrak{n})$ *and some S-regular sequence* $\underline{x} \subseteq \mathfrak{n}^2$ *of length c, where* $c \leq 2$. *Let M be a nonzero R-module, and let* $\mathfrak{a}$ *be an ideal of R. Assume M is generically free and torsion-free. Assume further M has bounded Betti numbers. Then it follows that* $\operatorname{depth}_R(\mathfrak{a},M) \leq \operatorname{depth}_R(\mathfrak{a},R)+1$.

Proof. Note that, as R is Cohen-Macaulay, M is generically free and torsion-free, we have that $M \cong \Omega_R N$ for some R-module N. Since M has bounded Betti numbers, so does N. Hence it follows that N is 2-Tor-rigid; see 2.7(iv). Furthermore, N is generically free because M is generically free. Thus the result follows from Corollary 3.4. □

Corollary 3.6. *Let R be a local ring and let* $\mathfrak{a}$ *be an ideal of R such that* $\operatorname{depth}_R(\mathfrak{a},R) \geq 1$. *Let N be a nonzero R-module such that N is generically free and* $\operatorname{depth}_R(N) \geq 1$. *If* $M=\Omega_R(\mathfrak{m}N) \neq 0$*, then it follows that* $\operatorname{depth}_R(\mathfrak{a},M) \leq \operatorname{depth}_R(\mathfrak{a},R)+1$.

Proof. Note that we may assume R is not Artinian. Hence, $\mathfrak{m}N$ is generically free. Moreover, $\mathfrak{m}N$ is 2-Tor-rigid; see 2.7(iv). Therefore, the claim follows from Corollary 3.4. □

Corollary 3.7. *Let R be a local ring,* $\mathfrak{a}$ *be an ideal of R and let* $\mathfrak{b}$ *is a Burch ideal of R. Assume* $\operatorname{depth}_R(\mathfrak{a},R) \geq 1$ *and* $\operatorname{depth}_R(\mathfrak{b},R) \geq 1$. *Then it follows that* $\operatorname{depth}_R(\mathfrak{a},\mathfrak{b}) \leq \operatorname{depth}_R(\mathfrak{a},R)+1$.

Proof. Note that $\mathfrak{b}=\Omega_R N$, where $N=R/\mathfrak{b}$ is 2-Tor-rigid; see 2.7(v). Moreover, N is generically free since $\operatorname{depth}_R(\mathfrak{b},R) \geq 1$; see 2.3(i). Hence, the result follows from Corollary 3.4. □

It is known that integrally closed ideals are Burch over local rings that have positive depth; see [19, 2.2 (3) and (4)]. Therefore, Corollary 3.7 yields:

Corollary 3.8. *Let R be a local ring, and let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be ideals of R. Assume* $\operatorname{depth}_R(\mathfrak{a},R) \geq 1$ *and* $\operatorname{depth}_R(\mathfrak{b},R) \geq 1$. *If* $\mathfrak{b}$ *is integrally closed, then it follows that* $\operatorname{depth}_R(\mathfrak{a},\mathfrak{b}) \leq \operatorname{depth}_R(\mathfrak{a},R)+1$. □

In the following corollaries, we show that condition (iv) of Theorem 3.3 holds if $\mathfrak{a}$ is a Cohen-Macaulay ideal, i.e., $R/\mathfrak{a}$ is a Cohen-Macaulay ring.

Corollary 3.9. *Let R be a Gorenstein local ring, N be an R-module, and let* $\mathfrak{a}$ *be an ideal of R. Set* $M=\Omega_R^n N$ *for some integer* $n \geq 1$ *and* $m=\operatorname{depth}_R(\mathfrak{a},R)$. *Assume the following conditions hold:*

(i) $\mathfrak{a}$ *is a Cohen-Macaulay ideal.*
(ii) $M \neq 0$ *and* $m \geq n$.
(iii) N *is locally free on* $\widetilde{X}^{n-1}(R)$.
(iv) N *is* $(n+1)$*-Tor-rigid.*

Then it follows that $\operatorname{depth}_R(\mathfrak{a}, M) \leq m+n$.

Proof. Note that, as $R/\mathfrak{a}$ is a Cohen-Macaulay ring, it follows $\operatorname{depth}(R/\mathfrak{a}) = \dim(R) - m$, and also $\operatorname{Ext}^i_R(R/\mathfrak{a}, R) = 0$ for $i \neq m$ by the local duality theorem; see [7, 3.5.8]. Therefore, $\operatorname{Tr}\Omega^m_R(R/\mathfrak{a})$ is an $(n-1)$st syzygy module since $\operatorname{Ext}^i_R(R/\mathfrak{a}, R) = 0$ for all $i = m+1, \ldots, m+n-1$; see 2.5(i). Now, since all the hypotheses of Theorem 3.3 hold, the required depth inequality follows from Theorem 3.3. □

The next corollary corroborates Corollary 3.5:

Corollary 3.10. *Let R be a local complete intersection ring of codimension c such that $\widehat{R} = S/(\underline{x})$ for some unramified regular ring $(S, \mathfrak{n})$ and some S-regular sequence $\underline{x} \subseteq \mathfrak{n}^2$ of length c, where $c \geq 2$. Let M be a nonzero R-module, and let $\mathfrak{a}$ be an ideal of R. Assume the following hold:*

(i) $\mathfrak{a}$ *is a Cohen-Macaulay ideal such that* $\operatorname{depth}(\mathfrak{a}, R) \geq c-1$.
(ii) $\operatorname{cx}_R(M) < c$.
(iii) M *satisfies* $(\widetilde{S}_{c-1})$.
(iv) M *is locally free on* $\widetilde{X}^{c-2}(R)$.

Then it follows that $\operatorname{depth}_R(\mathfrak{a}, M) \leq \operatorname{depth}_R(\mathfrak{a}, R) + c - 1$.

Proof. Note that, by 2.5(iii), we have $M = \Omega^{c-1}_R N$ for some R-module N, where $\operatorname{Ext}^i_R(N, R) = 0$ for all $i = 1, \ldots, c-1$. Let $\mathfrak{p} \in \widetilde{X}^{c-2}(R)$. Then, since M is locally free on $\widetilde{X}^{c-2}(R)$, it follows $\operatorname{pd}_{R_\mathfrak{p}}(N_\mathfrak{p}) \leq c-1$. As $\operatorname{Ext}^i_R(N, R) = 0$ for all $i = 1, \ldots, c-1$, we conclude that $N_\mathfrak{p}$ is free. This shows that N is locally free on $\widetilde{X}^{c-2}(R)$. Furthemore, as $\operatorname{cx}_R(N) = \operatorname{cx}_R(M) < c$, it follows that N is c-Tor-rigid; see 2.7(iv). Hence the result follows from Corollary 3.9 by setting $n = c-1$. □

Remark 3.11. Let us note that, if $c = 2$ in Corollary 3.10, then the Cohen-Macaulay assumption on the ideal $\mathfrak{a}$ is not needed due to Corollary 3.5. Moreover, the assumption $\operatorname{cx}_R(M) < c$ in Corollary 3.10 implies the vanishing of the eta function if R is an isolated singularity; in this case M would be a c-Tor-rigid module; see [14, 6.3 and 6.8]. In the appendix we recall the definition of the eta function and discuss some of its applications that are related to our results.

4. PROOF OF PROPOSITION 2.8

In this section we prove Proposition 2.8. For its proof we need some basic facts which we recall next for the convenience of the reader; see, for example, [38, 1.2, 1.4 and 3.2].

4.1. Let R be a ring, $x \in R$ and let A, B and C be R-modules. Set $\sigma = (0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0) \in \operatorname{Ext}^1_R(C, A)$.

(i) The connecting homomorphism $\operatorname{Hom}_R(C, C) \to \operatorname{Ext}^1_R(C, A)$ is given by the rule $\gamma \mapsto E$, where $E = (0 \to A \to Z \to C \to 0)$ is the short exact sequence obtained by the following pull-back diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] & C \arrow[r] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & Z \arrow[r] \arrow[u] \arrow[ur, phantom, "\text{PB}"] & C \arrow[r] \arrow[u, "\gamma"'] & 0
\end{tikzcd}$$

(ii) The multiplication homomorphism $A \xrightarrow{x} A$ induces a homomorphism $\operatorname{Ext}^1_R(C,A) \xrightarrow{x} \operatorname{Ext}^1_R(C,A)$ which sends σ to σ', where $\sigma' = (0 \to A \to W \to C \to 0)$ is the short exact sequence obtained by the following push-out diagram:

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "f"] \arrow[d, "x"'] \arrow[dr, phantom, "\text{PO}"] & B \arrow[r, "g"] \arrow[d] & C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & C \arrow[r] & 0 \\
 & A/xA \arrow[r, equal] \arrow[d] & A/xA \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}$$

Therefore, it follows that $\sigma' \in x \cdot \operatorname{Ext}^1_R(C,A)$.

Moreover, the diagram above induces the following commutative diagram where the leftmost square is a pushout square:

$$\begin{tikzcd}
0 \arrow[r] & \Omega_R A \arrow[r] \arrow[d, "x"'] \arrow[dr, phantom, "\text{PO}"] & \Omega_R B \arrow[r] \arrow[d] & \Omega_R C \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \Omega_R A \arrow[r] & \Omega_R W \arrow[r] & \Omega_R C \arrow[r] & 0
\end{tikzcd}$$

Therefore, it follows that the bottom short exact sequence $0 \to \Omega_R A \to \Omega_R W \to \Omega_R C \to 0$ belongs to $x \cdot \operatorname{Ext}^1_R(\Omega_R C, \Omega_R A)$.

(iii) The multiplication homomorphism $C \xrightarrow{x} C$ induces a homomorphism $\operatorname{Ext}^1_R(C,A) \xrightarrow{x} \operatorname{Ext}^1_R(C,A)$ which sends σ to σ'', where $\sigma'' = (0 \to A \to V \to C \to 0)$ is the short exact sequence obtained by the following pull-back diagram:

$$\begin{tikzcd}
 & & 0 & 0 & \\
 & & C/xC \arrow[u] \arrow[r, equal] & C/xC \arrow[u] & \\
0 \arrow[r] & A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[u] & C \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & V \arrow[r] \arrow[u] \arrow[ur, phantom, "\text{PB}"] & C \arrow[r] \arrow[u, "x"'] & 0
\end{tikzcd}$$

Therefore, it follows that $\sigma'' \in x \cdot \operatorname{Ext}^1_R(C,A)$.

Lemma 4.2. Let R be a ring, $x \in R$ and let N be an R-module. Then the following are equivalent.

(i) The multiplication map $N \xrightarrow{x} N$ factors through a free R-module.
(ii) $x \cdot \operatorname{Ext}^i_R(N, -) = 0$ for each $i \geq 1$.
(iii) $x \cdot \operatorname{Ext}^1_R(N, \Omega_R N) = 0$.

Furthermore, if one of these equivalent conditions holds and x is a non zero-divisor on N, then there is an isomorphism $\Omega_R(N/xN) \cong N \oplus \Omega_R N$.

Proof. Note that the implication (ii) $\Rightarrow$ (iii) is trivial. Hence we show (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i).

To establish (i) $\Rightarrow$ (ii), we assume $N \xrightarrow{x} N$ factors through a free R-module F, i.e., there exist R-module homomorphisms f and g such that $N \xrightarrow{f} F \xrightarrow{g} N$, where $gf = x \cdot 1_N$. Now let X be an R-module and $n \geq 1$ be an integer. Then f and g induce R-module homomorphisms f^* and g^* such that $\mathrm{Ext}^n_R(N,X) \xrightarrow{g^*} \mathrm{Ext}^n_R(F,X) \xrightarrow{f^*} \mathrm{Ext}^n_R(N,X)$, where $f^*g^* = x \cdot 1_{\mathrm{Ext}^n_R(N,X)}$. As $\mathrm{Ext}^n_R(F,X)$ vanishes, we conclude that $f^*g^* = 0$, i.e., $x \cdot \mathrm{Ext}^n_R(N,X) = 0$. This proves the implication (i) $\Rightarrow$ (ii).

Next consider the syzygy exact sequence $E = (0 \to \Omega_R N \to G \xrightarrow{p} N \to 0)$, where G is free. This induces the exact sequence $0 \to \mathrm{Hom}_R(N, \Omega_R N) \to \mathrm{Hom}_R(N,G) \xrightarrow{p_*} \mathrm{Hom}_R(N,N) \to \mathrm{Ext}^1_R(N, \Omega_R N)$. Note that $1_N \mapsto E$ under the connecting homomorphism $\mathrm{Hom}_R(N,N) \to \mathrm{Ext}^1_R(N, \Omega_R N)$; see 4.1(i). So the image of the map $N \xrightarrow{x} N$ under the connecting homomorphism belongs to $x \cdot \mathrm{Ext}^1_R(N, \Omega_R N)$.

Now assume $x \cdot \mathrm{Ext}^1_R(N, \Omega_R N) = 0$. Then the multiplication map $N \xrightarrow{x} N$ is in $\mathrm{im}(p_*)$, and hence it factors through the free module G. Consequently, (iii) $\Rightarrow$ (i) follows.

Next assume x is a non zero-divisor on N. Then we consider the multiplication map $N \xrightarrow{x} N$ and make use of 4.1(iii) with the exact sequence E, and obtain short exact sequences of R-modules:

$$E_1 = (0 \to V \to G \to N/xN \to 0) \text{ and } E_2 = (0 \to \Omega_R N \to V \to N \to 0) \in x \cdot \mathrm{Ext}^1_R(N, \Omega_R N) = 0.$$

Now E_2 splits so that E_1 yields the isomorphism $\Omega_R(N/xN) \cong V \cong N \oplus \Omega_R N$, as required. □

Next we use Lemma 4.2 and give a proof of Proposition 2.8. We also need the following fact:

4.3. Let R be a local ring and let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then there is a short exact sequence $0 \to \Omega_R C \to A \oplus H \to B \to 0$, where H is a free R-module; see, for example, [21, 2.2]. Therefore, if A is free, then $\Omega_R C \cong \Omega_R B$.

Proof of Proposition 2.8. Note that, since $\underline{x}$ is R-regular and M is an nth syzygy module, we see that $\underline{x}$ is also M-regular; see 2.5(ii). We proceed by induction on n. First assume $n = 1$.

As in the proof of Lemma 4.2, we look at the syzygy exact sequence $E = (0 \to \Omega_R N \to F \to N \to 0)$, where F is free. Then, by using the multiplication map $M \xrightarrow{x_1} M$ and 4.1(ii), we obtain short exact sequences of R-modules of the form

$$E_1 = (0 \to F \to W \to M/xM \to 0) \text{ and } E_2 = (0 \to \Omega_R N \to W \to N \to 0) \in x_1 \cdot \mathrm{Ext}^1_R(N, \Omega_R N) = 0.$$

Now E_2 splits, and hence E_1 yields the required short exact sequence.

Next we assume $n > 1$, and set $N' = N \oplus \Omega_R N$, $M' = \Omega^{n-1}_R N' \cong \Omega^{n-1}_R N \oplus \Omega^n_R N$, and $\underline{x}' = x_1, \dots, x_{n-1}$. Note that it follows:

$$\mathrm{Ext}^1_R(N', \Omega_R N') = \mathrm{Ext}^1_R(N, \Omega_R N) \oplus \mathrm{Ext}^1_R(N, \Omega^2_R N) \oplus \mathrm{Ext}^2_R(N, \Omega_R N) \oplus \mathrm{Ext}^2_R(N, \Omega^2_R N). \tag{2.8.2}$$

As $\underline{x} \cdot \mathrm{Ext}^1_R(N, \Omega_R N) = 0$, we see from Lemma 4.2 that $\underline{x} \cdot \mathrm{Ext}^i_R(N, -) = 0$ for all $i \geq 1$. Therefore, by (2.8.2), we conclude that $\underline{x}$, and hence $\underline{x}'$ annihilates the module $\mathrm{Ext}^1_R(N', \Omega_R N')$. Thus the following short exact sequence exists due to the induction hypothesis:

$$0 \to F' \to \bigoplus_{i=0}^{n-1} \Omega^{i+n-2}_R(N')^{\oplus \binom{n-1}{i}} \to \Omega^{n-2}_R(M'/\underline{x}'M') \to 0, \tag{2.8.3}$$

where F' is a free R-module. Furthermore, as $M' = \Omega^{n-1}_R N'$, we use 4.3 along with (2.8.3) and obtain:

$$\Omega^{n-1}_R(M'/\underline{x}'M') \cong \bigoplus_{i=0}^{n-1} \Omega^i_R(M')^{\oplus \binom{n-1}{i}} \tag{2.8.4}$$

Recall that $M = \Omega^n_R N$. Hence there is a short exact sequence $0 \to M \to F \to \Omega^{n-1}_R N \to 0$ for some free R-module F. It follows, since $\underline{x}'$ is R-regular, that $\underline{x}'$ is $\Omega^{n-1}_R N$-regular; see 2.5(ii). So we have a short exact sequence of the form:

$$0 \to M/\underline{x}'M \xrightarrow{\alpha} F/\underline{x}'F \to \Omega^{n-1}_R N/\underline{x}'\Omega^{n-1}_R N \to 0.$$

We take the pushout of α and the injective map $M/\underline{x}'M \xrightarrow{x_n} M/\underline{x}'M$, and obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & M/\underline{x}'M \arrow[r, "\alpha"] \arrow[d, "x_n"'] \arrow[dr, phantom, "\mathrm{PO}"] & F/\underline{x}'F \arrow[r] \arrow[d] & \Omega_R^{n-1}(N)/\underline{x}'\Omega_R^{n-1}(N) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & M/\underline{x}'M \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & \Omega_R^{n-1}(N)/\underline{x}'\Omega_R^{n-1}(N) \arrow[r] & 0 \\
 & M/\underline{x}M \arrow[r, equal] \arrow[d] & M/\underline{x}M \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd} \tag{2.8.5}$$

Note that the short exact sequence $0 \to \Omega_R^{n-1}(M/\underline{x}'M) \to \Omega_R^{n-1}W \to \Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right) \to 0$ belongs to $x_n \cdot \mathrm{Ext}^1(\Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right), \Omega_R^{n-1}(M/\underline{x}'M))$; see (2.8.5) and 4.1(ii).

Next note that we have the following isomorphisms:

$$\mathrm{Ext}_R^1(\Omega_R^{n-1}(M'/\underline{x}'M'), -) \cong \bigoplus_{i=0}^{n-1} \mathrm{Ext}_R^{i+1}(M', -)^{\oplus \binom{n-1}{i}} \cong \bigoplus_{i=n}^{2n} \mathrm{Ext}_R^{i}(N, -)^{\oplus r(i)}, \tag{2.8.6}$$

where $r(i)$ is a positive integer depending on i. The first isomorphism in (2.8.6) is due to (2.8.4), while the second one follows from the fact that $M' \cong \Omega_R^{n-1}N \oplus \Omega_R^{n}N$.

Recall that $\Omega_R^{n-1}N$ is a direct summand of M'. Therefore, $\Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right)$ is a direct summand of $\Omega_R^{n-1}(M'/\underline{x}'M')$. This implies, in view of (2.8.6), that $\mathrm{Ext}^1(\Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right), -)$ is a direct summand of $\bigoplus_{i=n}^{2n} \mathrm{Ext}_R^i(N,-)^{\oplus r(i)}$. It follows, since $\underline{x} \cdot \mathrm{Ext}_R^i(N,-) = 0$ for all $i \geq 1$, that x_n annihilates each direct summand of $\mathrm{Ext}_R^i(N,-)$ for each $i \geq 1$; in particular, we conclude that $x_n \cdot \mathrm{Ext}_R^1(\Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right), \Omega_R^{n-1}(M/\underline{x}'M)) = 0$. This implies that the bottom short exact sequence in (2.8.5) splits so that we have the following isomorphism:

$$\Omega_R^{n-1}W \cong \Omega_R^{n-1}(M/\underline{x}'M) \oplus \Omega_R^{n-1}\left(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\right). \tag{2.8.7}$$

Recall that, by (2.8.5), we have a short exact sequence $0 \to F/\underline{x}'F \to W \to M/\underline{x}M \to 0$. Hence, by taking syzygy and using (2.8.7), we obtain the exact sequence:

$$0 \to \Omega_R^{n-1}(F/\underline{x}'F) \to \Omega_R^{n-1}(M/\underline{x}'M) \oplus \Omega_R^{n-1}(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N) \to \Omega_R^{n-1}(M/\underline{x}M) \to 0. \tag{2.8.8}$$

The minimal free resolution $F_\bullet$ of $F/\underline{x}'F$ is of the form $0 \to F \to F^{\oplus n-1} \to \cdots \to F^{\oplus n-1} \to F \to 0$ since $\mathrm{H}_i(F_\bullet \otimes_R \mathrm{K}(\underline{x}';R)) = \mathrm{Tor}_i^R(F, R/\underline{x}'R) = 0$ for all $i \geq 0$, where $\mathrm{K}(\underline{x}';R)$ is the Koszul complex of R with respect to $\underline{x}'$. Therefore, it follows that:

$$\Omega_R^{n-1}(F/\underline{x}'F) \cong F. \tag{2.8.9}$$

We have the following isomorphisms about the middle module in the short exact sequence (2.8.8):

$$
\begin{aligned}
\Omega_R^{n-1}(M/\underline{x}'M) \oplus \Omega_R^{n-1}\big(\Omega_R^{n-1}N/\underline{x}'\Omega_R^{n-1}N\big) &\cong \Omega_R^{n-1}(M'/\underline{x}'M') \\
&\cong \bigoplus_{i=0}^{n-1} \Omega_R^{i}(M')^{\oplus\binom{n-1}{i}} \\
&\cong \bigoplus_{i=0}^{n-1} \big(\Omega_R^{i+n-1}N \oplus \Omega_R^{i+n}N\big)^{\oplus\binom{n-1}{i}} \\
&\cong \left[\bigoplus_{i=0}^{n-1} \big(\Omega_R^{i+n-1}N\big)^{\oplus\binom{n-1}{i}}\right] \bigoplus \left[\bigoplus_{i=1}^{n} \big(\Omega_R^{i+n-1}N\big)^{\oplus\binom{n-1}{i-1}}\right] \qquad (2.8.10)\\
&\cong \left[\big(\Omega_R^{n-1}N\big)^{\oplus\binom{n}{0}}\right] \bigoplus \left[\bigoplus_{i=1}^{n-1} \big(\Omega_R^{i+n-1}N\big)^{\oplus\binom{n}{i}}\right] \bigoplus \left[\big(\Omega_R^{2n-1}N\big)^{\oplus\binom{n}{n}}\right] \\
&\cong \bigoplus_{i=0}^{n} \big(\Omega_R^{i+n-1}N\big)^{\oplus\binom{n}{i}}.
\end{aligned}
$$

In (2.8.10), the first and the third isomorphisms follow since $M' \cong \Omega_R^{n-1}N \oplus \Omega_R^{n}N = \Omega_R^{n-1}N \oplus M$, while the second isomorphism is nothing but (2.8.4). The other isomorphisms are elementary.

Now, in view of (2.8.9) and (2.8.10), we conclude that the short exact sequence in (2.8.8) is the required one. This completes the induction argument and hence the proof of the proposition. □

We end this section with a consequence of Proposition 2.8 which corroborates [29, 2.1] and [45, 2.2].

Corollary 4.4. *Let R be a local ring, N a nonzero R-module, and let $M = \Omega_R^n N$ for some $n \geq 1$. Assume there is an R-regular sequence $\underline{x} = x_1, \ldots, x_n$ of length n such that $\underline{x} \cdot \operatorname{Ext}_R^1(N, \Omega_R N) = 0$. Then the following isomorphism holds:*

$$
\Omega_R^n(M/\underline{x}M) \cong \bigoplus_{i=0}^{n} \Omega_R^i(M)^{\oplus\binom{n}{i}} \qquad (4.4.1)
$$

Proof. It follows from Proposition 2.8 that we have the following short exact sequence:

$$
0 \longrightarrow F \longrightarrow \bigoplus_{i=0}^{n} \left(\Omega_R^{i+n-1}N\right)^{\oplus\binom{n}{i}} \longrightarrow \Omega_R^{n-1}(M/\underline{x}M) \longrightarrow 0,
$$

where $M = \Omega_R^n N$. Therefore 4.3 yields the short exact sequence

$$
0 \longrightarrow \Omega_R\big(\Omega_R^{n-1}(M/\underline{x}M)\big) \longrightarrow F \oplus G \longrightarrow \bigoplus_{i=0}^{n} \left(\Omega_R^{i+n-1}N\right)^{\oplus\binom{n}{i}} \longrightarrow 0,
$$

where G is a free R-module. Hence, we conclude that:

$$
\Omega_R^n(M/\underline{x}M) \cong \Omega_R\left(\bigoplus_{i=0}^{n}\left(\Omega_R^{i+n-1}N\right)\right)^{\oplus\binom{n}{i}} \cong \bigoplus_{i=0}^{n}\left(\Omega_R^{i+n}N\right)^{\oplus\binom{n}{i}} \cong \bigoplus_{i=0}^{n} \Omega_R^i(M)^{\oplus\binom{n}{i}}.
$$

□

Appendix A. On Tor-rigid modules over complete intersection rings

It is known that a module of finite projective dimension over a local ring is not necessarily Tor-rigid; see [27]. On the other hand, if the ring considered is a hypersurface that is quotient of an unramified regular local ring, then each R-module that has finite projective dimension turns out to be Tor-rigid; see 2.7(ii). This result was generalized by Dao by using the theta function $\theta^R(-,-)$; more precisely, Dao proved that, if R is a hypersurface as before and M and N are R-modules such that $\theta^R(M,N)=0$, then the pair (M,N) is Tor-rigid, where $\theta^R(M,N)=\operatorname{length}_R(\operatorname{Tor}_{2n}^R(M,N))-\operatorname{length}_R(\operatorname{Tor}_{2n-1}^R(M,N))$ for $n\gg 0$; see [15, 2.1] for the details. It should be noted that this function has been initially defined by Hochster [32] to study the direct summand conjecture, which is now a theorem [1].

The theta function is a natural extension of Serre's intersection multiplicity and it has been proved to be a very useful tool to study Tor-rigidity and other subtle problems. For example, Gabber [24] conjectured that the Picard group of the punctured spectrum of a complete intersection ring of dimension three is torsion-free; see also [13]. Dao [16] proved Gabber's aforementioned conjecture for the hypersurface case by using techniques that rely upon the usage of the theta function. More on the history, conjectures, applications, and results concerning the theta function can be found in the survey article [18] and also in [11, 17, 20, 40, 41, 46].

In this section we obtain another generalization of the fact that modules of finite projective dimension are Tor-rigid over hypersurfaces. We observe that modules that are eventually periodic of odd period are Tor-rigid over hypersurfaces that are quotient of unramified regular local rings. In fact, we show that such periodic modules are c-Tor-rigid over complete intersections of codimension c; see A.4. In particular, we conclude that modules that are eventually periodic of odd period satisfy the depth inequality of Theorem 1.1; see A.5.

The main tool we use in this section is the eta function $\eta^R(-,-)$ introduced by Dao [14]. We recall its definition next but first let us note that the eta function is an extension of the theta function discussed previously. In fact, the eta function equals, under some mild conditions, to Serre's intersection multiplicity over regular rings, to two times the theta function over hypersurface rings, and to a constant factor of a notion of Gulliksen over complete intersection rings; see [14, 4.4] for the details. We also refer the interested reader to [9] for a function which is defined in terms of the Ext functor and which is analogous to the eta function.

Throughout, R denotes a local complete intersection ring such that the $\mathfrak{m}$-adic completion $\widehat{R}$ of R is of the form $S/(\underline{x})$ for some unramified (or equi-characteristic) regular ring $(S,\mathfrak{n})$ and some S-regular sequence $\underline{x}\subseteq\mathfrak{n}^2$ of length c, where $c\geq 1$. Note that this setup does not necessarily imply that R itself can be expressed as such a quotient; see [28].

A.1. ([14, 4.2, 4.3(1), 5.4]; see also [9, 3.3]) Let M and N be R-modules such that $\operatorname{Tor}_i^R(M,N)$ has finite length for all $i\gg 0$. Set $f=\inf\{s:\operatorname{length}_R(\operatorname{Tor}_i^R(M,N))<\infty \text{ for all } i\geq s\}$. Then the *eta function* $\eta^R(M,N)$ is defined as follows:

$$\eta^R(M,N)=\lim_{n\to\infty}\frac{\sum_{i=f}^{n}(-1)^i\operatorname{length}_R(\operatorname{Tor}_i^R(M,N))}{n^c}$$

In the following we collect some properties of the eta function:

A.2. Let M and N be R-modules.

(i) If $\eta^R(M,N)=0$, then the pair (M,N) is c-Tor-rigid; see 2.7 and [14, 6.3]. For example, if $c=1$ and R is a simple hypersurface singularity of even dimension, then it follows that $\eta^R(M,N)=0$ for all R-modules M and N so that each module is Tor-rigid over R; see [14, 4.4] and [17, 3.16].

(ii) The eta function is additive whenever it is defined. Namely, if $0\to M'\to M\to M''\to 0$ is a short exact sequence of R-modules such that $\operatorname{Tor}_i^R(M',N)$ and $\operatorname{Tor}_i^R(M'',N)$ have finite length for all $i\gg 0$, then it follows that $\eta^R(M,N)=\eta^R(M',N)+\eta^R(M'',N)$; see [14, 4.3(2)]. □

We proceed to observe that modules that are eventually periodic of odd period are c-Tor-rigid over R. Note that this property is not true for eventually periodic modules of even period; when $c=1$, modules over R are eventually periodic of period two [22], but they are not necessarily Tor-rigid, in general.

A.3. Let N be an R-module such that N is eventually periodic of odd period, i.e., $\Omega_R^n N \cong \Omega_R^{n+q} N$ for some odd integer q and for all $n \gg 0$. If X is an R-module and $\operatorname{Tor}_i^R(N,X)$ has finite length for all $i \gg 0$, then the pair (N,X) is c-Tor-rigid over R.

To see this, first note that $\eta^R(N,X)$ is well-defined; see A.2. Moreover, for $n \gg 0$, the following equalities hold:

$$\begin{aligned}\eta^R(N,X) &= (-1)^n\, \eta^R(\Omega_R^n N, X)\\ &= (-1)^n\, \eta^R(\Omega_R^{n+q} N, X)\\ &= (-1)^n (-1)^{n+q}\, \eta^R(N,X)\\ &= -\,\eta^R(N,X).\end{aligned}$$

Here, the first and third equalities are due to A.2(ii), while the second one follows by the hypothesis. Consequently, we conclude $\eta^R(N,X)=0$, and this implies that the pair (N,X) is c-Tor-rigid; see A.2(i).

A.4. Let N be an R-module such that $\Omega_R^n N \cong \Omega_R^{n+q} N$ for some odd integer q and for all $n \gg 0$. Then it follows that N is c-Tor-rigid.

To see this, let X be an R-module with $\operatorname{Tor}_1^R(N,X) = \cdots = \operatorname{Tor}_c^R(N,X) = 0$. We set $r = \dim_R(N \otimes_R X)$ and proceed by induction on r to show that $\operatorname{Tor}_i^R(N,X) = 0$ for all $i \geq 1$.

If $r \leq 0$, then the claim follows from A.3. So we assume $r \geq 1$, and pick $\mathfrak{p} \in \operatorname{Supp}_R(N \otimes_R X)$ such that $\mathfrak{p} \neq \mathfrak{m}$. Note that $\Omega_{R_\mathfrak{p}}^n N_\mathfrak{p} \cong \Omega_{R_\mathfrak{p}}^{n+q} N_\mathfrak{p}$ for all $n \gg 0$. Then it follows by the induction hypothesis that $\operatorname{Tor}_i^R(N,X)_\mathfrak{p} = 0$ for all $i \geq 1$. This shows that $\operatorname{Tor}_i^R(N,X)$ has finite length for all $i \geq 1$. Hence, by A.3, the pair (N,X) is c-Tor-rigid over R. Thus, as $\operatorname{Tor}_1^R(N,X) = \cdots = \operatorname{Tor}_c^R(N,X) = 0$, we conclude that $\operatorname{Tor}_i^R(N,X)$ vanishes for each $i \geq 1$, as claimed.

A.5. Let R be a hypersurface ring, $\mathfrak{a}$ be an ideal of R, and let N be an R-module.

(i) Let N be an R-module which is eventually periodic of odd period. Then N is necessarily eventually periodic of period one as it is already eventually periodic of period two. Then it follows that N is Tor-rigid and hence $\operatorname{depth}_R(\mathfrak{a},N) \leq \operatorname{depth}_R(\mathfrak{a},R)$; see Theorem 1.1 and A.4.

(ii) If $\Omega_R N \cong M \oplus \Omega_R M$ for some R-module M, then it follows from part (i) that N is Tor-rigid over R and hence $\operatorname{depth}_R(\mathfrak{a},N) \leq \operatorname{depth}_R(\mathfrak{a},R)$: this is because M is eventually periodic of period at most two [22] and hence N is eventually periodic of period one. □

If R is hypersurface, then it is clear that modules of the form $M \oplus \Omega_R M$ are Tor-rigid over R; see 2.7(ii). On the other hand, the fact that modules as in A.5(ii) are Tor-rigid over R seems interesting to us since a module over a hypersurface ring need not be Tor-rigid in general, even if its syzygy module is Tor-rigid.

Appendix B. An example about Tor-rigidity

In this section we give an example of a ring and modules that do not satisfy the hypotheses of Theorem 1.4. Let us note that the ring we construct is a four-dimensional local domain that is not Cohen-Macaulay.

Example B.1. Let k be an algebraically closed field, $R = k[x_1,x_2,x_3]/(x_1^3+x_2^3+x_3^3)$, and let $S = k[y_1,y_2,y_3]$. Then R and S are standard graded rings of dimension 2 and 3, respectively. Moreover, both R and S are Cohen-Macaulay.

Let $T = R\#S = \bigoplus_{n\geq 0} R_n \otimes S_n$, the Segre product of R and S, which is a subring of $R \otimes_k S$. Then T is a graded normal domain; see [25, Remark 4.0.3(v)]. Set $M = R(1)\#S$, where $R(1)$ is the graded shift of R by one, that is, $R(1)_n = R_{n+1}$ for each $n \geq 0$. Then it follows that $\dim(T) = 4$ and $\operatorname{depth}(T) = 2$; see [25, 4.1.5 and 4.2.3] (note depth is computed here for the graded ring by using local cohomology in view of the fact that the a-invariants of R and S are 0 and -3, respectively). Then we see from [25, 4.4.13] that M is a small (that is, finitely generated) maximal Cohen-Macaulay T-module, i.e. $\operatorname{depth}_T(M) = 4$.

Set $f = x_1^3 + x_2^3 + x_3^3 \in k[x_1, x_2, x_3]$. Then one can check that

$$\begin{aligned} T &\cong \Big(k[x_1,x_2,x_3]\# k[y_1,y_2,y_3]\Big)/(f \otimes k[y_1,y_2,y_3]_3) \\ &\cong k[x_iy_j \mid 1 \le i,j \le 3]/(fy_1^3, fy_1^2y_2, fy_1^2y_3, fy_1y_2^2, fy_1y_2y_3, fy_1y_3^2, fy_2^3, fy_2^2y_3, fy_2y_3^2, fy_3^3) \end{aligned}$$

Note, as $R(1) = Rx_1 + Rx_2 + Rx_3$, it follows that $M = Tx_1 + Tx_2 + Tx_3$. So $y_1M = Ty_1x_1 + Ty_1x_2 + Ty_1x_3$, and hence there is an injective map $M \to T$ given by multiplication by y_1 as y_1x_1, y_1x_2, and y_1x_3 are elements of T. This implies that $M \cong y_1M$ and M is isomorphic to an ideal of T. Therefore M is a torsion-free T-module so that $M \cong \Omega_T(N)$ for some T-module N.

Next we consider the complete local ring $\widehat{T}$, which is obtained by taking the completion of T at its graded maximal ideal T_+. Note that $\dim(\widehat{T}) = 4$, $\operatorname{depth}(\widehat{T}) = 2$, $\widehat{M} \cong \Omega_{\widehat{T}}(\widehat{N})$, and $\widehat{M}$ is a small maximal Cohen-Macaulay $\widehat{T}$-module. Note also that, since M is generically free over T, we deduce that $\widehat{M}$ is generically free over $\widehat{T}$. Moreover, if $\mathfrak{a}$ is the maximal ideal of $\widehat{T}$, then we have

$$\operatorname{depth}_{\widehat{T}}(\mathfrak{a}, \widehat{M}) = 4 > 3 = \operatorname{depth}_{\widehat{T}}(\mathfrak{a}, \widehat{T}) + 1.$$

Therefore we conclude from Theorem 1.1 that $\widehat{M}$ is not Tor-rigid over $\widehat{T}$, and conclude from Theorem 1.4 that $\widehat{N}$ is not 2-Tor-rigid over $\widehat{T}$; see also Theorem 3.3.

Next we give an explicit description of T, M and N.

Note that $k[x_iy_j \mid 1 \le i,j \le 3] \cong k[z_{ij} \mid 1 \le i,j \le 3]/I$. Here the isomorphism is given by $x_iy_j \leftrightarrow z_{ij}$, and I is the ideal generated by 2-minors of the matrix $\begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{pmatrix}$. Therefore

$$T \cong k[z_{ij} \mid 1 \le i,j \le 3]/J,$$

where J is the ideal generated by

$$\begin{gathered} z_{11}^3 + z_{21}^3 + z_{31}^3,\ z_{11}^2z_{12} + z_{21}^2z_{22} + z_{31}^2z_{32},\ z_{11}^2z_{13} + z_{21}^2z_{23} + z_{31}^2z_{33},\ z_{11}z_{12}^2 + z_{21}z_{22}^2 + z_{31}z_{32}^2, \\ z_{11}z_{12}z_{13} + z_{21}z_{22}z_{23} + z_{31}z_{32}z_{33},\ z_{11}z_{13}^2 + z_{21}z_{23}^2 + z_{31}z_{33}^2,\ z_{12}^3 + z_{22}^3 + z_{32}^3,\ z_{12}^2z_{13} + z_{22}^2z_{23} + z_{32}^2z_{33}, \\ z_{12}z_{13}^2 + z_{22}z_{23}^2 + z_{32}z_{33}^2,\ z_{13}^3 + z_{23}^3 + z_{33}^3,\ z_{11}z_{22} - z_{21}z_{12},\ z_{11}z_{23} - z_{21}z_{13},\ z_{12}z_{23} - z_{22}z_{13}, \\ z_{11}z_{32} - z_{31}z_{12},\ z_{11}z_{33} - z_{31}z_{13},\ z_{12}z_{33} - z_{32}z_{13},\ z_{21}z_{32} - z_{31}z_{22},\ z_{21}z_{33} - z_{31}z_{23},\ z_{22}z_{33} - z_{32}z_{23}. \end{gathered}$$

Note that the T-module M is given by a presentation

$$T^{\oplus 15} \xrightarrow{A} T^{\oplus 3} \to M \to 0,$$

where A can be computed by Macaulay2 [26] as follows:

$$\begin{pmatrix} z_{23} & z_{22} & z_{21} & z_{13} & z_{12} & z_{11} & 0 & 0 & 0 & z_{33}^2 & z_{32}z_{33} & z_{31}z_{33} & z_{32}^2 & z_{31}z_{32} & z_{31}^2 \\ -z_{33} & -z_{32} & -z_{31} & 0 & 0 & 0 & z_{13} & z_{12} & z_{11} & z_{23}^2 & z_{22}z_{23} & z_{21}z_{23} & z_{22}^2 & z_{21}z_{22} & z_{21}^2 \\ 0 & 0 & 0 & -z_{33} & -z_{32} & -z_{31} & -z_{23} & -z_{22} & -z_{21} & z_{13}^2 & z_{12}z_{13} & z_{11}z_{13} & z_{12}^2 & z_{11}z_{12} & z_{11}^2 \end{pmatrix}.$$

Similarly N is given by a presentation

$$T^{\oplus 3} \xrightarrow{B} T^{\oplus 3} \to N \to 0,$$

where B can be computed by Macaulay2 [26] as follows:

$$\begin{pmatrix} z_{33} & z_{23} & z_{13} \\ z_{32} & z_{22} & z_{12} \\ z_{31} & z_{21} & z_{11} \end{pmatrix}.$$

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Availability of data and material

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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