The Geometry of Polygon Spaces

Victor Piercey and Matthew Thomas

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Moduli Spaces

Moduli spaces are parameter spaces of algebraic varieties. Moduli spaces of polygons are examples with a clear geometric meaning.

The moduli space of a given polygon with fixed side lengths is defined as the space of deformations in 3-dimensional Euclidean space.

The trivial case is the moduli space of a triangle. Since a triangle cannot be deformed while keeping side lengths fixed, the moduli space of a triangle is a point.

The General Quadrilateral

A simple example to keep in mind is a generic quadrilateral, shown below. The dotted line represents a diagonal. We refer to the length of this diagonal as the diameter and use the diameter as a parameter for the moduli space.
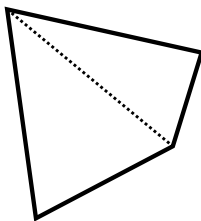
Given a specific diameter between the minimum and maximum, we may rotate the upper right point around the diameter, allowing a degree of freedom. Graphing the parameter space, we see that $\mathbb{S}^2$ is the parameter space of the quadrilateral shown.

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This will require our first side to lie along the z-axis.

The General Quadrilateral

- Each side is a vector in $\mathbb{R}^3$ that can be translated to the origin and normalized.
- Each side corresponds to a point on the Riemann sphere.
- We specify without loss of generality that our first such point is located at ∞ (the North pole) and the second at the origin (the South pole).
- Require that our first side length is the longest side.
- As a consequence, in order for our polygon to be closed, no other side can be parallel to the first side.

- Symplectic Geometry
- Geometric Invariant Theory
- The Kirwan-Kempf-Ness Theorem
- Polygon Spaces
- Stable Polygons/Curves
- Results

Symplectic Forms

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- A symplectic manifold is a manifold M with a symplectic form ω such that for each $p \in M$, $\omega_p : T_p M \times T_p M \to \mathbb{R}$ is skew-symmetric and bilinear.

Symplectic Geometry

A useful example here is $\mathbb{S}^2$ in $\mathbb{R}^3$.

- The symplectic form is induced by the map

$$\omega_p(u, v) := <p, u \times v>.$$

- This is a top degree form, hence closed.
- It is nondegenerate because if $u \neq 0$, then for $v = u \times p$, $<p, u \times v> \neq 0$.

The Moment Map

- Let (M, ω) be a symplectic manifold
- G be a Lie group acting symplectically on (M, ω) (This means that the group action preserves the symplectic form ω.)
- $\mathfrak{g}$ be its Lie algebra, and
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- In our case, it will suffice to think of the Lie group and its Lie algebra as matrix spaces.

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- As an example, consider the action of $\mathrm{SO}(3)$ on $\mathbb{S}^2$. The moment map is the inclusion of $\mathbb{S}^2$ into $\mathbb{R}^3$.

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- Actions that do admit moment maps are called Hamiltonian actions.

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- K acts freely on $\Phi_\omega^{-1}(0)$.
- The orbit space $\Phi_\omega^{-1}(0)/K$ is naturally a symplectic manifold by the Marsden-Weinstein-Meyer theorem

Symplectic Geometry on the Moduli Space of Polygons

- An n-gon will be determined by a set of n points in 3-dimensional Euclidean space, $v_1, v_2, ..., v_n$ which form the vertices of the polygon.
- The edges $e_1, e_2, ..., e_n$ will be formed be cyclically connecting the vertices. (For $1 \leq i \leq n-1$, e_i connects v_i and v_{i+1} and e_n connects v_n and v_1.)

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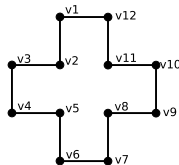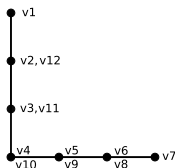- If we consider the edges to be vectors in $\mathbb{R}^3$, then we require

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- We identify polygons which can be obtained from each other by orientation-preserving isometries of $\mathbb{E}^3$.

- To do so, we first translate so that v_1 lies at the origin.

- The remaining idendtifications come from the natural action of $\mathrm{SO}(3)$ on $\mathbb{R}^3$.
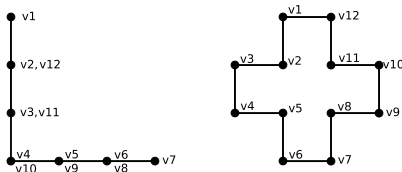
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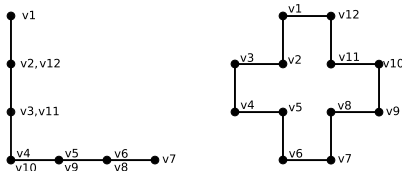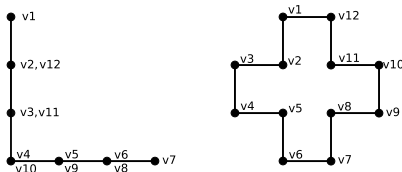
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- We can assume that parallel edges are adjacent, since we can always permute the edges and use the inverse of the permutation to return to where we started.
- We will also say that a set of edges is *degenerate* if that set of edges is a maximal set which is parallel in the polygon.

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- We make the choice to normalize r so that

$$\sum_{i=1}^n r_i = 2.$$

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- M_r is the fiber $\pi^{-1}(r)$ for $r \in \mathbb{D}_2^n$.
- We assume for the rest of the talk that the side-length vector r is in the interior of the hypersimplex $\mathbb{D}_2^n$.

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- This gives a homeomorphism

$$\epsilon : M_r \to \mathcal{M}_r$$

called the Gauss map.

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- For each $i = 1, \ldots, n$ let $p_i : (\mathbb{S}^2)^n \to \mathbb{S}^2$ be the projection onto the ith component. We then obtain a symplectic form (that is a Kähler form) Ω on $(\mathbb{S}^2)^n$ by

$$\Omega = \sum_{i=1}^{n} r_i p_i^*(\omega_i).$$

The space $(\mathbb{S}^2)^n$ with the symplectic form Ω is sometimes called the weighted configuration space.

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- Since $\mathrm{SO}(3)$ is a compact group, it follows that $\mathcal{M}_r = \mu^{-1}(0)/\mathrm{SO}(3)$ is the symplectic reduction of $(\mathbb{S}^2)^n$ via the action of $\mathrm{SO}(3)$.

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- The complex structure on $(\mathbb{S}^2)^n$ is compatible with the symplectic form Ω.
- As a consequence, the symplectic reduction $\mathcal{M}_r \setminus \Sigma$ is complex analytic.

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- It follows that away from line-gons, M_r has a complex analytic structure.
- The singularities are isolated, there are finitely many, and are contained in compatible complex analytic neighborhoods.
- This makes the moduli space M_r into a complex analytic space that is obtained as a result of symplectic reduction.

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- The problem is that the ordinary quotient may not be an algebraic variety - it may not even be Hausdorff.

- Suppose $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ where the action is

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- For $z_1 z_2 = 0$, there are three orbits - the orbit of $(0, z_2)$ where $z_2 \neq 0$, the orbit of $(z_1, 0)$ where $z_1 \neq 0$ and the orbit of $(0,0)$.
- The latter consists of the single point $\{(0,0)\}$ whereas the other two orbits are the z_2- and z_1-axes minus the origin, respectively.

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- The latter consists of the single point $\{(0, 0)\}$ whereas the other two orbits are the z_2- and z_1-axes minus the origin, respectively.
- Any open neighborhood of the orbit of $(0, 0)$ in the orbit space must intersect other orbits, and the orbit space is therefore not Hausdorff.

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- We begin with the affine case.

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- Let $\mathcal{R}$ be the ring of functions on X:

$$\mathcal{R} = \mathbb{C}[z_1, \ldots, z_n]/(p_1, \ldots, p_k).$$

This gives us a natural embedding $X \hookrightarrow \mathbb{C}^n$ as the space of all points $z \in \mathbb{C}^n$ where $p_i(z) = 0$ for all $i = 1, \ldots, k$.

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This gives us a natural embedding $X \hookrightarrow \mathbb{C}^n$ as the space of all points $z \in \mathbb{C}^n$ where $p_i(z) = 0$ for all $i = 1, \ldots, k$.

- In this setting, X is called the spectrum of $\mathcal{R}$ and we write

$$X = \mathrm{Spec}(\mathcal{R}).$$

(This is equivalent to the set of all prime ideals of $\mathcal{R}$.)

The Affine Case

- Suppose X is an affine variety defined by polynomials $p_1, \ldots, p_k$ in n complex variables.

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- If $f : \mathcal{R} \to \mathcal{S}$ is a ring homomorphism and $\mathfrak{p} \subset \mathcal{S}$ is a prime ideal, $f^{-1}(\mathfrak{p})$ is a prime ideal of $\mathcal{R}$, which induces a map from $\mathrm{Spec}(\mathcal{S})$ to $\mathrm{Spec}(\mathcal{R})$.

- Now suppose G is a complex Lie group that acts on our ring $\mathcal{R}$ by ring automorphisms.

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- Denote the set of all G-invariant functions on X by $\mathcal{R}^G$.
- $\mathcal{R}^G$ forms a ring (with operations inherited from $\mathcal{R}$) called the ring of invariants. We define $\mathrm{Spec}(\mathcal{R}^G)$ to be the GIT quotient of X by G:

$$X /\!/ G = \mathrm{Spec}(\mathcal{R}^G).$$

The Affine Case

- Let us return to the example above where $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$.
- Since X is an affine variety defined solely by the zero polynomial, the ring of functions is $\mathcal{R} = \mathbb{C}[z_1, z_2]$, and the ring of invariants is $\mathcal{R}^G$, generated by the polynomial $z_1 z_2$.

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- If we let $w = z_1 z_2$:
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- It follows that $X // G = \mathbb{C}$.
- What has happened geometrically is the orbits where $z_1 z_2 \neq 0$ have been left unaltered, but we have identified the three orbits where $z_1 z_2 = 0$.

The Projective Case

- To see why this construction isn't sufficient for the projective case, let $X = \mathbb{C}^2$ and $G = \mathbb{C}^*$ as above, but let the action be given by

$$\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda z_2).$$

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- The obvious quotient *should* be $\mathbb{CP}^1$, but projective space is not $\mathrm{Spec}(\mathcal{S})$ for any ring $\mathcal{S}$.
- We need to do more work to obtain a GIT quotient for a projective variety.

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- In the affine case, the corresponding embedding was handled using the ring of functions on X.
- We don't have that at our disposal in the projective case, since such functions would have to be constant.

The Projective Case

- Let $\mathcal{L}$ be a line bundle over X.
- Let $\Gamma(X, \mathcal{L})$ be the space of sections of $\mathcal{L}$.
- Let $(s_0, \ldots, s_n)$ be a set of its sections.

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- Identifying $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ with $\mathbb{CP}^n$, this gives us a closed embedding $X \hookrightarrow \mathbb{CP}^n$.

The Projective Case

- Suppose we have a Lie group G acting on X.
- Suppose the action lifts to an action of G on $\mathcal{L}$ that is linear on fibers and takes the fiber L_x to the fiber $L_{g \cdot x}$ for every $g \in G$ and $x \in X$.

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- We say that the action of G on $\mathcal{L}$ linearizes the action of G on X, and that $\mathcal{L}$ is a linearized line bundle.
- The linearization of the action preserves sections. As a consequence, we get an action of G on $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ such that the embedding $X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*)$ is G-equivariant.

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- If we let $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ be the natural projection, then $\pi^{-1}(X)$ is an affine variety in $\mathbb{C}^{n+1}$ that is invariant under dilations.

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- The action of G lifts to an action on $\pi^{-1}(X)$, so we can consider the ring of **homogeneous** invariants.
- Taking the quotient again gives us a projective variety, and provides us with our construction of $X//G$.

The Projective Case

- In order to be able to prove anything about projective GIT quotients, we need to make this construction more concrete.

- A point $x \in X$ is called *semi-stable* (with respect to $\mathcal{L}$) if there is some $m > 0$ and G-invariant section $s \in \Gamma(X, \mathcal{L}^{\otimes m})^G$ such that $s(x) \neq 0$ and the set $X_s = \{y \in X : s(y) \neq 0\}$ is affine.

- If additionally the orbit of G through x is closed, x is said to be *stable* (with respect to $\mathcal{L}$).

- Note that if x is stable with $s(x) \neq 0$, the orbit of G through any point $y \in X_s$ is closed.

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- The set of semistable, stable, and unstable points are denoted by

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  respectively.
- It is clear that $X^{\mathrm{ss}}(\mathcal{L})$ and $X^{\mathrm{s}}(\mathcal{L})$ are open subsets of X, although they may be empty.

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- We can cover X^{ss} with the affine open subsets X_{s_i}, take the affine GIT quotient of these open subsets and "glue them together" in an appropriate manner. This gives us the GIT quotient

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- This construction makes $X // S$ into a projective variety. What actually happens is that the orbits of stable points in X become single points in $X // G$, while orbits of points in $X^{ss} \setminus X^s$ are identified whenever the intersection of their closures is nonempty.

Kirwen-Kempf-Ness Theorem

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- We suppose X is a projective variety with symplectic form ω and a line bundle $\mathcal{L}$ such that $c_1(\mathcal{L}) = [\omega]$ in $H^2_{\mathrm{dR}}(X)$, where c_1 is the first Chern class. Suppose further that G is a Lie group acting on X that linearizes $\mathcal{L}$.

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- Let $\mathfrak{g}$ be the Lie algebra over G and suppose the action of G admits a moment map $\mu : X \to \mathfrak{g}^*$.

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- If $\mu^{-1}(0)/K$ is smooth, this is a diffeomorphism. If $\mu^{-1}(0)/K$ is complex analytic, this is a complex analytic equivalence.

- We would like to find a complex-analytic equivalence between the moduli space M_r and the weighted configuration space $((\mathbb{S}^2)^n, \Omega)$.

Polygon Spaces

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- We will form a GIT quotient structure on M. The construction will depend on our choice of r.
- We need to assume $r \in \mathbb{Q}_+^n$

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $G = \mathrm{PSL}(2,\mathbb{C})$ acts on $\mathbb{S}^2$, and thus extends to a diagonal action on M by

$$A \cdot (u_1, \ldots, u_n) = (A \cdot u_1, \ldots, A \cdot u_n).$$

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- Given the action and $r \in \mathbb{D}_2^n$, the semi-stable points in M are the points $\vec{u} \in (\mathbb{S}^2)^n$ such that

$$\sum_{u_j = v} r_j \leq 1$$

for every $v \in \mathbb{S}^2$.

- stable points in M are those that satisfy the same condition with a strict inequality.

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- Let $Q = M^{\mathrm{ss}}//\mathrm{PSL}(2, \mathbb{C})$.
- Q is the GIT quotient $M//G$ and is therefore a projective variety.

- Define the cusp points to be $M^{\mathrm{cusp}} = M^{\mathrm{ss}} \setminus M^{\mathrm{s}}$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- Define the cusp points to be $M^{\mathrm{cusp}} = M^{\mathrm{ss}} \setminus M^{\mathrm{s}}$.
- Every point in M^{cusp} is determined by a partition of $S = \{1, \ldots, n\}$ into two disjoint sets $S_1 = \{i_1, \ldots, i_j\}$ and $S_2 = \{j_1, \ldots, j_{n-k}\}$ such that $r_{i_1} + \cdots + r_{i_k} = 1$ (and whence $r_{j_1} + \cdots + r_{j_{n-k}} = 1$).

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- In the quotient, the cusp points are *uniquely* determined by the corresponding partition. Therefore, the subspace $Q^{\mathrm{cusp}} = M^{\mathrm{cusp}}/G \subset Q$ is in one-to-one correspondence with a subset of partitions of n, which in turn is finite.

- Thus, in the quotient, there are at most finitely many cusp points.

- The closing condition

$$\sum_{i=1}^{n} r_i u_i = 0$$

implies that $\tilde{\mathcal{M}}_r \subset M^{\mathrm{ss}}$.

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- This is the first part of Kirwan-Kempf-Ness: $\mu^{-1}(0) \subset M^{\mathrm{ss}}$.

The Weighted GIT Quotient of the Configuration Space $(\mathbb{S}^2)^n$

- $SO(3) \subset PSL(2, \mathbb{C})$ is a maximal compact subgroup so there is a complex analytic equivalence

$$\mathcal{M}_r \cong Q$$

from the second part of the Kirwan-Kempf-Ness, which induces, via the Gauss map ϵ, a complex analytic equivalence

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- Under ϵ, the cusp points Q^{cusp} correspond to the degenerate polygons in M_r.
- It follows that M_r, and hence $\mathcal{M}_r$, have at most finitely many singularities.

- Consider the moduli spaces for special vectors $r \in \mathbb{D}_2^n$.

Moduli Spaces for Favorable Chambers

- Consider the moduli spaces for special vectors $r \in \mathbb{D}_2^n$.
- The hypersimplex is divided into chambers by walls of the form

$$W_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{D}_2^n : \sum_{i \in J} x_i = 1 \right\}$$

where J runs over all proper subsets of $\{1, \ldots, n\}$.

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- Two points x and y are in the same chamber if and only if for every proper subset $J \subset \{1, \ldots, n\}$

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- If the chamber is maximal (with respect to inclusion), the inequalities are strict

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- A theorem of Hu states that if r is in the interior of Δ_i, the moduli space M_r is isomorphic to $\mathbb{CP}^{n-3}$.
- The quadrilateral example in the introduction is a moduli space isomorphic to $\mathbb{CP}^{n-3}$. Since $n = 4$ for a quadrilateral, this is the same as $\mathbb{CP}^1$, which is isomorphic to $\mathbb{S}^2$.

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- The quadrilateral example in the introduction is a moduli space isomorphic to $\mathbb{CP}^{n-3}$. Since $n = 4$ for a quadrilateral, this is the same as $\mathbb{CP}^1$, which is isomorphic to $\mathbb{S}^2$.
- The discussion in the introduction shows that any $r \in \mathbb{R}_+^n$ where there is some i such that r_i is sufficiently larger then the other r_j's is contained in $\Delta_{\{i\}}$, hence $M_r \cong \mathbb{CP}^{n-3}$.

Stable Polygons and Stable Curves

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- The mechanism grows out of the notion of stable polygons, which is connected to the moduli space of stable n-pointed curves of genus zero.

Stable Curves of Genus Zero

- Start with the moduli space of n distinct points on $(\mathbb{CP}^1)^n$ modulo the action of $\mathrm{PSL}(2,\mathbb{C})$, denoted by $\mathcal{M}_{0,n}$. This will contain all the generic points in the moduli space we will construct.

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- View a point in $\mathcal{M}_{0,n}$ as n distinct points on a single copy of $\mathbb{CP}^1$ and we let these points vary. As a pair of points get close to one another, introduce a new copy of $\mathbb{CP}^1$ intersecting the original $\mathbb{CP}^1$ at the point of intersection.

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- The colliding points will then proceed along the new $\mathbb{CP}^1$.

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- The end result will be a collection of intersecting copies of $\mathbb{CP}^1$ with the extra restriction that each irreducible component (each copy of $\mathbb{CP}^1$) has at least three points consisting of the original points that are left on that copy of $\mathbb{CP}^1$ and intersections with other copies.

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- $\overline{\mathcal{M}}_{0,n}$ is a compactification of $\mathcal{M}_{0,n}$ that illustrates a general procedure called the Deligne-Mumford compactification

Stable Curves of Genus Zero, an Example

- As an example, consider four points on $\mathbb{CP}^1$. A stable curve is illustrated in the cartoon below, where each line represents a copy of $\mathbb{CP}^1$ and each circle either represents a point or an intersection.
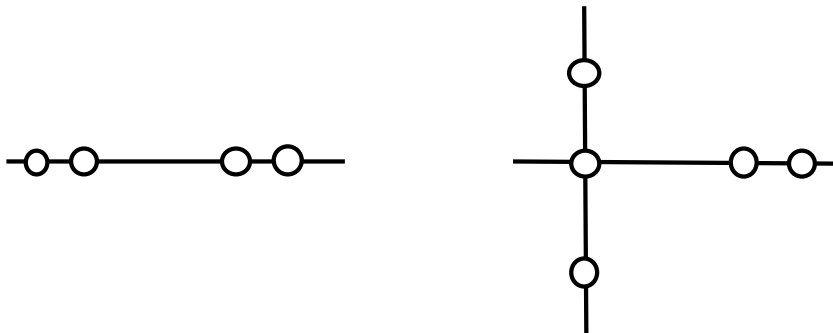
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- Denote the subspace of generic polygons by M_r^0, which can be identified with $\mathcal{M}_{0,n}$ and so is an open complex analytic space
- Compactify M_r^0 by adding in appropriate limiting objects
- The basic idea is that whenever a set of edges become parallel, we resolve by introducing a "bubble" polygon

Stable Polygons

- Suppose a polygon P is a limit of a sequence of polygons in M_r^0 and P degenerates at a subset I of cardinality k

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- Suppose a polygon P is a limit of a sequence of polygons in M_r^0 and P degenerates at a subset I of cardinality k
- Pair P with a generic $(k+1)$-gon P' whose first k-sides inherit the lengths of the original parallel edges of P but whose last side length is $r_{i_1} + r_{i_2} + \cdots + r_{i_k} - \epsilon_{i_1,\ldots,i_k}$.
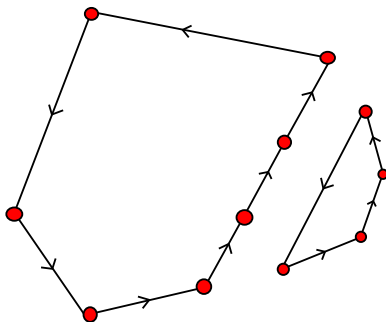
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Stable Polygons

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- It will suffice to use the same ϵ in all cases, so we let $\epsilon = \min\{r_1, \ldots, r_n\}$.

Stable Polygons

- As a consequence of this choice, if we define

$$r_I = \left(r_{i_1}, \ldots, r_{i_k}, \sum_I r_i - \epsilon \right)$$

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- P' can degenerate at other edges, in which case we pair (P, P') with a third polygon P'' constructed similarly.
- Our choice of ϵ guarantees that there are only finitely many degenerations that are possible, and hence this process terminates after a finite number of steps.

Stable Polygons

- A proper subset $I \subset \{1, \ldots, n\}$ with $|I| \geq 2$ and r_I as defined above give a pair

$$(P, P') \in M_r \times M_{r_I}$$

called a **bubble pair** (We call P' a **bubble of** P if P degenerates at the edges e_I.)

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- We say that a proper subset $I \subset 1, \ldots, n$ with $|I| \geq 2$ is a **relevant** subset if r satisfies (1). The collection of all relevant subsets of $\{1, \ldots, n\}$ will be denoted by $\mathcal{R}(r)$.

Stable Polygons

- A **stable** *n*-**gon** with respect to the side length vector r is a collection of labeled (but not ordered) polygons

$$\mathbf{P} := (P_0, P_1, \ldots, P_m) \in M_r \times M_{r_{I_1}} \times \cdots \times M_{r_{I_m}}$$

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- The moduli space of stable polygons is denoted $\mathfrak{M}_{r,\epsilon}$.

Results

The main results of Hu that bring all of these ideas together:

Theorem

The moduli space $\mathfrak{M}_{r,\epsilon}$ carries a natural smooth, compact complex analytic structure.

Theorem

The moduli spaces $\mathfrak{M}_{r,\epsilon}$ and $\overline{\mathcal{M}}_{0,n}$ are biholomorphic. As a consequence, the complex structure of $\mathfrak{M}_{r,\epsilon}$ is independent of r and ϵ

the Kähler structure may depend on the choices of r and ϵ. It follows that one may use the moduli spaces of stable polygons to study the Kähler cone in $H^2(\overline{\mathcal{M}}_{0,n})$.

Results

Back to our original moduli spaces of polygons M_r:

Theorem

By forgetting all the bubbles, we obtain a natural projection

$$\pi_{r,\epsilon} : \mathfrak{M}_{r,\epsilon} \to M_r$$

that is holomorphic and bimeromorphic. It is the iterated blow up of M_r along (the proper transforms of) some explicitly described subvarieties Y_α when M_r is smooth. When M_r is singular, $\pi_{r,\epsilon} : \mathfrak{M}_{r,\epsilon} \to M_r$ is the composite of a canonical resolution of singularities followed by explicit iterated blowups.

This result allows us to view the moduli spaces M_r as a single family of smooth complex-analytic spaces $\mathfrak{M}_{r,\epsilon}$ whether M_r started out as singular or not.

Conclusion

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- The moduli space of n-gons with a prescribed length is dominated by $\overline{\mathcal{M}_{0,n}}$.

Conclusion

Finally, we extend a thanks and wishes for a happy birthday to our advisor, Dr. Philip Foth.