Probability Models for Rolling Irregular Dice in 2 Dimensions with Special Cases in 3 Dimensions

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Abstract

The probability of any surface of an ordinary playing die rolling face-down is 1/6. Unknown however, is the probabilities of a die with a side shaved down a little; no mathematical model exists that gives the expected probability for these kinds of dice.

Our research investigated this question and found mathematical models for the probability of any side of an irregular polygon landing face-down (2D). Future research will extend these models to irregular polyhedra (3D) and their surfaces. The bulk of our research went into proving mathematically that our models are true, and we also provide empirical evidence with sample objects.

Keywords: Probability, Shaved Dice Problem, Irregular Polyhedra

1. Background and Motivation

The problem of finding the face probabilities of an irregular polyhedron is often referred to as the Shaved Dice Problem. Statistical approaches to this problem have been done by rolling shaved dice thousands of times, yet there is no standard model that, given any polyhedron, will compute each surface's theoretical probability of landing face-down [1]. If solved, dice can be created with predetermined, chosen probabilities, and optimal strategies for games of chance, such as *Pass the Pigs*, can be formulated.

2. Strategy

When presented with such a problem, it is natural to think of rolling everyday objects, such as a white board eraser or cup. From these initial observations, a certain obvious restriction on the kind of polyhedra we are allowed to roll comes to light. Consider rolling a cup-like polyhedron. If the face probabilities depended on the surface area of the face, then it would seem that the inside of the cup has equal probability of landing face-down as the outside of the cup, which of course, is not what we intended.

The polyhedra we choose to study are convex, meaning a line connecting any two points on the surface always lies in the interior or surface of the polyhedron. For the rest of this paper, when we mention polyhedra, we mean convex polyhedra.

Beginning approaches to solving this problem involved rolling 3D printed irregular polyhedra and examining factors that could be influencing the face probabilities. In the end, we concluded that three-dimensional objects are difficult to analyze. We then decided to simplify the problem by studying two-dimensional objects, i.e. convex irregular polygons, and see if we can extend the results into three dimensions. For the rest of this paper, when we mention polygons, we mean convex polygons.

3. Rolling a 2-Dimensional Die

When we roll a die in three dimensions, we take a polyhedron, throw it up in the air, and record the side that lies face down on the floor. In two dimensions, we make adjustments to this concept; namely, instead of rolling polyhedra, we roll polygons. Also, we imagine a two-dimensional gravitational force that constantly pushes these polygons towards the floor. To clarify any further ambiguity, consider the following definitions:

Definition 3.1 (Floor). *The floor is the line we are concerned with the polygon falling towards.*

Definition 3.2 (State). *If the following conditions are met, there is a state between the floor and a polygon:*

1. *There exists some point on the boundary of the polygon such that the distance between the floor and this point is zero.*
2. *The intersection between the floor and the interior of the polygon is empty.*

Definition 3.3 (Final State). *A state is final if you can construct a line segment perpendicular to the floor that is entirely contained by the interior of the die and goes through the die's center of mass.*

Definition 3.4 (Initial State/Collision)**.** *An initial state is the first state that a polygon enters such that the polygon will, from then on, be in some state.*

Definition 3.5 (Vertex and Side States)**.** *A polygon with vertex K is said to be in vertex state K if the polygon is in a state such that the distance between K and the floor is zero. If the polygon has two adjacent vertices, say K_1 and K_2, such that it is in both vertex state K_1 and vertex state K_2, then the polygon is said to be in side state $\overline{K_1 K_2}$.*

Axiom 3.1 (Falling Direction)**.** *An object will fall along the shortest path between the center of mass and the floor. The shortest path is the line orthogonal to the floor that intersects the center of mass.*

For the purposes of our model, we are concerned about the steps between the initial and final states of the die. First, we need to know the probabilities associated with the initial collision. Then, we will determine the probabilities associated with passing through various states until we reach a final state.

Definition 3.6 (Convex Polygon)**.** *A polygon is said to be convex if all its interior angles are less than $180°$. Alternatively, a polygon is convex if all line segments joining any two points of the polygon are contained in the polygon.*

Theorem 3.2 (Probabilities of Initial States)**.** *The probabilities associated with a convex polygon's initial collision with the floor are as follows:*

1. *The probability of an initial state being a side state is zero.*
2. *Given a polygon has a vertex A, the probability of the initial state being vertex state A is given by A's exterior angle divided by the sum of the polygon's exterior angles. This gives the formula:*

$$P(A) = \frac{180° - A_{interior}}{360°}$$

Proof. Let K be a convex polygon with n sides. Observe that there are both n possible angles and n possible side angles that can collide with the floor. Since there are only n ways a side can land on the floor out of an infinite amount of orientations, the probability of a side landing on the floor is 0.
Now, the only other possible way for K to land is on a vertex V. Thus $\sum P(V) = 1$. Now, observe that the maximum angle that a leg of V can have with the floor is its exterior angle. (see Figure 3). If the figure rotates more

than this exterior angle, it transfers to another vertex state. So, we add up all the exterior angles for the possible orientations of all the states. Then, the probability of V landing on the floor is V's exterior angle divided by this sum. $\square$

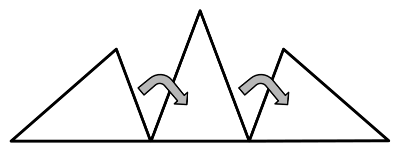

Figure 1: Possible faces an isosceles triangle can land on

Theorem 3.3 (Vertex to Side). *The probability of a convex polygon rolling from a vertex V_{AB} to its adjacent sides A and B can be found using the following process:*

1. *Construct the line that goes between the center of mass and the vertex.*
2. *Construct a perpendicular line to this segment that goes through V_{AB}.*
3. *Take the smallest angle between A and this line and call it α. Do the same for B and call it β. If either of these angles are contained within the polygon, set that angle equal to zero.*
4. $P(A|V_{AB}) = \frac{\alpha}{\alpha+\beta}$ *and* $P(B|V_{AB}) = \frac{\beta}{\alpha+\beta}$.

Proof. Let us assume that we have a state associated with the vertex V_{AB}. Calculate α and β as the instructions dictate. Note that $\alpha + \beta$ is equal to the exterior angle of V_{AB}. We will call the angles between side A and the floor and side B and the floor γ_A and γ_B respectively. Now note that if $\gamma_A < \alpha$ then the line perpendicular to the floor that goes through the center of mass intersects A. Thus, by Axiom 3.1, we will pass onto side A. Similarly, if $\gamma_B < \beta$ then the the state will pass onto side B. Thus, $P(A|V_{AB}) = \frac{\alpha}{\alpha+\beta}$ and $P(B|V_{AB}) = \frac{\beta}{\alpha+\beta}$. $\square$

Now that we have Theorem 3.3, we can examine some sample computations.

Example 3.1. *Consider the following polygon:*

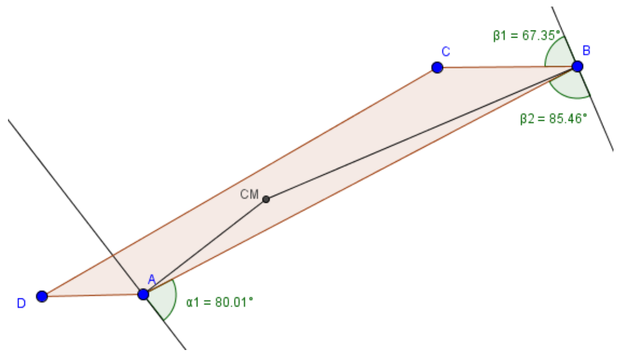

Figure 2: Passing from a vertex state to a side state

Consider being in vertex state A.
By Theorem 3.3, $P(\overline{AB}|A) = \frac{80.01}{0+80.01} = 1$ *and* $P(\overline{AD}|A) = \frac{0}{80.01} = 0.$

Now, consider being in vertex state B.
Then, $P(\overline{AB}|B) = \frac{85.46}{67.35+85.46} = \frac{85.46}{152.81} \approx 0.559.$
Also, $P(\overline{BC}|B) = \frac{67.35}{67.35+85.46} = \frac{67.35}{152.81} \approx 0.441$

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Theorem 3.4 (Side to Vertex)**.** *Let A be a side of a polygon where the endpoints of A are V_1 and V_2 and let C be the center of mass. Then this state is final if and only if the angles $\angle CV_1V_2$ and $\angle CV_2V_1$ are both acute. Moreover, if one of these angles is obtuse, the state will transfer to the associated vertex state.*

Proof. Construct $\angle CV_1V_2$ and $\angle CV_2V_1$. If both of these angles are acute, then $\angle V_2CV_1$ is also acute. We have formed an acute triangle. Thus there exists a line that intersects and is perpendicular to $\overline{V_1V_2}$ that passes through C. By definition, we are in a final state.

Now assume that we are in a final state and $\overline{V_1V_2}$ is parallel to the floor. Then, by definition, we can construct a line that intersects and is perpendicular to $\overline{V_1V_2}$ and C. By so doing, we form two right triangles. Thus $\angle CV_1V_2$ and $\angle CV_2V_1$ are both acute.

Now, without loss of generality, let us assume $\angle CV_1V_2$ is obtuse. Let's construct the line perpendicular to the floor that goes through C. Because $\angle CV_1V_2$ is obtuse and our polygon is convex, this line is not entirely contained in the polygon. Thus, it must move to either state V_1 or state V_2.

Observe the line perpendicular to the floor that goes through C is closer to V_1. This makes it clear, from Axiom 3.1, that we must transfer to V_1's state. $\square$

Example 3.2. *Given a* $6-5-5$ *isosceles triangle, find the probabilities of each face landing as a final state.*

We will begin by using computer software that generates the interior angles of the polygon. We also calculate and label where the center of mass is. Using the generated angles (see Figure 3 below) and Theorem 3.2, we compute the following initial state probabilities.

$$P(A) = P(C) \approx \tfrac{180-53.13}{360} \approx .3524$$
$$P(B) \approx \tfrac{180-73.74}{360} \approx .2952$$

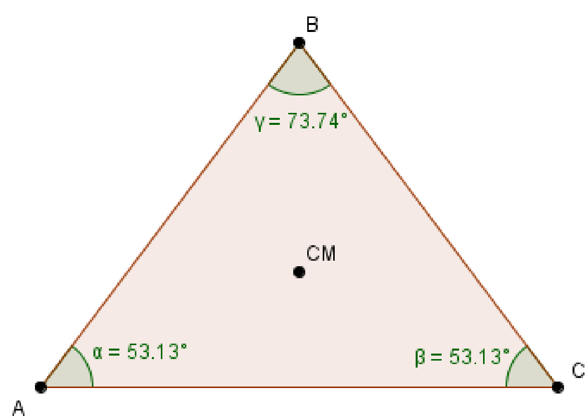


Figure 3: Angles and Center of Mass of a $6-5-5$ Isosceles Triangle

Now we compute the probabilities transferring from our vertex states to each side state by using another computer generated image (Figure 4) and Theorem 3.3.

$$P(\overline{AB}|A) = P(\overline{BC}|C) \approx \tfrac{60.83}{126.87} \approx .4795$$

$$P(\overline{AB}|B) = P(\overline{BC}|B) = \frac{53.13}{106.26} = 0.5$$
$$P(\overline{AC}|A) = P(\overline{AC}|C) \approx \frac{66.04}{126.87} \approx .5205$$

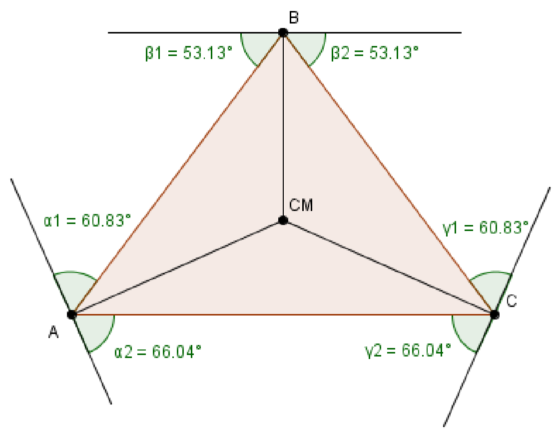
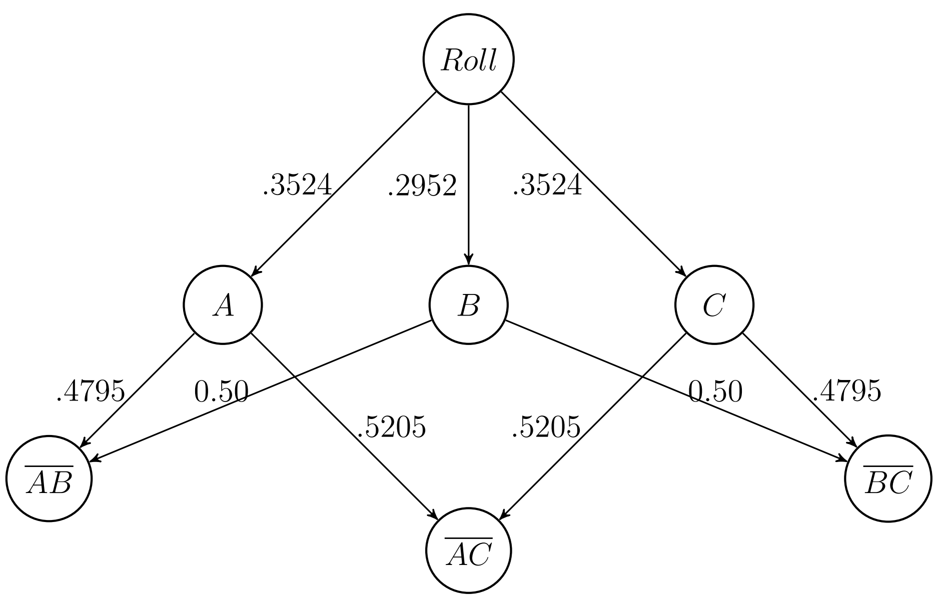


Figure 4: Floor-Vertex Angles in a 6 – 5 – 5 Isosceles Triangle

Note that by Theorem 3.4 all thee side states are final states. We can now plug all our probabilities into a chain.



With the help of this chain, we compute the probabilities of each final state.

$$P(\overline{AB}) = P(\overline{BC}) \approx .3524(.4795) + .2952(.5) = .3166$$
$$P(\overline{AC}) = 2(.3524)(.5205) = .3668$$

Thus, the probability of landing on the longest side is approximately .3668 whereas the probability of landing on one of of the shorter sides is .3166.

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Example 3.3. *Find the probabilities of each final state when rolling a parallelogram with points $A(0,0)$, $B(1.5,2)$, $C(2.5,2)$, and $D(1,0)$.*

We shall begin by using Theorem 3.2 to calculate probabilities of initial states. The various exterior angles are shown in the diagram below:

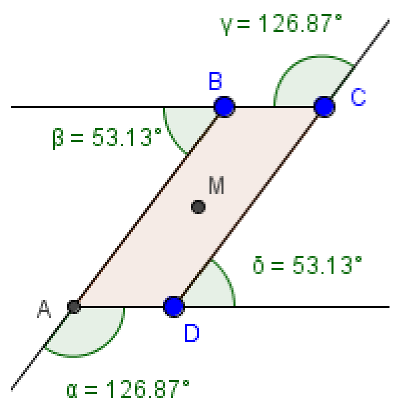


Figure 5: Exterior angles of $ABCD$

We get the initial probabilities as follows:

$$P(A) = P(C) = \tfrac{126.87}{360}$$
$$P(B) = P(D) = \tfrac{53.13}{360}$$

Now we will use Theorem 3.3 to compute the probabilities of passing to each side from a given vertex. Using the diagram below, we get:

$$P(\overline{AB}|A) = P(\overline{CD}|C) = \tfrac{75.53}{126.87}$$
$$P(\overline{AD}|A) = P(\overline{BC}|C) = \tfrac{51.34}{126.87}$$
$$P(\overline{AB}|B) = P(\overline{CD}|D) = 1$$
$$P(\overline{AD}|D) = P(\overline{BC}|C) = 0$$

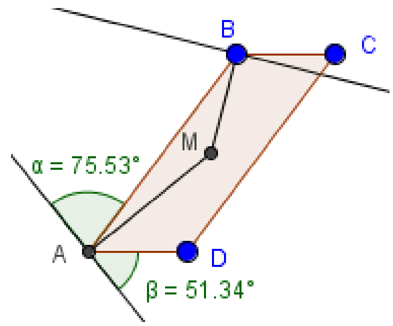
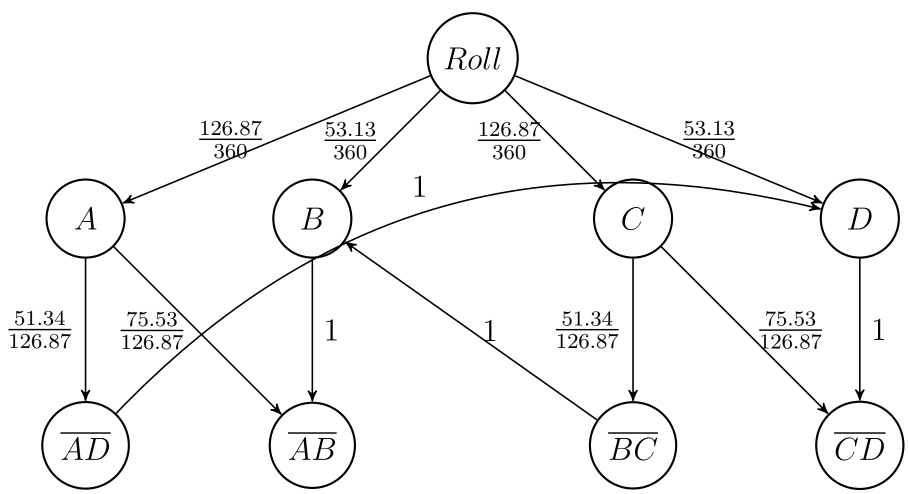


Figure 6: Exterior angles of $ABCD$

After noting that, by Theorem 3.4, $P(D|\overline{AD}) = P(B|\overline{BC}) = 1$, We can plug all our conditional probabilities into a chain.



$P(\overline{AD}) = P(\overline{BC}) = 0.$

$$P(\overline{AB}) = P(\overline{CD}) = \frac{126.87}{360} \cdot \frac{51.34}{126.87} \cdot 1 \cdot 1 + \frac{126.87}{360} \cdot \frac{75.53}{126.87} + \frac{53.13}{360} \cdot 1 = 0.5.$$

We can conclude that the probability of landing on $\overline{AB}$ and $\overline{CD}$ is the same.

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4. Special Cases in 3-Dimensions

We have demonstrated how to roll a two-dimensional die. There are shapes that we can build in 3-dimensions that demonstrate similar probabilities. We will call this special polyhedron 3-dimensional footballs.

Theorem 4.1 (3-Dimensional Football)**.** *For a given polygon, construct a polyhedron using the following:*

1. *Take two distinct points in space, A and B.*
2. *Find the midpoint of the line segment between these points. Call this CM.*
3. *Build the polygon and position it such that $\overline{AB}$ is orthogonal to it and CM is the center of mass.*
4. *Construct a polyhedron K such that the set of vertices of K and the set of A,B, and vertices of our polygon are equal.*

If $\overline{V_1V_2}$ is a side of our polygon and S the surface(s) enclosed by AV_1BV_2A, then $P(\overline{V_1V_2}) = P(S)$.

Proof. By following the instructions listed above, we chose the center of mass to be the midpoint of the $\overline{AB}$ axis. Now consider the cross sections of the constructed die that are orthogonal to $\overline{AB}$. We will call this set of cross-sections $\bigcup_{\alpha\in\mathcal{F}}\mathcal{P}_\alpha$ where each $\mathcal{P}_\alpha$ is a unique cross-section. Observe that every polygon in $\bigcup_{\alpha\in\mathcal{F}}\mathcal{P}_\alpha$ is similar. Moreover, observe that the center of mass of each $\mathcal{P}_\alpha$ is along the $\overline{AB}$ axis. Now choose a side to our original polygon. Call it $\mathcal{S}$. Let $\mathcal{S}_\alpha$ be the set of sides corresponding to $\mathcal{S}$ in each polygon $\mathcal{P}_\alpha$.

Let $\beta \in \mathcal{F}$. Then $\mathcal{P}_\beta \in \bigcup\limits_{\alpha\in\mathcal{F}}\mathcal{P}_\alpha$ and $\mathcal{S}_\beta \in \bigcup\limits_{\alpha\in\mathcal{F}}\mathcal{S}_\alpha$. Observe $P(\mathcal{S}_\beta | \bigcup\limits_{\substack{\alpha\in\mathcal{F}\\ \alpha\neq\beta}}\mathcal{S}_\alpha) = 1$. Thus, by Bayes's formula, $P(\mathcal{S}_\beta) = P(\mathcal{S}_\beta \cap \bigcup\limits_{\substack{\alpha\in\mathcal{F}\\ \alpha\neq\beta}}\mathcal{S}_\alpha) = P(\bigcup\limits_{\alpha\in\mathcal{F}}\mathcal{S}_\alpha)$.

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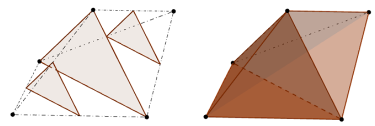

Figure 7: Die made by Theorem 4.1

5. Future Research

We have shown that it is possible to find face probabilities of some non-Platonic dice. However, these methods have not led us to a solution for the Shaved Die Problem. We hope further investigation into two dimensional dice as a way to analyze irregular polyhedra will bring about a solution.

6. Acknowledgments

Thank you Dr. Paul Hurst for presenting us with the shaved die problem. Also, thank you Dr. Russel Carlson for giving us the idea of the three-dimensional football. Thanks be to you both for mentoring us through our research.

7. References

[1] M.G. Pavlides and M.D. Perlman, *On Estimating the Face Probabilities of Shaved Dice with Partial Data*, https://www.stat.washington.edu/research/reports/2009/tr566.pdf, 2017.