Say whether the following is true or false and support your answer by a proof.

$$(\exists m \in N)(\exists n \in N)(3m + 5n = 12)$$

If $3m + 5n = 12$,then $n = \dfrac{12\text{-}3m}{5}$

$n \in N$ if $12\text{-}3m=5k$, $k \in N$

If, $m=1$, $12\text{-}3 =9 \neq 5k$
$m=2$, $12\text{-}5 =7 \neq 5k$
$m=3$, $12\text{-}9=3 \neq 5k$
$m=4$, $12\text{-}12=0 \notin N$

$\forall m>4$,$12\text{-}3m \notin N$
$\therefore \forall m \in N, \nexists\ n \in N$ s.t $3m\text{-}5n=12$

So the statement is False ■

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

Five consecutive integers: $n,(n+1),(n+2),(n+3),(n+4)$,

$$n+(n+1)+(n+2)+(n+3)+(n+4) = 5n+10 = 5k \quad \text{(let k be "something")}$$
$$5n+10 = 5k \;\; \forall n \in Z$$

By PMI

First case:

If $n=1$, $5(1)+10 = 15 = 5(3)$, so $5n+10 = 5k$

Hypothesis:

$n=p$, $5p+10 = 5k_1$

Thesis:
For $p+1$, $5(p+1)+10 = 5k_2$

$$5(p+1)+10= 5p+5+10 =\underbrace{5p+10}_{\boxed{\substack{5k_1 \\ \text{our hypothesis}}}}+5 = 5k_1+5 = 5(k_1+1) = 5\,k \text{ (for some other k aside k1 and k2)} \;\blacksquare$$

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number n 2 + n + 1 is odd.

By PMI, we show that : ∀n∈Z, n^2 + n + 1 has the form 2a+1

We take the expression n^2 + n + 1 as n(n+1)+1 , and we focus in the term n(n+1).
We know that the product of two consecutive integers is an even number.

First case:
n=1, (1)(1+1) +1=2+1=3 = 2(1)+1 , so it has the form 2a+1

Hypothesis:
n=q ; q^2 + q + 1=q(q+1)+1 and assume that $2a_1+1$

Thesis:
For q+1
$(q+1)^2+(q+1)+1 = q^2+2q+1+q+1+1=\underbrace{q^2+q}_{\text{this is part of our hypothesis wich equals 2a}}+2q+2+1= 2a_1+2q+2+1$

$2a_1+2q+2+1=2(a_1+q+1)+1=2a_2+1$

So, for any integer n n^2 + n + 1 has the form 2k (k be something) plus 1, wich means is odd $\blacksquare$

4. Prove that every odd natural number is of one of the forms $4n + 1$ or $4n + 3$, where n is an integer.

Since we are looking for every odd natural number,
$(\forall n \in Z)(n \geq 0)\ \exists r \geq 0\ [(4n+1 \wedge 4n+3) \Rightarrow (2r+1)]$

We have three forms types of numbers, 0, 2k,2k+1 or 2k-1

If,
$n=0$; $4(0)+1 = 1$; $4(0)+3 = 3$

$n=2k$; $4(2k)+1 = 8k+1 = 2(4k)+1_{\text{(form } 2r+1)}$; $4(2k)+3 = 8k+2+1 = 2(2k+1)+1_{\text{(form } 2r+1)}$

$n=2k+1$; $4(2k+1)+1 = 8k+4+1 = 2(4k+2)+1_{\text{(form } 2r+1)}$
$4(2k+1)+3 = 8k+4+3 = 8k+6+1 = 2(4k+3)+1_{\text{(form } 2r+1)}$

So, every odd in natural set can be in the form $4n+1$ or $4n+3$ $\blacksquare$

5. Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.

By PMI

$\forall n \in Z[n \vee (n+2) \vee (n+4)=3x]$ *$\vee$ as inclusive or

First case:
r=remainder
n=1; 1/3 r=1 False ; (1+2)/3 r=0 True
so, statement pass the first test

Hypothesis:
Assume that , $\exists n \in Z\ [(n=3a) \vee (n+2=3b) \vee (n+4=3c)]$

Thesis:
For n+1
$3|n+1 \vee 3|((n+1)+2) \vee ((n+1)+4)$

$n+1 = 3a+1$, r=1 False,

$n+3= \underbrace{3a}_{\substack{\text{from hypothesis} \\ 3a=n}}+3 =3(a+1)$, r=0 True,

$n+5=\underbrace{3b}_{\substack{\text{from hypothesis} \\ 3b=n+2}}+3=3(b+1)$, r=0 True.

$\therefore$ allways we can find some n in set Z for n,n+2 and n+4 with the form 3x wich makes the statement $\forall n \in Z[n \vee (n+2) \vee (n+4)=3x]$ True $\blacksquare$

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes',pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

We know that the first triplet is 3,5,7
If , $3+5+7=15=3(5)=3q$
seen as variables

$$p+(p+2)+(p+4)=3p+6=3(p+2)=3q_1$$

By the problem 5 we know that $(n \vee (n+2) \vee (n+4))=3q$, this means that at least one is divisible by 3. $(n+(n+2)+(n+4)=3n+6=3(n+2)=3k)$
So for any other triplet where number 3 is not involved, we will still get the 3q form from at least one of his members.
Since number 3 cant be involved anymore in any other possible triplet, but still one of his members is divisible by 3 this shows that there is not another possible prime triplet aside of 3,5,7 ■

7. Prove that for any natural number n,

$$2 + 2^2 + 2^3 + \dots + 2^n = 2^{n+1} - 2$$

By PMI

First case:

$$\sum_{i=1}^{1} 2^i = 2 = 2^{1+1} - 2 = 2$$

Hypothesis:

Assume: $$\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$$

Thesis:

$$\sum_{i=1}^{n+1} 2^i = 2^{n+2} - 2$$

By hypothesis , $\sum_{i=1}^{n} 2^i = \underbrace{\boxed{2^{n+1} - 2}}_{\boxed{2+2^2 \dots + 2^n}}$ and we add the next number in the sequence wich is 2^{n+1}

So, $2^{n+1} - 2 + 2^{n+1} = 2(2^{n+1}) - 2$

By law of exponents,

$$2^1(2^{n+1}) = 2^{1+n+1} = 2^{n+2}$$

Thus, $2^{n+1} - 2 + 2^{n+1} = 2^{n+2} - 2$ wich proves the statement in our thesis ■

Prove (from the definition of a limit of a sequence) that if the sequence $\{a_n\}_{n=1}^{\infty}$ tends to limit L as $n \to \infty$, then for any fixed number $M > 0$, the sequence $\{Ma_n\}_{n=1}^{\infty}$ tends to the limit ML.

We have the sequence $\{a_n\}_{n=1}^{\infty}$

$\{a_n\}_{n=1}^{\infty} = a_1, a_2, a_3 \ldots$ to "some a" as $n \to \infty$ in other words, we get closer to some value a

This means that the value of the sequence get arbitrarily closer and closer to some value a

Therefore "some a" is the limit, that is L

"Some a"=L

Considering M as a value that does not change, let say M=2, we have the following:

$\{Ma_n\}_{n=1}^{\infty} = \{2a_n\}_{n=1}^{\infty} = 2a_1, 2a_2, \ldots 2a_n$

The sequence goes toward "some a" multiplied by 2. We can see this as:

"some a" multiplied by M

If "Some a"=L and $\{Ma_n\}_{n=1}^{\infty}$ goes to "some a" multiplied by M

Then $\{Ma_n\}_{n=1}^{\infty}$ tends to ML

By definition:

$(\exists\varepsilon>0)(\exists n\in N)(\forall m\geq n)[|a_m$-"some a"$|<\varepsilon]$

Hypothesis: $|a_m$-"some a"$|<\varepsilon_1$

Thesis: $|Ma_m$ -M("some a")$| < \varepsilon_2$

$|Ma_m$- M("some a")$|=|$M(a_m-"some a")$|=|M|*|a_m$-"some a"$|=$ M*$\underbrace{|a_m\text{-"some a"}|}_{\boxed{<\varepsilon_1 \text{ from our hypothesis}}} <$ M

taken $M\varepsilon_1$ as ε_2

$|Ma_m$ -M("some a")$| < \varepsilon_2$ ■

9. Given an infinite collection A_n , n = 1, 2, . . . of intervals of the real line, their intersection is defined to be

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

Give an example of a family of intervals $A_n, n = 1, 2, \ldots$, such that $A_{n+1} \subset A_n$ for all n and $\bigcap_{n=1}^{\infty} A_n = \emptyset$. Prove that your example has the stated property.

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid (\forall n)(x \in A_n)\}$$

A_2={2,4,8....}={x/x=2k} n=1
A_4={4,8,12...}={x/x=4k} n=2
A_6={6,12,18...}={x/x=6k} n=3

A_{2n}={2_n,2*2_n,3*2_n,....} "some n" goes to infinty

$$\bigcap_{n=1}^{\infty} A_{2n} = \{x / \forall x \in A_{2kn}\}\ ,k\in N$$

Suppose that n is infinetely large with n∈N
let n=10 A_2={2,4,6,8,10,12,14,16,18,20}
A_4={4,8,12,16,20,24,28,32,36,40}
A_6={6,12,18,24,30,36,42,48,54,60}
A_8={8,16,24,32,40,48,56,64,72,80}

$A_2 \cap A_4$={4,8,12,16,20}
$A_2 \cap A_4 \cap A_6$={12}
$A_2 \cap A_4 \cap A_6 \cap A_8$={}

With this example we can say that A (with n goes to infinity)will have at least one element that is not in the previus sequences

Thus, in some point we get a empty set. $\blacksquare$

10. Give an example of a family of intervals An, n = 1, 2, . . ., such that An+1 ⊂ An for all n and
T∞
n=1 An consists of a single real number. Prove that your example has the stated property

$A_2 \cap A_4 \cap A_6$={12}

Proved in problem 9