UNIVERSITÀ DI TRENTO

SINGLE TRACK MODEL AND STEERING BEHAVIOR

Vehicle Dynamics, Plannig and Control of Robotic cars

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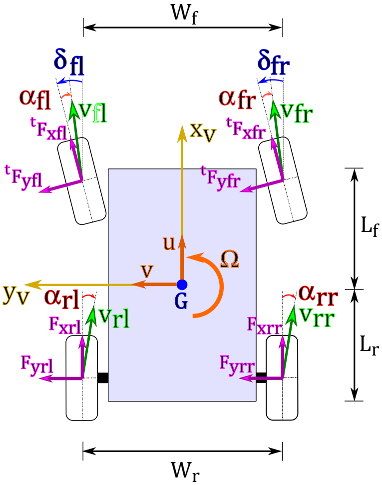
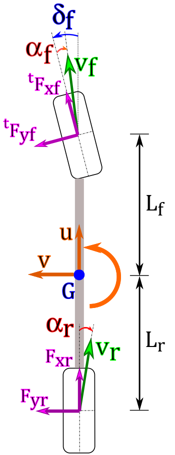
Overview

1. Overview and assumptions
2. Single track model
3. Steering and handling behavior
4. Handling diagram

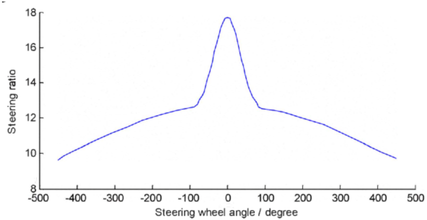
Overview and assumptions

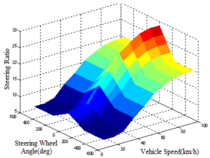
From double track to single track





Single track model assumptions - 1/2





- In general, the **steering angle** δ_D applied by the driver produces a steering angle $\delta_f = \delta$ at the front wheel of a single track model: $\delta_D = \tau_D \delta$.
- τ_D is a transmission ratio usually $\in [10, 20]$
- Since δ is *small*, the steering angles for front right and front left wheels $\{\delta_{fr}, \delta_{fl}\}$ are almost the same and equal to δ

- Starting from $\alpha_{ij} = -\arctan\left(\frac{v_{cy_{ij}}}{v_{cx_{ij}}}\right)$, $i \in \{f, r\}$, $j \in \{r, l\}$
- If δ and the chassis side slip angle β are *small*.
- $|u| >> |\Omega W_r/2|$, $|u| >> |\Omega W_f/2|$, while $\delta(L_f\Omega)$ is negligible
- linearized expressions of side slip angles are (refer to the double track model slides):

$$\begin{cases} \alpha_{rr} = \alpha_{rl} = \alpha_r = -\beta + \frac{L_r\Omega}{u} \\ \alpha_{fr} = \alpha_{fl} = \alpha_f = -\beta - \frac{L_f\Omega}{u} + \delta \end{cases}$$

Notice that linearized side slip angles are equal for both wheels within the same axle (rear or front)

Single track model

Single track model equations - 1/2

- Total **axle forces**

$$
\begin{cases}
F_{xf} = \overbrace{{}^tF_{x_{fl}}\cos(\delta) - {}^tF_{y_{fl}}\sin(\delta)}^{\hat{F}_{x_{fl}}} + \overbrace{{}^tF_{x_{fr}}\cos(\delta) - {}^tF_{y_{fr}}\sin(\delta)}^{\hat{F}_{x_{fr}}} \\
F_{xr} = \hat{F}_{x_{rr}} + \hat{F}_{x_{rl}} \\
F_{yf} = \overbrace{{}^tF_{y_{fl}}\cos(\delta) + {}^tF_{x_{fl}}\sin(\delta)}^{\hat{F}_{y_{fl}}} + \overbrace{{}^tF_{y_{fr}}\cos(\delta) + {}^tF_{x_{fr}}\sin(\delta)}^{\hat{F}_{y_{fr}}} \\
F_{yr} = \hat{F}_{y_{rr}} + \hat{F}_{y_{rl}}
\end{cases}
$$

- **Dynamic equations** balance of force

$$
\begin{cases}
ma_x = m(\dot{u} - \Omega v) = F_{xr} + F_{xf} - F_{Ax} \\
ma_y = m(\dot{v} + \Omega u) = F_{yr} + F_{yf}
\end{cases}
$$

- **Dynamic equations** of balance of moments

$$\begin{cases} I_{zz}\dot{\Omega} = F_{yf}L_f - F_{yr}L_r + \cancelto{0}{\Delta F_{xf}W_f} + \cancelto{0}{\Delta F_{xr}W_r} + \sum_{i=1}^{4} M_{i,z} \end{cases}$$

- **Wheel dynamics**:

$$\begin{cases} (I_{w_{fl}} + I_{w_{fr}})\left(\frac{\mathrm{d}\omega_f(t)}{\mathrm{d}t}\right) = -F_{x_f}R_f + (T_{w_{fl}} + T_{w_{fr}}) \\ (I_{w_{rl}} + I_{w_{rr}})\left(\frac{\mathrm{d}\omega_r(t)}{\mathrm{d}t}\right) = -F_{x_r}R_r + (T_{w_{rl}} + T_{w_{rr}}) \end{cases}$$

Steady state single track model - Lateral Forces

- At *steady-state* it is $(\dot{u}, \dot{v}, \dot{\Omega}) = (0, 0, 0)$, so that

$$\begin{cases} ma_x = m\Omega v = F_{xr} + F_{xf} - F_{Ax} \\ ma_y = m\Omega u = F_{yr} + F_{yf} \\ 0 = F_{yf} L_f - F_{yr} L_r + \sum_{i=1}^{4} M_{i,z} \end{cases}$$

- Neglecting self-aligning torques $M_{i,z}$, the steady-state lateral axle forces are

$$\begin{cases} F_{yf} = F_{yf}(a_y) = m a_y \frac{L_r}{L_f + L_r} \\ F_{yr} = F_{yr}(a_y) = m a_y \frac{L_f}{L_f + L_r} \end{cases}$$

- The **lateral force** (F^*_{yf}, F^*_{yr}) comes from the tire (see Pacejka)

$$\begin{cases} F_{yf} \approx F^*_{yf} = \text{pacejka\_model}(\alpha_f) \\ F_{yr} \approx F^*_{yf} = \text{pacejka\_model}(\alpha_r) \end{cases}$$

Steady state single track model - Vertical Forces

- The simplified **vertical loads** for each axle are

$$
\begin{cases}
F_{zf} = mg\frac{L_r}{L} + F_{Azf} - \cancelto{0}{F_x\frac{h_G}{L}} + \cancelto{0}{\frac{I_{yz}}{L}\dot{\Omega}} \\
F_{zr} = mg\frac{L_f}{L} + F_{Azr} + \cancelto{0}{F_x\frac{h_G}{L}} - \cancelto{0}{\frac{I_{yz}}{L}\dot{\Omega}}
\end{cases}
$$

- We define **vertical static loads** for each axle are

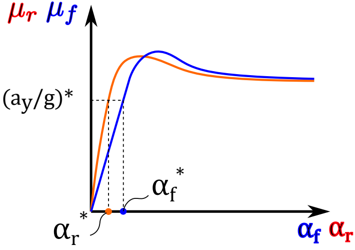
$$
\begin{cases}
F_{zf0} = mg\frac{L_r}{L_f+L_r} \\
F_{zr0} = mg\frac{L_f}{L_f+L_r}
\end{cases}
$$

Steering and handling behavior

- We can normalize the steady-state F^*_{yr} and F^*_{yf} with the static vertical loads

$$\begin{cases} \dfrac{F^*_{yf}}{F_{zf0}} = \mu_f(\alpha_f) \approx \dfrac{F_{yf} = ma_y \frac{L_r}{L_f+L_r}}{mg\frac{L_r}{L_f+L_r}} = \dfrac{a_y}{g} \\ \dfrac{F^*_{yr}}{F_{zr0}} = \mu_r(\alpha_r) \approx \dfrac{F_{yf} = ma_y \frac{L_f}{L_f+L_r}}{mg\frac{L_f}{L_f+L_r}} = \dfrac{a_y}{g} \end{cases}$$

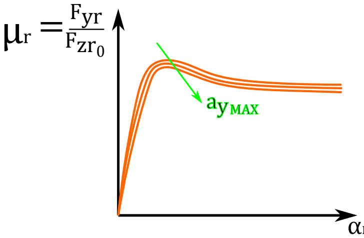
- μ_r and μ_f are called **axle characteristic curves**, and they are non linear functions of side slips α_r and α_f.



- **Axle characteristics** depend not only on tire performance (e.g. modeled with Pacejka formulas), but also on suspensions and other effects
- Given $\{\mu_r, \mu_f\}$ curves, it is possible to (graphically or numerically) compute the axle side slips $\{\alpha_r, \alpha_f\}$ in correspondence of a certain value of a_y/g

Axle characteristic curves - 3/3

- Notice that the linearized expressions of side slip angles $\{\alpha_r, \alpha_f\}$ are employed, but the dependency of μ_r and μ_f upon α_r and α_f is *non* linear.
- $\{\mu_r, \mu_f\}$ also depend on $a_{y_{MAX}}$, u etc.... As $a_{y_{MAX}}$ increases, lateral load transfer increases, but the total lateral force within front and rear axles $\{F_{yr}, F_{yf}\}$ slightly decreases. This is due to the saturation effect of lateral tire forces with vertical tire load.



Steering characteristic

- Target → find the relation between the **driver steering angle** $\delta_D = \tau_D \delta$ and the **curvature radius** of vehicle trajectory
- Starting from the linearized side slip angles

$$\begin{cases} \alpha_r = -\beta + \frac{L_r \Omega}{u} \\ \alpha_f = -\beta - \frac{L_f \Omega}{u} + \delta \end{cases}$$

- The **steering characteristic** is obtained by doing

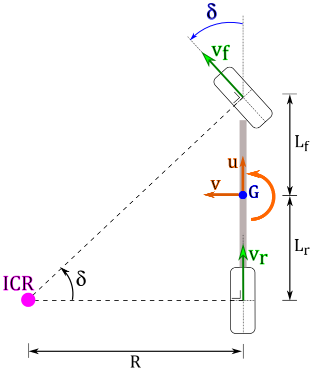
$$\alpha_r - \alpha_f = \frac{\Omega(L_r + L_f)}{u} - \delta = \frac{\Omega L}{u} - \delta$$

Remember that, from kinematic analysis, $\frac{\Omega}{u}$ is the steady-state vehicle trajectory curvature ρ

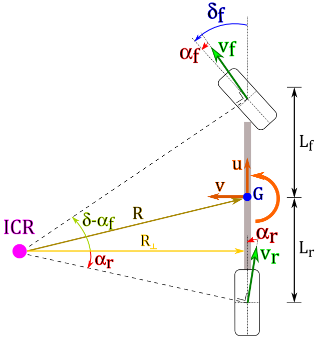
- The **steering characteristic** can be rewritten as

$$\rho L = \delta + \alpha_r - \alpha_f$$

- $\Delta\alpha = \alpha_r - \alpha_f \rightarrow$ if $\Delta\alpha = 0$ then **perfect kinematic steering** law $\rightarrow \rho = \delta/L, \quad \delta = L/R$



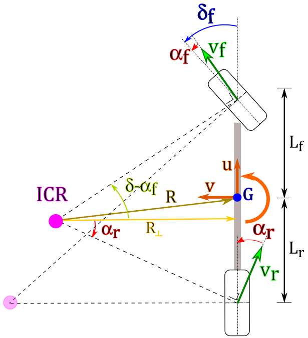
- Notice that $\Delta\alpha$ can be zero even if $\alpha_r = \alpha_f \neq 0$
- If $\Delta\alpha = 0 \longrightarrow \delta = \frac{L}{R}$ is the kinematic or **Ackerman steering angle**, and the vehicle behavior is **neutral** $\longrightarrow$ desirable for the driver since Ω is directly proportional to $\delta \longrightarrow \Omega = \frac{\delta u}{L}$



- If $\alpha_r > \alpha_f$, i.e. $\Delta\alpha > 0$, then the vehicle is **oversteering** (OS)

$$\frac{L}{R} = \delta + \alpha_r - \alpha_f \implies R = \frac{L}{\delta + \alpha_r - \alpha_f}$$

- For the same steering angle δ, the trajectory radius R is reduced if $\Delta\alpha > 0 \rightarrow$ the vehicle *steers more than expected*

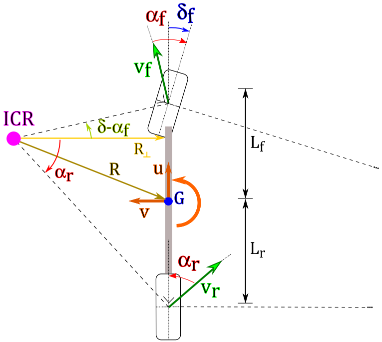


Oversteering vehicle - 2/3

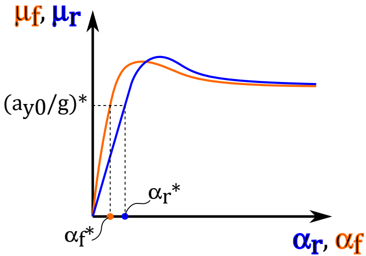
- Equivalently, to achieve a certain circular trajectory radius, a smaller steering angle δ is needed

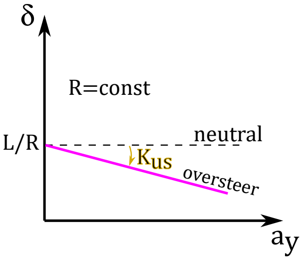
$$\delta = \frac{L}{R} - \Delta\alpha$$

- $\frac{L}{R} - \Delta\alpha$ can even be $< 0 \rightarrow$ the driver needs to counter-steer to avoid losing adherence



- **Understeering gradient** K_{us} → it is < 0 if oversteering vehicle → $\delta - \frac{L}{R} = -\Delta\alpha = \alpha_f - \alpha_r = K_{us} a_y$
- K_{us} is the slope of the curve describing the evolution of δ as a_y increases. The behavior is usually ≈ linear only up to a certain value of a_y

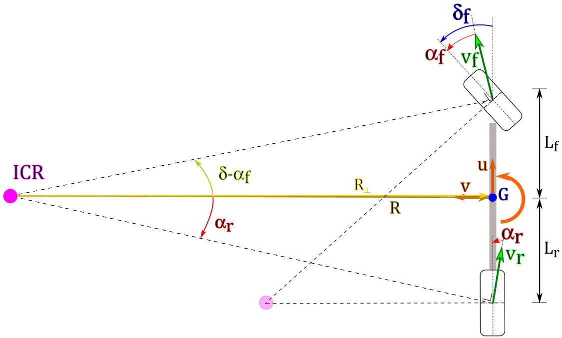




- If $\alpha_r < \alpha_f$, i.e. $\Delta\alpha < 0$, then the vehicle is **understeering** (US)

$$\frac{L}{R} = \delta + \alpha_r - \alpha_f \Longrightarrow R = \frac{L}{\delta + \alpha_r - \alpha_f}$$

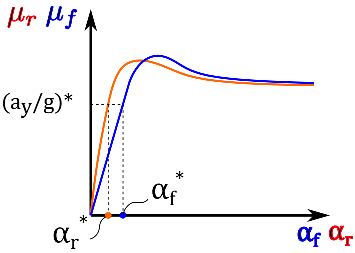
- For the same steering angle δ, the trajectory radius R is increased if $\Delta\alpha < 0 \longrightarrow$ the vehicle *steers less than expected*

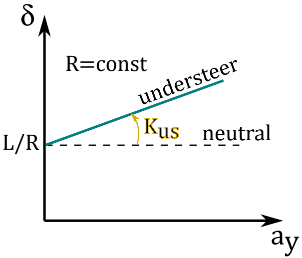


- Equivalently, to achieve a certain circular trajectory radius, a higher steering angle δ is needed

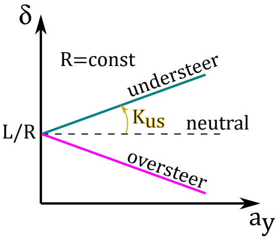
$$\delta = \frac{L}{R} - \Delta\alpha$$

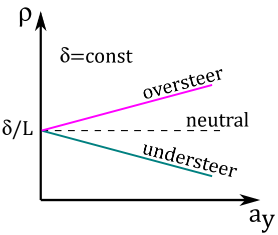
- Understeering gradient K_{us} is > 0 if understeering vehicle



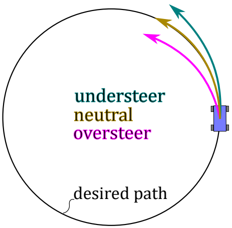


Oversteer vs. understeer





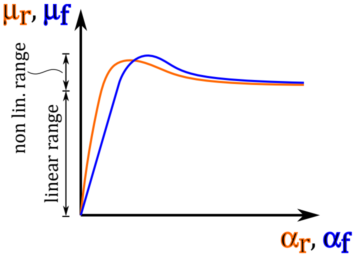
For the same δ, $R_{OS} < R_{US}$



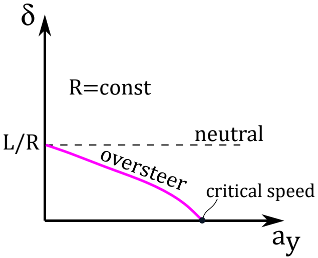
https://www.youtube.com/watch?v=lIUPYUjSNJI

Non linear steering behavior - 1/4

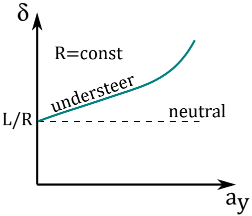
- Remember that the analysis for the steering behavior only refers to **steady-state** conditions → transient effects are not included
- Axle characteristics (like Pacejka tire curves) show an almost linear behavior up to certain values of axle side slips $\{\alpha_r, \alpha_f\}$

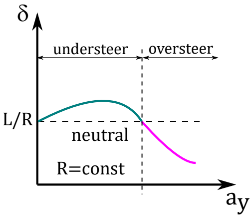


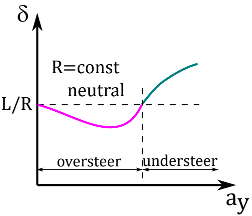
- As the lateral acceleration a_y increases, the steering behavior may change, and K_{us} becomes a non linear function of $a_y \rightarrow \delta - \frac{L}{R} = K_{us}(a_y)$
- For example, for high a_y an oversteering vehicle may become even more oversteering $\rightarrow$ a critical speed may exist, at which the driver needs to counter-steer to keep turning $\rightarrow u_{cr} = \sqrt{a_{y_{cr}} R} \rightarrow$ the vehicle gets *unstable*



- It is also possible that an understeering vehicle gets more understeering at high a_y, or that an oversteering vehicle becomes understeering for large a_y (or viceversa)

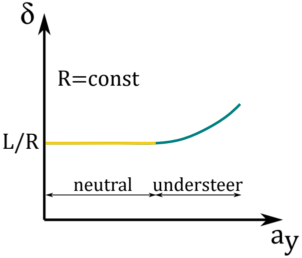


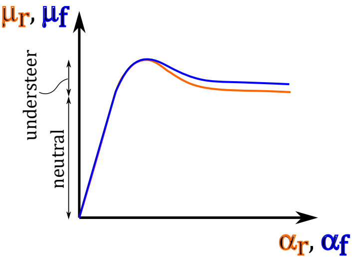




Non linear steering behavior - 4/4

- It may even happen that a vehicle has a neutral steering behavior for low a_y, while it becomes e.g. understeerig for high a_y



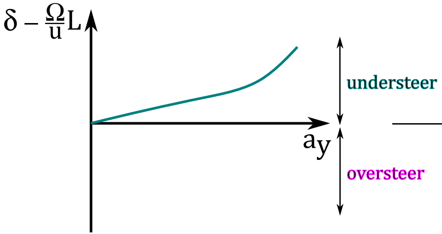


Comments about under/oversteer

- Under/oversteer depends on suspensions, tires, aerodynamics, differential and other factors
- Identifying the vehicle steering behavior is fundamental to control a self-driving car $\longrightarrow$ **predict vehicle yaw rate** Ω when an **input steering angle** is applied
- The **neutral steering behavior** is always **preferable** (since more predictable) even though it may be hard to obtain
- **Oversteer** makes the vehicle **unstable**, so it is to be avoided in passenger vehicles. In racing vehicles, oversteer may be used to achieve the max performance since it makes the vehicle more reactive

Handling diagram

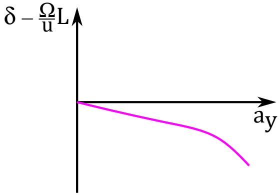
- The **handling diagram** is an important tool that allows to assess the steering behavior → it shows the deviation of the real steering angle δ from the perfect kinematic steering angle $\frac{\Omega}{u}L = \frac{L}{R} = \delta$, as a function of a_y
- The diagram can be obtained by means of experimental tests (steering maneuvers) carried out with a real vehicle or a vehicle model in simulations

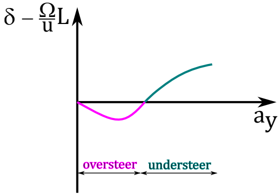


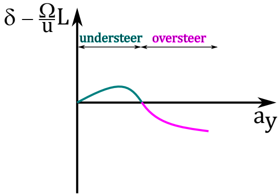
→ handling diagram for an understeering vehicle

Handling diagram - 2/4

- Different vehicles may have very different handling diagrams, depending on their under/oversteering tendency. Some examples are here shown

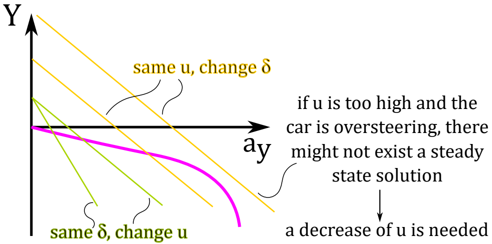






- The **handling diagram** (like all the steering analysis) refers to *steady-state* conditions, for which $a_y = \Omega u$, and $\delta - \frac{\Omega}{u}L = -\Delta\alpha = K_{us}(a_y) \longrightarrow$ it is possible to fit the diagram using a non linear function $K_{us}(a_y)$, e.g. n[th] order polynomial $K_{us}(a_y) = K_{us_1} a_y + K_{us_2} a_y^2 + \cdots + K_{us_n} a_y^n$
- Often the handling curve also needs to be fitted as a function of u and, especially if the vehicle is endowed with a mechanical/electronic differential, also as a function of a_x. In the most general case we have a *4D handling surface*, with $K_{us} = K_{us}(a_y, u, a_x)$

- The **handling diagram** can be used to determine steady-state cornering operating conditions, which are provided by the intersection of lines described by $Y = \delta - \frac{\Omega}{u}L = \delta - \frac{a_y}{u^2}L$ with the handling curve $K_{us}(a_y)$



Identification of the handling diagram

- The handling diagram can be identified with suitable **maneuvers**, to be possibly carried out in simulation and then compared with the real vehicle data
- Maneuvers have to be designed so that the vehicle operates in conditions very **close to steady-state** $\longrightarrow$ avoid abrupt variations of δ and u
- Two possible maneuvers are the *constant steer US test* and the *sine steer US test*
- These maneuvers can be performed by a human driver or by a steering robot

Constant steer US test

- This test is performed at **constant** steering angle, while vehicle speed is **smoothly increased**, for example with a smooth linear ramp (trying to preserve steady-state and avoid transient effects)
- Remember that δ is the steering angle at the front wheel (single track vehicle model), while the steering angle at the steering wheel applied by the driver is $\delta_D = \tau_D \delta$
- The quantities $\{\delta_D, \Omega, u, a_y\}$ are measured throughout the simulation/test (a_y may also be estimated as Ωu, since the maneuver preserves steady-state). The handling diagram is obtained by plotting a_y vs. $\delta - \frac{\Omega}{u} L = \frac{\delta_D}{\tau_D} - \frac{\Omega}{u} L$.
- Experimental points can then be fitted with a non linear function (e.g. a polynomial) K_{us}, as a function of $\{a_y, u, a_x\}$

Sine steer US test

- This test is performed with **sinusoidal steering angle**, while vehicle speed is kept constant. Sine frequency should be low to almost preserve steady-state and avoid transient effects
- The quantities $\{\delta_D, \Omega, u, a_y\}$ are measured throughout the simulation/test (a_y may also be estimated as Ωu)
- The handling diagram is obtained by plotting a_y vs. $\delta - \frac{\Omega}{u}L = \frac{\delta_D}{\tau_D} - \frac{\Omega}{u}L$.
- Experimental points can then be fitted with a non linear function (e.g. a polynomial) K_{us}, as a function of $\{a_y, u, a_x\}$. Notice that, in general, if a_y is *small* then the handling diagram should be fitted quite well with a line.

References

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