# ELEC-E8103 - Modelling, Estimation and Dynamic Systems

## Assignment 1

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# 1 Exercise 1: Car Suspension System

## 1.1 Part a

The output of the system is $z_2$. The input is the displacement of the road $d$. The constants of the systems are $m_1$, $m_2$, $k_w$, $k_s$ and $b$. The internal time varying variable of the system is $z_1$.

## 1.2 Part b & c

Starting with mass 2 $m_s$ and the spring force $F_s$ and the damper force $F_d$, the following equations are derived:

$$m_2 \cdot \ddot{z_2} = -k_s \cdot (z_2 - z_1) - b \cdot (\dot{z_2} - \dot{z_1}) \tag{1}$$

While for mass 1 $m_1$ the force related to the spring $k_w$ needs to be taken into account as well. Therefore the following equation is derived:

$$m_1 \cdot \ddot{z_1} = -k_w \cdot (z_1 - d) + k_s \cdot (z_2 - z_1) + b \cdot (\dot{z_2} - \dot{z_1}) \tag{2}$$

However for the state space representation a system of first order differential equations is required, thus two new variables are introduced.
$z_3 = \dot{z_2}$ & $z_4 = \dot{z_1}$
Therefore a system of four first order differential equations can be derived, after rearranging Eq.[1] and Eq.[2], The following system of equations is obtained:

$$\begin{cases} \dot{z_1} = z_4 \\ \dot{z_2} = z_3 \\ \dot{z_3} = \frac{-k_s \cdot z_2}{m_2} + \frac{k_s \cdot z_1}{m_2} - \frac{b \cdot z_3}{m_2} + \frac{b \cdot z_4}{m_2} \\ \dot{z_4} = \frac{-k_w \cdot z_1}{m_1} + \frac{k_w \cdot d}{m_1} + \frac{k_s \cdot z_2}{m_1} - \frac{k_s \cdot z_1}{m_1} + \frac{b \cdot z_3}{m_1} - \frac{b \cdot z_4}{m_1} \end{cases} \tag{3}$$

From the aforementioned system of differential equations it is possible to derive its state space model. The general form of the state space model is the following:

$$\begin{cases} \dot{x} = A \cdot x + B \cdot u \\ y = C \cdot x + D \cdot u \end{cases}$$

In this particular case:

$$\dot{x} = \begin{bmatrix} \dot{z_1} \\ \dot{z_2} \\ \dot{z_3} \\ \dot{z_4} \end{bmatrix} \tag{4}$$

$$x = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \tag{5}$$

$$u = \begin{bmatrix} d \end{bmatrix} \tag{6}$$

$$y = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{7}$$

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ k_s/m_2 & -k_s/m_2 & -b/m_2 & b/m_2 \\ -(k_w + k_s)/m_1 & k_s/m_1 & b/m_1 & -b/m_1 \end{bmatrix} \tag{8}$$

$$B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ k_w/m_1 \end{bmatrix} \tag{9}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{10}$$

$$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{11}$$

### 1.3 Part d

When the input of the system is the following:

$$d(t) = e^{-t} \cdot sin(t) \tag{12}$$

The system response showing the displacement of the unsprung mass $z_1$ and the displacement of the sprung mass $z_2$ for a 10 seconds simulation time can be observed in the figure below:

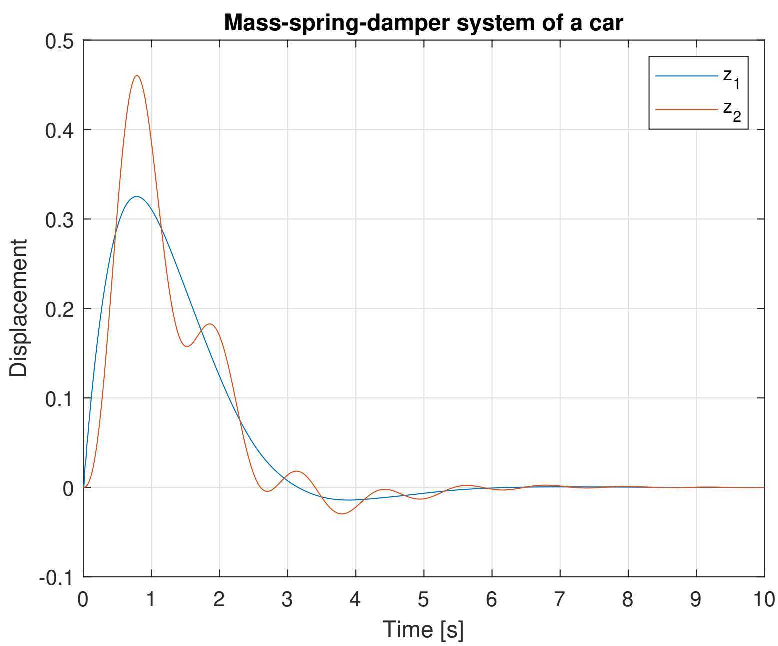


Figure 1: $z_1$ and $z_2$ behaviors over a 10 seconds time

The transfer function of the system can be found in the MatLab script.

## 2 Exercise 2: Water tank system

### 2.1 Part a

The output of the system is the output signal of the sensor measuring the water level in the tank $y(t)$. The input is the input signal $u(t)$ which can be controlled to regulate the inflow. The constants of the systems are represented by the cross-sectional areas $E$ and $F$. The internal time varying variables of the system are the disturbance $z(t)$, the outflow $q_{out}$ and the height of the water in the tank $h(t)$

## 2.2 Part b

The volumetric flow balance of the water is described by the following:

$$\begin{cases} \dot{h} = \frac{-E\cdot\sqrt{2g\cdot h}}{F} + \frac{u(t)\cdot q_{umax}}{F} + \frac{z(t)\cdot q_{zmax}}{F} \\ \dot{y} = \frac{1}{\tau}(h-y) \end{cases} \tag{13}$$

## 2.3 Part c

The state variables of the system are $h(t)$, as well as $u(t)$ and the disturbance $z(t)$ as those are the variables needed to predict the future behavior of the system.
Furthermore the system is non-linear as the output is not proportional to the input $u(t)$ due to the fact that the disturbance $z(t)$ is present.

## 2.4 Part d

As it is required that the water level is stationary around $h_{eq} = 1\ m$ , the amount of inflow water needs to be equal to the outflow. Therefore:

$$q_{Ueq} = q_{Oeq} - q_{Zeq} \tag{14}$$

where, $q_{Ueq}$ represents the water inflow from the valve controlled by the signal $u(t)$, $q_{Oeq}$ represents the water outflow rate and $q_{Zeq}$ represents the inflow of water caused by the disturbance signal $z(t)$. As the derivative of a constant is 0, it is possible to calculate the inflow rate through Eq.[13]. Thus it was calculated an inflow rate $q_{Ueq}$ to be equal to 0.012463480562321 $m^3/s$.

## 2.5 Part e

In order to obtain the linearized system around the equilibrium point, the following system of equations needs to be used:

$$\begin{cases} \Delta\dot{x} = A\Delta x + B\Delta u \\ \Delta y = C\Delta + D\Delta u \end{cases} \tag{15}$$

Where $\dot{x}$ is represented by $\dot{h}$ and $\dot{y}$ as those are related to the differential equations of the system, as described through Eq.[13]. Renaming the variables to adapt to the linearization form:
$h = x_1$ , $y = x_2$ , $u(t) = u_1$ and $z(t) = u_2$. Therefore the following is obtained: $\dot{h} = \dot{x_1}$ and $\dot{y} = \dot{x_2}$ Furthermore the output of the system is described by the following equation:

$$y = x_2 \tag{16}$$

The matrices A, B, C and D are Jacobian matrices of partial derivatives obtained following the relations below:

$$A = \begin{bmatrix} \frac{\delta\dot{x_1}}{\delta x_1} & \frac{\delta\dot{x_1}}{\delta x_2} \\ \frac{\delta\dot{x_2}}{\delta x_1} & \frac{\delta\dot{x_2}}{\delta x_2} \end{bmatrix} = \begin{bmatrix} \frac{-2E\sqrt{2g}x_1^{-1/2}}{F} & 0 \\ \frac{1}{\tau} & \frac{1}{\tau} \end{bmatrix} \tag{17}$$

$$B = \begin{bmatrix} \frac{\delta\dot{x_1}}{\delta u_1} & \frac{\delta\dot{x_1}}{\delta u_2} \\ \frac{\delta\dot{x_2}}{\delta u_1} & \frac{\delta\dot{x_2}}{\delta u_2} \end{bmatrix} = \begin{bmatrix} \frac{q_{Umax}}{F} & \frac{q_{Zmax}}{F} \\ 0 & 0 \end{bmatrix} \tag{18}$$

$$C = \begin{bmatrix} \frac{\delta y}{\delta x_1} & \frac{\delta y}{\delta x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{19}$$

$$D = \begin{bmatrix} \frac{\delta y}{\delta u_1} & \frac{\delta y}{\delta u_2} \end{bmatrix} \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{20}$$

Furthermore the matrix A is evaluated at the point. $x_1 = 1$. Moreover the following relations are needed to linearize the system:

$$
\begin{cases}
\Delta x_1 = x_1 - 1 \\
\Delta x_2 = x_2 - 1 \\
\Delta y = y - 1 \\
\Delta u_1 = u_1 - \frac{q_{Umax}}{q_{Ueq}} \\
\Delta u_2 = u_2 - 0.5
\end{cases}
\tag{21}
$$

Thus the final system of equations of the linearized system is the following:

$$
\begin{cases}
\Delta \dot{x_1} = -0.0886 \cdot x_1 + u_1 \cdot \frac{u_{eq} \cdot q_{Umax}}{F} + u_2 \cdot \frac{q_{Zmax}}{F} - \frac{q_{Ueq}}{F} - \frac{0.5 \cdot q_{Zmax}}{F} - \frac{-2E\sqrt{2g}}{F} \\
\Delta \dot{x_2} = 0.2x_1 - 0.2x_2 \\
\Delta y = x_2
\end{cases}
\tag{22}
$$

where $u_{eq}$ represents the input signal when the water level is stationary at 1 m.

## 2.6 Part f

The state space of the linearized system should be:

$$
\begin{cases}
\Delta \dot{x} = A \cdot \Delta x + B \cdot \Delta u + K_1 \\
\Delta y = C \cdot \Delta x + D \cdot \Delta u + K_2
\end{cases}
$$

Deriving it from Eq.[22], the matrices below are obtained:

$$
A = \begin{bmatrix} -0.0886 & 0 \\ 0.2 & 0.2 \end{bmatrix} \tag{23}
$$

$$
B = \begin{bmatrix} q_{Umax}/F & q_{Zmax}/F \\ 0 & 0 \end{bmatrix} \tag{24}
$$

$$
K_1 = \begin{bmatrix} -0.0593 \\ 0 \end{bmatrix} \tag{25}
$$

$$
C = \begin{bmatrix} 0 & 1 \end{bmatrix} \tag{26}
$$

$$
D = \begin{bmatrix} 0 & 0 \end{bmatrix} \tag{27}
$$

$$
K_2 = \begin{bmatrix} 0 \end{bmatrix} \tag{28}
$$

## 2.7 Part g

The transfer function of the system can be found in the MatLab script.

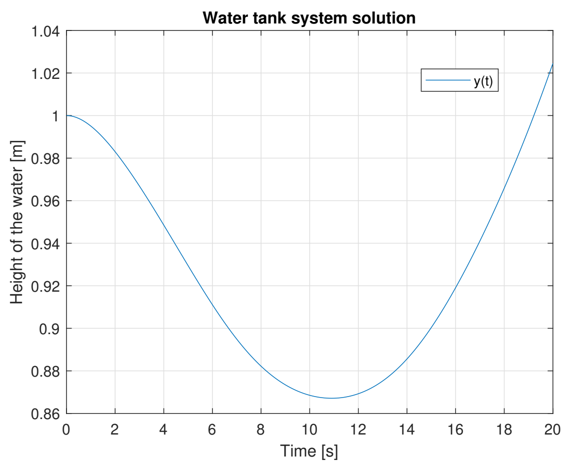


Figure 2: water level behavior over a 20 seconds time