Assignment 1

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1 Let D be a partial order. Prove or disapprove the following statements.

1.1 If $S_1 \subseteq D$ and $S_2 \subseteq D$ are directed, then so is $S_1 \cup S_2$.

This statement is false. Consider a partial order D as displayed in Figure 1. The $x \rightarrow y$ relation depicts a $x \leq y$ order.

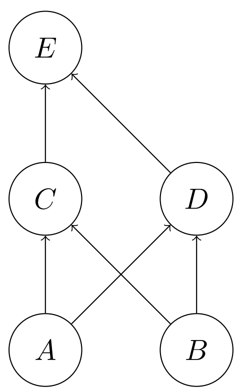
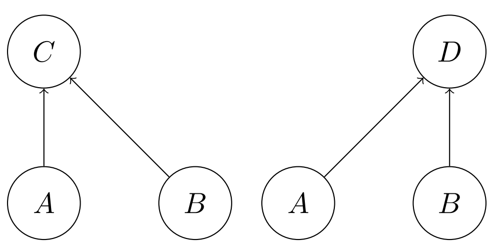


Figure 1: Partially orderd set D

Let there be two subsets S_1 = {A, B, C} and S_2 = {A, B, D} as displayed in Figures (a) and (b).



(a) Subset S_1 (b) Subset S_2

The union of S_1 and S_2 would not directed, be since (C, D) does not have an upper bound in $S_1 \cup S_2$ as displayed in Figure 2

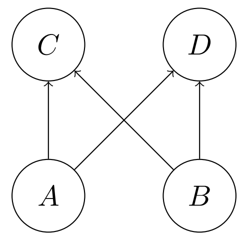


Figure 2: Union of S_1 and S_2

1.2 If $S_1 \subseteq D$ and $S_2 \subseteq D$ are directed, then so is $S_1 \cap S_2$.

This statement is false. Considering the two subsets S_1 and S_2 from 1a, the intersection would be {A, B} as displayed in Figure 3. Obviously, these two elements do not have an upper bound in $S_1 \cap S_2$.

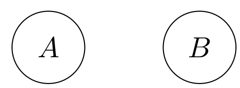


Figure 3: Intersection of S_1 and S_2

1.3 For every $d \in D$, the set $\{s \in D | s \leq d\}$ is directed.

This statement is true. Let S be the set $\{s \in D | s \leq d\}$. Let for this proof $\cap C$ stand for "*the set of upper bounds of C*" and $\cap_D C$ for "*the set of upper bounds of C in D*" for $C \subseteq D$. Assuming that S is not directed, it must be that

$$\exists (a, b) \mid a, b \in S \tag{1.1}$$

such that

$$\cap_S \{a, b\} = \emptyset \tag{1.2}$$

Since

$$S = \{s \in D | s \leq d\} \mid d \in D \tag{1.3}$$

it must be that

$$\begin{aligned} &\exists d \in D \mid \forall s_i \in S, d \leq s_i && (1.4)\\ \Rightarrow &d \leq a \text{ and } d \leq b && (1.5)\\ \Rightarrow &d \in \cap_S \{a, b\} && (1.6)\\ \Rightarrow &\cap_S \{a, b\} \neq \emptyset && (1.7) \end{aligned}$$

contradicting equation 1.2. Consequently, S must be directed!

1.4 For every $d \in D$, the set $\{s \in D | d \leq s\}$ is directed.

This statement if false. Consider the subset S_3 of the poset D from Figure 1 containing {A, C, D} as displayed in Figure 4 The statement does allow choosing A for d which would result in the entire poset. Consequently, the poset itself would need to be directed. Obviously, C and D have no upper bound, thus the poset is not directed.

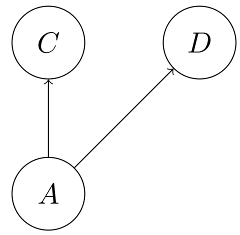


Figure 4: Subset S_3

2 For a cpo D, consider the function $Apply : [D \to D] \times D \to D$ for which $Apply(f, x) = f(x)$. Prove that $Apply$ is continuous.

As established in class, $[D \to D]$ denotes a ccpo of continuous functions.
Let F a be a subset of $[D \to D]$ such that it is a chain. Let furthermore C be a subset of D such that it is a chain. Since F is a set of continuous functions, it must be true that

$$\sqcup \{f\ C\} = f\ \sqcup C \mid \forall f \in F \tag{2.1}$$

When comparing function spaces such as F it is for all $f_i, f_{i+1} \in F$ that

$$f_i \leq f_{i+1} \text{ if } f_i\ x \leq f_{i+1}\ x \mid \forall x \in D \tag{2.2}$$

Since F is a ccpo of functions, it must be that

$$\exists \sqcup F \text{ such that } f \leq \sqcup F \mid \forall f \in F \tag{2.3}$$
$$\Rightarrow f\ x \leq \sqcup F\ x \mid \forall f \in F, \forall x \in D \tag{2.4}$$
$$\Rightarrow \sqcup \{f\ x \mid \forall f \in F\} = \sqcup F)\ x \mid \forall x \in D \tag{2.5}$$
$$\Rightarrow \sqcup \{F\ x\} = (\sqcup F)\ x \mid \forall x \in D \tag{2.6}$$

In order for *Apply* to be continuous, it must be that for all $f \in F$ and $x \in C$

$$\sqcup\, Apply\ (f, x) = Apply\ \sqcup (f, x) \tag{2.7}$$

It is given that a function is continuous if it is continuous in each of its parameters. Therefore, if *Apply* is continuous in its parameters, it must be continuous itself as well. By fixing single parameters, partially applied functions can be constructed that have the same properties (such as the continuity) for the remaining parameters as *Apply*. Consequently, by fixing f, the partially applied function $Apply_f$ can be constructed, that allows the evaluation of the continuity of *Apply* in x. Respectively, $Apply_x$ allows the evaluation of the continuity of *Apply* in f by fixing x. If both $Apply_f$ and $Apply_x$ are contiuous, *Apply* must be continuous, too. Now in order for $Apply_f$ and $Apply_x$ to be continuous, it must be true that

$$\sqcup \{Apply_f\ C\} = Apply_f\ \sqcup C \mid \forall f \in F \tag{2.8}$$
$$\sqcup \{Apply_x\ F\} = Apply_x\ \sqcup F \mid \forall x \in C \tag{2.9}$$

Since $Apply\ (f, x) = f\ x$, any $Apply_f\ x$ can also be expressed as $f\ x$. Thus, equation 2.8 results to

$$\sqcup \{f\ C\} = f\ \sqcup C \mid \forall f \in F \tag{2.10}$$

which is provided by equation 2.1. Consequently it is that $Apply_f$ is continuous, implying that $Apply$ **is continuous in** x. Furthermore, $Apply\ (f, x) = f\ x$ requires that equation 2.9 results to

$$\sqcup \{F\ x\} = (\sqcup F)\ x \mid \forall x \in D \tag{2.11}$$

which is provided to equation 2.6. Consequently it is that $Apply_x$ is continuous, implying that $Apply$ **is continuous in** f. Thus it is proven that $Apply$ is continuous both in f and in x, Subsequentially, it must be true that $Apply$ **is continuous**.

3 Let D be a cpo. An element $x \in D$ is called compact if for every ascending chain $S \subseteq D$ one has: $x \sqcup S$ implies $x \leq s$ for some $s \in S$. Show that if x and y are compact and they have a least upper bound in D, then the least upper bound is also compact.

Since x and y are compact, it is given that for all ascending chains $C \subseteq D$

$$\text{if } x \leq \sqcup C \text{ then } \exists c \in C \text{ such that } x \leq c \tag{3.1}$$

with the same being true for y instead of x.
Let's say there is a least upper bound p for {x, y} such that

$$x \leq p \wedge y \leq p \tag{3.2}$$

Assume p is not compact, it must be that $\exists C_i \subseteq D$ so that

$$p \leq \sqcup C_i \tag{3.3}$$
$$\nexists c \in C_i \text{ such that } p \leq c \tag{3.4}$$

Due to equation 3.2 and 3.3, it must be that

$$x \leq \sqcup C_i \wedge y \leq \sqcup C_i \tag{3.5}$$

Since x and y are compact and due to equation 3.1, it must also be true that

$$\exists c_1 \in C_i \text{ such that } x \leq c_1 \tag{3.6}$$
$$\exists c_2 \in C_i \text{ such that } y \leq c_2 \tag{3.7}$$

With c_1 and c_2 being elements of the same ascending chain, it is that

$$\exists c_i \in C_i = \sqcup\{c_1, c_2\} \tag{3.8}$$
$$\Rightarrow x \leq c_i \wedge y \leq c_i \tag{3.9}$$
$$\Rightarrow c_i \text{ is an upper bound of } \{x, y\} \tag{3.10}$$

Since p is established to be the least upper bound, it must be that

$$p \leq c_i \text{ with } c_i \in C_i \tag{3.11}$$
$$\Rightarrow \exists c_i \in C_i \text{ such that } p \leq c_i \tag{3.12}$$

This contradicts equation 3.4, **thus forcing p to be compact**!

4 A partially ordered set D is a complete lattice if every subset S of D has a least upper bound $\sqcup S$. Show that in a complete lattice D, an element e is compact if and only if for all $S \subseteq D$: $e \leq \sqcup S$ implies $e \leq \sqcup X$ for some finite $X \subseteq S$.

5 Provide a denotational semantics for "for" loops given in Assignment 1.

The for loop can for this scenario be rewritten as follows:

$$\begin{aligned} for \rightarrow& \text{for } i := E_0 \text{ to } E_1 \text{ step } E_2 \text{ do } St && (5.1)\\ \rightarrow& init; loop && (5.2)\end{aligned}$$

where

$$\begin{aligned} init \rightarrow& i := 0 && (5.3)\\ inc \rightarrow& i = i + E_2 && (5.4)\\ loop \rightarrow& \text{while } i <= E_1 \text{ do } (St; inc) && (5.5)\end{aligned}$$

Let S be the set of all statuses. The attribute "sc" is the statechange of elements, while "eval" evaluates expressions based on a given status. Consequently it must be that

$$for.sc = \lambda s \in S : (loop.sc \circ init.sc)(s) \tag{5.6}$$

with

$$\begin{aligned} init.sc =& \lambda s \in S : s[i \leftarrow 0] && (5.7)\\ inc.sc =& \lambda s \in S : s[i \leftarrow s(i) + E_2.eval(s)] && (5.8)\\ loop.sc =& \lambda s \in S : \text{cond}(i <= E_1, (inc.sc \circ loop.sc \circ St.sc)(s), s) && (5.9)\\ =& \text{FIX } F && (5.10)\\ where\ F =& \lambda g : \lambda s \in S : \text{cond}(i <= E_1, (inc.sc \circ g \circ St.sc)(s), s) && (5.11)\\ =& \lambda g : \lambda s \in S : \text{cond}(i <= E_1, ((\lambda x \in S : x[i \leftarrow x(i) + E_2.eval(x)]) \circ g \circ St.sc)(s), s) && (5.12)\end{aligned}$$

Putting these into equation 5.6 results to the denotational semantics for the *for* loop

$$\begin{aligned} for.sc =& \lambda s \in S : (\text{FIX } F)(s[i \leftarrow 0]) && (5.13)\\ where\ F =& \lambda g : \lambda s \in S : \text{cond}(i <= E_1, ((\lambda x \in S : x[i \leftarrow x(i) + E_2.eval(x)]) \circ g \circ St.sc)(s), s) && (5.14)\end{aligned}$$

6 Provide a denotational semantics for "case" statements as specified in Assignment 1.

6.1 I $\rightarrow$ n : St

$$I.statechange = \lambda s.\lambda m. \begin{cases} m = n & St.statechange(s) \\ m \neq n & s \end{cases} \tag{6.1}$$

6.2 Cl $\rightarrow$ I

$$Cl.statechange = I.statechange \tag{6.2}$$

6.3 $\mathsf{Cl}_0 \rightarrow \mathsf{Cl}_1$; I

$$Cl_0.statechange = \lambda s.\lambda m.(I.statechange \circ Cl_1.statechange(s))(m) \tag{6.3}$$

6.4 <u>case</u> E <u>of</u> Cl

$$Cs.statechange = \lambda s.(Cl.statechange(s) \circ E.evaluate)(s) \tag{6.4}$$