Assignment 1

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Introduction

Let for this assignment be

$$\begin{aligned}
\cup S &: \text{all upper bounds of } S \\
&\Rightarrow \forall u_i \in \cup S : \forall s_i \in S, s_i \leq u_i \\
\cup_D S &: \text{all upper bounds of } S \text{ in } D \\
&\Rightarrow \forall u_i \in \cup_D S \exists u_i \in D : \forall s_i \in S, s_i \leq u_i
\end{aligned}$$

1 Let D be a partial order. Prove or disapprove the following statements.

1.1 If $S_1 \subseteq D$ and $S_2 \subseteq D$ are directed, then so is $S_1 \cup S_2$.

This statement is false. Consider a partial order D as displayed in Figure 1. The $x \rightarrow y$ relation depicts a $x \leq y$ order.

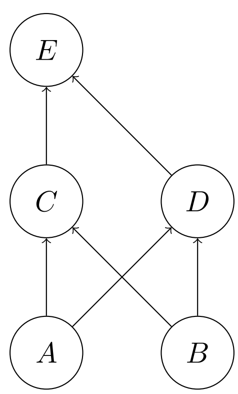
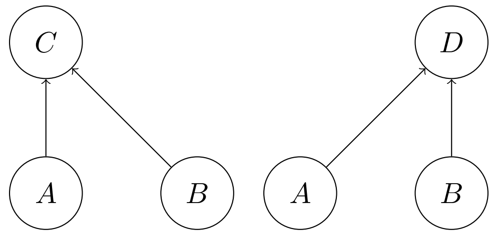


Figure 1: Partially orderd set D

Let there be two subsets $S_1 = \{A, B, C\}$ and $S_2 = \{A, B, D\}$ as displayed in Figures (a) and (b). The union of S_1 and S_2 would not directed, be since (C, D) does not have an upper bound in $S_1 \cup S_2$ as displayed in Figure 2



(a) Subset S_1 (b) Subset S_2

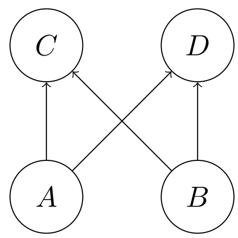


Figure 2: Union of S_1 and S_2

1.2 If $S_1 \subseteq D$ and $S_2 \subseteq D$ are directed, then so is $S_1 \cap S_2$.

This statement is false. Considering the two subsets S_1 and S_2 from 1a, the intersection would be {A, B} as displayed in Figure 3. Obviously, these two elements do not have an upper bound in $S_1 \cap S_2$.

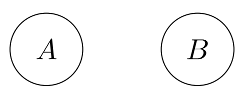


Figure 3: Intersection of S_1 and S_2

1.3 For every $d \in D$, the set $\{s \in D | s \leq d\}$ is directed.

This statement is true. Let S be the set $\{s \in D | s \leq d\}$. Assuming that S is not directed, it must be that

$$\exists (a, b) \mid a, b \in S \tag{1.1}$$

such that

$$\cap_S \{a, b\} = \emptyset \tag{1.2}$$

Since

$$S = \{s \in D | s \leq d\} \mid d \in D \tag{1.3}$$

it must be that

$$\begin{aligned} &\exists d \in D \mid \forall s_i \in S, d \leq s_i && (1.4)\\ \Rightarrow &d \leq a \text{ and } d \leq b && (1.5)\\ \Rightarrow &d \in \cap_S \{a, b\} && (1.6)\\ \Rightarrow &\cap_S \{a, b\} \neq \emptyset && (1.7) \end{aligned}$$

contradicting equation 1.2. Consequently, S must be directed!

1.4 For every $d \in D$, the set $\{s \in D | d \leq s\}$ is directed.

This statement if false. Consider the subset S_3 of the poset D from Figure 1 containing {A, C, D} as displayed in Figure 4 The statement does allow choosing A for d which would result in the entire

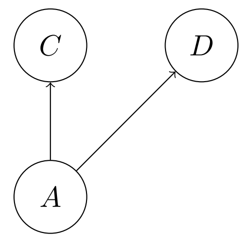


Figure 4: Subset S_3

poset. Consequently, the poset itself would need to be directed. Obviously, C and D have no upper bound, thus the poset is not directed.

2 For a cpo D, consider the function $Apply : [D \to D] \times D \to D$ for which $Apply(f, x) = f(x)$. Prove that $Apply$ is continuous.

As established in class, $[D \to D]$ denotes a ccpo of continuous functions.
Let F a be a subset of $[D \to D]$ such that it is a chain. Let furthermore C be a subset of D such that it is a chain. Since F is a set of <u>continuous</u> functions, it must be true that

$$\sqcup \{f\ C\} = f\ \sqcup C \mid \forall f \in F \tag{2.1}$$

When comparing function spaces such as F it is for all $f_i, f_{i+1} \in F$ that

$$f_i \leq f_{i+1} \text{ if } f_i\ x \leq f_{i+1}\ x \mid \forall x \in D \tag{2.2}$$

Since F is a <u>ccpo</u> of functions, it must be that

$$\exists \sqcup F \text{ such that } f \leq \sqcup F \mid \forall f \in F \tag{2.3}$$
$$\Rightarrow f\ x \leq \sqcup F\ x \mid \forall f \in F, \forall x \in D \tag{2.4}$$
$$\Rightarrow \sqcup \{f\ x \mid \forall f \in F\} = \sqcup F)\ x \mid \forall x \in D \tag{2.5}$$
$$\Rightarrow \sqcup \{F\ x\} = (\sqcup F)\ x \mid \forall x \in D \tag{2.6}$$

In order for $Apply$ to be continuous, it must be that for all $f \in F$ and $x \in C$

$$\sqcup\ Apply\ (f, x) = Apply\ \sqcup (f, x) \tag{2.7}$$

It is given that a function is continuous if it is continuous in each of its parameters. Therefore, if $Apply$ is continuous in its parameters, it must be continuous itself as well. By fixing single parameters, partially applied functions can be constructed that have the same properties (such as the continuity) for the remaining parameters as $Apply$. Consequently, by fixing f, the partially applied function $Apply_f$ can be constructed, that allows the evaluation of the continuity of $Apply$ in x. Respectively, $Apply_x$ allows the evaluation of the continuity of $Apply$ in f by fixing x. If both $Apply_f$ and $Apply_x$ are contiuous, *Apply* must be continuous, too. Now in order for $Apply_f$ and $Apply_x$ to be continuous, it must be true that

$$\sqcup \{Apply_f \ C\} = Apply_f \ \sqcup C \mid \forall f \in F \tag{2.8}$$
$$\sqcup \{Apply_x \ F\} = Apply_x \ \sqcup F \mid \forall x \in C \tag{2.9}$$

Since $Apply \ (f, x) = f \ x$, any $Apply_f \ x$ can also be expressed as $f \ x$. Thus, equation 2.8 results to

$$\sqcup \{f \ C\} = f \ \sqcup C \mid \forall f \in F \tag{2.10}$$

which is provided by equation 2.1. Consequently it is that $Apply_f$ is continuous, implying that *Apply* **is continuous in** x. Furthermore, $Apply \ (f, x) = f \ x$ requires that equation 2.9 results to

$$\sqcup \{F \ x\} = (\sqcup F) \ x \mid \forall x \in D \tag{2.11}$$

which is provided to equation 2.6. Consequently it is that $Apply_x$ is continuous, implying that *Apply* **is continuous in** f. Thus it is proven that *Apply* is continuous both in f and in x, Subsequentially, it must be true that *Apply* **is continuous**.

3 Let D be a cpo. An element $x \in D$ is called compact if for every ascending chain $S \subseteq D$ one has: $x \sqcup S$ implies $x \leq s$ for some $s \in S$. Show that if x and y are compact and they have a least upper bound in D, then the least upper bound is also compact.

Since x and y are compact, it is given that for all ascending chains $C \subseteq D$

$$\text{if } x \leq \sqcup C \text{ then } \exists c \in C \text{ such that } x \leq c \tag{3.1}$$

with the same being true for y instead of x.
Let's say there is a least upper bound p for {x, y} such that

$$x \leq p \wedge y \leq p \tag{3.2}$$

Assume p is not compact, it must be that $\exists C_i \subseteq D$ so that

$$p \leq \sqcup C_i \tag{3.3}$$
$$\nexists c \in C_i \text{ such that } p \leq c \tag{3.4}$$

Due to equation 3.2 and 3.3, it must be that

$$x \leq \sqcup C_i \wedge y \leq \sqcup C_i \tag{3.5}$$

Since x and y are compact and due to equation 3.1, it must also be true that

$$\exists c_1 \in C_i \text{ such that } x \leq c_1 \tag{3.6}$$
$$\exists c_2 \in C_i \text{ such that } y \leq c_2 \tag{3.7}$$

With c_1 and c_2 being elements of the same ascending chain, it is that

$$\exists c_i \in C_i = \sqcup\{c_1, c_2\} \tag{3.8}$$
$$\Rightarrow x \leq c_i \wedge y \leq c_i \tag{3.9}$$
$$\Rightarrow c_i \text{ is an upper bound of } \{x, y\} \tag{3.10}$$

Since p is established to be the least upper bound, it must be that

$$p \leq c_i \text{ with } c_i \in C_i \tag{3.11}$$
$$\Rightarrow \exists c_i \in C_i \text{ such that } p \leq c_i \tag{3.12}$$

This contradicts equation 3.4, **thus forcing** p **to be compact**!

4 A partially ordered set D is a complete lattice if every subset S of D has a least upper bound $\sqcup S$. Show that in a complete lattice D, an element e is compact if and only if for all $S \subseteq D$: $e \leq \sqcup S$ implies $e \leq \sqcup X$ for some finite $X \subseteq S$.

4.1 Introduction

Let A be the set of all subsets of D and C be the set of all subsets of D that are ascending chains such that

$$A \subseteq D \tag{4.1}$$
$$C \subseteq D \tag{4.2}$$
$$C \subseteq A \tag{4.3}$$

It has to be proven that

$$\textbf{iff } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.4}$$
$$\textbf{then } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.5}$$

for this to be true, two theorems can be deducted that have to be proven:

4.2 Theorem A

$$\textbf{if } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.6}$$
$$\textbf{then } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.7}$$

4.3 Theorem B

$$\textbf{if } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.8}$$
$$\textbf{then } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.9}$$

4.4 Proof

4.4.1 Proof of Theorem A

Due to equation 4.3, it must be that by satisfying a condition for all $S_A \subseteq A$, it will be satisfied for all $S_C \subseteq C$, too. Consequently it must be that

$$\textbf{if } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.10}$$
$$\textbf{then } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists \text{ finit } X \subseteq S_C \text{ such that } e \leq \sqcup X \tag{4.11}$$
$$\tag{4.12}$$

Since C is a set of ascending chains, it must be that for all finite $X \subseteq S_C$

$$\exists x \in X = \sqcup X \tag{4.13}$$
$$\Rightarrow \exists x \in S_C = \sqcup X \tag{4.14}$$

Therefore

$$e \leq \sqcup X \text{ for some finite } X \subseteq S_C \tag{4.15}$$
$$\Rightarrow e \leq x \text{ for some } x \in \text{ some finite } X \subseteq S_C \tag{4.16}$$
$$\Rightarrow e \leq x \text{ for some } x \in S_C \tag{4.17}$$

such that

$$\textbf{if } \forall S_C \in C : \forall e \leq \sqcup S_C, exists \text{ finit } X \subseteq S_C \text{ such that } e \leq \sqcup X \tag{4.18}$$
$$\textbf{then } \forall S_C \in C : \forall e_C \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.19}$$

Due to equation 4.10 to 4.12, it must therefore be that

$$\textbf{if } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.20}$$
$$\textbf{then } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.21}$$

Q.E.D. Consequently, Theorem A is true!

4.4.2 Proof of Theorem B

For Theorem B to be true it must be that

$$\textbf{if } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.22}$$
$$\textbf{then } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.23}$$

For every subset $S_A \in A$: Let Y be the set of all $\{s_i, s_j\}$ with $s_i, s_j \in S_A$ such that $\{s_i, s_j\} \subseteq S_A$. Let furthermore be $Y_b = \bigcup_{\forall y \in Y} \{\sqcup y\}$. It must then be that

$$\sqcup\, S_A = \sqcup Y_b \tag{4.24}$$

It must furthermore be that $(Y_b \cup \sqcup S_A)$ is a directed set since D being a complete lattice results in

$$\forall d_i, d_j \in D, \exists \sqcup \{d_i, d_j\} \in D \tag{4.25}$$
$$\Rightarrow \forall y_i, y_j \in Y_u, \exists \sqcup \{y_i, y_j\} \in (Y_u \cup \sqcup S_A) \tag{4.26}$$

Since every directed set always contains at least one ascending chain with the same supremum, it must be that there exists an ascending chain $Y_c \subseteq Y_b$ such that

$$\sqcup\, Y_c = \sqcup S_A \tag{4.27}$$
$$S_A \in C \tag{4.28}$$

and due to equation 4.22 it must then be that

$$\forall e \leq \sqcup S_A, e \leq \sqcup Y_c \tag{4.29}$$
$$\Rightarrow \forall e \leq \sqcup S_A, \exists s \in Y_c \text{ such that } e \leq s \tag{4.30}$$
$$\Rightarrow \forall e \leq \sqcup S_A, \exists s \in Y_b \text{ such that } e \leq s \tag{4.31}$$

Since $Y_b = \bigcup_{\forall y \in Y} \sqcup y$, it must be that

$$\forall s \in Y_b, \exists y \in Y \text{ such that } s \in y \tag{4.32}$$
$$\Rightarrow \forall e \leq \sqcup S_A, \exists y \in Y, \text{ such that } \sqcup\, y \leq e \tag{4.33}$$

and since every entry $y \in Y$ is a finite $\{s_i, s_j\} \subseteq S_A$, it must be that

$$\exists \text{ finite } X \subseteq S_A = y \text{ such that:} \tag{4.34}$$
$$\Rightarrow \forall e \leq \sqcup S_A, e \leq X \tag{4.35}$$
$$\Rightarrow \forall e \leq \sqcup S_A, \exists \text{ finite } X \text{ such that } e \leq \sqcup X \tag{4.36}$$

Consequently, equation 4.23 is true under the condition of equation 4.22 such that

$$\textbf{if } \forall S_C \in C : \forall e \leq \sqcup S_C, \exists s \in S_C \text{ such that } e \leq s \tag{4.37}$$
$$\textbf{then } \forall S_A \in A : \forall e \leq \sqcup S_A, \exists \text{ finit } X \subseteq S_A \text{ such that } e \leq \sqcup X \tag{4.38}$$

Q.E.D.. Consequently, Theorem B is true!

4.5 Conclusion

Since both Theorem A and Theorem B were proven to be true, it original statement from the introduction is correct!

5 Provide a denotational semantics for "for" loops given in Assignment 1.

The for loop can for this scenario be rewritten as follows:

$$\begin{aligned} for \rightarrow & \text{for } i := E_0 \text{ to } E_1 \text{ step } E_2 \text{ do } St && (5.1)\\ \rightarrow & init; loop && (5.2) \end{aligned}$$

where

$$\begin{aligned} init \rightarrow & i := E_0 && (5.3)\\ inc \rightarrow & i = i + E_2 && (5.4)\\ loop \rightarrow & \text{while } i <= E_1 \text{ do } (St; inc) && (5.5) \end{aligned}$$

Let S be the set of all statuses. The attribute "sc" is the statechange of elements, while "eval" evaluates expressions based on a given status. Consequently it must be that

$$for.sc = \lambda s \in S : (loop.sc \circ init.sc)(s) \quad (5.6)$$

with

$$\begin{aligned} init.sc =& \lambda s \in S : s[i \leftarrow E_0.eval(s)] && (5.7)\\ inc.sc =& \lambda s \in S : s[i \leftarrow s(i) + E_2.eval(s)] && (5.8)\\ loop.sc =& \lambda s \in S : \text{cond}(s(i) <= E_1.eval(s), (loop.sc \circ inc.sc \circ St.sc)(s), s) && (5.9)\\ =& \text{FIX } F && (5.10)\\ where\ F =& \lambda g : \lambda s \in S : \text{cond}(s(i) <= E_1.eval(s), (g \circ inc.sc \circ St.sc)(s), s) && (5.11)\\ =& \lambda g : \lambda s \in S : \text{cond}(s(i) <= E_1.eval(s), (g \circ (\lambda x \in S : x[i \leftarrow x(i) + E_2.eval(x)]) \circ St.sc)(s), s) && (5.12) \end{aligned}$$

Putting these into equation 5.6 results to the denotational semantics for the *for* loop

$$\begin{aligned} for.sc =& \lambda s \in S : (\text{FIX } F)(s[i \leftarrow E_0.eval(s)]) && (5.13)\\ where\ F =& \lambda g : \lambda s \in S : \text{cond}(s(i) <= E_1.eval(s), (g \circ (\lambda x \in S : x[i \leftarrow x(i) + E_2.eval(x)]) \circ St.sc)(s), s) && (5.14) \end{aligned}$$

6 Provide a denotational semantics for "case" statements as specified in Assignment 1.

6.1 I $\rightarrow$ n : St

$$I.statechange = \lambda s.\lambda m. \begin{cases} m = n & St.statechange(s) \\ m \neq n & s \end{cases} \quad (6.1)$$

6.2 Cl $\rightarrow$ I

$$Cl.statechange = I.statechange \quad (6.2)$$

6.3 $\mathsf{Cl}_0 \rightarrow \mathsf{Cl}_1$; I

$$Cl_0.statechange = \lambda s.\lambda m.(I.statechange \circ Cl_1.statechange(s))(m) \tag{6.3}$$

6.4 <u>case</u> E <u>of</u> Cl

$$Cs.statechange = \lambda s.(Cl.statechange(s) \circ E.evaluate)(s) \tag{6.4}$$