Assignment 3

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1 Prove lamdba expressions

1.1 $\mathsf{I}M = M$

$$\begin{aligned} \mathbf{I} &= \lambda x.x && (1.1)\\ \Rightarrow \mathbf{I}M &= (\lambda x.x)M && (1.2)\\ &= x[x \leftarrow M] && (1.3)\\ &= M && (1.4) \end{aligned}$$

1.2 $\mathsf{K}MN = M$

$$\begin{aligned} \mathbf{K} &= \lambda x.\lambda y.x && (1.5)\\ \Rightarrow \mathbf{K}MN &= (\lambda x.\lambda y.x)MN && (1.6)\\ &= (\lambda y.x)[x \leftarrow M]N && (1.7)\\ &= (\lambda y.M)N && (1.8)\\ &= M[y \leftarrow N] && (1.9)\\ &= M && (1.10) \end{aligned}$$

1.3 $\mathsf{S}MNL = ML(NL)$

$$\begin{aligned} \mathbf{S} &= \lambda x.\lambda y.\lambda z.xz(yz) && (1.11)\\ \Rightarrow \mathbf{S}MNL &= (\lambda x.\lambda y.\lambda z.xz(yz))MNL && (1.12)\\ &= (\lambda y.\lambda z.xz(yz))[x \leftarrow M]NL && (1.13)\\ &= (\lambda y.\lambda z.Mz(yz))NL && (1.14)\\ &= (\lambda z.Mz(yz))[y \leftarrow N]L && (1.15)\\ &= (\lambda z.Mz(Nz))L && (1.16)\\ &= Mz(Nz)[z \leftarrow L] && (1.17)\\ &= ML(NL) && (1.18) \end{aligned}$$

2 Show that, for all lambda expressions M, N $(\lambda y.(\lambda x.M))N = \lambda x.((\lambda y.M)N)$

It is to be proven that

$$(\lambda y.(\lambda x.M))N \stackrel{?}{=} \lambda x.((\lambda y.M)N) \tag{2.1}$$

For the left side it is that

$$\begin{aligned}(\lambda y.(\lambda x.M))N &= (\lambda y.\lambda x.M)N && (2.2)\\ &= (\lambda x.M)[y \leftarrow N] && (2.3)\\ &= \lambda x.M[y \leftarrow N] && (2.4)\end{aligned}$$

For the right side it is that

$$\begin{aligned}\lambda x.((\lambda y.M)N) &= \lambda x.(M[y \leftarrow N]) && (2.5)\\ &= \lambda x.M[y \leftarrow N] && (2.6)\end{aligned}$$

Since equation 2.4 equals to equation 2.6, it is that equation 2.1 must be true.

3 Show that there exists a lambda expression M suc that $MN = MM$ holds for all lambda expressions N

4 Exercise 11.47

4.1 Prove the successor function for zero to two

For (successor zero) = one

$$\begin{aligned}
zero &= \lambda f.\lambda x.x && (4.1)\\
successor &= \lambda n.\lambda f.\lambda x.(f((nf)x)) && (4.2)\\
successor(zero) &= (\lambda n.\lambda f.\lambda x.(f((nf)x)))(\lambda f.\lambda x.x) && (4.3)\\
&= (\lambda f.\lambda x.(f((nf)x)))[n \leftarrow (\lambda f.\lambda x.x)] && (4.4)\\
&= \lambda f.\lambda x.(f(((\lambda f'.\lambda x'.x')f)x)) && (4.5)\\
&= \lambda f.\lambda x.(f((\lambda x'.x')[f' \leftarrow f]x)) && (4.6)\\
&= \lambda f.\lambda x.(f((\lambda x'.x')x)) && (4.7)\\
&= \lambda f.\lambda x.(f(x'[x' \leftarrow x])) && (4.8)\\
&= \lambda f.\lambda x.(f(x)) && (4.9)\\
&= \lambda f.\lambda x.(fx) && (4.10)\\
&= one && (4.11)
\end{aligned}$$

Q.E.D.
For (successor one) = two

$$\begin{aligned}
one &= \lambda f.\lambda x.(fx) && (4.12)\\
successor &= \lambda n.\lambda f.\lambda x.(f((nf)x)) && (4.13)\\
successor(one) &= (\lambda n.\lambda f.\lambda x.(f((nf)x)))(\lambda f.\lambda x.(fx)) && (4.14)\\
&= (\lambda f.\lambda x.(f((nf)x)))[n \leftarrow (\lambda f.\lambda x.(fx))] && (4.15)\\
&= \lambda f.\lambda x.(f(((\lambda f'.\lambda x'.(f'x'))f)x)) && (4.16)\\
&= \lambda f.\lambda x.(f(((\lambda x'.(f'x'))[f' \leftarrow f])x)) && (4.17)\\
&= \lambda f.\lambda x.(f((\lambda x'.(fx'))x)) && (4.18)\\
&= \lambda f.\lambda x.(f((fx')[x' \leftarrow x])) && (4.19)\\
&= \lambda f.\lambda x.(f(fx)) && (4.20)\\
&= two && (4.21)
\end{aligned}$$

Q.E.D.

4.2 Generalization of the successor function

Let the general lambda abstraction for numbers be

$$church_n = \lambda f.\lambda x.(f^n x) \tag{4.22}$$

Such that

$$\begin{aligned}
f^0(x) &= x && (4.23)\\
f^1(x) &= f(x) && (4.24)\\
f^2(x) &= f(f(x)) && (4.25)\\
f^3(x) &= f(f(f(x))) && (4.26)\\
&\dots && (4.27)\\
f^{n+1}(x) &= f(f^n(x)) && (4.28)
\end{aligned}$$

For the successor function this requires

$$successor(church_n) = church_{n+1} \tag{4.29}$$

such that

$$
\begin{aligned}
church_n &= \lambda f.\lambda x.(f^n x) && (4.30)\\
successor(church_n) &= \lambda n.\lambda f.\lambda x.(f((nf)x)) && (4.31)\\
&= (\lambda f.\lambda x.(f((nf)x)))(\lambda f.\lambda x.(f^n x)) && (4.32)\\
&= (\lambda f.\lambda x.(f((nf)x)))[n \leftarrow (\lambda f.\lambda x.(f^n x))] && (4.33)\\
&= \lambda f.\lambda x.(f(((\lambda f'.\lambda x'.(f'^n x'))f)x)) && (4.34)\\
&= \lambda f.\lambda x.(f(((\lambda x'.(f'^n x'))[f' \leftarrow f])x)) && (4.35)\\
&= \lambda f.\lambda x.(f((\lambda x'.(f^n x'))x)) && (4.36)\\
&= \lambda f.\lambda x.(f((f^n x')[x' \leftarrow x])) && (4.37)\\
&= \lambda f.\lambda x.(f(f^n x)) && (4.38)\\
&= \lambda f.\lambda x.(f^{n+1} x) && (4.39)\\
&= church_{n+1} && (4.40)
\end{aligned}
$$

Q.E.D.

4.3 Define addition and multiplication for Church numbers

The addition for Church numbers can be defined as

$$add = \lambda m.\lambda n.\lambda f.\lambda x.(mf)(nfx) \tag{4.41}$$

Due to equation 4.22, it must be that

$$
\begin{aligned}
church_m + church_n &= church_{m+n} && (4.42)\\
\Rightarrow \lambda f.\lambda x.(f^m x) + \lambda f.\lambda x.(f^n x) &= \lambda f.\lambda x.(f^{n+m} x) && (4.43)\\
&= \lambda f.\lambda x.(f^n(f^m x)) && (4.44)
\end{aligned}
$$

In order for equation 4.41 to be true, it must be that

$$add(church_m, church_n) = church_{m+n} \tag{4.45}$$

This can be proven as follows:

$$
\begin{aligned}
church_m &= \lambda f.\lambda x.(f^m x) && (4.46)\\
church_n &= \lambda f.\lambda x.(f^n x) && (4.47)\\
add(church_m, church_n) &= (\lambda m.\lambda n.\lambda f.\lambda x.(mf)(nfx))(\lambda f.\lambda x.(f^m x))(\lambda f.\lambda x.(f^n x)) && (4.48)\\
&= (\lambda n.\lambda f.\lambda x.(mf)(nfx))[m \leftarrow (\lambda f.\lambda x.(f^m x))](\lambda f.\lambda x.(f^n x)) && (4.49)\\
&= (\lambda n.\lambda f.\lambda x.((\lambda f'.\lambda x'.(f'^m x'))f)(nfx))(\lambda f.\lambda x.(f^n x)) && (4.50)\\
&= (\lambda n.\lambda f.\lambda x.((\lambda x'.(f'^m x'))[f' \leftarrow f])(nfx))(\lambda f.\lambda x.(f^n x)) && (4.51)\\
&= (\lambda n.\lambda f.\lambda x.(\lambda x'.(f^m x'))(nfx))(\lambda f.\lambda x.(f^n x)) && (4.52)\\
&= (\lambda f.\lambda x.(\lambda x'.(f^m x'))(nfx))[n \leftarrow (\lambda f.\lambda x.(f^n x)) && (4.53)\\
&= \lambda f.\lambda x.(\lambda x'.(f^m x'))((\lambda f''.\lambda x''.(f''^n x''))fx) && (4.54)\\
&= \lambda f.\lambda x.(\lambda x'.(f^m x'))((\lambda x''.(f''^n x''))[f'' \leftarrow f]x) && (4.55)\\
&= \lambda f.\lambda x.(\lambda x'.(f^m x'))((\lambda x''.(f^n x''))x) && (4.56)\\
&= \lambda f.\lambda x.(\lambda x'.(f^m x'))((f^n x'')[x'' \leftarrow x]) && (4.57)\\
&= \lambda f.\lambda x.(\lambda x'.(f^m x'))(f^n x) && (4.58)\\
&= \lambda f.\lambda x.(f^m x')[x' \leftarrow (f^n x)] && (4.59)\\
&= \lambda f.\lambda x.(f^m (f^n x)) && (4.60)\\
&= \lambda f.\lambda x.(f^{m+n} x) && (4.61)\\
&= church_{m+n} && (4.62)
\end{aligned}
$$

Q.E.D.
The multiplication for Church numbers can be easily defined using the *add* function

$$mult = \lambda m.\lambda n.\lambda f.\lambda x.(m(add(nfx)))x \qquad (4.63)$$

The proof for this function can be provided analog to the one for the addition. However, since it requires verbose equations, it will be omitted at this point.