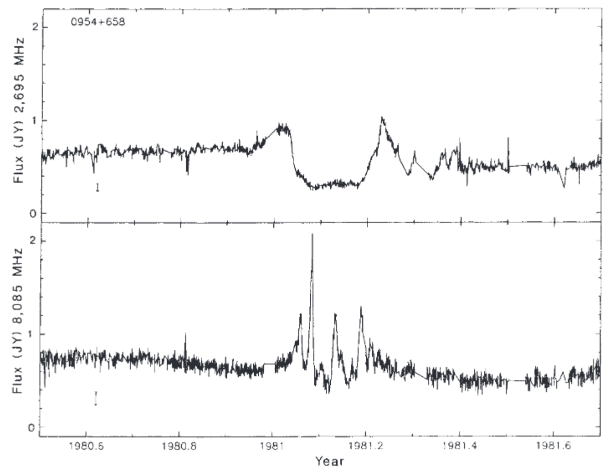
ASTP 720 - Homework 1 - Extreme Scattering Events

Due September 3, 2020

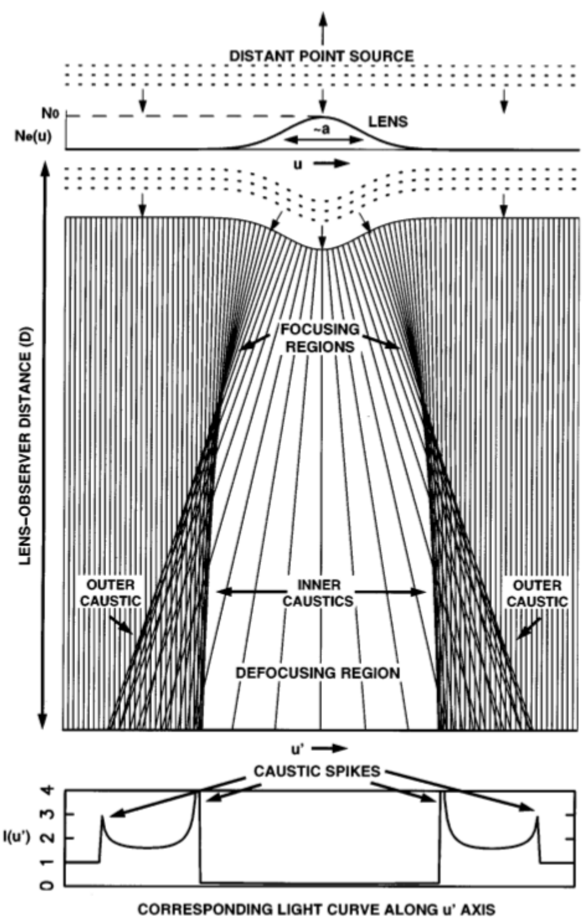
The History of Extreme Scattering Events

A growing area of interest in astronomy has been that of understanding the population of so called “extreme scattering events” seen in radio observations in the direction of quasars and pulsars. The first was described in Fiedler et al. (1987), shown at right. The flux of the quasar at two radio frequencies was seen to spike, then fade, then spike, and then return to the baseline. Since then, observations have indicated more of these, both in flux variations and in scintillation parameters that change how radio point sources “twinkle” (stars twinkle, planets do not, though this scintillation is due to optical rays traveling through the turbulent atmosphere rather than radio rays traveling through the turbulent interstellar plasma).



Fiedler et al. 1987 (Figure 1)

One model for these observations is of *plasma lenses*, where light rays bend around a lens of plasma in the interstellar medium (ISM), causing focusing and defocusing of the emission, as seen at right. As the radio emission from a distant point source (quasar, pulsar) passes by the overdensity of material in the ISM, the bending means that you get more or fewer rays hitting your telescope over time, with “caustic spikes” occurring symmetrically for their toy model of a Gaussian-shaped lens (see bottom panel). You get changes over time because the geometry of the line of sight, you, the lens, and the source if it is a pulsar (quasars not so much, too far away) is changing. The time, of course, is related to this effective velocity and the distance traveled.



Clegg, Fey, and Lazio 1998 (Figure 2)

While these lenses are of course 3D structures, rather than consider the 2D projected density (as with gravitational lensing), you will consider the case of a 1D lens as in the cartoon since it is simpler.

For reference, in the next sections, I am following the formalism of Clegg, Fey, and Lazio (1998) below, with some slight additions and modifications.

Background: Propagation through a Plasma

In a propagating electromagnetic field, a free electron feels a force from the electric field component and begins to oscillate. One can work through Maxwell's equations in this scenario and find that the index of refraction of the medium will change, and as a function of frequency (in cgs units, sorry):

$$n_r(\nu) = \sqrt{1-\left(\frac{\nu_p}{\nu}\right)^2} = \sqrt{1-\frac{\lambda^2 r_e n_e}{\pi}} \tag{1}$$

where

$$\nu_p = \sqrt{\frac{e^2 n_e}{\pi m_e}} \approx 8.979 \text{ kHz } \left(\frac{n_e}{\text{cm}^{-3}}\right)^{1/2} \tag{2}$$

is known as the plasma frequency. The constants are the electron charge e, mass m_e, and classical radius $r_e = e^2/(m_e c^2)$ (in cgs units), while the number density is given by n_e. A rule of thumb for the ISM is about 1 particle per cubic centimeter, though in many areas it will be less (maybe 0.1). Since for all radio astronomical observations, $\nu \gg \nu_p$ (ν_p for the ionosphere is 10 MHz, which means that we cannot see below that frequency or the index of refraction becomes imaginary and the wave will not propagate), we have by the binomial approximation

$$n_r(\nu) \approx 1-\frac{\lambda^2 r_e n_e}{2\pi}. \tag{3}$$

The phase velocity of the light is given by $v_\phi = c/n_r$, and since $n_r \leq 1$ inside some material, then the phase will *advance* as it exits compared to what it would have done if it was traveling through the rest of the ambient space. The time difference the phase takes is

$$\begin{aligned} \tau &= t_{\text{ambient}} - t_{\text{lens}} \\ &= \frac{L}{c} - \frac{L}{v_\phi} \\ &= \frac{L}{c} - \frac{L n_r}{c} \\ &= \frac{L}{c}(1-n_r) \\ &\approx \frac{L}{c}\frac{\lambda^2 r_e n_e}{2\pi}. \end{aligned} \tag{4}$$

The phase itself will be advanced by an amount

$$\phi = \omega\tau = 2\pi\nu\tau = \frac{2\pi c}{\lambda} \times \frac{L}{c}\frac{\lambda^2 r_e n_e}{2\pi} = \lambda r_e n_e L = \boxed{\lambda r_e N_e}, \tag{5}$$

where $N_e = n_e L$ is the column density through the lens.

Ray Optics and Lensing

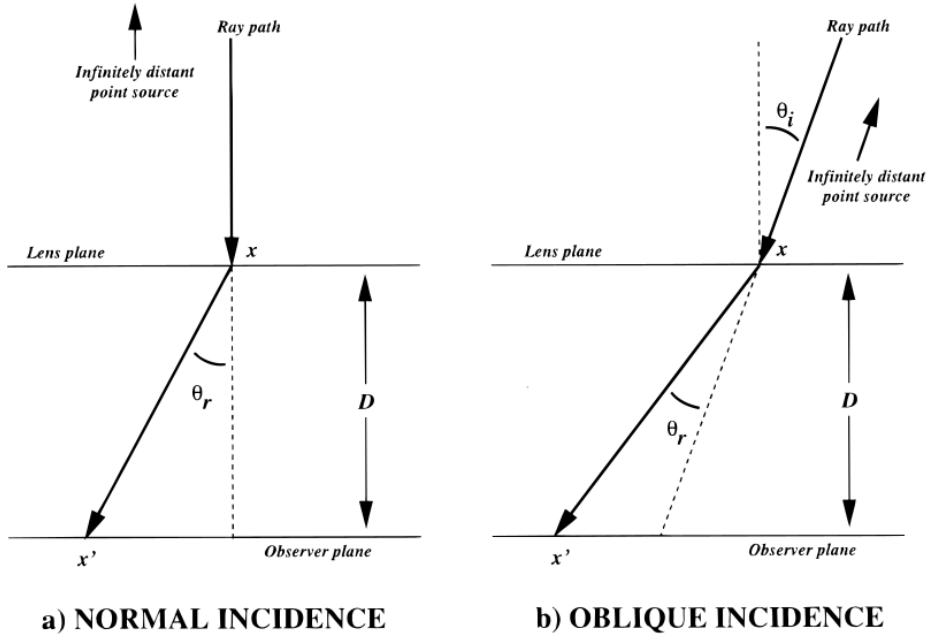
Without good justification except to say that in the geometric optics description, light rays will propagate normal to surfaces of constant phase. Therefore, since the phases are changing, the light rays will be refracted by an amount

$$\theta_r(x) = \frac{\lambda}{2\pi}\frac{d}{dx}\phi(x) = \frac{\lambda^2 r_e}{2\pi}\frac{d}{dx}N_e(x), \tag{6}$$

where x is the 1D coordinate in the lens plane. The ray at coordinate x is refracted by this angle $\theta_r(x)$ and so hits the observer's plane at the coordinate (assuming small angles)

$$x' = x - \theta_r(x)D. \tag{7}$$

In general, if the rays hit the lens at some incidence angle θ_i, the equation becomes $x' = x - [\theta_r(x) + \theta_i]\, D$. The geometry is shown below: For simplicity, you can assume that $\theta_i = 0$.



Clegg, Fey, and Lazio (1998), Figure 1

The Gaussian Spherical Lens

In the work of Clegg, Fey, and Lazio, they calculate the relevant quantities for a Gaussian lens of size scale a, with central column density N_0, at a distance D. For convenience, they are:

$$\begin{aligned}
N_e(x) &= N_0 e^{-(x/a)^2} && (8)\\
\phi(x) &= \lambda r_e N_0 e^{-(x/a)^2} && (9)\\
\theta_r(x) &= -\frac{\lambda^2 r_e N_0}{\pi a^2} x e^{-(x/a)^2} && (10)\\
x' &= x\left[1 + \frac{\lambda^2 r_e N_0 D}{\pi a^2} e^{-(x/a)^2}\right] && (11)
\end{aligned}$$

You can use these expressions to check your functions if you so choose, but do not feel that you need to.

The Isothermal and Pseudo-Isothermal Spherical Lens

There are many other common types of density profiles in astrophysics depending on the situation. Let's use one that probably isn't quite plausible but will at least have a solution you can check, namely that of the isothermal sphere, which has $\rho(r) = \rho_0 r_c^2/r^2$. To get the surface density, we need to integrate along z. Moving to cylindrical coordinates via $r = \sqrt{z^2 + R^2}$ (using R as the polar radius instead of ρ to avoid confusion), the surface density profile is

$$\Sigma(R) = \int_{-\infty}^{+\infty} \rho(r(z))\, dz = \pi \rho_0 r_c^2 / R \tag{12}$$

if I did that correctly. This isn't ideal since at $R = 0$, looking straight down the center of the lens, we have an infinite surface density. So one correction that can be employed is what's called a pseudo-isothermal sphere, with $\rho(r) = \rho_0 / \left[1 + (r/r_c)^2\right]$. In this case, we can work out that

$$\Sigma(R) = \frac{\pi \rho_0 r_c^2}{\sqrt{r_c^2 + R^2}}. \tag{13}$$

Since we're looking along a 1D trajectory, then we have $R^2 = x^2 + y^2$ with $y = 0$, so we have in reduced and simplified form:

$$N_e(x) = N_0 \left[1 + \left(\frac{x}{r_c}\right)^2\right]^{-1/2}. \tag{14}$$

If you work it out, you should get

$$\frac{d}{dx} N_e(x) = -\frac{N_0 x}{r_c^2 \left[1 + (x/r_c)^2\right]^{3/2}}. \tag{15}$$

Your Tasks:

[1.] Write a library (i.e., in a separate file that you can call) for the three root-finding algorithms we discussed in class: Bisection, Newton, Secant. These functions should each take *functions* rather than data points. Make sure that each takes an optional argument the threshold and that it also takes a variable that allows the user to print out or return the number of iterations it took to hit that threshold.

[2.] For the pseudo-isothermal sphere, using your root-finding algorithms, numerically calculate the full width at half maximum, i.e., what is the width (in terms of r_c) when $N_e(x) = N_0/2$, half the amplitude. Drawing pictures for yourself might be useful! Do so with each of your root-finding algorithms and show how many iterations each takes as a function of your threshold. Please plot the results.

[3.] You can probably see from the Gaussian Lens equations that if you have a light ray hitting x, and you know the other parameters of the lens (a, N_0, D, etc.), then you know what x' is. But that's boring and not what you actually observe. Let's instead say you are an observer in a "circular orbit" along the x' axis with radius 1 AU and a period of 1 year but centered at $x' = 1$ AU. Then, you know where your position x' is but not where the light rays from the source are intersecting the lens plane at x - as expected, analytically solving for x is not really an option.

Using one of your root-finding algorithms, solve the lens equation for each value of x' and make a ray-tracing plot as on the first page. Assume $D = 1$ kpc, $a = 1$ AU, $\lambda = 21$ cm, and $N_0 = 0.01$ pc cm^{-3} (these are observer units, probably best to convert to something like cm^{-2}).

[4.] Repeat but for the pseudo-isothermal sphere with the same parameters but $r_c = 1$ AU.

[5.] Write a library for piecewise linear interpolation, given a set of x and y data points. This should return a function f that one can use to calculate a new point $f(x_{\text{new}}) \to y_{\text{new}}$.

[6.] In the file `lens_density.txt` are a series of values of x and $N_e(x)$ for some shape. Use your interpolator to plot the values of $N_e(x)$ halfway in between all of the given x values, i.e., when $x = 0.5, 1.5, \cdots$.

A few reminders

- Please submit a short write-up in LaTeXwith figures. Just explain what you did and what is going on in the figures. Please provide a link to your code on github.
- A tutorial on making understandable figures is available on `myCourses`.
- To keep track of units, it is recommended to use `Astropy`.
- You must document your code!