# ASTP 720 - Homework 4 - The Structure of Stellar Remnants

Due September 24th, 2020

### The Mass-Radius Relationship for Degenerate Objects

The most basic quantity that describes a stellar object is its mass $M$, which largely determines the life cycle of a star. For most objects, as you increase the mass, the size of the object (radius $R$) increases. For degenerate matter, the radius $R$ actually *decreases* as you add mass. One can demonstrate, for example, that $R \propto M^{-1/3}$ in the case of *non-relativistic* degenerate matter. Come ask me if you want to see some very simple order-of-magnitude estimates.

Rather than order-of-magnitude estimates, however, we have computers. Let's take a look at this relationship further.

### Hydrostatic Equilibrium

As a reminder, for objects where an internal pressure counteracts the force of gravity, we have the equations of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{GM_{\text{enc}}(r)\rho(r)}{r^2} \tag{1}$$

$$\frac{dM_{\text{enc}}(r)}{dr} = 4\pi r^2 \rho(r). \tag{2}$$

For stars, one can also calculate temperature, luminosity, and chemical gradient, but these quantities will all derive from the two above so we will ignore them here.

In order to solve these two equations, you will need a relation between pressure and density. In addition, as initial value problems, you will need the initial values to start your integrations (in contrast to boundary value problems). You can take these to be $M_{\text{enc}}(0) = 0$ and $P(0) = P_c$. When the pressure goes to zero, we define this condition as $r = R$ and $M_{\text{enc}} = M$, i.e., given the two initial conditions above, you're able to calculate $M$ and $R$ for a star. Numerically speaking, your pressure may not reach zero exactly, so just make a close approximation. Don't forget that at the first step of $r = 0$, we require $dP/dr = 0$ so that there is no cusp in the solution.

### Polytropic Models

For stars in hydrostatic equilibrium, there often exists a *polytropic relation* within the internal material, namely that the equation of state simplifies as $\rho(P, T) \to \rho(P)$, where $\rho$ is the density, $P$ is the pressure, and $T$ is the temperature of the fluid. The polytropic relation takes the form

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}} = K\rho^{(n+1)/n}, \tag{3}$$

where $K$, $\gamma$, and $n$ are constants. In degenerate objects, where the Heisenberg Uncertainty principle applies, temperature does not factor in unlike with nuclear fusion, so the polytropic model *does* represent the true equation of state.

For convenience, in the cases you will consider, you will be given $\rho_c$ rather than $P_c$ since it is a little more intuitive. You can convert between them using the polytropic equation above.

**White Dwarfs**

At the end of their lives, lower-mass stars like the Sun will leave behind a degenerate core: a white dwarf. A low-mass white dwarf (i.e., not close to the Chandrasekhar Mass) can be described by a polytrope with $n = 3/2$, the result of looking at a non-relativistic degenerate electron gas. This is given by[1]

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_e m_u^{5/3}} \left(\frac{\rho}{\mu_e}\right)^{5/3} = 1.0 \times 10^{13} \left(\frac{\rho}{\mu_e}\right)^{5/3} \text{ (in cgs)}, \tag{4}$$

where[2] $h = 6.63 \times 10^{-27}$ erg s is Planck's constant, $m_e = 9.11 \times 10^{-28}$ g is the electron mass, $m_u = 1.66 \times 10^{-24}$ g = 1 amu (atomic mass unit), $\mu_e$ is the mean molecular weight per free electron, or alternatively the number of nucleons per electron. Roughly speaking, since there will be an equal number of protons, neutrons, and electrons (e.g., for a Carbon-Oxygen core, both of these elements have equal numbers), then $\mu_e = 2$.

**Neutron Stars**

For massive enough objects, electron degeneracy pressure will no longer support the stellar remnant against collapse and so the next form of internal support against gravity is *neutron degeneracy pressure*. The simplest alteration to our equation of state above for neutrons is that $m_e \to m_n$, the mass of the neutron, which itself is approximately $m_u$. In the case of a neutron star, the majority of the interior is made of neutrons, and so the equivalent $\mu$ is simply 1. Thus, we can write this expression as:

$$P = \frac{1}{20}\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{m_n^{8/3}} \rho^{5/3} = 5.4 \times 10^{9}\ \rho^{5/3} \text{ (in cgs)}. \tag{5}$$

Real neutron stars are approximated by polytropic indices of around $n = 0.5$ to 1 but realistic models need to account for a significant number of other types of pressure support (other internal interactions, rotation, etc.), so this approximation is still reasonable for use here[3].

However, one "easier" modification we can make to the above is to consider hydrostatic equilibrium in the framework of General Relativity, where one can assume a perfectly fluid, spherically symmetric star. The result is what's known as the Tolman-Oppenheimer-Volkoff (TOV) equation, given by (dropping the functional dependences on $r$ for clarity):

$$\frac{dP}{dr} = -\frac{GM_{\rm enc}}{r^2}\rho\left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{4\pi r^3 P}{M_{\rm enc} c^2}\right)\left(1 - \frac{2GM_{\rm enc}}{rc^2}\right)^{-1}. \tag{6}$$

One can expand the products of those and keep terms $\mathcal{O}(c^{-2})$ in order to find the post-Newtonian approximation; keeping terms $\mathcal{O}(1)$ gives us the original Newtonian derivation.

Assuming you have your ODEs tested against the above, please use those. Otherwise, please use scipy's `odeint`.

[1] Kippenhahn and Weigert (1990), Eq 15.23
[2] In cgs units because that's what everyone else uses...
[3] Shapiro and Teukolsky (1983), §9.2

**[1.]** For central densities in the range $\rho_c = 10^4 - 10^6$ g/cm$^3$, solve via your RK4 solver the hydrostatic equilibrium equations above for white dwarfs and plot the mass-radius curve. That is, chose a central density. You will then integrate over radial slices $dr$, and once you have $M$ and $R$, you will store that and move onto the next central density. Since you are feeding your solver a list of radial slices to solve at, you can either determine some stopping criterion internally or just give the code a list of slices out to a few $R_\oplus$ (since we know that white dwarfs can't be much larger than this), thereby ensuring that you will hit $P \approx 0$ somewhere in the array. Once you have your whole list of $M$s and $R$s, plot $M$ versus $R$ and that is the mass-radius curve.

**[2.]** For central densities in the range $\rho_c = 10^{14} - 10^{16}$ g/cm$^3$, solve the relativistic hydrostatic equilibrium equations above for neutron stars and plot the mass-radius curve. In this case, your slices only need to go out to maybe 20 km.

**[3.]** The Neutron Star Interior Composition Explorer (NICER) measured the radius and mass of PSR J0030+0451 to be $R = 13.02^{+1.24}_{-1.06}$ km and $M = 1.44^{+0.15}_{-0.14} M_\odot$, respectively. Numerically calculate the expected mass of a neutron star with an equation of state governed by the TOV equation and with a radius of 13.02 km. It will likely not be as observed!

As usual, please document your code. Feel free to not explicitly unit test your code since you are writing specific testing code, but if you find it useful to put it in the unit test framework, that's okay too. Note that for any previous functionality you have written , you may use functions from other packages. Please hand in a usual write-up.