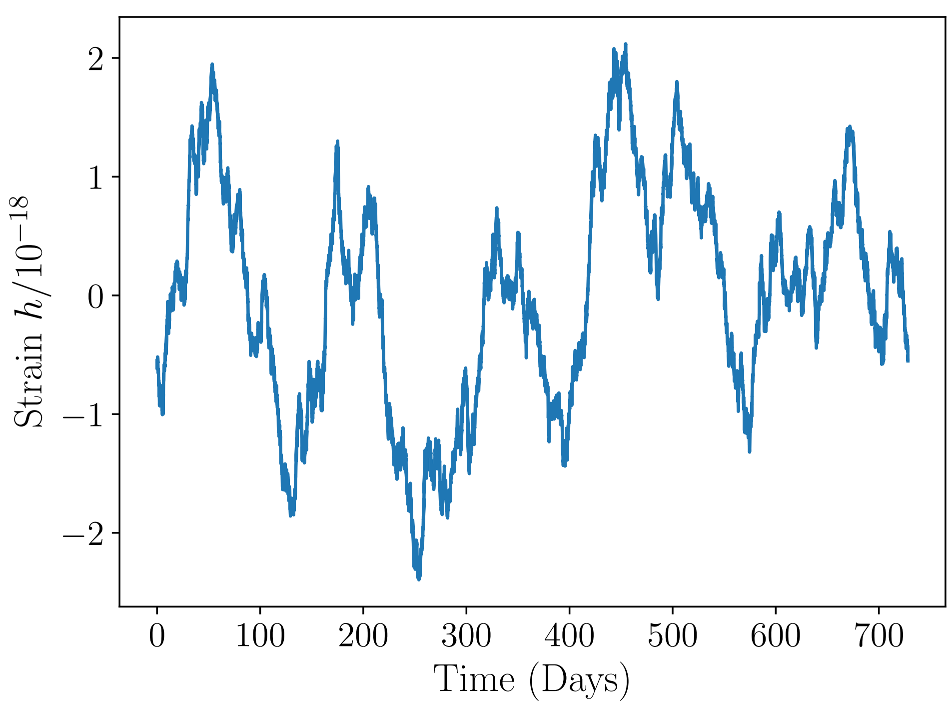
ASTP 720 - Homework 8 - Double White Dwarf Binaries with LISA

Due Date: November 5th, 2020

Unlike ground-based interferometers like the Laser Interferometer Gravitational-Wave Observatory (LIGO) and VIRGO, and pulsar timing arrays like the North American Nanohertz Observatory for Gravitational Waves (NANOGrav), the space-based Laser Interferometer Space Antenna (LISA) is expected to be so sensitive that they will have to disentangle many signals from one another rather than signals from pure noise. LISA will operate in the $10^{-5} - 1$ Hz, equivalent to periodicities of 1 to 10^5 seconds ($\sim$1.2 days).

One of the dominant signals will be that of galactic double white dwarf (WD) binaries in tight orbits. These will fall into two classes: resolvable and unresolvable. Resolvable ones will be binaries that we can localize and determine the properties of (position, gravitational-wave frequency, chirp mass, etc.) may be of order $\mathcal{O}(10^3 - 10^4)$. The unresolvable ones will form a noisy background from which we can extract ensemble information from the total statistics. We can think of the two in the following analogy: in a crowded room, the unresolvable ones will form the murmur of noise while the resolvable ones are individual conversations we can pick out.

In `strain.npy`, you will find simulated data of the strain h, the fractional change in distance as a gravitational wave passes, as a function of time. There are $2^{20} \approx 10^6$ regularly-sampled measurements, each one minute apart, for a total of approximately two years of data. The strain curve looks as follows:



The simulated data contains no instrumental noise and only the following two signals: a double WD binary background, characterized by a strain spectrum $S_h(f) \propto f^{-7/3}$. Note that this is different from the *characteristic strain*[1] $h_c \propto f^{-2/3}$.

[1] See Moore, Cole, and Berry 2015, Classical and Quantum Gravity, 32, 015014 for more information between these, where $fS_h(f) = [h_c(f)]^2$

In addition to the background, there is one simulated binary. In a rough back-of-the-envelope calculation, one can work out that the gravitational wave strain for a mass scale M with a separation R over a timescale T is approximately

$$h \sim \frac{G}{Dc^4}\ddot{I} \sim \frac{G}{Dc^4}\frac{MR^2}{T^4} \sim \frac{G}{Dc^2}MR^2 f^2 \sim \frac{G}{4\pi^2 Dc^2}MR^2\Omega^2, \tag{1}$$

where I is the moment of inertia, $f = 1/T$ is the orbital frequency scale, and $\Omega = 2\pi f$ is the angular orbital frequency scale. The division by the distance to the source D comes from the far-field limit, analogous to in electromagnetic radiation $(1/r)$. The second-time-derivative of the moment of intertia is the first non-zero component of the mass multipole expansion (e.g., mass monopole is the sum of the masses, which is constant, mass dipole is related to the momentum, which is conserved, etc.). The G/c^4 are needed to cancel out the units.

In the Newtonian limit, we can use the Keplerian orbital frequency to relate $\Omega^2 \sim GM/R^3$. Therefore, we can write

$$\begin{aligned} h &\sim \frac{(GM)^2}{4\pi^2 c^4 DR} \\ &\approx 2.6 \times 10^{-21} \left(\frac{M}{M_\odot}\right)^2 \left(\frac{D}{\mathrm{pc}}\right)^{-1} \left(\frac{R}{R_\odot}\right)^{-1}. \end{aligned} \tag{2}$$

Again, M you can take to be the total mass of the system, D is the distance to the system, and R is the *separation* of the binary (recall that there are approximately $215R_\odot$ in 1 AU). The gravitational wave frequency is given by

$$\begin{aligned} f_{\mathrm{GW}} &\sim \frac{1}{2\pi}\left(\frac{GM}{R^3}\right)^{1/2} \\ &\approx 10^{-4}\ \mathrm{Hz} \left(\frac{M}{M_\odot}\right)^{1/2} \left(\frac{R}{R_\odot}\right)^{-3/2}. \end{aligned} \tag{3}$$

In each equation, fiducial values are given in convenient units for the LISA ranges.

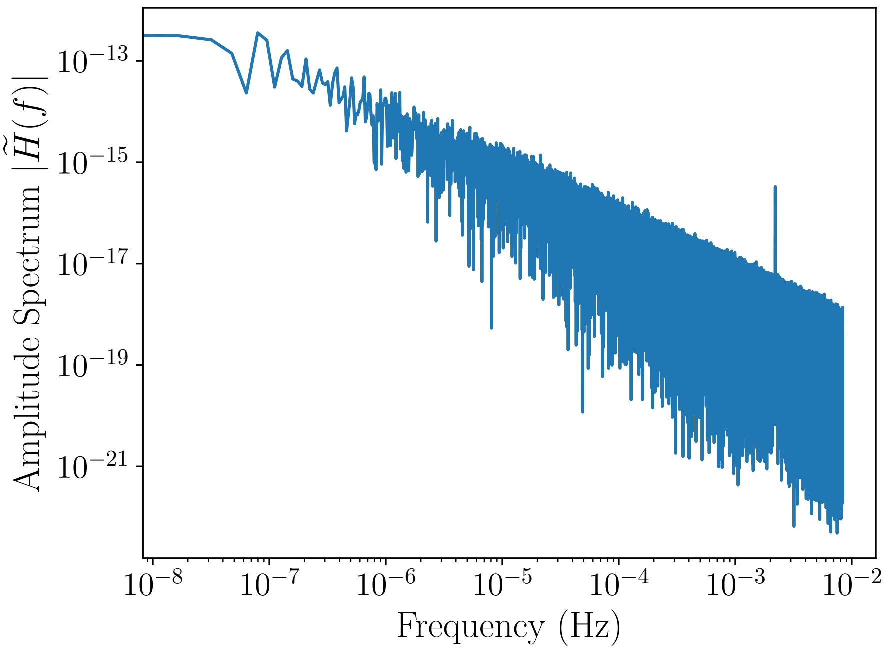
The Amplitude Spectrum

If you take a Discrete Fourier Transform (DFT) of the data, you can determine the gravitational wave frequency as a single spike in the amplitude spectrum, $|\widetilde{H}(f)|$, or alternatively the power spectrum $S_h(f) = |\widetilde{H}(f)|^2$. Again recall that a sine wave will appear twice: once at positive frequencies and one at negative frequencies. Since this is symmetric, you can determine the frequency of the sine wave from the spike at positive frequencies only.

You can determine the amplitude of the sine wave, i.e., the strain of the sinusoidal signal, from the amplitude of the spike in the amplitude spectrum $|\widetilde{H}(f)|$ with some caveats. A sine wave with a single frequency will not show up as a perfect spike but instead a sinc function because of how the DFT operates; one can interpolate the peak with a sinc function or even a quadratic but for your purposes you can just take the peak. You can probably convince yourself that from the definition of the DFT that you need to divide the amplitude of that peak by the number of samples N to get something close to the true amplitude of the sine wave. Lastly, you should multiply by two if you are considering a *one-sided spectrum*, i.e., you are only looking at positive frequencies. Remember that variance is conserved between domains, and so if half of the variance is going into the negative frequencies and half is going into the positive frequencies, then the height of the spikes will be half what they should be. A true one-sided spectrum takes the power from the negative half and actually adds it to the positive half.

In summary: the frequency of the gravitational wave signal is found as the frequency of the spike. The amplitude of the gravitational wave signal is found as the height of the spike, divided by N, and multiplied by 2.

In order to visualize the amplitude spectrum, you should plot the positive frequencies on a log-log graph. What you might expect to see then is some kind of sloped line from the power-law background and one spike at the frequency of the resolved white dwarf binary. An appropriate calculation usin `numpy`'s FFT package produces the following:



Note the x-axis is in Hz (1/s) and not inverse days!

Your Tasks

Your goal is to calculate the physical properties of the resolved binary system in the simulated data. In order to do so, you should program the Cooley-Tukey Algorithm to perform an FFT. At what point you stop the recursive step is up to you.

Once you have the $f_{\rm GW}$ and h measurements from the amplitude spectrum, assume that the system is at the distance of WD 0727+482, a known double WD system[2] determined at a distance of 12 pc. Determine the total mass M and separation R.

If you find yourself struggling with the FFT step, I recommend just using `numpy`'s FFT package. You will not receive full credit for this but it will allow you to finish the astrophysical calculation. You can also use this to compare with your FFT, and it should produce a plot similar to the one above.

[2] As far as I can tell, it is spectroscopically identified and not actually resolved. If so, this is a great LISA candidate!