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An adjoint approach to optimal design of turbine blades

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Abstract

In power plants, the aerodynamic optimization of turbine blades is crucial for efficiency considerations. The profile of the turbine blade is described by Bézier polynomials, where the coefficients are used as design variables in a nonlinear optimization procedure. The fluid-mechanics are modelled by the 2D Euler equations. This article deals with the application of an adjoint method for the optimization algorithm in the stationary case.

Keywords: Adjoint method; Sensitivity method; Turbine design

1. Introduction

In power plants the last blade row of turbines supplies 7% of the total power output. In order to enhance the efficiency of the last blade row, it is necessary to study the corresponding transonic flow field in more detail. The gas flow suffers from the occurrence of shock-waves, which produce high losses of energy and therefore of efficiency. By optimizing the blade profile, shock-waves can nearly be avoided or remarkably reduced in their strengths.

This technical problem leads to a restricted optimization problem. The restrictions are of geometrical or mechanical origin for the blade design and are also due to the conservation laws for the gas flow. The best approved optimization techniques like sequential quadratic programming (SQP)-methods are based on gradient information. There are three possible techniques for the gradient computation: numerical

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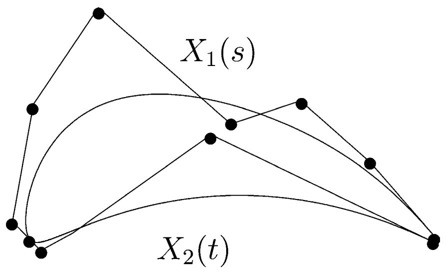

Fig. 1. Profile and Bézier polygon.

differentiation, sensitivity equation and adjoint method. This article will focus on the latter. In order to demonstrate the adjoint approach, in the following the 2D case is discussed in detail for a stationary model problem.

2. Modelling of the profile geometry

The profile of the turbine blades is described by two Bézier curves $X_1(s)$, $X_2(t)$ which are geometrically C^2-connected at the nose

$$X_1(s) = \begin{pmatrix} x_1(s) \\ y_1(s) \end{pmatrix} = \sum_{i=0}^{m} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} B_i^m(s), \quad s \in [0, 1], \tag{1}$$

and

$$X_2(t) = \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix} = \sum_{i=0}^{n} \begin{pmatrix} \alpha_{i+m} \\ \beta_{i+m} \end{pmatrix} B_i^n(t), \quad t \in [0, 1], \tag{2}$$

$$(\alpha_m, \beta_m)^{\mathrm{T}} = X_1(1) = X_2(0).$$

The basis functions B_i^n are the Bernstein-polynomials

$$B_i^n(t) = \begin{pmatrix} n \\ i \end{pmatrix} t^{n-i} t^i. \tag{3}$$

The $2(n+m+1)$ Bézier coefficients (α_i, β_i) are the design parameters for the geometry of the blade and are changed during the optimization process. Fig. 1 shows the profile of a turbine blade and its corresponding Bézier polygon.

3. 2D-Euler gas equations

The flow around the turbine blade is described by the Euler equations. With the conservative variables density ρ, momentum in x-direction $m = \rho u$, momentum in y-direction $n = \rho v$ and total energy E the gas equations are written as a conservation law of hyperbolic type

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{\partial}{\partial x} F(\mathbf{q}) + \frac{\partial}{\partial y} G(\mathbf{q}) = 0 \tag{4}$$

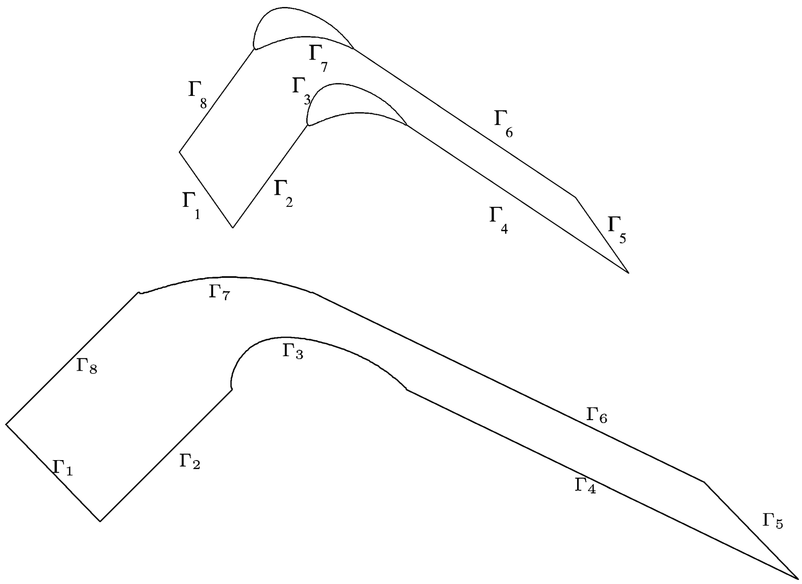

Fig. 2. Simulation area Ω.

with

$$\mathbf{q} = \begin{pmatrix} \rho \\ m \\ n \\ E \end{pmatrix}, \qquad F(\mathbf{q}) = \begin{pmatrix} m \\ mu + p \\ mv \\ (E+p)u \end{pmatrix}, \qquad G(\mathbf{q}) = \begin{pmatrix} n \\ nu \\ nv + p \\ (E+p)v \end{pmatrix}. \tag{5}$$

In order to close the set of equations additional conditions are needed, e.g., for an ideal gas

$$p = p(\rho, T):\ p = \frac{1}{\gamma}\rho T \quad \text{where } \gamma = \frac{c_p}{c_V} \approx 1.3, \tag{6}$$

$$e = e(p, T):\ e = c_V T. \tag{7}$$

The total energy can be described by

$$E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho\left(u^2 + v^2\right). \tag{8}$$

u and v are the velocities in x- and y-direction. c_p and c_V are the specific heat coefficients under constant pressure and constant volume.

Fig. 2 shows the simulation domain between two blades, which is extended in flow direction to guarantee stationary flow at the inlet Γ_1 and outlet Γ_5. At Γ_2, Γ_8 and Γ_4, Γ_6 periodic boundary conditions are prescribed. At the inlet Γ_1 the temperature T, the total pressure P_t and the angle of the flow $\angle(u, v)^{\mathrm{T}}$ are given. At the outlet Γ_5 the static pressure p is given. At the profile sections Γ_3, Γ_7 the slip condition with normal vector $\mathbf{n}$ holds

$$\langle w, \mathbf{n} \rangle = 0. \tag{9}$$

The geometry of Γ_3, Γ_7 is determined by the shape of the profile, i.e., the Bézier coefficients. The total pressure P_t is discussed in the next section.

4. Optimization problem

According to [7], a good parameter to measure the efficiency of a turbine is the total pressure, a direct function of the fluid variables

$$P_t = p\left(\frac{\gamma-1}{2}M^2+1\right)^{\frac{\gamma}{\gamma-1}}, \quad M = \sqrt{\frac{(u^2+v^2)\rho}{\gamma p}}. \tag{10}$$

M is the Mach number. The difference between the total pressure given at the inlet Γ_1 and the computed pressure at the outlet Γ_5 defines the total pressure loss, which has mainly three reasons:

(i) The loss due to the occurrence of shocks.
(ii) The loss due to the boundary layer.
(iii) The loss due to the inner friction of the fluid.

For transonic flows the approach presented here works fine, as the total pressure loss is dominated by shocks. For subsonic flows a combination of the stationary Euler equation with Prandtl's equation for the boundary layer is used. This approach can be found in the MIT manual [8]. Thus, also for subsonic fluids a total pressure loss is defined. Due to the high Reynolds number, the inner friction of the fluid may be neglected in gas turbines. Therefore the Euler equation is considered a good approximation.

The total pressure loss defines the objective function

$$\min_{\alpha} P_t\big(\mathbf{q}(\alpha)\big) := P_t|_{\Gamma_1} - \frac{\int_{\Gamma_5} \rho P_t \langle (u,v), \mathbf{n}_5\rangle \,\mathrm{d}s}{\int_{\Gamma_5} \rho \langle (u,v), \mathbf{n}_5\rangle \,\mathrm{d}s}. \tag{11}$$

$\mathbf{n}_5$ is the outer normal direction at the outlet Γ_5 and α stands for the design vector $(\ldots,\alpha_i,\beta_i,\ldots)^{\mathrm{T}}$. Eq. (11) has to be minimized under additional mechanical constraints, like limitations for the surface area, the center of gravity and the moment of inertia. Additionally the curvature of the blade at the leading and trailing edge are prescribed.

This minimization problem is solved by the sequential quadratic programming (SQP) method, see [11,15]. In order to make use of this efficient algorithm for nonlinear optimization problems under constraints the Jacobian of the constraints and the gradient of the objective function $\nabla P_t(\mathbf{q}(\alpha))$ must be provided. The Jacobian can be achieved by explicit differentiation, whereas the computation of the gradient is subject to further discussions.

The total pressure is an explicit function of the state variables. Assuming the states are continuous, its gradient with respect to α is given by

$$\nabla P_t\big(\mathbf{q}(\alpha)\big) = \frac{\partial P_t}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \alpha}. \tag{12}$$

In principle we have three approaches to compute the derivatives of the state variables with respect to the design parameters:

(i) $\frac{\partial P_t}{\partial \mathbf{q}}$ can be computed analytically, but the evaluation of $\frac{\partial \mathbf{q}}{\partial \alpha}$ by finite differences is too imprecise and inefficient.

Table 1
Boundary conditions

	Euler gas equation	Sensitivity equation
Γ_1	$T,\ P_t,\ \angle(u,v)^{\mathrm{T}}$	$T_\alpha = 0,\ P_{t,\alpha} = 0,\ \angle(u,v)^{\mathrm{T}}$
Γ_5	p	$p_\alpha = 0$
Γ_3, Γ_7	$\langle w, \mathbf{n}\rangle = 0$	$\langle w, \mathbf{n}\rangle = -\langle w_x x_\alpha + w_y y_\alpha, \mathbf{n}\rangle - \langle w, \mathbf{n}_\alpha\rangle$
Γ_2, Γ_8	$\mathbf{q}_{\Gamma_2} = \mathbf{q}_{\Gamma_8}$	$\mathbf{q}_{\alpha,\Gamma_2} = \mathbf{q}_{\alpha,\Gamma_8}$
Γ_4, Γ_6	$\mathbf{q}_{\Gamma_4} = \mathbf{q}_{\Gamma_6}$	$\mathbf{q}_{\alpha,\Gamma_4} = \mathbf{q}_{\alpha,\Gamma_6}$

(ii) Explicit differentiation of the Euler gas equations with respect to α results in the sensitivity equation, see [1,2,4,5]

$$\frac{\partial \mathbf{q}_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\mathrm{d}F(\mathbf{q})}{\mathrm{d}\mathbf{q}}\mathbf{q}_\alpha\right) + \frac{\partial}{\partial y}\left(\frac{\mathrm{d}G(\mathbf{q})}{\mathrm{d}\mathbf{q}}\mathbf{q}\right) = 0, \tag{13}$$

with

$$\mathbf{q}_\alpha = \frac{\partial \mathbf{q}}{\partial \alpha} = \begin{pmatrix} \partial\rho/\partial\alpha \\ \partial m/\partial\alpha \\ \partial n/\partial\alpha \\ \partial E/\partial\alpha \end{pmatrix} = \begin{pmatrix} \rho_\alpha \\ m_\alpha \\ n_\alpha \\ E_\alpha \end{pmatrix}. \tag{14}$$

The derivatives of the initial and boundary conditions with respect to α are summed up in Table 1.

(iii) The third approach by an adjoint method for the stationary case is discussed in the following, see also [10,13,16].

5. The adjoint method

5.1. The dual problem

Applying the adjoint method means solving a dual problem, which leads to the same result as the original problem. This procedure will be clarified shortly for linear equation systems. In order to avoid the solution x of a linear equation system occurring in a scalar product $g^{\mathrm{T}}x$, one possibility is to solve its dual problem. $g^{\mathrm{T}}x$ depends on the solution of

$$Ax = b, \quad A \in \mathbb{R}^{n\times n} \text{ regular}, \ x, b \in \mathbb{R}^n. \tag{15}$$

Coupling (15) to the scalar product via the dual problem $A^{\mathrm{T}}y = g$ gives

$$g^{\mathrm{T}}x = g^{\mathrm{T}}x - y^{\mathrm{T}}(Ax - b) \tag{16}$$

$$= g^{\mathrm{T}}x - \big(A^{\mathrm{T}}y\big)^{\mathrm{T}}x + y^{\mathrm{T}}b \tag{17}$$

$$= y^{\mathrm{T}}b. \tag{18}$$

So $g^{\mathrm{T}}x$ can be computed via $y^{\mathrm{T}}b$ without the knowledge of x. In this example the advantages of the adjoint method are only apparent for s different right-hand sides b_r, $r = 1, \ldots, s$. Instead of solving s decomposed LR-systems to receive x_r, one has to compute s scalar products $y^{\mathrm{T}}b_r$ for one system.

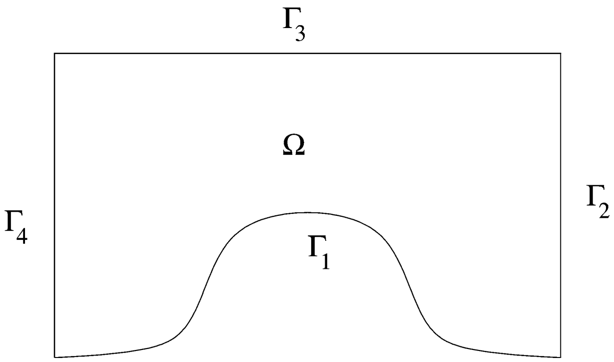

Fig. 3. Two-dimensional model problem.

To extend this approach to functionals the scalar product (g, x_r) under the restriction of the partial differential equation (PDE) $Lx_r = f_r$ can be replaced by the dual problem of solving (y, f_r) under restriction of $L^* y = g$ with the adjoint operator L^*. Here homogeneous boundary conditions are assumed for clarity reasons.

5.2. The Lagrange view

The Lagrange view allows the handling of boundary conditions, which is crucial for practical implementations. The lengthy formalism is demonstrated for a model problem, the stationary 2D-Euler equations for a flow through a nozzle with variable geometry Ω, see Fig. 3.

$$F(\mathbf{q})_x + G(\mathbf{q})_y = 0. \tag{19}$$

At the inlet Γ_4 density $\rho = 1.3$, angle $\angle(u, v)^{\mathrm{T}}$, velocity $v = 0$ and pressure $p = 0.9$ are given. At the outlet Γ_2 pressure $p = 0.9$ is prescribed. At the boundaries Γ_1 and Γ_3 $w^{\mathrm{T}}\mathbf{n} = 0$ holds for the velocity in normal direction $\mathbf{n}$. The upper wall is fixed whereas the lower wall Γ_1 can be optimized via

$$y(x) = \begin{cases} 0\colon\ -0.5 \leqslant x < 0, \\ \sum_{i=1}^{4} \alpha_i B_i^4\colon\ 0 \leqslant x < 1, \\ 0: 1 \leqslant x < 1.5, \end{cases} \tag{20}$$

α_i, $i = 1, \ldots, 4$ are the design parameters. The functions B_i^4 are the Bernstein-polynomials as in (3). At Γ_1 a pressure distribution is prescribed as nominal pressure p^d. Find α_i such that the functional J is minimized

$$J = \frac{1}{2} \int\limits_{\Gamma_1} \left(p - p^d\right)^2 \mathrm{d}s. \tag{21}$$

This requires the calculation of $\frac{\partial J}{\partial \alpha_i}$. To identify the problem with Section 5.1 use

$$(g, x_r) \leftrightarrow \frac{\partial J}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \alpha_i}. \tag{22}$$

$\frac{\partial \mathbf{q}}{\partial \alpha_i} = \mathbf{q}_{\alpha_i}$ is the solution of $L\mathbf{q}_{\alpha_i} = f$, with

$$\frac{\partial}{\partial x}\left(\frac{\partial F(\mathbf{q})}{\partial \mathbf{q}}\mathbf{q}_{\alpha_i}\right) + \frac{\partial}{\partial y}\left(\frac{\partial G(\mathbf{q})}{\partial \mathbf{q}}\mathbf{q}_{\alpha_i}\right) = 0. \tag{23}$$

The boundary conditions for every α_i are

$$\frac{\partial}{\partial \alpha_i}\left(w^{\mathrm{T}}\mathbf{n}\right) = 0. \tag{24}$$

In a general case no homogeneous boundary conditions as in Section 5.1 are assumed, therefore the boundary conditions in the current section correspond with the term f_r from Section 5.1.

To achieve the full information for the adjoint equation, the Euler equations and the boundary conditions are coupled to the functional J via Lagrange multipliers $\Lambda \in \mathbb{R}^4$ and $\mu \in \mathbb{R}$.

$$J = \frac{1}{2}\int_{\Gamma_1} \left(p - p^d\right)^2 \mathrm{d}s + \int_{\Omega} \Lambda^{\mathrm{T}}\left(F(\mathbf{q})_x + G(\mathbf{q})_y\right) \mathrm{d}\Omega + \int_{\Gamma_1} \mu w^{\mathrm{T}}\mathbf{n}\,\mathrm{d}s. \tag{25}$$

Differentiation by α_i and using Reynolds' transport theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V(t)} f(t)\,\mathrm{d}V = \iiint_{V(t)} \frac{\partial f(t)}{\partial t}\,\mathrm{d}V + \iint_{\delta V} f(V)\langle u, \mathbf{n}\rangle\,\mathrm{d}s$$

gives

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_i}J = \int_{\Gamma_1(\alpha_i)} \frac{\partial p}{\partial \mathbf{q}}\left(p - p^d\right)\frac{\partial \mathbf{q}}{\partial \alpha_i}\,\mathrm{d}s + \frac{1}{2}\int_{-1}^{1}\left(p - p^d\right)^2 \frac{\mathrm{d}B_i^4}{\mathrm{d}x}\,\mathrm{d}x \tag{26}$$

$$+ \int_{\Omega(\alpha_i)} \Lambda^{\mathrm{T}}\left(\left(\frac{\partial F}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \alpha_i}\right)_x + \left(\frac{\partial G}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \alpha_i}\right)_y\right)\mathrm{d}\Omega \tag{27}$$

$$+ \frac{\partial}{\partial \alpha_i}\int_{\Omega(\alpha_i)} \Lambda^{\mathrm{T}}\left(F(\mathbf{q})_x + G(\mathbf{q})_y\right)\mathrm{d}\Omega \tag{28}$$

$$+ \int_{\Gamma_1(\alpha_i)} \mu\mathbf{n}\frac{\partial w}{\partial \mathbf{q}}\frac{\partial \mathbf{q}}{\partial \alpha_i}\,\mathrm{d}s + \int_{\Gamma_1(\alpha_i)} \mu w^{\mathrm{T}}\frac{\partial \mathbf{n}}{\partial \alpha_i}\,\mathrm{d}s + \int_{-1}^{1} \mu w^{\mathrm{T}}\mathbf{n}\frac{\mathrm{d}B_i^4}{\mathrm{d}x}\,\mathrm{d}x. \tag{29}$$

By partial integration (27) is equivalent to

$$\sum_k \int_{\Gamma_k} \Lambda^{\mathrm{T}}\left(\frac{\partial F}{\partial \mathbf{q}}n_x + \frac{\partial G}{\partial \mathbf{q}}n_y\right)\frac{\partial \mathbf{q}}{\partial \alpha_i}\,\mathrm{d}s - \int_{\Omega}\left(\Lambda_x^{\mathrm{T}}\frac{\partial F}{\partial \mathbf{q}} + \Lambda_y^{\mathrm{T}}\frac{\partial G}{\partial \mathbf{q}}\right)\frac{\partial \mathbf{q}}{\partial \alpha_i}\,\mathrm{d}\Omega. \tag{30}$$

This finally leads to

$$\frac{\mathrm{d}}{\mathrm{d}\alpha_i} J = \int_{\Gamma_1} \frac{\partial p}{\partial \mathbf{q}} \left(p - p^d\right) \frac{\partial \mathbf{q}}{\partial \alpha_i} \, \mathrm{d}s + \frac{1}{2} \int_{-1}^{1} \left(p - p^d\right)^2 \frac{\mathrm{d}B_i^4}{\mathrm{d}x} \, \mathrm{d}x \tag{31}$$

$$\times \sum_k \int_{\Gamma_k} \Lambda^{\mathrm{T}} \left(\frac{\partial F}{\partial \mathbf{q}} n_x + \frac{\partial G}{\partial \mathbf{q}} n_y \right) \frac{\partial \mathbf{q}}{\partial \alpha_i} \, \mathrm{d}s \tag{32}$$

$$- \int_{\Omega} \left(\Lambda_x^{\mathrm{T}} \frac{\partial F}{\partial \mathbf{q}} + \Lambda_y^{\mathrm{T}} \frac{\partial G}{\partial \mathbf{q}} \right) \frac{\partial \mathbf{q}}{\partial \alpha_i} \, \mathrm{d}\Omega \tag{33}$$

$$+ \frac{\partial}{\partial \alpha_i} \int_{\Omega} \Lambda^{\mathrm{T}} \left(F(\mathbf{q})_x + G(\mathbf{q})_y \right) \mathrm{d}\Omega \tag{34}$$

$$+ \int_{\Gamma_1} \mu \mathbf{n} \frac{\partial w^{\mathrm{T}}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \alpha_i} \, \mathrm{d}s + \int_{\Gamma_1} \mu w^{\mathrm{T}} \frac{\partial \mathbf{n}}{\partial \alpha_i} \, \mathrm{d}s + \int_{-1}^{1} \mu w^{\mathrm{T}} \mathbf{n} \frac{\mathrm{d}B_i^4}{\mathrm{d}x} \, \mathrm{d}x. \tag{35}$$

The adjoint equation has to be solved only once for the Lagrange multiplier Λ, which is identified with

$$-\left(\frac{\partial F}{\partial \mathbf{q}} \right)^{\mathrm{T}} \Lambda_x - \left(\frac{\partial G}{\partial \mathbf{q}} \right)^{\mathrm{T}} \Lambda_y = 0 \quad \text{in } \Omega, \tag{36}$$

and the boundary conditions

$$\Lambda^{\mathrm{T}} \left(\frac{\partial F}{\partial \mathbf{q}} n_x + \frac{\partial G}{\partial \mathbf{q}} n_y \right) \frac{\partial \mathbf{q}}{\partial \alpha_i} = 0 \quad \text{on } \Lambda_k,\ k = 2, 3, 4, \tag{37}$$

$$\Lambda^{\mathrm{T}} \left(\frac{\partial F}{\partial \mathbf{q}} n_x + \frac{\partial G}{\partial \mathbf{q}} n_y \right) \frac{\partial \mathbf{q}}{\partial \alpha_i} + \frac{\partial p}{\partial \mathbf{q}} \left(p - p^d\right) \frac{\partial \mathbf{q}}{\partial \alpha_i} + \mu \mathbf{n} \frac{\partial w^{\mathrm{T}}}{\partial \mathbf{q}} \frac{\partial \mathbf{q}}{\partial \alpha_i} = 0 \quad \text{on } \Gamma_1. \tag{38}$$

Here $g = \frac{\mathrm{d}I}{\mathrm{d}\mathbf{q}}$ (see (22)) adds to the boundary conditions instead of the PDE. To solve the adjoint equation (36) numerically, the asymptotic limit of the instationary equation

$$\Lambda_t - \left(\frac{\partial F}{\partial \mathbf{q}} \right)^{\mathrm{T}} \Lambda_x - \left(\frac{\partial G}{\partial \mathbf{q}} \right)^{\mathrm{T}} \Lambda_y = 0 \tag{39}$$

is calculated.

Now $\frac{\mathrm{d}J}{\mathrm{d}\alpha}$ reduces to

$$\frac{\mathrm{d}J}{\mathrm{d}\alpha} = \frac{1}{2} \int_{-1}^{1} \left(p - p^d\right)^2 \frac{\mathrm{d}B_i^4}{\mathrm{d}x} \, \mathrm{d}x + \int_{-1}^{1} \mu w^{\mathrm{T}} \mathbf{n} \frac{\mathrm{d}B_i^4}{\mathrm{d}x} \, \mathrm{d}x + \int_{\Gamma_1} \mu w^{\mathrm{T}} \frac{\partial \mathbf{n}}{\partial \alpha_i} \, \mathrm{d}s. \tag{40}$$

Formula (40) can be identified with (y, f_r) from Section 5.1.

The connection between the Lagrange multipliers Λ and μ can be computed from the boundary conditions at Γ_1. Consider Eq. (38) with $k = \frac{\gamma - 1}{2}$, $n_x u + n_y v = 0$ and

$$\frac{\partial p}{\partial \mathbf{q}} = 2k \left(\frac{u^2 + v^2}{2}, -u, -v, 1 \right), \tag{41}$$

$$\frac{\partial w^{\mathrm{T}}}{\partial \mathbf{q}} = \frac{1}{\rho} \begin{pmatrix} -u & 1 & 0 & 0 \\ -v & 0 & 1 & 0 \end{pmatrix}. \tag{42}$$

Eq. (38) leads to the four components

$$2k\frac{\|w\|_2}{2}\left(p-p^d\right)+n_x k\|w\|_2\lambda_2+n_y k\|w\|_2\lambda_3=0, \tag{43}$$

$$\begin{aligned}&-2ku\left(p-p^d\right)+n_x\lambda_1+(2-\gamma)n_x u\lambda_2\\&\quad+(n_x v-2n_y ku)\lambda_3+n_x\left(\gamma\frac{E}{\rho}-k\|w\|_2\right)\lambda_4+\frac{n_x}{\rho}\mu=0,\end{aligned} \tag{44}$$

$$\begin{aligned}&-2kv\left(p-p^d\right)+n_y\lambda_1+(n_y u-2n_x kv)\lambda_2\\&\quad+n_y(2-\gamma)v\lambda_3+n_y\left(\gamma\frac{E}{\rho}-k\|w\|_2\right)\lambda_4+\frac{n_y}{\rho}\mu=0,\end{aligned} \tag{45}$$

$$2k\left(p-p^d\right)+2kn_x\lambda_2+2kn_y\lambda_3=0. \tag{46}$$

Eqs. (43) and (46) reduce to

$$\left(p-p^d\right)+n_x\lambda_2+n_y\lambda_3=0. \tag{47}$$

The connection between μ and Λ follows from Eqs. (44) and (45)

$$\mu=-\rho\left(\lambda_1+u\lambda_2+v\lambda_3+\left(\gamma\frac{E}{\rho}-k\|w\|_2\right)\lambda_4\right). \tag{48}$$

5.3. *Properties of the adjoint equation*

In general, differentiability of conservation laws due to the parameters cannot be assumed. Even for continuous flux function and continuous initial values nonlinear conservation laws only possess solutions in a weak sense.

Special concepts like shift differentiability have been introduced to deal with these difficulties, see [6,17]. The sensitivity and adjoint equations have to be studied at each shock location of the states. Special limiters can be constructed to catch the shock for the sensibility equation, see [17]. These new developments are in an infant state and so far not applied in practical implementations. The adjoint equation itself is continuous and constant at a discontinuity of the states. The inner boundary conditions are not discretized, but they can be forced explicitly by a shock-tracking method.

In industrial environment the discontinuity problem is avoided by introducing artificial numerical diffusion. The error of this approach is not controlled: the comparison between simulation results and measurements is the only criteria.

5.4. *Numerical example*

As a first step for the adjoint method only numerical simulation results are presented. The numerical solution of the adjoint equation was calculated by a standard first order upwind scheme, see Appendix A. Figs. 4 and 5 were calculated for a fixed Γ_1 with $\alpha_1=2$, $\alpha_i=0$, $i=2,3,4$. The nominal pressure distribution p^d holds for Γ_1^d with $\alpha_1=1$, $\alpha_i=0$, $i=2,3,4$. This pressure distribution is projected from Γ_1^d to Γ_1. Fig. 4 shows the resulting density and the y-component of the velocity as solution of the Euler equations. The connection between the adjoint variables and the physical variables is not straight–forward. One can summarize. The larger the values of the adjoint variables the more they influence the

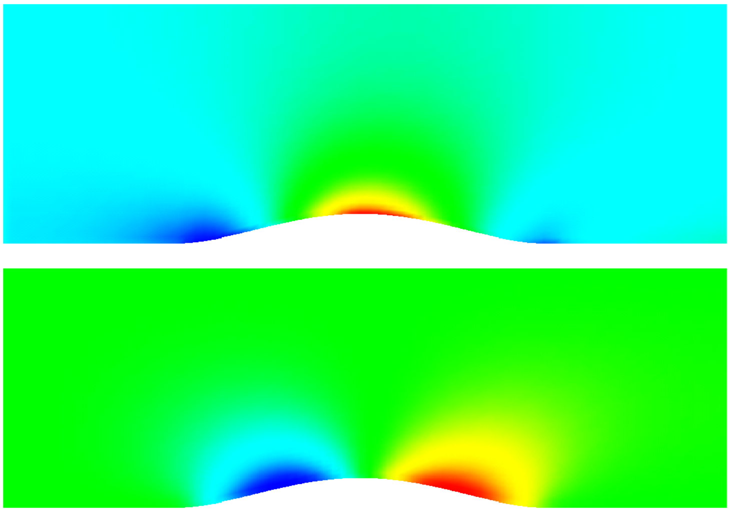

Fig. 4. Density distribution and y-component of velocity distribution.

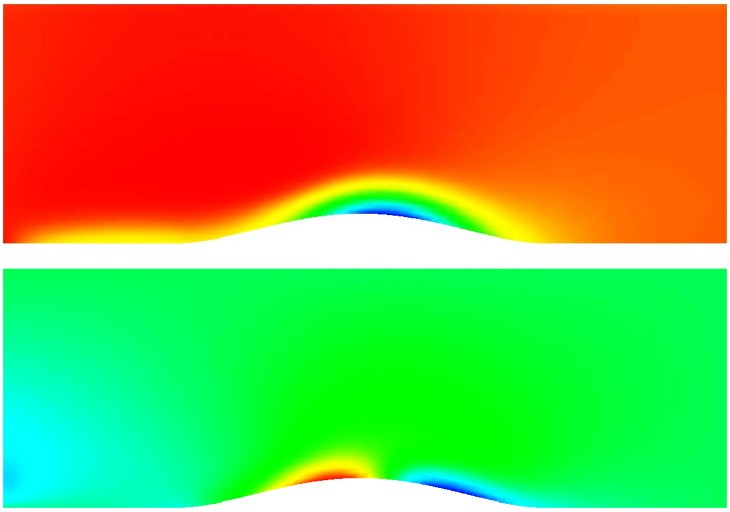

Fig. 5. 1st and 3rd component of the adjoint variable.

objective function. The 1st and 3rd component of the adjoint variable have been chosen arbitrarily for display in Fig. 5.

5.5. *Optimal turbine blade*

Using the sensitivity equation approach and an adopted SQP solver, optimal configurations of turbine blades were computed with our industrial partner [3]. Fig. 6 shows the starting profile and an optimal profile of a turbine blade.

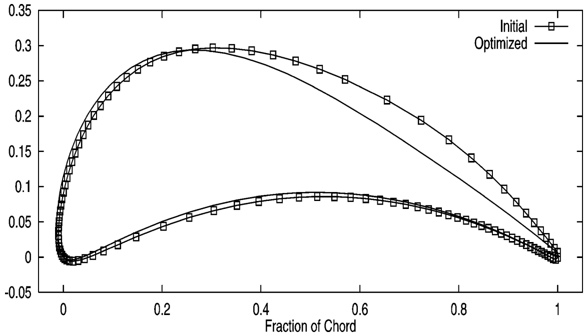

Fig. 6. Starting profile and optimal profile.

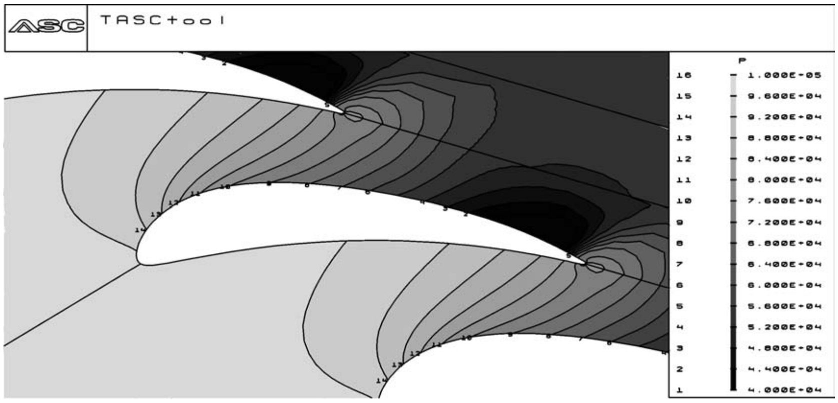

Fig. 7. Pressure distribution for the starting profile.

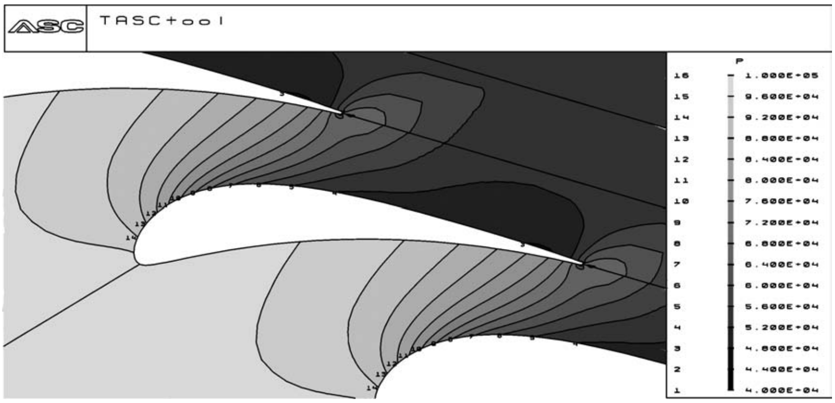

Fig. 8. Pressure distribution for the optimal profile.

The corresponding Figs. 7 and 8 give the pressure distribution. The darker the region the higher are the pressure values indicating shock regions. The optimal profile in Fig. 8 shows a significantly lower pressure value (dark grey: initial profile, light grey: optimal profile).

6. Conclusion

From three possibilities of gradient computation the first, the numerical differentiation is too imprecise and too costly. Notice that each Bézier parameter variation requires a regridding of the net, since the net is not parameterized.

So far the sensitivity equation is well approved in turbine simulations, see also our simulation results in Section 5.5. The sensitivity equation has the advantage that it is independent of the objective functional J (cf. 5.1). Therefore different objective functions can be considered without high additional costs. The main disadvantage of the sensitivity method is, that a linear system of PDEs has to be solved for every design parameter α_i which is very costly.

The adjoint method is still in development for turbine simulations. Several authors [9,10,12,14] report promising results from different flow applications. In the adjoint method the objective functional J enters the boundary conditions of the adjoint equation, which is a disadvantage if several objective functions are considered. The huge advantage of the adjoint method is that the system of PDEs has to be solved only once. Instead of solving a system of PDEs for every design parameter α_i a less expensive scalar product can be calculated. This has been demonstrated in the model problem of Section 5.2. The advantage of the adjoint method is underlined by the fact that nearly all solvers compute the solution of the stationary PDE as the asymptotic limit of the instationary equation.

So far the adjoint equation seems superior to the other techniques. Nevertheless in special cases the sensitivity approach can be competitive. If the computations of the sensitivity equation and the Euler equations are decoupled, in the stationary case the discretized sensitivity equation reduces to a system of linear equations. Then for each parameter α_i only a forward–backward substitution is necessary, this is utilized in MISES, see [8]. In this case the adjoint and the sensitivity equation approach are equivalent.

Appendix A. Numerical technique for the adjoint equation

To solve the adjoint equation a standard first order upwind-scheme was applied

$$
\begin{aligned}
\Lambda_{ij}^{n+1} = \Lambda_{ij}^{n} &+ \frac{h}{k}\Big(\big(A^{\mathrm{T}}\big)^{+}_{i-\frac{1}{2}j}\big(\Lambda_{ij}^{n} - \Lambda_{i-1j}^{n}\big) + \big(A^{\mathrm{T}}\big)^{-}_{i+\frac{1}{2}j}\big(\Lambda_{i+1j}^{n} - \Lambda_{ij}^{n}\big)\Big) \\
&+ \frac{h}{k}\Big(\big(B^{\mathrm{T}}\big)^{+}_{ij-\frac{1}{2}}\big(\Lambda_{ij}^{n} - \Lambda_{ij-1}^{n}\big) + \big(B^{\mathrm{T}}\big)^{-}_{ij+\frac{1}{2}}\big(\Lambda_{ij+1}^{n} - \Lambda_{ij}^{n}\big)\Big), \\
A = \frac{\partial F}{\partial \mathbf{q}}, &\qquad B = \frac{\partial G}{\partial \mathbf{q}}.
\end{aligned}
$$

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