SECTION C

THREE-DIMENSIONAL FLOW IN TURBOMACHINES

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C,1. Introduction. The flow through a turbomachine blade row may be separated conceptually into two parts: that which is due to the gross influence of the blade row and that which is associated with the details of blade shape. The general idea of decomposing the flow field in this manner is a direct extension of classical propeller theory which, in turn, was motivated by the success of the Lanchester-Prandtl wing theory. In wing theory, one first computes the downwash at a given point of the wing induced by the entire wing and there analyzes the detailed flow over the element in question. In turbomachine theory, one computes the flow induced by all blades or blade rows at a given point and then analyzes the detailed flow over the particular blade element in question.

It was Ruden [*1*] who first gave the intuitive description of turbomachine flow in the language of wing theory. In spite of the fact that he did not pursue analytical development of this picture, Ruden did recognize some essential relationships between linearized wing and turbomachine theory. One of these was the fact that, for a linearized theory, the axial velocity distortion in the plane of a blade row is half that induced by the blade row far downstream in much the same manner that the downwash in the plane of a finite wing is half that in the Trefftz plane.

The computational difficulties of the full three-dimensional flow field caused investigators to seek a simpler but less general description of the interesting portion of the flow. This search led many to develop independently the so-called radial equilibrium theory. In the radial equilibrium theory one looks only at the flow far upstream and far downstream of the blade row where radial accelerations have vanished and assumes a completely axially symmetric flow where no trace of the individual blades remains. In fact this description became so widespread by 1942, at least among workers in Germany, Switzerland, United States, and England, and the publication of these elementary results was so casual or nonexistent, that it is impossible to cite accurately any priority in the matter. The work of Traupel [*2*] is one of the earliest complete accounts in which the radial equilibrium idea is employed, but it was used by others at an earlier date. One of particular interest, developed and used extensively in the period of 1940, was due to von Kármán and Rannie although it was not published in detail until 1949 [3]. This particular treatment differs from others by being somewhat more than a linear theory.

The drawback of the radial equilibrium theory is that it gives no account of how the changes in velocity distribution take place across a blade row. One has no idea whether the change in profile is complete immediately behind or many blade lengths behind the blade row. To rectify this difficulty, Meyer [4] studied the detailed potential flow through a stationary row of a finite number of blades, using a technique which was originally introduced by Ackeret for analysis of flow through cascades. More recently Wu [5,6,7] developed a fairly general method of treating turbomachine problems from a purely numerical viewpoint. The difficulty experienced here is the general complexity of results, the time consumed in calculation of a specific example and the relative impossibility of extracting from these theories some essential features that may be extended to apply to technological problems. On the other hand the complete three-dimensional theories do offer the possibility of investigating in detail very particular problems if they arise.

The three-dimensional problem simplifies greatly when it is assumed that the flow is axially symmetric, as was done in the radial equilibrium treatment. Physically this implies that the blade row must consist of an infinite number of infinitely thin blades. To achieve the most realistic representation, these blades may be given their proper chordwise load distribution but averaged circumferentially. A three-dimensional flow theory based on these ideas was introduced by the present author [8] and expanded to cover a variety of typical examples by Marble [9] and Marble and Michelson [10]. The idea of this axially symmetric throughflow is exactly that outlined in the introductory paragraph: to compute an induced flow field in which the individual blades may be considered operating. The assumption of axial symmetry indicates merely that the induced flow in the plane of the blade row will be computed as if the trailing vortex field were circumferentially uniform rather than separated into discrete sheets.

The actual flow pattern over a turbomachine blade row differs from the axially symmetric throughflow because of the finite peripheral spacing of the blades and the detailed geometrical shape. The detailed blade geometry is the main concern of cascade studies. In applying the results of cascade theory and experiments to the three-dimensional flow problem it is usual to approximate the effect of blade geometry, at a given radius, by the local two-dimensional flow about the blade; the approach stream direction and magnitude are given by the axially symmetric throughflow theory. In this manner the blade angle of attack, the blade camber and thickness, the critical Mach number, etc., may be selected or deduced from two-dimensional cascade information, the process being repeated at as many radii as required to define the blade. In this process it is tacitly assumed that the flow about the blade at one radius does not influence appreciably the flow about the blade at another radius.

The calculations involved in axially symmetric throughflow theory are still sufficiently complex that application of the theory is restricted to examples requiring special attention and is not very useful in routine design. In fact the axially symmetric throughflow theory would be of only modest help to the designer were it not for the fact that it leads to a very simple and useful approximation. In his 1948 paper [*8*] the present author introduced the so-called "exponential approximation" to the throughflow. In essence this is a method of describing the transition from the known radial equilibrium patterns far upstream and downstream of the blade row; it employs the ideas of the axially symmetric throughflow theory in obtaining this approximate transition. These ideas were subsequently applied by the author [*9,10*] and later independently by Railly [*11*] and by Horlock [*12,13*] to describe a wide variety of technological problems including mutual interference of neighboring blade rows. This latter problem was also treated in a somewhat less general manner, and by a completely different technique, by Wu and Wolfenstein [*14*]. At present the radial equilibrium theory, the exponential approximation, and its extensions given in the following work constitute a method of sufficient accuracy and simplicity for utilization in the majority of turbomachine problems.

There are effects, sometimes significant ones, that are in no way accounted for in the combination of axially symmetric throughflow theory and two-dimensional cascade theory. One of these is the fact that the vorticity trailing from the actual blade row is not distributed but is concentrated in the blade row wakes. It would be expected that this modification to the induced velocity field is significant for wide blade spacings and for large spanwise gradients of shed vorticity. This problem has not, in fact, been investigated to a significant extent. A second property of the actual flow, which is unaccounted for in the throughflow-cascade theory, is the effect of boundary layers. These phenomena were discussed physically by Weske [*15*]. Of particular importance is the interaction of the individual blades with thick boundary layers on the hub and tip casings of the turbomachine. This problem of "secondary flow" has been analyzed by Hawthorne [*16*] and several others as a source of losses and as an influence upon blade angle setting. Finally there is the rather singular problem of the transonic compressor, that is a blade row in which the relative velocities are supersonic at the tip and subsonic at the root. Here the question is whether cascade theory can be applied locally to determine blade performance and required angles. It is not unreasonable to expect that the flow about a blade element in the transonic region would alter significantly the flow about blade elements at different radii and hence negate the assumption that blade elements at different radii are independent. A detailed analysis by McCune [*17*] has recently indicated that the interference is indeed so large that cascade theory is not applicable in this instance.

Of the various flow processes comprising the complete three-dimensional flow field, only the axially symmetric throughflow and the two-dimensional cascade flow have been developed to the point of usefulness in the design of turbomachine components. Consequently it is with the axially symmetric throughflow that the present section is principally concerned.

C,2. Formulation of Axially Symmetric Throughflow. The throughflow is described by the axially symmetric motion of an ideal

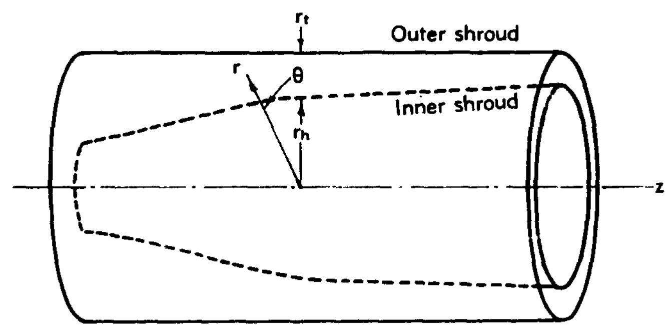


Fig. C,2a. Schematic diagram of turbomachine.

gas passing through a prescribed force field representing the blade row and moving tangentially to hub and tip boundaries, Fig. C,2a, of given shape. There are, in general, two types of problem of interest to turbomachine designers differing according to the information prescribed about the blade row, the direct and inverse problems. The *inverse problem* considers the flow field generated when a prescribed distribution of enthalpy or angular momentum is added by the blade row. Here the aim is to proportion the blades and to choose their angles appropriately to meet conditions at a certain design point. The *direct problem,* on the other hand, considers the flow field induced by blades of given geometric configuration. Here it is desired to determine the blade row performance at operating conditions other than those for which it was designed.

The aerothermodynamic equations. It is most convenient to describe the throughflow in a cylindrical coordinate system, Fig. C,2b, where u, v, and w are the velocity components in the r, θ, and z directions. Denoting the axially symmetric force components corresponding to the blade row by F_r, F_θ, and F_z, the equations of motion are

$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + F_r \tag{2-1}$$

$$u\frac{\partial (rv)}{\partial r} + w\frac{\partial (rv)}{\partial z} = rF_\theta \tag{2-2}$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + F_z \tag{2-3}$$

In writing the equations of motion, account has been taken of the general axial symmetry of the flow, and consequently all peripheral

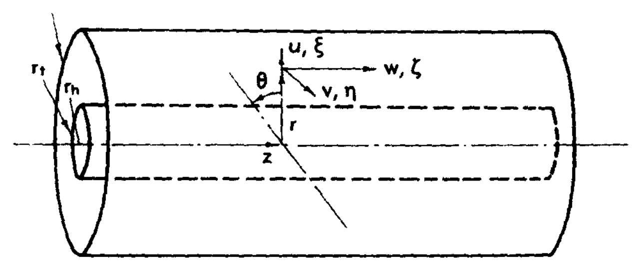


Fig. C,2b. Notation for axes, velocity components, and vorticity components.

derivatives vanish. The continuity equation may be written similarly as

$$\frac{1}{r}\frac{\partial}{\partial r}(r\rho u) + \frac{\partial}{\partial z}(\rho w) = 0 \tag{2-4}$$

It will often be convenient to describe the flow field in terms of the radial, tangential, and axial vorticity components defined respectively as

$$\xi = -\frac{\partial v}{\partial z} \tag{2-5}$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \tag{2-6}$$

$$\zeta = \frac{1}{r}\frac{\partial (rv)}{\partial r} \tag{2-7}$$

The equations of motion, Eq. 2-1, 2-2, and 2-3, may be rewritten in terms of vorticity components as

$$w\eta - v\zeta = F_r - \left[\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\partial}{\partial r}\left(\frac{q^2}{2}\right)\right] \tag{2-8}$$

$$u\zeta - w\xi = F_\theta \tag{2-9}$$

$$v\xi - u\eta = F_z - \left[\frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left(\frac{q^2}{2}\right)\right] \tag{2-10}$$

where q is the magnitude of the velocity, $q^2 = u^2 + v^2 + w^2$.

For some analysis it will be convenient to write the equations of motion in terms of the enthalpy and entropy gradients rather than the pressure gradient. Denote the general pressure variation δp as

$$\frac{1}{\rho}\,\delta p = \delta h - T\delta s \tag{2-11}$$

where s and h are respectively the entropy and thermodynamic enthalpy of the gas. Then defining the stagnation enthalpy

$$h^0 = h + \frac{q^2}{2} \tag{2-12}$$

equations of motion in the radial and axial direction, Eq. 2-8 and 2-10, mav be rewritten

$$w\eta - v\zeta = F_r - \frac{\partial h^0}{\partial r} + T\frac{\partial s}{\partial r} \tag{2-13}$$

$$v\xi - u\eta = F_z - \frac{\partial h^0}{\partial z} + T\frac{\partial s}{\partial z} \tag{2-14}$$

It is worth noting also, at this point, that the combined first and second laws of thermodynamics, Eq. 2-11, may be written to give the entropy variation of the gas along a stream surface in the form

$$T\left(u\frac{\partial s}{\partial r} + w\frac{\partial s}{\partial z}\right) = u\frac{\partial e}{\partial r} + w\frac{\partial e}{\partial z} + p\left(u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left(\frac{1}{\rho}\right) \tag{2-15}$$

Although the heat transfer to the gas is usually negligible, the losses in the blades make the flow process a nonisentropic one. Consequently it is necessary to calculate the entropy variation explicitly.

Conditions at the blades. In the absence of any shear stresses or losses the local blade force is normal to the local blade surface. When losses such as surface shear stresses are present, however, the complete force system must also have a component tangential to the local blade surface. It is appropriate then to express the blade forces in two parts: one set, $F_r^{(1)}$, $F_\theta^{(1)}$, $F_z^{(1)}$, the resultant of which is normal to the blade surface; and a second set, $F_r^{(2)}$, $F_\theta^{(2)}$, $F_z^{(2)}$, which is tangential to the blade surface.

The first set of blade forces, normal to the blade surface, bears a simple relation to the velocity vector. For since the relative velocity u, $v - \omega r$, w is tangential to the blade surface locally, it is likewise normal to the force $F_r^{(1)}$, $F_\theta^{(1)}$, $F_z^{(1)}$, that is, the scaler product vanishes.

$$uF_r^{(1)} + (v - \omega r)F_\theta^{(1)} + wF_z^{(1)} = 0 \tag{2-16}$$

The second set of blade forces, representing the blade drag and leading to losses, will be considered of a specialized form. It will be assumed that this component is not only tangential to the blade surface but parallel with the gas velocity vector $(u, v - \omega r, w)$ relative to the blade. This restriction is clearly appropriate when the force $F_r^{(2)}, F_\theta^{(2)}, F_z^{(2)}$ arises only from simple skin friction. It is an oversimplification of fact, however, when there exists a cross flow within the boundary layer, i.e. when the local direction of boundary layer flow deviates from the direction of flow just outside the boundary layer. These values of transverse skin friction are generally of minor importance and there is a question, moreover, whether the effect of such flow may properly be evaluated within the axially symmetric throughflow theory. Under these restrictions this second force system may be written as

$$(F_r^{(2)}, F_\theta^{(2)}, F_z^{(2)}) = -\frac{1}{2} C_D[u^2 + (v - \omega r)^2 + w^2] \frac{(u, v - \omega r, w)}{\sqrt{u^2 + (v - \omega r)^2 + w^2}} \tag{2-17}$$

where C_D is an appropriate local drag coefficient which can be considered a prescribed function $C_D(r, z)$ within the blade row.

There exists an additional restriction [18] upon the blade force system $F_r^{(1)}, F_\theta^{(1)}, F_z^{(1)}$ arising from the fact that it is generated by a continuous, generally smooth, blade surface. Suppose the blade surface is defined by a relation,

$$\beta(r, \theta, z) = \text{const} \tag{2-18}$$

so normalized that

$$\left(\frac{\partial \beta}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \beta}{\partial \theta}\right)^2 + \left(\frac{\partial \beta}{\partial z}\right)^2 = 1$$

Then the unit vector normal to the blade surface has components

$$\frac{\partial \beta}{\partial r}, \frac{\partial \beta}{r\partial \theta}, \frac{\partial \beta}{\partial z}$$

and if the blade loading is given by a function $\Lambda(r, z)$, the blade force vector $(F_r^{(1)}, F_\theta^{(1)}, F_z^{(1)})$ is given by

$$(F_r^{(1)}, F_\theta^{(1)}, F_z^{(1)}) = \Lambda(r, z)\left(\frac{\partial \beta}{\partial r}, \frac{\partial \beta}{r\partial \theta}, \frac{\partial \beta}{\partial z}\right) \tag{2-19}$$

Taking the curl operation of Eq. 2-19 gives curl $\mathbf{F}$ = curl (Λ grad β). If the right side of this relation be expanded and note is taken of the fact that curl grad $\beta \equiv 0$, then

$$\text{curl } \mathbf{F}^{(1)} = \text{grad } \Lambda \times \text{grad } \beta \tag{2-20}$$

Furthermore it is clear from Eq. 2-19 that the force vector $\mathbf{F}^{(1)}$ is parallel to grad β with the consequence that $\mathbf{F}^{(1)} \cdot (\text{grad } \Lambda \times \text{grad } \beta) = 0$. Thus the left-hand side of Eq. 2-20 yields the condition that the force must satisfy the relation

$$\mathbf{F}^{(1)} \cdot \text{curl } \mathbf{F}^{(1)} = 0 \tag{2-21}$$

in order that the flow follow a blade surface that is constructable and continuous.

The condition given by Eq. 2-21 may be written down explicitly for the present special case of axial symmetry, and gives

$$-F_r^{(1)} \frac{\partial F_\theta^{(1)}}{\partial z} + F_\theta^{(1)} \left(\frac{\partial F_r^{(1)}}{\partial z} - \frac{\partial F_z^{(1)}}{\partial r} \right) + F_z^{(1)} \frac{1}{r} \frac{\partial}{\partial r}(r F_\theta^{(1)}) = 0 \quad (2\text{-}22)$$

This may be written in the more compact form

$$\frac{\partial}{\partial z} \left(\frac{F_r^{(1)}}{r F_\theta^{(1)}} \right) = \frac{\partial}{\partial r} \left(\frac{F_z^{(1)}}{r F_\theta^{(1)}} \right) \quad (2\text{-}23)$$

The restriction imposed by Eq. 2-23 is of mathematical significance in that it reduces somewhat the freedom of choice in selecting the force field to represent a blade row. From a strictly utilitarian point of view, however, there is some doubt as to how seriously Eq. 2-23 should be taken. In the first place the axially symmetric throughflow itself is an approximation which makes the concept of detailed blade shape rather dubious. Secondly, in the preponderant number of cases concerned in jet propulsion application, the radial force component F_r is negligible. It is, therefore, very defensible to assume $F_r^{(1)} = 0$ and neglect the condition given by Eq. 2-23 or its equivalent. In the present work Eq. 2-23 will be retained in the general theoretical consideration but will be dropped in particular appropriate examples.

Variations in stagnation enthalpy and entropy. The so-called Euler turbine equation is one of the few, very general results of turbomachine theory. In its usual form it relates the stagnation enthalpy rise through a blade row to the change in angular momentum of the fluid imparted by a moving blade row. Since it will be advantageous to follow fluid mass along the streamlines it is appropriate to denote by l the length measured, from an arbitrary origin, along the axially symmetric stream surface in a fixed meridional plane. Then the derivative along the stream surface, moving with the fluid, is

$$V_l \frac{\partial}{\partial l} \equiv u \frac{\partial}{\partial r} + w \frac{\partial}{\partial z} \quad (2\text{-}24)$$

where $V_l = \sqrt{u^2 + w^2}$ is the meridional velocity along the stream surface. For example, dynamic equilibrium in the tangential direction, Eq. 2-2, becomes

$$V_l \frac{\partial (rv)}{\partial l} = rF_\theta \quad (2\text{-}25)$$

The product rv is referred to as the angular momentum (per unit mass). Eq. 2-25 states physically that, along a stream surface, the angular momentum changes at a rate proportional to the moment of the applied tangential force.

The Euler turbine equation must follow from the first law of thermodynamics, given by Eq. 2-15. The second term on the right-hand side, representing the work associated with volume changes, may be rewritten as

$$p\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{1}{\rho}\right)=\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{p}{\rho}\right)-u\frac{1}{\rho}\frac{\partial p}{\partial r}-w\frac{1}{\rho}\frac{\partial p}{\partial z}\tag{2-26}$$

The pressure derivatives $(1/\rho)\partial p/\partial r$ and $(1/\rho)\partial p/\partial z$ may be eliminated through use of the first and third equations of motion, Eq. 2-1 and 2-3. Substituting from these into Eq. 2-26 gives, after some simplification,

$$p\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{1}{\rho}\right)=\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{p}{\rho}+\frac{q^2}{2}\right)-\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{v^2}{2}\right)$$
$$-u\frac{v^2}{r}-uF_r-wF_z\tag{2-27}$$

From the tangential equation of motion, Eq. 2-2, it follows that

$$vF_\theta=u\frac{v^2}{r}+\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{v^2}{2}\right)$$

which provides, upon substitution into Eq. 2-27,

$$p\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{1}{\rho}\right)=\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)\left(\frac{p}{\rho}+\frac{q^2}{2}\right)-(uF_r+rF_\theta+wF_z)\tag{2-28}$$

Now using Eq. 2-28 and noting that the quantity $e+(p/\rho)+(q^2/2)$ is just the stagnation enthalpy of the gas, the combined first and second laws of thermodynamics, Eq. 2-15, may be written in the form,

$$\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)h^0-\omega rF_\theta=T\left(u\frac{\partial}{\partial r}+w\frac{\partial}{\partial z}\right)$$
$$+[uF_r+(v-\omega r)F_\theta+wF_z]\tag{2-29}$$

It will be recalled now that the force field associated with the blades has been divided into two parts: that part $F_r^{(1)}, F_\theta^{(1)}, F_z^{(1)}$ which acts normal to the relative motion and hence satisfies the condition that

$$uF_r^{(1)}+(r-\omega r)F_\theta^{(1)}+wF_z^{(1)}=0\tag{2-30}$$

and that part $F_r^{(2)}, F_\theta^{(2)}, F_z^{(2)}$ which is parallel to the relative motion, according to Eq. 2-17, and is associated with the losses. In view of Eq. 2-30, therefore, Eq. 2-29 may be written

$$V_l\frac{\partial h^0}{\partial l}-\omega rF_\theta=TV_l\frac{\partial s}{\partial l}+[uF_r^{(2)}+(v-\omega r)F_\theta^{(2)}+wF_z^{(2)}]\tag{2-31}$$

In the absence of losses the right-hand side of the equation vanishes identically, for then the shear force system $F_r^{(2)}, F_\theta^{(2)}, F_z^{(2)}$ and the entropy variation are both zero. The result of setting the left-hand side to zero gives

$$V_l \frac{\partial h^0}{\partial l} = \omega r F_\theta \tag{2-32}$$

and may be recognized as a gross application of the first law of thermodynamics. The term $V_l \partial h^0/\partial l$ represents the rate at which stagnation enthalpy of the fluid is rising and the term $\omega r F_\theta$ represents the rate at which torque applied to a particular radial element of the blade row is doing work on the system. Since the heat transfer is usually a negligible quantity, it is clear that this work must go into increasing the stagnation enthalpy of the gas. Furthermore it is evident that Eq. 2-32, being an over-all thermodynamic relation, holds regardless of whether or not losses are present. Then, of course, F_θ is the total tangential force, the sum of the normal $F_\theta^{(1)}$ and dissipative $F_\theta^{(2)}$ tangential forces. As a result of this observation it follows that the entropy variation along stream surfaces may be calculated by setting the right-hand side of Eq. 2-31 to zero. Thus

$$TV_l \frac{\partial s}{\partial l} = -[uF_r^{(2)} + (v - \omega r)F_\theta^{(2)} + wF_z^{(2)}] \tag{2-33}$$

It is clear now that only the second set of forces tends to change the entropy. Substituting from Eq. 2-17 the entropy variation may be expressed in terms of the velocity components

$$TV_l \frac{\partial s}{\partial l} = \tfrac{1}{2}C_D[u^2 + (v - \omega r)^2 + w^2]^{\frac{3}{2}} \tag{2-34}$$

so that the rate of entropy increase along the stream surface is proportional to the local values of C_D within the blade row and vanishes outside of the blade rows. In principle, at least, this relation allows computation of the entropy distribution that enters into determination of the tangential vorticity through Eq. 2-43.

Returning now to the variation of stagnation enthalpy, the moment of blade force may be eliminated from Eq. 2-32 by using Eq. 2-25, giving the change of angular momentum along a stream surface. Thus

$$V_l \frac{\partial h^0}{\partial l} = V_l \frac{\partial}{\partial l}(\omega r v) \tag{2-35}$$

which is essentially the Euler turbine equation in differential form. Clearly this may be integrated along any stream surface where the blade row in question has a fixed angular velocity ω. Across a blade row with angular velocity ω, the change Δh^0 in stagnation enthalpy is given by

$$\Delta h^0 = \omega\Delta(rv) \tag{2-36}$$

where $\Delta(rv)$ is the change in angular momentum of the fluid and all changes are measured along a fixed stream surface. When ω and $\Delta(rv)$ are of the same sign, as in a compressor, the stagnation enthalpy rises; when ω and $\Delta(rv)$ are of opposite sign, as in a turbine, the stagnation enthalpy of the gas is reduced. Across a stator, a stationary blade row, the stagnation enthalpy remains constant since $\omega = 0$. Finally in a space that is free of blades, the tangential force F_θ vanishes and consequently, according to Eq. 2-25, the angular momentum of the gas remains constant along stream surfaces. Obviously the stagnation enthalpy remains constant under this circumstance.

The tangential vorticity. A glance at the expressions for vorticity components [*8*], given by Eq. 2-5, 2-6, and 2-7 shows that, while the radial and axial vorticity components are given in terms of only the tangential velocity, the tangential vorticity components include both the radial and axial velocities. In other words the meridional velocities, the radial and axial components which make up the throughflow, may be related to the tangential vorticity. It is appropriate, then, to investigate the propagation tangential vorticity. In carrying out this analysis it will prove necessary to calculate the variation of quantities normal to the stream surfaces, and hence a length must be introduced to measure distance normal to the stream surface. This may be accomplished using the stream function ψ itself, which is defined by the properties that

$$\rho u = \rho_0 \frac{1}{r}\frac{\partial \psi}{\partial z} \tag{2-37}$$

$$\rho w = -\rho_0 \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{2-38}$$

where ρ_0 is a constant reference density. It follows that, if $\delta\psi$ is the small difference in stream function between two nearby stream surfaces, then the normal distance between them is $-(1/rV_l)(\rho_0/\rho)\delta\psi$. The differential operator normal to a stream surface is just

$$\frac{w}{V_l}\frac{\partial}{\partial r} - \frac{u}{V_l}\frac{\partial}{\partial z}$$

so that, in a more compact form,

$$-\frac{\rho}{\rho_0} r V_l \frac{\partial}{\partial \psi} = \frac{w}{V_l}\frac{\partial}{\partial r} - \frac{u}{V_l}\frac{\partial}{\partial z} \tag{2-39}$$

Now it is a simple matter to calculate the variation of stagnation enthalpy normal to the stream surface, by taking $\partial h^0/\partial r$ from Eq. 2-12 and $\partial h^0/\partial z$ from Eq. 2-13. Substituting these values into the right-hand side of Eq. 2-39 gives

$$-\frac{\rho}{\rho_0} r V_l \frac{\partial h^0}{\partial \psi} = \frac{w}{V_l}\left(F_r + T\frac{\partial s}{\partial r} - w\eta + v\zeta\right) - \frac{u}{V_l}\left(F_z + T\frac{\partial s}{\partial z} - v\xi + u\eta\right)$$

$$= -V_l\eta + \frac{w}{V_l}v\zeta + \frac{u}{V_l}v\xi + \frac{w}{V_l}F_r - \frac{u}{V_l}F_z - \frac{\rho}{\rho_0} r V_l T \frac{\partial s}{\partial \psi} \quad (2\text{-}40)$$

Eq. 2-40 may be interpreted as a relation for the tangential vorticity η. The terms involving radial and axial vorticity components may, as was suggested previously, be expressed in terms of the tangential velocity. Referring again to the vorticity components, Eq. 2-5 and 2-7, it follows that

$$\begin{aligned} \frac{w}{V_l}v\zeta + \frac{u}{V_l}v\xi &= \frac{v}{r}\left[\frac{w}{V_l}\frac{\partial(rv)}{\partial r} - \frac{u}{V_l}\frac{\partial(rv)}{\partial z}\right] \\ &= -\frac{\rho}{\rho_0} V_l v \frac{\partial(rv)}{\partial \psi} \end{aligned} \quad (2\text{-}41)$$

Furthermore it appears that the force term $(w/V_l)F_r - (u/V_l)F_z$ is simply the force component normal to the stream surfaces and consequently it is convenient to define

$$F_\psi \equiv \frac{w}{V_l}F_r - \frac{u}{V_l}F_z \quad (2\text{-}42)$$

Using the results of Eq. 2-41 and 2-42, the tangential vorticity may be written, from Eq. 2-40, as

$$\eta = \frac{\rho}{\rho_0} r \left[\frac{\partial h^0}{\partial \psi} - \frac{v}{r}\frac{\partial(rv)}{\partial \psi} - T\frac{\partial s}{\partial \psi}\right] + \frac{1}{V_l}F_\psi \quad (2\text{-}43)$$

A restricted form of this was originally obtained by Bragg and Hawthorne [*19*] and the complete expression was given by Marble and Michelson [*10*]. Although this is not, in its present form, a very useful relation for determining the tangential vorticity since differentiation occurs with respect to the unknown stream function, it is, however, a very convenient guide for physical reasoning. The tangential vorticity associated with the force component F_ψ is essentially a "bound vorticity" and is of the same origin as the bound vorticity connected with the lift of a wing. If the angular momentum were invariant with ψ and the tangential vorticity depended upon h^0 and s only, then the flow outside of the blade row becomes relatively simple. Under these circumstances h^0 and s are constant along stream surfaces and hence $\eta/\rho r$ is constant along a stream surface. It is easily shown that this result follows from the fact that the circulation about a physical annular vortex tube remains constant as it moves outside of a force field.

On the other hand, if the enthalpy and entropy were uniform, and again the space outside a blade row is considered, the tangential vorticity arises only from the angular momentum. But since the angular momentum is constant along stream surfaces, then clearly the quantity $\eta r/\rho$ is constant along a stream surface. Physically this fact is related to the constancy of circulation along a helical stream tube which may have a greater or smaller component in the tangential direction depending upon how the stream tube is deformed by the flow. Within a blade row, where force components act on the fluid, the tangential vorticity responds in a manner that, while fairly complex, may be determined from Eq. 2-40 when the appropriate values of h^0, rv, and s are known.

The mathematical problem. Sufficient development of the gas dynamic details has been undertaken so that the mathematical problem of throughflow calculation can be formulated in a more or less satisfactory manner. Since the velocity components u and w are of central interest, the tangential vorticity (Eq. 2-43) will be the focus of attention. Since the stream function ψ has already been introduced it is natural to express the tangential vorticity in terms of the stream function (see Gravalos [*20*] and Marble [*10*]). From Eq. 2-6 and the definition of the stream function, it follows that

$$\eta \equiv \frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \frac{\partial}{\partial z}\left(\frac{\rho_0}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{\rho_0}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \tag{2-44}$$

To eliminate the density derivatives that naturally arise in expanding Eq. 2-44 it is necessary to introduce the equations of motion in the radial and axial direction (Eq. 2-1 and 2-3). Again utilizing the stream function Eq. 2-1 becomes

$$u\frac{\partial}{\partial r}\left(\frac{\rho_0}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + w\frac{\partial}{\partial z}\left(\frac{\rho_0}{\rho}\frac{1}{r}\frac{\partial\psi}{\partial z}\right) - \frac{v^2}{r} = \left.\frac{\partial p}{\partial\rho}\right|_s \frac{\rho}{\rho_0}\frac{\partial}{\partial r}\left(\frac{\rho_0}{\rho}\right) - \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial r} + F_r \tag{2-45}$$

Here, of course, $\partial p/\partial\rho\big|_s$ is just a^2 the square of the sonic velocity and for a perfect gas

$$\frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho = (\gamma - 1)T \tag{2-46}$$

Expanding Eq. 2-45 and collecting derivatives of ρ_0/ρ gives

$$\begin{aligned}(a^2 - u^2)\frac{\partial}{\partial r}\left(\frac{\rho_0}{\rho}\right) - uw\frac{\partial}{\partial z}\left(\frac{\rho_0}{\rho}\right) = \left(\frac{\rho_0}{\rho}\right)^2\bigg[u\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) \\ + w\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\bigg] - \frac{\rho_0}{\rho}\left[F_r + \frac{v^2}{r} - (\gamma - 1)T\frac{\partial s}{\partial z}\right]\end{aligned} \tag{2-47}$$

Treating Eq. 2-3 in an entirely similar manner yields

$$-uw\frac{\partial}{\partial r}\left(\frac{\rho_0}{\rho}\right) + (a^2 - w^2)\frac{\partial}{\partial z}\left(\frac{\rho_0}{\rho}\right)$$
$$= -\left(\frac{\rho_0}{\rho}\right)^2\left[u\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) + w\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] - \frac{\rho_0}{\rho}\left[F_z - (\gamma - 1)T\frac{\partial s}{\partial z}\right] \tag{2-48}$$

Eq. 2-47 and 2-48 may be solved simultaneously for $(\partial/\partial r)(\rho_0/\rho)$ and $(\partial/\partial z)(\rho_0/\rho)$ and the results employed when the right-hand side of Eq. 2-44 is expanded, with the result that

$$\frac{\rho}{\rho_0}\eta = \left(1 - \frac{u^2}{a^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{uw}{a^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]$$
$$+\left(1 - \frac{w^2}{a^2}\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) - \frac{v^2}{ra^2}\frac{1}{r}\frac{\partial\psi}{\partial r} - \frac{\rho}{\rho_0}\frac{V_l^2}{a^2}\frac{F_\psi}{V_l} + \left(\frac{\rho}{\rho_0}\right)^2\frac{V_l^2}{a^2}(\gamma - 1)T\frac{\partial s}{\partial\psi} \tag{2-49}$$

Finally inserting this expression for the tangential vorticity into Eq. 2-40, the partial differential equation for the stream function is obtained

$$\left(1 - \frac{u^2}{a^2}\right)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) - \frac{uw}{a^2}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right]$$
$$+\left(1 - \frac{w^2}{a^2}\right)\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial\psi}{\partial z}\right) - \frac{1}{r}\frac{v^2}{a^2}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)$$
$$= \left(\frac{\rho}{\rho_0}\right)^2\left\{r\frac{\partial h^0}{\partial r} - v\frac{\partial rv}{\partial\psi} - \left[1 + (\gamma - 1)\frac{V_l^2}{a^2}\right]Tr\frac{\partial s}{\partial\psi}\right\} - \frac{\rho}{\rho_0}\left(1 - \frac{V_l^2}{a^2}\right)\frac{F_\psi}{V_l} \tag{2-50}$$

The stagnation enthalpy h^0, the angular momentum rv, and the entropy s are described by Eq. 2-32, 2-25, and 2-34, respectively. The local speed of sound a, which appears in Eq. 2-50, is related to the stagnation enthalpy and the velocity components through the enthalpy integral

$$\frac{1}{\gamma - 1}a^2 = h^0 - \tfrac{1}{2}(u^2 + v^2 + w^2) \tag{2-51}$$

The density ρ, which is involved in the principle equation (Eq. 2-50) and elsewhere, follows simply from the combined first and second laws of thermodynamics

$$\frac{\gamma - 1}{\rho}\frac{\partial\rho}{\partial l} = \frac{1}{a^2}\frac{\partial a^2}{\partial l} - \frac{1}{c_v}\frac{\partial s}{\partial l} \tag{2-52}$$

The physical description of the flow field is thus completed.

So far as the stagnation enthalpy h^0, the angular momentum rv, and the entropy s are concerned, the stream surfaces themselves constitute parabolic characteristics. The differential operator for the stream function, given by Eq. 2-50, requires a little more analysis. Expanding the left-hand side of Eq. 2-50 gives the differential operator as

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial^2\psi}{\partial r^2}-\frac{2uw}{a^2}\frac{\partial^2\psi}{\partial r\partial z}+\left(1-\frac{w^2}{a^2}\right)\frac{\partial^2\psi}{\partial z^2}-\left(1-\frac{u^2}{a^2}\right)\frac{1}{r}\frac{\partial u}{\partial r}-\frac{uw}{a^2}\frac{1}{r}\frac{\partial\psi}{\partial z}+\frac{v^2}{a^2}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right) \tag{2-53}$$

The slopes of the characteristics (see Monroe [21]) may be written down directly from the coefficients of the second order terms as

$$-\frac{dz}{dr}=\frac{\frac{2uw}{a^2}\pm\sqrt{\left(\frac{2uw}{a^2}\right)^2-4\left(1-\frac{u^2}{a^2}\right)\left(1-\frac{w^2}{a^2}\right)}}{2\left(1-\frac{u^2}{a^2}\right)}$$

$$=\frac{M^2\sin\varphi\cos\varphi\pm\sqrt{M^2-1}}{1-M^2\sin^2\varphi} \tag{2-54}$$

where the meridional Mach number V_l/a has been denoted M and φ is the angle between the stream surface and the axis of symmetry, $\tan\varphi = u/w$. The angles between the characteristic surfaces and the stream surfaces are also easily found. For if α is the angle between characteristic surfaces and the symmetry axis, then $\tan\alpha = dr/dz$ and

$$\tan\mu = \tan(\alpha-\beta) \tag{2-55}$$

where μ is now the angle between characteristic surfaces and stream surfaces. Using $dr/dz = \tan\alpha$ from Eq. 2-54, a little calculation yields

$$\tan\mu = \pm\frac{1}{\sqrt{M^2-1}} \tag{2-56}$$

It is of main interest that the differential equation (Eq. 2-50) changes from elliptic to hyperbolic type accordingly as the *meridional* Mach number is less than or greater than unity. The nature of the flow then depends upon the meridional velocity component, the total velocity $\sqrt{u^2+v^2+w^2}$ entering only through the manner in which it affects the velocity of sound according to Eq. 2-51. Whether or not $\sqrt{u^2+v^2+w^2}/a$ exceeds unity has no bearing on the present problem. The characteristics, defined by Eq. 2-56, have all the properties familiar from usual axially symmetric flow problems where there is no tangential motion, so long as the Mach number M is interpreted in the appropriate manner. In the present work it will be assumed that the meridional Mach number is less than unity throughout the turbomachine.

It is now possible to make some statements, although not very conclusive ones, regarding the formulation of the inverse problem, that is where something other than the blade geometry is prescribed. They are not really conclusive because the nonlinear nature of Eq. 2-50 precludes the guarantee of a well-behaved solution under boundary conditions of any generality. The results are of interest, however, because they do delineate the maximum amount of information that must be prescribed.

If one prescribes

1. $h^0(r, z)$ or $rv(r, z)$ or $F_\theta(r, z)$ throughout the turbomachine (cf. Eq. 2-25, 2-32, and 2-35).
2. The blade shape or loading at the leading edge (cf. Eq. 2-23).
3. That the flow be tangential to inner and outer walls of given shap :.
4. The values of the stream function, enthalpy, entropy, and angular momentum far upstream of any blade row.
5. That the stream function, enthalpy, entropy, and angular momentum be regular far downstream.

Then the details of the throughflow can be determined through solution of Eq. 2-50, together with those of Eq. 2-25, 2-32, 2-35, 2-43, 2-51, and 2-52 that are required for the quantity given under (1) above.

A word of further explanation may be added concerning item (2) above. Referring to Eq. 2-23 relating the three blade forces, this equation has a direct analogue when the "intrinsic" coordinates are employed and the components denoted $F_\psi^{(1)}, F_\theta^{(1)}, F_l^{(1)}$,

$$\frac{\partial}{\partial l}\left(\frac{F_\psi^{(1)}}{rF_\theta^{(1)}}\right) = \frac{\rho}{\rho_0} rV_l \frac{\partial}{\partial \psi}\left(\frac{F_l^{(1)}}{rF_\theta^{(1)}}\right) \tag{2-57}$$

But from the kinematical condition that the blade forces be normal to the relative velocity,

$$V_l F_l^{(1)} + (v - \omega r)F_\theta^{(1)} = 0 \tag{2-58}$$

and hence

$$\frac{\partial}{\partial l}\left(\frac{F_\psi^{(1)}}{rF_\theta^{(1)}}\right) = -\frac{\rho}{\rho_0} rV_l \frac{\partial}{\partial \psi}\left(\frac{v - \omega r}{rV_l}\right) \tag{2-59}$$

Now considering $F_\theta^{(1)}$ and rv as prescribed or calculable quantities according to statement (1) above, Eq. 2-59 may be integrated along a stream surface to give

$$\frac{F_\psi^{(1)}}{rF_\theta^{(1)}} - \frac{F_\psi^{(1)}}{rF_\theta^{(1)}}\left(\psi, l_0\right) = -\int_{l_0}^{l} \frac{\rho}{\rho_0} r V_l \frac{\partial}{\partial\psi}\left(\frac{v - \omega r}{rV_l}\right) dl \tag{2-60}$$

where l_0 represents the leading edge of the blade on the stream surface in question. Therefore if $F_\psi^{(1)}$ is prescribed at the blade leading edge it may, in principle, be determined through Eq. 2-60 at other points of the blade. It is clear, furthermore, that the ratio of force $F_\psi^{(1)}/F_\theta^{(1)}$ represents the inclination of the leading edge with respect to a meridional plane and hence the prescription of leading edge shape will suffice also.

For the direct problem, that is the physical situation where the blade shape is prescribed, it may be considered that the components n_r, n_θ, n_z of the unit normal to the blade surface are known. In the intrinsic coordinate system this vector has components n_ψ, n_θ, n_l. Clearly this vector is normal to the velocity vector relative to the blades, so that

$$(v - \omega r)n_\theta + V_l n_l = 0 \tag{2-61}$$

or the angular momentum must be

$$rv = r\left(\omega r - V_l \frac{n_l}{n_\theta}\right) \tag{2-62}$$

Furthermore, since the force $F_\psi^{(1)}$, $F_\theta^{(1)}$, $F_l^{(1)}$ is parallel with the unit vector n_ψ, n_θ, n_l, consequently

$$F_\psi^{(1)} = F_\theta^{(1)} \frac{n_\psi}{n_\theta} \tag{2-63}$$

Utilizing now the definition of $F_\theta^{(2)}$ from Eq. 2-17 and the conservation of angular momentum from Eq. 2-25, and noting that $F_\psi^{(2)} = 0$ since the velocity component normal to the stream surfaces vanishes, it follows that

$$F_\psi = \frac{n_\psi}{n_\theta}\left[\frac{V_l}{r}\frac{\partial rv}{\partial l} + \frac{1}{2} C_D(v - \omega r)\sqrt{(v - \omega r)^2 + V_l^2}\right] \tag{2-64}$$

Consequently knowledge of n_l/n_θ and n_ψ/n_θ permits calculation of the angular momentum rv and hence the force component F_ψ normal to the stream surfaces. The direct problem is reduced, therefore, to one formally identical with the inverse problem, the solution of which has been discussed.

C,3. Linearized Treatment of Throughflow. The general throughflow problem as outlined in the previous articles is a strongly nonlinear one and, as a consequence, this exact formulation is seldom treated by other than numerical methods. The nonlinearity results from the facts that (1) the conservation relations for angular momentum, stagnation enthalpy, and entropy hold along stream surfaces and these surfaces are not known in advance of solution; (2) the coefficients in the differential equation (Eq. 2-50) depend upon the velocity components of the solution because the gas is compressible; and (3) in the direct problem the force components themselves, the angular momentum, etc., are given in terms of the final solution through their relation to the given blade surface shape. Fortunately it is usually the case, in axial turbomachine problems, that a useful linearization of the problem may be effected. The possibility of this arises because (1) the blades are relatively lightly loaded in the sense that the change in angular momentum across a blade row is small in comparison with the mean value of the angular momentum; (2) the geometrical boundaries are sufficiently simple that a reasonably accurate assumption of stream surface shape may be made in advance of detailed solution; and (3) the perturbations to mean velocity, density, and velocity of sound are sufficiently small that the compressibility terms may be calculated from conditions in the undisturbed flow.

The linearizing assumptions. The linearized problem of turbomachine throughflow may be formulated in a satisfactorily unified manner by assuming that the blades are lightly loaded. The linearization given by Marble [*8,10*] was arrived at by an iteration procedure and is inconsistent in the order of some terms neglected. A consistent linearization was first given by Rannie [*22*] for the region outside of the blade rows. In analytic form it is assumed that the force field associated with the blade row is of the form

$$\begin{aligned} F_r &= \epsilon f_r \\ F_\theta &= \epsilon f_\theta \\ F_z &= \epsilon f_z \end{aligned} \tag{3-1}$$

where $\epsilon \ll 1$ and f_r, f_θ, f_z are functions of order unity in the independent variables. It will prove convenient to employ the radial velocity component as the dependent variable in place of the stream function ψ.

The basic flow field will be assumed to be that which exists in the absence of a blade row, satisfies the boundary conditions on inner and outer surfaces, and takes on the appropriate initial values far upstream. If the inner and outer surfaces were concentric cylinders, the stream surfaces of this undisturbed flow would also be concentric cylinders. For usual axial turbomachine configurations it is adequate to assume that the inner and outer surfaces deviate from circular cylinders only in the order ϵ, so that indeed the undisturbed flow will take place along concentric cylindrical surfaces. Therefore the radial velocity component u is at most of order ϵ.

It is assumed now that all dependent variables may be expanded in powers of the small parameter ϵ which is indicative of the magnitude of the applied forces (Eq. 3-1). Denote the order of the terms by a superscript, the superscript "0" being the basic undisturbed flow, "1" the first order perturbation, etc.

$$\begin{aligned} u &= \epsilon u^{(1)} + \cdots & s &= s^{(0)} + \epsilon s^{(1)} + \cdots \\ v &= v^{(0)} + \epsilon v^{(1)} + \cdots & a^2 &= a^{2(0)} + \epsilon a^{2(1)} + \cdots \\ w &= w^{(0)} + \epsilon w^{(1)} + \cdots & \rho &= \rho^{(0)} + \epsilon \rho^{(1)} + \cdots \\ h_0 &= h^{(0)} + \epsilon h^{(1)} + \cdots & T &= T^{(0)} + \epsilon T^{(1)} + \cdots \end{aligned} \tag{3-2}$$

For consistency of the force field it is necessary to assume also that

$$C_D = \epsilon \xi_D \tag{3-3}$$

where ξ_D is a function of r and z of magnitude unity.

First order perturbation. Although it is possible to obtain the desired perturbation relations for radial velocity by differentiation of Eq. 2-50 and substitution from expansions in powers of ϵ (Eq. 3-2), it is generally simpler to accomplish this directly from the equations of motion themselves. It is convenient first, however, to derive the linearized relations for the stagnation enthalpy, the entropy, angular momentum, etc.

Recalling that $F_\theta = F_\theta^{(1)} + F_\theta^{(2)}$ it follows that to the first order in ϵ

$$F_\theta = \epsilon f_\theta^{(1)} - \tfrac{1}{2}\epsilon \xi_D (v^{(0)} - \omega r)\sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2}} \tag{3-4}$$

where the expansion for $F_\theta^{(2)}$ is obtained from Eq. 2-17 using the expression for C_D given in Eq. 3-3. Then the second equation of motion, Eq. 2-2, which expresses the conservation of angular momentum of the fluid, may be written

$$\begin{aligned} (\epsilon u^{(1)} + \cdots) \frac{\partial}{\partial r}[r(v^{(0)} + \epsilon v^{(1)} + \cdots)] \\ + (w^{(0)} + \epsilon w^{(1)} + \cdots) \frac{\partial}{\partial z}[r(v^{(0)} + \epsilon v^{(1)} + \cdots)] \\ = r[\epsilon f_\theta^{(1)} - \tfrac{1}{2}\epsilon \xi_D (v^{(0)} - \omega r)\sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2} + \cdots}] \end{aligned} \tag{3-5}$$

where it is assumed that $f_\theta^{(1)}(r, z)$ and $\xi_D(r, z)$ are prescribed functions. The zeroeth order relation (terms not containing the small parameter ϵ) gives simply

$$w^{(0)} \frac{\partial}{\partial z}(rv^{(0)}) = 0 \tag{3-6}$$

so that the initial undisturbed value of angular momentum $rv^{(0)}$ does not change along the axis of the turbomachine, but is transported unchanged along circular cylinders. The first order relation (terms proportional to the small parameter ϵ) may be written, taking account of Eq. 3-6, as

$$u^{(1)} \frac{\partial r v^{(0)}}{\partial r} + w^{(0)} \frac{\partial r v^{(1)}}{\partial r} = r f_\theta^{(1)} - \tfrac{1}{2} \xi_D r (v^{(0)} - \omega r) \sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2}} \tag{3-7}$$

From this relation, the first order perturbation in angular momentum is seen to consist of two parts. The first part arises from the moment of tangential forces applied to the fluid, the right-hand side of Eq. 3-7, and the second part $u^{(1)} \partial r v^{(0)} / \partial r$ is due to the radial transport of the initial angular momentum $r v^{(0)}$.

Treating Eq. 2-32 for the stagnation enthalpy in exactly the same manner gives similar results for the zeroeth and first order relations:

$$w^{(0)} \frac{\partial h^{(0)}}{\partial z} = 0 \tag{3-8}$$

and

$$u^{(1)} \frac{\partial h^{(0)}}{\partial r} + w^{(0)} \frac{\partial h^{(1)}}{\partial z} = \omega r f_\theta^{(1)} - \tfrac{1}{2} \omega r \xi_D (v^{(0)} - \omega r) \sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2}} \tag{3-9}$$

Similarly the entropy variations follow from Eq. 2-43 as

$$T^{(0)} w^{(0)} \frac{\partial s^{(0)}}{\partial z} = 0 \tag{3-10}$$

and

$$u^{(1)} \frac{\partial s^{(0)}}{\partial r} + w^{(0)} \frac{\partial s^{(1)}}{\partial z} = \frac{1}{2T^{(0)}} \xi_D [(v^{(0)} - \omega r)^2 + w^{(0)2}]^{\frac{3}{2}} \tag{3-11}$$

It is to be noticed in the preceding equations (Eq. 3-6, 3-7, 3-8, 3-9, 3-10, and 3-11) that, to calculate the first order perturbations of angular momentum, stagnation enthalpy, and entropy, only the first order perturbation in radial velocity, $u^{(1)}$, is required. Therefore it is necessary to develop a differential equation which may be solved for the radial velocity perturbation.

If the equation of motion in the radial direction be perturbed in accordance with Eq. 3-2, the resulting zeroeth order relation is

$$\frac{v^{(0)2}}{r} = \frac{1}{\rho^{(0)}} \frac{\partial p^{(0)}}{\partial r} \tag{3-12}$$

while the first order relation is just

$$w^{(0)} \frac{\partial u^{(1)}}{\partial r} = \frac{2 v^{(0)} v^{(1)}}{r} + \frac{v^{(0)2}}{r} \frac{\rho^{(1)}}{\rho^{(0)}} - \frac{1}{\rho^{(0)}} \frac{\partial p^{(1)}}{\partial r} + f_r \tag{3-13}$$

The equation of motion in the axial direction, perturbed in the same manner, yields only a first order part

$$u^{(1)}\frac{\partial w^{(0)}}{\partial r} + w^{(0)}\frac{\partial w^{(1)}}{\partial z} = -\frac{1}{\rho^{(0)}}\frac{\partial p^{(1)}}{\partial z} + f_z \tag{3-14}$$

The pressure perturbation $p^{(1)}$ may be eliminated between Eq. 3-13 and 3-14 through cross-differentiation, to give

$$\begin{aligned} w^{(0)}\left(\frac{\partial^2 u^{(1)}}{\partial z^2} - \frac{\partial^2 w^{(1)}}{\partial r \partial z}\right) - \frac{\partial w^{(0)}}{\partial r}\frac{\partial u^{(1)}}{\partial r} - \frac{\partial^2 w^{(0)}}{\partial r^2}u^{(1)} - \frac{\partial w^{(0)}}{\partial r}\frac{\partial w^{(1)}}{\partial z} \\ - \frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}\left(u^{(1)}\frac{\partial w^{(0)}}{\partial r} + w^{(0)}\frac{\partial w^{(1)}}{\partial z}\right) \\ = \frac{2v^{(0)}}{r}\frac{\partial v^{(1)}}{\partial z} + \frac{v^{(0)2}}{r}\frac{1}{\rho^{(0)}}\frac{\partial \rho^{(1)}}{\partial z} - \frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}f_z + \frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r} \end{aligned} \tag{3-15}$$

To express his relation in a useable form it is necessary to eliminate (1) the perturbation axial velocity $w^{(1)}$, (2) the perturbation density $\rho^{(1)}$, (3) the perturbation angular momentum $rv^{(1)}$ and (4) the force components f_r and f_z.

The axial velocity perturbation may be eliminated through use of the continuity equation (Eq. 2-4) which gives, to the first order,

$$\frac{1}{r}\frac{\partial}{\partial z}(ru^{(1)}) + \frac{\partial w^{(1)}}{\partial z} + \frac{1}{\rho^{(0)}}\left(u^{(1)}\frac{\partial \rho^{(0)}}{\partial r} + w^{(0)}\frac{\partial \rho^{(1)}}{\partial z}\right) = 0 \tag{3-16}$$

The density perturbation occurs as $\partial\rho^{(1)}/\partial z$ and this may be eliminated through use of the first order equation of motion in the axial direction, Eq. 3-14. In particular the pressure p, considered as a function of ρ and s in the thermodynamic sense, may be expanded

$$\begin{aligned} \frac{1}{\rho}\frac{\partial p}{\partial z} &= \frac{1}{\rho}\left.\frac{\partial p}{\partial \rho}\right|_s \frac{\partial \rho}{\partial z} + \frac{1}{\rho}\left.\frac{\partial p}{\partial s}\right|_\rho \frac{\partial s}{\partial z} \\ &= a^2\frac{1}{\rho}\frac{\partial \rho}{\partial z} + \frac{\mathcal{R}T}{c_v}\frac{\partial s}{\partial z} \end{aligned} \tag{3-17}$$

where the speed of sound is defined as $\sqrt{\partial p/\partial \rho\big|_s} = \sqrt{\gamma\mathcal{R}T}$. To the first order then, the pressure variation of Eq. 3-14 may be written

$$\frac{1}{\rho^{(0)}}\frac{\partial p^{(1)}}{\partial z} = \frac{a^{2(0)}}{\rho^{(0)}}\frac{\partial \rho^{(1)}}{\partial z} + \frac{a^{2(0)}}{c_p}\frac{\partial s^{(1)}}{\partial z} \tag{3-18}$$

Thus Eq. 3-14, written in the form

$$u^{(1)}\frac{\partial w^{(0)}}{\partial r} + w^{(0)}\frac{\partial w^{(1)}}{\partial z} = -\frac{a^{2(0)}}{\rho^{(0)}}\frac{\partial \rho^{(1)}}{\partial z} - \frac{a^{2(0)}}{c_p}\frac{\partial s^{(1)}}{\partial z} + f_z \tag{3-19}$$

may be considered an expression for $\partial \rho^{(1)}/\partial z$ and employed to eliminate the density variation that arises in Eq. 3-15. The entropy perturbation that appears in Eq. 3-19 is given by the entropy relations, Eq. 3-11, while the term $\partial v^{(1)}/\partial z$ follows from the angular momentum relation, Eq. 3-7. These substitutions into Eq. 3-15, together with some simplification, give the result:

$$
\begin{aligned}
&\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial r u^{(1)}}{\partial r}\right) + \frac{\partial^2 u^{(1)}}{\partial z^2} - \left(\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\right)\frac{\partial u^{(1)}}{\partial r} - \left(\frac{1}{w^{(0)}}\frac{\partial^2 w^{(0)}}{\partial r^2}\right)u^{(1)} \\
&\quad + \frac{\partial}{\partial r}\left[\frac{1}{1 + (w^{(0)2}/a^{2(0)})}\left(\frac{v^{(0)2}}{ra^{2(0)}}u^{(1)} - \frac{w^{(0)}}{a^{2(0)}}\frac{\partial w^{(0)}}{\partial r}u^{(1)} + \frac{w^{(0)2}}{a^{2(0)}}\frac{1}{r}\frac{\partial r u^{(}}{\partial r}\right)\right] \\
&\quad - \frac{1}{w^{(0)}}\left(\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r} - \frac{v^{(0)2}}{ra^{2(0)}} - \frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}\right)\left[\frac{1}{1 - a^{2(0)}/(w^{(0)2})}\right] \\
&\quad \left(-\frac{\partial w^{(0)}}{\partial r}u^{(1)} + \frac{v^{(0)2}}{rw^{(0)}}u^{(1)} + \frac{a^{2(0)}}{rw^{(0)}}\frac{\partial r u^{(1)}}{\partial r}\right) \\
&\quad + \left[\frac{2v^{(0)}}{r^2 w^{(0)}}\frac{\partial r v^{(0)}}{\partial r} + \frac{v^{(0)2}}{rw^{(0)2}}\frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r} - \frac{v^{(0)}}{ra^{2(0)}}\frac{\partial r v^{(0)}}{\partial r}\frac{v^{(0)2}}{rw^{(0)2}}\right. \\
&\quad \left. - \frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r} + \frac{v^{(0)2}}{ra^{2(0)}w^{(0)2}}\frac{\partial}{\partial r}\left(\frac{v^{(0)2} + w^{(0)2}}{2}\right)\right]u^{(1)} \\
&\quad = \frac{\partial}{\partial r}\left[\frac{1}{1 - (w^{(0)2}/a^{2(0)})}\left\{\frac{\gamma - 1}{2}\frac{\xi_D}{a^{2(0)}}[(v^{(0)} - \omega r)^2 + w^{(0)2}]^{1/2} - \frac{w^{(0)}}{a^{2(0)}}f_z\right\}\right. \\
&\quad - \frac{1}{w^{(0)}}\left(\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r} - \frac{v^{(0)2}}{ra^{2(0)}} + \frac{1}{\rho^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}\right)\left[\frac{1}{1 - (w^{(0)2}/a^{2(0)})}\right] \\
&\quad \left.\left\{\frac{\gamma - 1}{2}\frac{\xi_D}{w^{(0)}}[(v^{(0)} - \omega r)^2 + w^{(0)2}]^{1/2} - f_z\right\}\right] \\
&\quad - \frac{v^{(0)2}}{rw^{(0)2}}\left\{\frac{\gamma}{2}\frac{\xi_D}{a^{2(0)}}[(v^{(0)} - \omega r)^2 + w^{(0)2}]^{1/2} - \frac{\omega r f_\theta}{a^{2(0)}} + \frac{v^{(0)}}{a^{2(0)}}f_\theta\right\} \\
&\qquad\qquad + \frac{2v^{(0)}}{rw^{(0)2}}f_\theta - \frac{1}{\rho^{(0)}w^{(0)}}\frac{\partial \rho^{(0)}}{\partial r}f_z + \frac{1}{w^{(0)}}\left(\frac{\partial f_r}{\partial z} - \frac{\partial f_z}{\partial r}\right) \qquad (3\text{-}20)
\end{aligned}
$$

This is a second order linear partial differentia' equation with variable coefficients and a nonhomogeneous term. The nonhomogeneous term, the right-hand side, is given in terms of the force component, loss coefficient, and known zero order conditions. Now it is known from the investigations of Art. 2, that all force components cannot be prescribed independently but, strictly speaking, only one of them. It is necessary to express, within the present approximation f_r and f_z in terms of f_θ and ξ_D, considered prescribed, and known zeroeth order conditions.

From Eq. 2-17, giving the loss forces, it is clear that, to the present approximation,

$$f_r^{(2)} \cong 0 \tag{3-21}$$

$$f_\theta^{(2)} = -\tfrac{1}{2}\xi_D(v^{(0)} - \omega r)\sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2}} \tag{3-22}$$

$$f_z^{(2)} = -\tfrac{1}{2}\xi_D w^{(0)}\sqrt{(v^{(0)} - \omega r)^2 + w^{(0)2}} \tag{3-23}$$

and hence these are known. Also to the first order, Eq. 2-16 becomes

$$(v^{(0)} - \omega r)f_\theta^{(1)} + w^{(0)}f_z^{(1)} = 0 \tag{3-24}$$

and hence, in detail, the axial force component may be written

$$f_z = f_z^{(1)} + f_z^{(2)} = -\frac{v^{(0)} - \omega r}{w^{(0)}} f_\theta^{(1)} - \left[1 + \left(\frac{v^{(0)} - \omega r}{w^{(0)}}\right)^2\right]^{\frac{1}{2}} \frac{w^{(0)2}}{2}\xi_D \tag{3-25}$$

Finally if Eq. 2-23, the condition of blade continuity, be written to the first order it follows, employing the ratio $f_z^{(1)}/f_\theta^{(1)}$ from Eq. 3-24, that

$$\frac{\partial}{\partial z}\left(\frac{f_r^{(1)}}{rf_\theta^{(1)}}\right) = -\frac{\partial}{\partial r}\left(\frac{v^{(0)} - \omega r}{rw^{(0)}}\right) \tag{3-26}$$

The right-hand side of Eq. 3-26 depends only upon the radius. Consequently integration from the leading edge z_0 to some arbitrary value of z gives

$$\frac{f_r^{(1)}(r, z)}{rf_\theta^{(1)}(r, z)} - \frac{f^{(1)}(r, z_0)}{rf_\theta^{(1)}(r, z_0)} = -(z - z_0)\frac{\partial}{\partial r}\left(\frac{v^{(0)} - \omega r}{rw^{(0)}}\right)$$

or

$$f_r \equiv f_r^{(1)}(r, z) = f_\theta^{(1)}\left[\frac{f_r(r, z_0)}{f_\theta^{(1)}(r, z_0)} - (z - z_0)r\frac{\partial}{\partial r}\left(\frac{v^{(0)} - \omega r}{rw^{(0)}}\right)\right] \tag{3-27}$$

where $f_r(r, z_0)/f_\theta^{(1)}(r, z_0)$ represents the inclination of the leading edge with respect to a meridional plane.

Eq. 3-25 and 3-27 are satisfactory expressions for f_z and f_r to be employed in the right-hand side of Eq. 3-20; it should be noted, in particular, that these expressions contain nothing involving the radial perturbation velocity $u^{(1)}$ and consequently do not modify the left-hand side of Eq. 3-20.

The linearized second order partial differential equation for the radial perturbation velocity may now be written in the form

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + p(r)\frac{\partial u^{(1)}}{\partial r} + q(r)u^{(1)} + (1 - M_z^2)\frac{\partial^2 u^{(1)}}{\partial z^2} = \phi(r, z) \tag{3-28}$$

where $p(r)$, $q(r)$ and $\phi(r, z)$ are known and equal to

$$p(r) = \frac{1}{r}\left(1 + \gamma M_\theta^2 - \frac{r}{1 - M_z^2}\frac{1}{a^{2(0)}}\frac{\partial a^{2(0)}}{\partial r} + \frac{2rM_z^2}{1 - M_z^2}\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\right) \tag{3-29}$$

$$\begin{aligned} q(r) = \frac{1}{r^2}\Bigg\{&-1 + \frac{v^{(0)2}}{w^{(0)2}}[2 - (4 - \gamma)M_z^2 + (\gamma - 1)M_\theta^2] - \frac{r^2}{w^{(0)}}\frac{\partial^2 w^{(0)}}{\partial r^2} \\ &+ \frac{2rv^{(0)}}{w^{(0)2}}\frac{\partial v^{(0)}}{\partial r} - \frac{v^{(0)2} + w^{(0)2}}{w^{(0)2}(1 - M_z^2)}\frac{r}{a^{2(0)}}\frac{\partial a^{2(0)}}{\partial r} \\ &+ \frac{1}{1 - M_z^2}\left[1 + M_z^2 + M_\theta^2(2 - \gamma + \gamma M_z^2) + \frac{r}{a^{2(0)}}\frac{\partial a^{2(0)}}{\partial r}\right. \\ &\left. - 2M_z^2\frac{r}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\right]\frac{r}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\Bigg\} \end{aligned} \tag{3-30}$$

$$\begin{aligned} \phi(r, z) = \frac{f_z}{w^{(0)}}\Bigg[&\frac{1}{a^{2(0)}}\frac{\partial a^{2(0)}}{\partial r}\left(\frac{1}{1 - M_z^2}\right) - \frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\left(\frac{2M_z^2}{1 - M_z^2}\right) \\ &- (\gamma - M_z^2)\frac{M_\theta^2}{r}\Bigg] - \frac{1}{w^{(0)}}\left[\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}(1 - M_z^2)\right] \\ &+ (1 - M_z^2)\frac{v^{(0)}}{rw^{(0)2}}f_\theta\left[2 + M_\theta^2\left(\frac{\omega r}{v^{(0)}} - 1\right)\right] \\ &+ \frac{\gamma - 1}{2}M_z^2\frac{\partial}{\partial r}\left\{\frac{\xi_D}{w^{(0)2}}[(v^{(0)} - \omega r)^2 + w^{(0)2}]^{\frac{1}{2}}\right\} \\ &+ \frac{\gamma - 1}{2}\frac{\xi_D}{w^{(0)2}}[(v^{(0)} - \omega r)^2 + w^{(0)2}]^{\frac{1}{2}}\left\{\frac{1}{w^{(0)}}\frac{\partial w^{(0)}}{\partial r}\left[\frac{M_z^2(3 - M_z^2)}{1 - M_z^2}\right]\right. \\ &\left. - \frac{1}{a^{2(0)}}\frac{\partial a^{2(0)}}{\partial r}\left[\frac{M_z^2(2 - M_z^2)}{1 - M_z^2}\right] + \frac{M_\theta^2}{r}\left[\frac{\gamma^2 - \gamma + 1}{\gamma - 1}M_z^2 - \frac{\gamma}{\gamma - 1}\right]\right\} \end{aligned} \tag{3-31}$$

The values of M_z^2 and M_θ^2 introduced above are defined

$$\begin{aligned} M_\theta^2 &= \frac{v^{(0)2}}{a^{2(0)}} \\ M_z^2 &= \frac{w^{(0)2}}{a^{2(0)}} \end{aligned} \tag{3-32}$$

and hence are functions of the radius in general.

To complete the problem it is necessary to prescribe boundary conditions on the differential equation (Eq. 3-28). In the first place the radial velocity disturbances should vanish far upstream and downstream of the blade row, so that

$$u^{(1)}(r, -\infty) = u^{(1)}(r, \infty) = 0 \tag{3-33}$$

Secondly, the inner and outer radii are given as functions of the axial distance down the machine and, since they can differ from circular cylindrical surfaces only by a quantity of small order (say α where $\alpha \ll 1$), they can be written

$$r_1 = r_1^{(0)} + \alpha r_1^{(1)}(z) \tag{3-34}$$

$$r_2 = r_2^{(0)} + \alpha r_2^{(1)}(z) \tag{3-35}$$

Within the linearization employed, the boundary conditions at the inner and outer radii may then be given as

$$\epsilon \frac{u^{(1)}(r_1, z)}{w^{(0)}} = \alpha \frac{dr_1^{(1)}(z)}{dz} \tag{3-36}$$

and

$$\epsilon \frac{u^{(1)}(r_2, z)}{w^{(0)}} = \alpha \frac{dr_2^{(1)}(z)}{dz} \tag{3-37}$$

Finally, the complete state of the gas is given far upstream of the blade row; the quantities $v^{(0)}(r)$, $w^{(0)}(r)$, $h^{(0)}(r)$, $s^{(0)}(r)$, etc. are assumed to be known. The formulation of the linearized problem is complete.

C,4. Incompressible Flow through Single Blade Rows with Constant Hub and Tip Radii. It is often the case that the Mach number of the flow (in the sense of Art. 2) is sufficiently low that the gas may be considered incompressible in calculating the throughflow. Furthermore, near the design point it is often a fact that the axial velocity $w^{(0)}$ far upstream of the blade row differs from a constant value only by a quantity of order ϵ. Then $w^{(0)}$ may be considered constant and any variations incorporated as perturbation quantities. These conditions, although valid only for operating points near the design, simplify the mathematical problem enormously and constitute a good starting point for investigation of more complex conditions.

When the fluid is incompressible, the losses negligible, and the axial velocity differs only slightly from a constant value $w^{(0)}$, the partial differential equation (Eq. 3-28) becomes

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}\left[1 - \frac{1}{rw^{(0)2}}\frac{\partial}{\partial r}(rv^{(0)})^2\right]u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = \phi(r, z) \tag{4-1}$$

where

$$\phi(r, z) = -\frac{1}{w^{(0)}}\left(\frac{\partial f_z}{\partial r} - \frac{\partial f_r}{\partial z}\right) + \frac{2v^{(0)}}{rw^{(0)2}} f_\theta \tag{4-2}$$

Consider a single blade row with hub and tip shrouds consisting of concentric circular cylinders having radii r_h and r_t, respectively (see Fig. C,4a). Assume furthermore that far upstream of the blade row the axial velocity is uniform and the tangential velocity vanishes. Then the

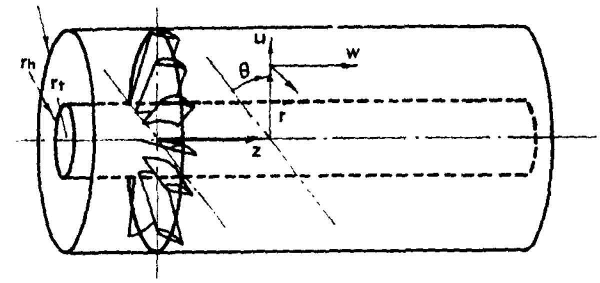


Fig. C,4a. Single blade row with hub and tip shrouds of constant radii.

tangential velocity and the axial velocity variation are only those introduced by the blade row itself so that the radial velocity perturbation is described by

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = -\frac{1}{w^{(0)}}\frac{\partial f_r}{\partial r} = -\frac{1}{w^{(0)}}\frac{\partial}{\partial z}\left(\frac{\partial h^{(1)}}{\partial r}\right) \tag{4-3}$$

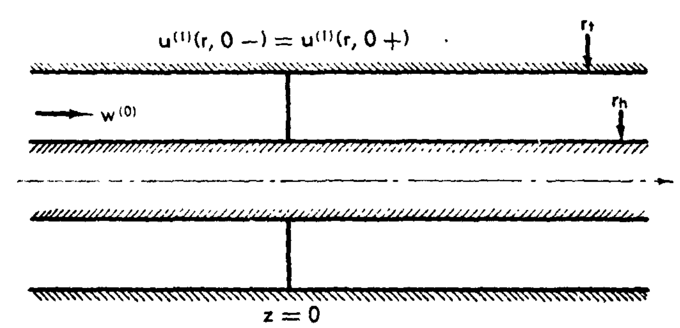
where enthalpy perturbation $h^{(1)}(r, z)$ will be assumed known. The boundary conditions to be satisfied are that

$$u^{(1)}(r_h, z) = u^{(1)}(r_t, z) = 0 \tag{4-4}$$

and

$$u^{(1)}(r, -\infty) = u^{(1)}(r, \infty) = 0 \tag{4-5}$$

Theory of the actuator disk. Suppose for the moment, following [8], that the blade row be shrunk axially into a discontinuity as in Fig. C,4b so that the change in enthalpy entering into the right-hand side of Eq. 4-3 is concentrated at $z = 0$. The radial velocity perturbation is therefore a



$$\left[\frac{\partial u^{(1)}}{\partial z}\right] = -\frac{1}{w^{(0)}}\left[\frac{\partial H^{(1)}}{\partial r}\right]$$

Fig. C,4b. The simple actuator disk.

solution of the homogeneous equation

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = 0 \tag{4-6}$$

both upstream and downstream of the discontinuity. Furthermore, the solution is symmetric with respect to $z = 0$ since both the differential equation and boundary conditions are symmetric. To determine the solution two matching conditions across the blade row or actuator disk must be established.

From the radial equation of motion (Eq. 2-1), it follows that the radial velocity $u^{(1)}$ can change across the actuator disk only as a result of a concentrated radial force. Within the linearized theory, therefore, the jump in radial velocity $[u^{(1)}]$ across the actuator disk may be written

$$[u^{(1)}] = \frac{1}{w^{(0)}}\int_{0-}^{0+} f_r dz \tag{4-7}$$

where we think here of the integrand as being very large, the interval very small, and the integral the total radial force of the blade row at a given point of the actuator disk. As a general rule in axial turbomachinery, the radial force component of the blades exerts a negligible influence due simply to the fact that the blade surfaces themselves are so nearly radial. This observation is so general that it is preferred to assume here that $[u^{(1)}] = 0$ and simply to note that, if a peculiar construction alters this situation, the following analysis would require modification in an obvious manner. The second matching condition follows directly from the partial differential equation (Eq. 4-3). Since the axial and radial velocity components are continuous across the actuator disk, the differential equation may be evaluated on each side of the discontinuity, integrated once with respect to z, and the results subtracted to give

$$\left[\frac{\partial u^{(1)}}{\partial z}\right]_{0-}^{0+} = -\frac{1}{w^{(0)}}\left[\frac{\partial h^{(1)}}{\partial r}\right]_{0-}^{0+} \tag{4-8}$$

a condition on the difference between values of $\partial u^{(1)}/\partial z$ evaluated on each side of the actuator disk.

Now a solution of the differential equation,

$$u^{(1)} = \sum_{1}^{\infty} C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - Y_1(\kappa_n r)J_1(\kappa_n r_h)]e^{\kappa_n z} \qquad z < 0 \tag{4-9}$$

and

$$u^{(1)} = \sum_{1}^{\infty} C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]e^{-\kappa_n z} \qquad z > 0 \tag{4-10}$$

satisfies the boundary conditions at $z = \pm\infty$ and at $r = r_h$, as well as gives a continuous value of $u^{(1)}$ across the actuator disk as required by the first matching condition. The boundary condition at the outer boundary, $u^{(1)}(r_t, z) = 0$, is satisfied identically by taking the characteristic values κ_n to be the roots of

$$J_1(\kappa_n r_t) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n r_t) = 0 \tag{4-11}$$

and that there is a countable infinity of roots ordered as to increasing magnitude is guaranteed by the Sturm-Liouville theorem. The values of the constants follow directly from the orthogonality property of the Bessel functions and the matching condition given by Eq. 4-8. Using the solutions (Eq. 4-9 and 4-10),

$$-2 \sum_1^\infty C_n \kappa_n [J_1(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n r)] = -\frac{1}{w^{(0)}} \left[\frac{\partial h^{(1)}}{\partial r}\right] \tag{4-12}$$

from which

$$C_n = \int_{r_h}^{r_t} \frac{\alpha(J_1(\kappa_n \alpha) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n \alpha)}{2\kappa_n \nu_n^2 w^{(0)}} \left[\frac{\partial h^{(1)}}{\partial \alpha}\right] d\alpha \tag{4-13}$$

where ν_n is the norm of $J_1(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n r)$ and is given by

$$\begin{aligned}\nu_n^2 &= \int_{r_h}^{r_t} \alpha[J_1(\kappa_n \alpha) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n \alpha)]^2 d\alpha \\ &= \tfrac{1}{2}\{r_t^2[J_0(\kappa_n r_t) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_0(\kappa_n r_t)]^2 \\ &\qquad - r_h^2[J_0(\kappa_n r_h) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_0(\kappa_n r_h)]^2\}\end{aligned} \tag{4-14}$$

The radial velocity distribution is now directly calculable from the foregoing results substituted into Eq. 4-9 and 4-10 after the integrals of Eq. 4-13 have been evaluated.

The axial velocity perturbation is related to the radial velocity through the continuity condition,

$$\frac{\partial w^{(1)}}{\partial z} = -\frac{1}{r} \frac{\partial (r u^{(1)})}{\partial r} \tag{4-15}$$

which from Eq. 4-10 and 4-11 yields

$$\frac{\partial w^{(1)}}{\partial z} = -\sum_1^\infty C_n \kappa_n [J_0(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_0(\kappa_n r)] e^{\pm \kappa_n z} \tag{4-16}$$

the plus or minus sign applying according to whether the point in question is upstream or downstream of the actuator disk. The integration to obtain $w^{(1)}(r, z)$ may be carried out easily to give

$$w^{(1)} = -\sum_1^\infty C_n [J_0(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_0(\kappa_n r)] e^{\kappa_n z} \qquad z < 0 \tag{4-17}$$

solution of the homogeneous equation

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = 0 \tag{4-6}$$

both upstream and downstream of the discontinuity. Furthermore, the solution is symmetric with respect to $z = 0$ since both the differential equation and boundary conditions are symmetric. To determine the solution two matching conditions across the blade row or actuator disk must be established.

From the radial equation of motion (Eq. 2-1), it follows that the radial velocity $u^{(1)}$ can change across the actuator disk only as a result of a concentrated radial force. Within the linearized theory, therefore, the jump in radial velocity $[u^{(1)}]$ across the actuator disk may be written

$$[u^{(1)}] = \frac{1}{w^{(0)}}\int_{0-}^{0+} f_r dz \tag{4-7}$$

where we think here of the integrand as being very large, the interval very small, and the integral the total radial force of the blade row at a given point of the actuator disk. As a general rule in axial turbomachinery, the radial force component of the blades exerts a negligible influence due simply to the fact that the blade surfaces themselves are so nearly radial. This observation is so general that it is preferred to assume here that $[u^{(1)}] = 0$ and simply to note that, if a peculiar construction alters this situation, the following analysis would require modification in an obvious manner. The second matching condition follows directly from the partial differential equation (Eq. 4-3). Since the axial and radial velocity components are continuous across the actuator disk, the differential equation may be evaluated on each side of the discontinuity, integrated once with respect to z, and the results subtracted to give

$$\left[\frac{\partial u^{(1)}}{\partial z}\right]_{0-}^{0+} = -\frac{1}{w^{(0)}}\left[\frac{\partial h^{(1)}}{\partial r}\right]_{0-}^{0+} \tag{4-8}$$

a condition on the difference between values of $\partial u^{(1)}/\partial z$ evaluated on each side of the actuator disk.

Now a solution of the differential equation.

$$u^{(1)} = \sum_1^\infty C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - Y_1(\kappa_n r)J_1(\kappa_n r_h)]e^{\kappa_n z} \qquad z < 0 \tag{4-9}$$

and

$$u^{(1)} = \sum_1^\infty C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]e^{-\kappa_n z} \qquad z > 0 \tag{4-10}$$

satisfies the boundary conditions at $z = \pm\infty$ and at $r = r_h$, as well as gives a continuous value of $u^{(1)}$ across the actuator disk as required and

$$w^{(1)} = -2\sum_{1}^{\infty} C_n[J_0(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r)] + \sum_{1}^{\infty} C_n[J_0(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r)]e^{-\kappa_n z} \qquad z > 0 \quad (4\text{-}18)$$

It is of interest to note here that there is a residual axial velocity perturbation far downstream of the blade row equal to $-2\sum_1^\infty C_n[J_0(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r)]$ and that the axial velocity perturbation is just twice that at the actuator disk itself. That this change should take place half upstream and half downstream of the blade row is a direct result of symmetry in the radial velocity distribution.

Since it has been assumed that the distribution of enthalpy rise is given, it follows that the first order perturbation in tangential velocity is also known. From Eq. 3-7 and 3-9 it may be inferred that

$$\omega r[v^{(1)}] = [h^{(1)}] \qquad (4\text{-}19)$$

and since this jump takes place only across the actuator disk, the tangential velocity is constant ($\equiv 0$) upstream of the blade row and constant ($\equiv v^{(1)}(r)$) downstream of the blade row. This first order solution for the velocity components is complete and agrees with that given originally in [8].

Since the foregoing example of the linearized treatment is one of the simplest that may be found, it is worthwhile to explore its characteristics in some detail. Take as a special case the enthalpy jump to be

$$[h^{(1)}(r)] = aw^{(0)}\omega r_t\left(\frac{r}{r_t}\right)^2 \qquad (4\text{-}20)$$

Referring to Eq. 4-19, this corresponds to imparting a tangential velocity of magnitude $aw^{(0)}r/r_t$ where a is a constant. Such a tangential velocity distribution is referred to as a solid body rotation since its magnitude is proportional to the radius.

To obtain an explicit solution, evaluation of the integrals for C_n is required:

$$C_n = \frac{aw^{(0)}\omega r_t}{\kappa_n \nu_n^2 w^{(0)} r_t}\int_{r_h}^{r_t} \alpha^2[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)]d\alpha \qquad (4\text{-}21)$$

which integrates directly to give

$$C_n = \frac{-2a(\omega r_t)}{(\kappa_n r_t)^2}\left\{\frac{r_t^2[J_0(\kappa_n r_t)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r_t)] - r_h^2[J_0(\kappa_n r_h)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r_h)]}{r_t^2[J_0(\kappa_n r_t)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r_t)]^2 - r_h^2[J_0(\kappa_n r_h)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r_h)]}\right\}$$

Substitution of these coefficients into Eq. 4-9, 4-10, 4-17, and 4-18 yields the solution for radial and axial velocity perturbation upstream and downstream of a blade row represented by a discontinuity. As an illustration of the results obtained, a blade row satisfying the above enthalpy jump has been chosen having a ratio of hub radius to tip radius $r_h/r_t = 0.60$. The characteristic values may be evaluated from Eq. 4-11, and from these all necessary factors of the solution follow. The radial velocity distribution is shown in Fig. C,4c for various distances upstream (or downstream by

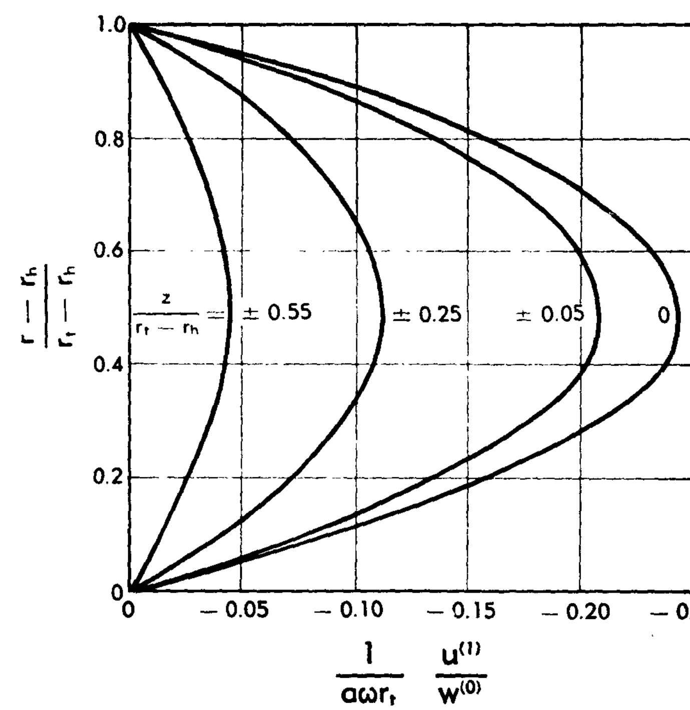


Fig. C,4c. Radial velocity patterns upstream and downstream of an actuator disk, moving with angular velocity ω and imparting tangential velocity $aw^{(0)}r/r_t$, $r_h/r_t = 0.6$.

symmetry) from the actuator disk. As is intuitively logical, the largest values of radial velocity occur near the middle of the channel. Actually, however, none of these values are large enough themselves to be of significance in turbomachine blade design. Rather it is the changes in axial velocity that are induced by this radial velocity that are of importance. The axial velocity perturbation computed from Eq. 4-17 and 4-18, using the coefficients given in Eq. 4-21, is shown in Fig. C,4d. The blade row has a quite sensible effect on the axial velocity profile at distances upstream of more than half a blade length. By virtue of the same symmetry noticed in connection with the radial velocities, distortion of the axial velocity continues to increase until it reaches a point about half a blade length downstream where the maximum and constant perturbation is achieved. These magnitudes are sufficient to distort the blade velocity diagram significantly from that which would be estimated neglecting the three-dimensional throughflow.

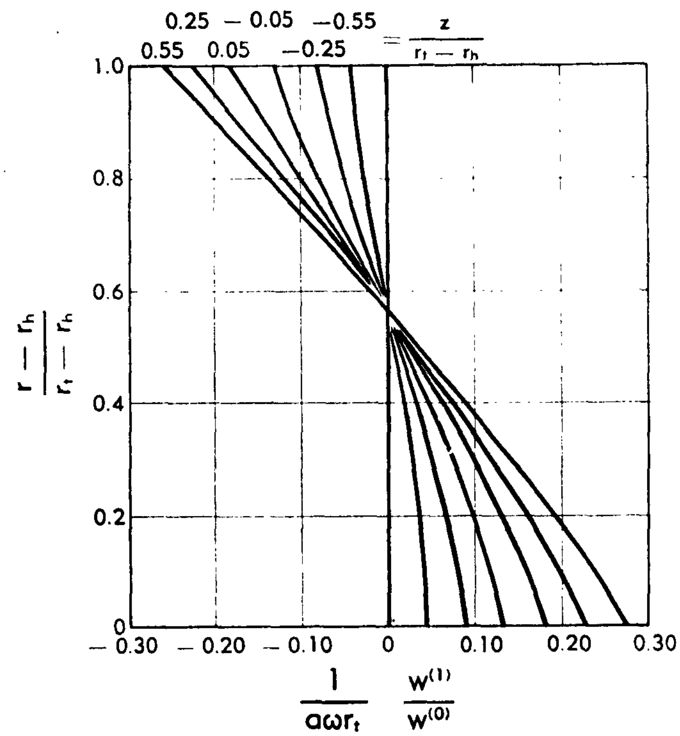


Fig. C,4d. Axial velocity perturbation patterns upstream and downstream of an actuator disk, moving with angular velocity ω and imparting tangential velocity $aw^{(0)}r/r_t$, $r_h/r_t = 0.6$.

Theory for blade row of finite chord. It is obvious that the results of actuator disk theory do not describe the flow accurately in the immediate vicinity of the blade row. The description may certainly be improved by developing the theory for a blade row of finite chord. Conceptually the blade of finite chord may be thought of as a sequence of actuator disks and, as a matter of fact, this idea has certain merit as an approximation. Mathematically the solution for the actuator disk may be employed as a unit out of which to construct the solution for a blade of finite chord by integration over the blade chord of actuator disks of the appropriate infinitesimal strength.

Consider then an actuator disk located at a point $z = \beta$ along the axis of the turbomachine. Referring to Eq. 4-9 and 4-10 it is clear that the radial velocity induced by this actuator disk may be written as

$$u^{(1)} = \sum_{1}^{\infty} C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]e^{\kappa_n(z-\beta)}$$

upstream of the actuator disk, and as

$$u^{(1)} = \sum_{1}^{\infty} C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]e^{-\kappa_n(z-\beta)}$$

downstream of the actuator disk. Now it will be assumed that such actuator disks are distributed continuously along the z axis at values of β between the leading and trailing edges of the blade, as shown in Fig. C,4e.

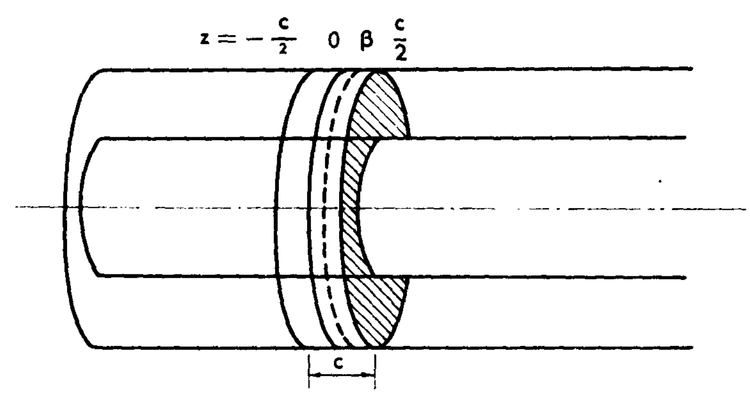


Fig. C,4e. Continuous blade row as a sequence of actuator disks.

Then again, if $h^{(1)}(r, z)$ is the perturbation of enthalpy imparted by the blade row, the function $[\partial(\partial h^{(1)}/\partial r)/\partial z]dz$ replaces the function $[\partial h^{(1)}/\partial r]$ in the determination of coefficients according to Eq. 4-13. For the elementary actuator disk situated at $z = \beta$, the flow can be written, substituting the values of C_n from Eq. 4-13,

$$u^{(1)} = \int_{r_h}^{r_t} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta}\, d\beta \sum_{1}^{\infty} \frac{\alpha[J_1(\kappa_n\alpha)Y(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]}{2\,\kappa_n \nu_n^2}\, e^{-\kappa_n|z-\beta|}d\alpha \qquad (4\text{-}22)$$

where the order of integration and summation have been interchanged with the assurance that the series possesses the appropriate convergence properties. One of these infinitesimal discontinuities exists at each point of the interval over which the blade row exists and, since the problem is a linear one, the complete solution of the original homogeneous problem can be obtained by summing them over the blade chord. This sum takes the form of an integration from $z = -c/2$ to $z = c/2$ where, as shown in Fig. C,4e, the blade chord has the magnitude c. The complete solution is thus

$$u^{(1)} = \int_{r_h}^{r_t}\int_{-c/2}^{c/2} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} G(r, z; \alpha, \beta) d\alpha d\beta \tag{4-23}$$

where the function $G(r, z; \alpha, \beta)$ is just the infinite series

$$\sum_1^\infty \frac{\alpha[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]}{2\kappa_n\nu_n^2} e^{-\kappa_n|z-\beta|} \tag{4-24}$$

The function $G(r, z; \alpha, \beta)$ may be interpreted as proportional to the radial velocity induced at a point r, z of the turbomachine by an element of tangential vorticity bound at a point α, β of the blade row.

The complete solution for the axial velocity perturbation may be constructed in a very similar manner, for referring to Eq. 4-17 and 4-18 it is clear the axial velocity perturbation induced by an actuator disk of axial length $d\beta$ is just

$$w^{(1)} = \int_{r_h}^{r_t} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} d\beta$$
$$\sum_1^\infty \frac{\alpha[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]}{2\kappa_n\nu_n^2} e^{-\kappa_n|z-\beta|} d\alpha$$
$$z < \beta \tag{4-25}$$

and

$$w^{(1)} = 2\int_{r_h}^{r_t} \frac{1}{w^{(0)}} \frac{\partial^2 h}{\partial\alpha\partial\beta} d\beta$$
$$\sum_1^\infty \frac{\alpha[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]}{2\kappa_n\nu_n^2} d\alpha$$
$$- \int_{r_h}^{r_t} \frac{1}{w^{(0)}} \frac{\partial^2 h}{\partial\alpha\partial\beta} d\beta \sum_1^\infty \frac{\alpha[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]}{2\kappa_n\nu_n^2} e^{-\kappa_n|z-\beta|} d\alpha$$
$$z > \beta \tag{4-26}$$

Again the complete solution for the blade row of finite chord is given by summing the above solution across the blade chord. For a point with any axial location z, solutions of the type given by Eq. 4-26 are summed over all values of $\beta < z$ and solutions of the type given by Eq. 4-25 for all

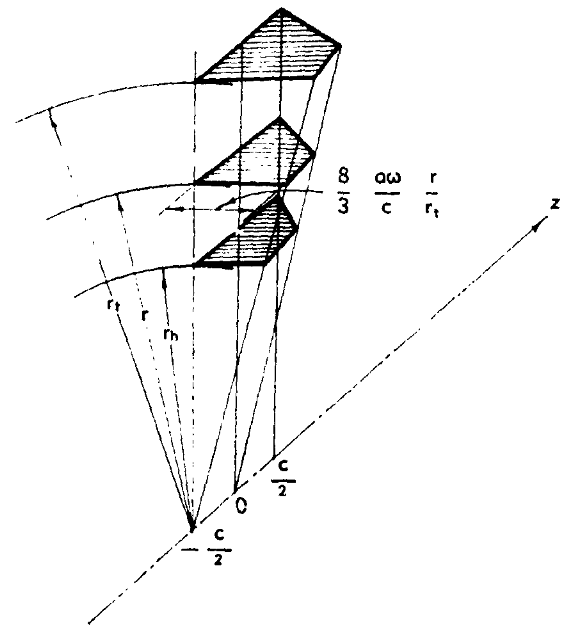


Fig. C,4f. Distribution of load on blade row of finite chord.

values of $\beta > z$. Therefore the complete solution is written as

$$w^{(1)} = \int_{r_h}^{r_t}\int_{-\infty}^{z} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} K(r, z; \alpha, \beta)d\alpha d\beta$$

$$- \int_{r_h}^{r_t}\int_{z}^{\infty} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} K(r, z; \alpha, \beta)d\alpha d\beta$$

$$+ 2\int_{r_h}^{r_t}\int_{-\infty}^{z} \frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} K(r, \beta; \alpha, \beta)d\alpha d\beta \quad (4\text{-}27)$$

where the function $K(r, z; \alpha, \beta)$ is defined as

$$\sum_{1}^{\infty} \frac{\alpha[J_1(\kappa_n\alpha)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n\alpha)][J_0(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r)]\, e^{-\kappa_n|z-\beta|}}{2\,\kappa_n \nu_n^2} \quad (4\text{-}28)$$

These results complete the formal solution for radial and axial velocity perturbations induced by a blade row of finite chord. It is instructive to use the example employed for the actuator disk as an illustration of the present theory for finite blade chord so that the results may be compared. Therefore in addition to the radius ratio $r_h/r_t = 0.60$ consider a blade row of chord c with a chordwise load distribution as shown in Fig. C,4f. For

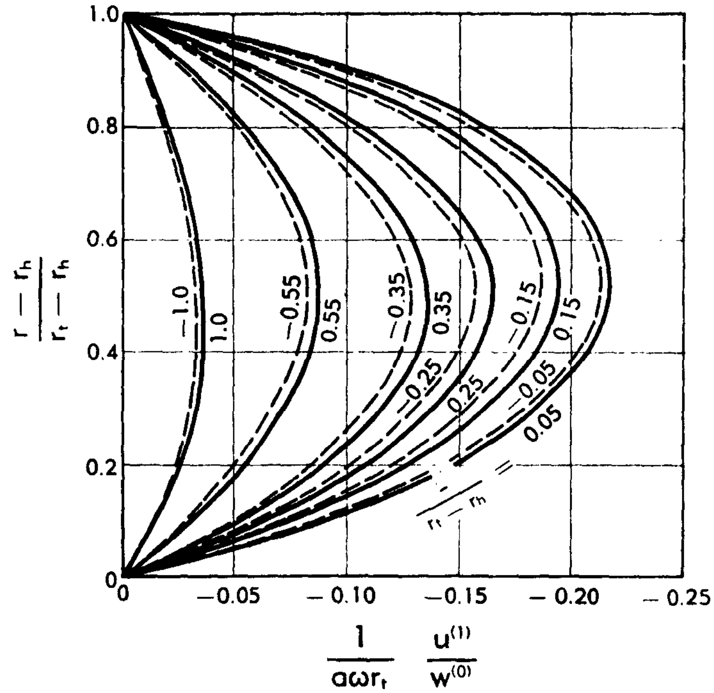


Fig. C,4g. Radial velocity patterns upstream, downstream, and within a blade row of finite chord, moving with angular velocity ω and loaded as indicated in Fig. C,4f, $r_h/r_t = 0.6$.

the first half of the blade row the value of $(1/w^{(0)})(\partial^2 h^{(1)}/\partial\alpha\partial\beta)$ is given by

$$\frac{1}{w^{(0)}}\frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} = \frac{8}{3}a\frac{\omega}{c}\frac{\alpha}{r_t} \qquad -\frac{c}{2} \leqq \beta \leqq 0 \tag{4-29}$$

and for the second half of the blade row,

$$\frac{1}{w^{(0)}}\frac{\partial^2 h^{(1)}}{\partial\alpha\partial\beta} = \frac{8}{3}a\frac{\omega}{c}\frac{\alpha}{r_t}\left(1 - \frac{2\beta}{c}\right) \qquad 0 \leqq \beta \leqq \frac{c}{2} \tag{4-30}$$

The total work added by this distribution is identical with the total work added in the actuator disk discussed previously. The results may be obtained explicitly by integration of Eq. 4-23 and 4-27 employing the values of enthalpy given by Eq. 4-29 and 4-30. This operation involves only algebraic complexity and the formulas, because of their length, will not be quoted here. The radial velocity distributions are given in Fig. C,4g for axial stations within the blade row as well as both upstream and downstream from it. The slight asymmetry of the flow pattern caused by the unsymmetrical loading of the blade row is noticeable. Of somewhat more technical interest are the distributions of axial velocity perturbation shown in Fig. C,4h. For comparison the axial velocity perturbations for the actuator disk are shown on the same curve. The differences are certainly noticeable but not marked. Remembering then that the present results

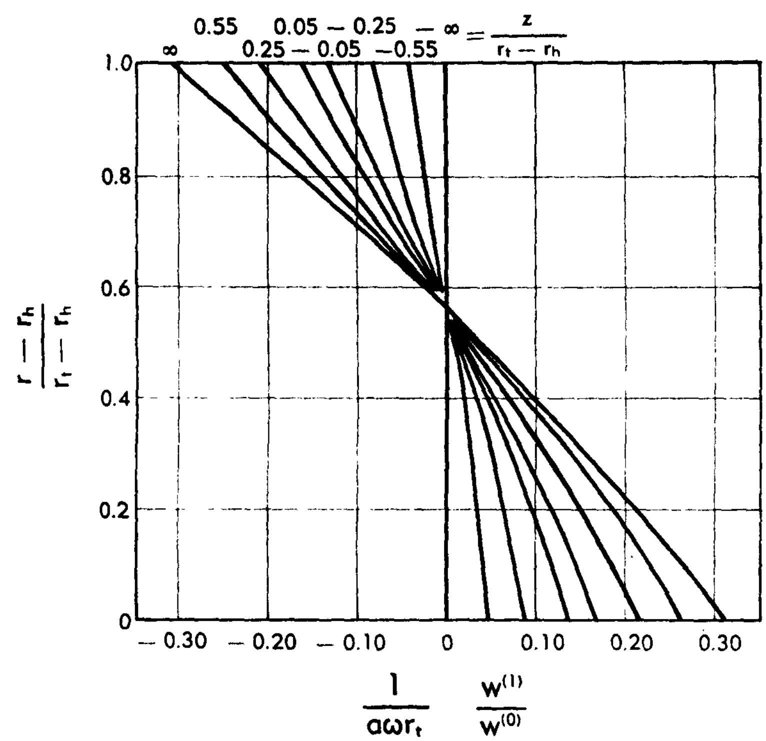


Fig. C,4h. Axial velocity perturbations upstream, downstream, and within a blade row of finite chord, moving with angular velocity ω and loaded as indicated in Fig. C,4f, $r_h/r_t = 0.6$.

hold for a blade whose ratio of length to axial extent is 2.0, it appears that for blades of high aspect ratio, such as those in the first few stages of current axial flow compressors, ordinary needs do not require the treatment of the finite blade chord. On the other hand, turbomachine blade rows with an aspect ratio of 1.0 or less do require consideration of the finite blade chord for accurate construction of velocity diagrams.

Effect of large tangential velocities. The discussion so far has been restricted to examples where the tangential velocities of the fluid were at most of the first order. This fact permitted simplification of the differential operator on the left-hand side of Eq. 4-1 and reduced the nonhomogeneous term on the right-hand side to the very simple result given in Eq. 4-3. It is usually the case in multistage axial turbomachinery, however, that although the change in tangential velocity across a blade row is small, a zeroeth order tangential velocity $v^{(0)}$ exists which is of the same general magnitude as the undisturbed throughflow velocity $w^{(0)}$. The differential equation describing the radial velocity in this case is then given by

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} + \left[\frac{1}{r^3 w^{(0)2}}\frac{\partial}{\partial r}(rv^{(0)})^2 - \frac{1}{r^2}\right] + \frac{\partial^2 u^{(1)}}{\partial z^2} = \frac{1}{w^{(0)}}\frac{\partial}{\partial z}\left[\frac{1}{r^2}\frac{\partial}{\partial r}(rv^{(0)}rv^{(1)})\right] - \frac{1}{w^{(0)}}\frac{\partial^2 h^{(1)}}{\partial r\partial z} \qquad (4\text{-}31)$$

The differential operator is complicated in an essential manner by introducing an additional variable coefficient of $u^{(1)}$; the addition of a term depending upon the tangential velocity perturbation introduces a new function on the right-hand side but does not really change the problem.

In the particular case when the zeroeth order tangential velocity is of the vortex type, that is $v^{(0)} \sim 1/r$, the term $rv^{(0)}$ is a constant and the expression $(1/r^3 w^{(0)2})\partial(rv^{(0)})^2/\partial r$ vanishes in Eq. 4-31. This leaves the differential operator, and hence the formal solution to the problem, exactly as it was for $v^{(0)} = 0$ except that now the inhomogeneous term of Eq. 4-31 replaces the simpler one of Eq. 4-3. A more interesting and significant example of large zeroeth order tangential velocity occurs when the tangential velocity is of the solid body type. Assume that the tangential velocity is given by

$$v^{(0)} = bw^{(0)}\frac{r}{r_t} \qquad (4\text{-}32)$$

where b is a numerical constant. The differential equation (Eq. 4-31) then becomes

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} + \left[\left(\frac{2b}{r_t}\right)^2 - \frac{1}{r^2}\right]u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = b\frac{\partial}{\partial z}\left[\frac{1}{r^2 r_t}\frac{\partial}{\partial r}(r^3 v^{(1)})\right] - \frac{1}{w^{(0)}}\frac{\partial^2 h^{(1)}}{\partial r\partial z} \qquad (4\text{-}33)$$

It should be noticed in particular that the additional coefficient on the left-hand side multiplying $u^{(1)}$ is a constant $(2b/r_t)^2$. The solution may be worked out in very much the same way as it was for Eq. 4-3. It is readily determined that the radial velocity for an actuator disk located at $z = 0$ is given by

$$u^{(1)} = \sum_1^\infty C_n[J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r)]e^{-\lambda_n|z|} \qquad (4\text{-}34)$$

where now the κ_n are still the roots of Eq. 4-11 and the λ_n are given by

$$\lambda_n = \sqrt{\kappa_n^2 - \left(\frac{2b}{r_t}\right)^2} \tag{4-35}$$

The constants C_n are given by

$$C_n = -\int_{r_b}^{r_t} \frac{\alpha(J_1(\kappa_n\alpha)Y_1(\kappa_n r_b) - J_1(\kappa_n r_b)Y_1(\kappa_n\alpha)}{2\lambda_n \nu_n^2 w^{(0)}} \left[\frac{b}{\alpha^2 r_t}\frac{\partial}{\partial\alpha}(\alpha v^{(1)}) - \frac{\partial h^{(1)}}{\partial\alpha}\right] d\alpha \tag{4-36}$$

The modification to the constants C_n is obvious inasmuch as it simply makes use of the new inhomogeneous term appearing in Eq. 4-33. The significant change is the modification of the exponents $e^{-\lambda_n|z|}$ appearing in the solution, Eq. 4-34. In aircraft gas turbine practice it is almost invariably true that $|2b/r_t| < \kappa_1$ and consequently all of the λ_n are real and nonvanishing. Thus the effect of a solid body rotation imposed upon the fluid far upstream of the blade row is to cause any disturbances generated by the blade row to decay more slowly upstream and downstream of the blade row than they would in the absence of a zeroeth order rotation. This effect is particularly pronounced upon the lower Bessel components corresponding to κ_1, κ_2, and κ_3 since the exponents associated with them are reduced in magnitude proportionally more than the higher Bessel components.

After having observed the nature of the modification introduced by solid body rotation it is simple to complete the solution for both the actuator disk and the blade of finite chord. For the actuator disk the axial velocity distribution is given by

$$w^{(1)} = \sum_1^\infty \frac{C_n\kappa_n}{\lambda_n}[J_0(\kappa_n r)Y_1(\kappa_n r_b) - J_1(\kappa_n r_b)Y_0(\kappa_n r)]e^{\lambda_n z} \qquad z < 0 \tag{4-37}$$

and

$$w^{(1)} = -2\sum_1^\infty \frac{C_n\kappa_n}{\lambda_n}[J_0(\kappa_n r)Y_1(\kappa_n r_b) - J_1(\kappa_n r_b)Y_0(\kappa_n r)] + \sum_1^\infty \frac{C_n\kappa_n}{\lambda_n}[J_0(\kappa_n r)Y_1(\kappa_n r_b) - J_1(\kappa_n r_b)Y_0(\kappa_n r)]e^{-\lambda_n z} \quad z > 0 \tag{4-38}$$

These demonstrate results that were intuitively clear from the previous calculation of radial velocity distribution. The over-all perturbation to the axial velocity, from far upstream of the blade row to a point far downstream, is changed because the values of $C_n\kappa_n/\lambda_n$ entering into the term

$$-2 \sum_{1}^{\infty} \frac{C_n \kappa_n}{\lambda_n} [J_0(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_0(\kappa_n r)]$$

are changed due to the function $(b/r^2 r_t)\partial/\partial r(rv^{(1)})$ that enters into the integral for C_n given by Eq. 4-36. Referring still to Eq. 4-36, the term $[\partial h^{(1)}/\partial r]$ is a measure of the tangential vorticity generated at the actuator disk due to actual work being done by the blade row. The new term $(b/r^2 r_t)\partial/\partial r(rv^{(1)})$ is of different origin. It arises from the fact that some of the axial vorticity, associated with the solid body rotation, is turned by the perturbation tangential velocity so that it becomes tangential vorticity. The second result is that this change of axial velocity profile starts farther upstream and completes farther downstream in the presence of solid body rotation than it does when no zeroeth order tangential velocity is present.

The resulting radial and axial velocities for a blade row of finite chord can be obtained immediately through simple modification of Eq. 4-23, 4-24, 4-27, and 4-28. Wherever the term

$$\frac{1}{w^{(0)}} \frac{\partial^2 h^{(1)}}{\partial r \partial z}$$

appears, it should be replaced by the term

$$\frac{1}{w^{(0)}} \frac{\partial}{\partial z} \left[- \frac{b w^{(0)}}{r^2 r_t} \frac{\partial}{\partial r} (r^3 v^{(1)}) + \frac{\partial h^{(1)}}{\partial r} \right]$$

wherever the characteristic value κ_n appears in the exponent $e^{-\kappa_n|z-\beta|}$, it should be replaced by λ_n to give $e^{-\lambda_n|z-\beta|}$ where λ_n is defined through Eq. 4-35. The gross results discussed above for the actuator disk apply also to the blade of finite chord. These effects are generally the same, in fact, whether the zeroeth order tangential velocity is of the solid body type or whether it is different. The particular example used is especially significant, however, since a mean solid body rotation is so frequently employed in axial compressors. Methods for treating more general distributions of zeroeth order tangential velocity will be deferred until later when the appropriate asymptotic expansions are discussed.

The entrance vane. Second order theory. The results discussed so far permit adequate treatment of every blade row in an axial turbomachine with the exception of the entrance vanes or guide vanes. Because the guide vanes are a stationary blade row the perturbation enthalpy $h^{(1)}$ vanishes and since there is no zeroeth order tangential motion upstream of the guide vanes, the term $rv^{(0)}$ vanishes also. Consequently the right-hand side of Eq. 4-31 vanishes so that the radial velocity and axial velocity perturbation vanish identically. The first order perturbation theory is inadequate therefore to deal with the guide vane problem. A second order calculation must be made, therefore, explicitly for the purposes of the guide vane. The fact that the radial velocity and axial velocity perturbation are of the second order carries no implication that these effects are small. Actually the effects are as large as or larger than those induced by ordinary rotor or stator blades; the fact that they show up as second order effects is due, as will be seen, to the quadratic nature of the perturbation term. It is also an indication that, since the change in tangential velocity across the guide vane is considerably larger than that across rotor or stator blade rows, the guide vane problem is not treated with the same accuracy of approximation by the first order analysis as are the other blade rows.

To investigate the guide vane flow, restrict conditions to a uniform zeroeth order axial velocity, $w^{(0)} = \text{const}$, along with the previous assumptions of cylindrical inner and outer boundaries, no losses, and incompressible fluid. From calculations of the previous section it is clear that the first order radial velocity and axial velocity disturbance vanish identically, that is,

$$\begin{aligned} u^{(1)} &\equiv 0 \\ w^{(1)} &\equiv 0 \end{aligned} \tag{4-39}$$

in spite of the fact that $v^{(1)} \neq 0$ and may, in fact, be rather large. The flow field is defined again through Eq. 2-1, 2-2, 2-3, and 2-4 and for purposes of the perturbation we call

$$\begin{aligned} u &= \epsilon^2 u^{(2)} \\ v &= \epsilon v^{(1)} + \epsilon^2 v^{(2)} \\ w &= w^{(0)} + \epsilon^2 w^{(2)} \\ p &= p^{(0)} + \epsilon p^{(1)} + \epsilon^2 p^{(2)} \\ F_r &= \epsilon f_r \\ F_\theta &= \epsilon f_\theta \\ F_z &= \epsilon f_z \end{aligned} \tag{4-40}$$

where account has been taken of the fact that the first order perturbation of radial and axial velocity vanish identically. Substituting into the equations of motion yields both first and second order parts. In the axial direction the first order relation is

$$\frac{1}{\rho}\frac{\partial p^{(1)}}{\partial r} = f_r \tag{4-41}$$

whereas the second order part is

$$w^{(0)}\frac{\partial u^{(2)}}{\partial z} - \frac{v^{(1)2}}{r} = -\frac{1}{\rho}\frac{\partial p^{(2)}}{\partial r} \tag{4-42}$$

From the tangential equilibrium relation the first and second order parts are respectively

$$w^{(0)} \frac{\partial}{\partial z} (rv^{(1)}) = rf_\theta \tag{4-43}$$

and

$$w^{(0)} \frac{\partial}{\partial z} (rv^{(2)}) = 0 \tag{4-44}$$

In the axial direction the first and second order parts are respectively

$$\frac{1}{\rho} \frac{\partial p^{(1)}}{\partial z} = f_z \tag{4-45}$$

and

$$w^{(0)} \frac{\partial w^{(2)}}{\partial z} = -\frac{1}{\rho} \frac{\partial p^{(2)}}{\partial z} \tag{4-46}$$

It is interesting to note that the radial and axial force field produces a first order pressure field within the blade row itself but no radial or axial velocity field. Moreover the tangential velocity distribution is entirely accounted for by the first order relation, Eq. 4-43, and the second order part $v^{(2)}$ vanishes identically according to Eq. 4-44 and the initial condition that the tangential velocity vanishes identically ahead of the guide vane. The second order velocity field is then defined by Eq. 4-42 and 4-46 in addition to the continuity equation

$$\frac{\partial u^{(2)}}{\partial r} + \frac{u^{(2)}}{r} + \frac{\partial w^{(2)}}{\partial z} = 0 \tag{4-47}$$

It is a simple matter to eliminate the second order pressure $p^{(2)}$ and axial velocity perturbation to give

$$\frac{\partial^2 u^{(2)}}{\partial r^2} + \frac{1}{r} \frac{\partial u^{(2)}}{\partial r} - \frac{u^{(2)}}{r^2} + \frac{\partial^2 u^{(2)}}{\partial z^2} = \frac{1}{w^{(0)}} \frac{\partial}{\partial z} \left(\frac{v^{(1)2}}{r} \right) \tag{4-48}$$

where the tangential velocity distribution $v^{(1)}$ is either prescribed or is known from prescribed tangential blade force or blade shape. The mathematical problem is therefore exactly that treated in the solution of Eq. 4-3 where now the known function

$$\frac{1}{w^{(0)}} \frac{\partial}{\partial z} \left(\frac{v^{(1)2}}{r} \right)$$

replaces the term

$$-\frac{1}{w^{(0)}} \frac{\partial}{\partial z} \left(\frac{\partial h^{(1)}}{\partial r} \right)$$

The corresponding solutions for the actuator disk and for the blade row of finite chord may be carried over directly with the above substitution.

For example consider a guide vane, approximated by an actuator disk, which imparts a solid body rotation to the fluid. The jump in tangential velocity across the blade row may then be written in the form,

$$[v^{(1)}] = bw^{(0)}\frac{r}{r_t} \tag{4-49}$$

The term of interest here, corresponding to the right-hand side of Eq. 4-8, is

$$\frac{1}{w^{(0)}}\left[\frac{v^{(1)2}}{r}\right] = \frac{b^2w^{(0)}r}{r_t^2} \tag{4-50}$$

so that the equation for the coefficient C_n to be employed in expansions (Eq. 4-9, 4-10, 4-17, and 4-18) may be written as

$$C_n = \frac{-b^2w^{(0)}}{2\kappa_n\nu_n^2r_t^2}\int_{r_b}^{r_t}\alpha^2[J_1(\kappa_n\alpha)Y_1(\kappa_nr_b) - J_1(\kappa_nr_b)Y_1(\kappa_n\alpha)]d\alpha \tag{4-51}$$

These coefficients differ only in the multiplicative constant from the example of a rotating blade row treated earlier, the results of which are shown in Fig. C,4c and C,4d. Comparison with the appropriate coefficients of that example, given by Eq. 4-21, shows that by replacing $a\omega r_t$ in Fig. C,4c and C,4d by the product $-b^2w^{(0)}$ from the present problem, the curves in these figures apply equally well to the present example of a guide vane. The radial and axial velocity distortions are therefore similar in the two situations but of opposite sign. Exactly the same parallel exists between the solutions of Eq. 4-48 for a continuous blade row imparting solid body rotation and detailed solutions for a rotating blade row given in Fig. C,4g and C,4h.

Asymptotic expansion of the Bessel functions. In a great number of instances the hub ratio r_b/r_t is sufficiently large that the characteristic values κ_n, the roots of Eq. 4-11, may be approximated by their asymptotic representations with good accuracy. This technique offers considerable simplicity in the calculation of some complicated summations involved.

The asymptotic representations of the Bessel function for large values of the argument κ_nr may be written as

$$J_1(\kappa_nr) = \frac{\sin(\kappa_nr) - \cos(\kappa_nr)}{\sqrt{\pi\kappa_nr}}\left[1 + \frac{15}{128(\kappa_nr)^2} + \cdots\right] - \frac{\sin(\kappa_nr) + \cos(\kappa_nr)}{\sqrt{\pi\kappa_nr}}\left[\frac{3}{8(\kappa_nr)} + \cdots\right] \tag{4-52}$$

and

$$Y_1(\kappa_nr) = -\frac{\sin(\kappa_nr) + \cos(\kappa_nr)}{\sqrt{\pi\kappa_nr}}\left[1 - \frac{15}{128(\kappa_nr)^2} + \cdots\right] + \frac{\sin(\kappa_nr) - \cos(\kappa_nr)}{\sqrt{\pi\kappa_nr}}\left[\frac{3}{8(\kappa_nr)} + \cdots\right] \tag{4-53}$$

Furthermore the Bessel functions of zero order may be written in asymptotic form

$$J_0(\kappa_n r) = \frac{\cos(\kappa_n r) + \sin(\kappa_n r)}{\sqrt{\pi \kappa_n r}}\left[1 + \frac{1}{128(\kappa_n r)^2} + \cdots\right] + \frac{\sin(\kappa_n r) - \cos(\kappa_n r)}{\sqrt{\pi \kappa_n r}}\left[\frac{1}{8\kappa_n r} + \cdots\right] \tag{4-54}$$

$$Y_0(\kappa_n r) = \frac{\sin(\kappa_n r) - \cos(\kappa_n r)}{\sqrt{\pi \kappa_n r}}\left[1 + \frac{1}{128(\kappa_n r)^2} + \cdots\right] - \frac{\cos(\kappa_n r) + \sin(\kappa_n r)}{\sqrt{\pi \kappa_n r}}\left[\frac{1}{8\kappa_n r} + \cdots\right] \tag{4-55}$$

In general it will be sufficient to retain only the terms in the brackets that are independent of the argument. On this basis the function that occurs in the solution for the radial velocity may be written

$$J_1(\kappa_n r) Y_1(\kappa_n r_h) - J_1(\kappa_n r_h) Y_1(\kappa_n r) \approx \frac{-2}{\pi \kappa_n \sqrt{r r_h}} \sin \kappa_n (r - r_h) \tag{4-56}$$

The characteristic values for the asymptotic calculations are then, according to Eq. 4-11, given by

$$\frac{-2}{\pi \kappa_n \sqrt{r_t r_h}} \sin \kappa_n (r_t - r_h) = 0 \tag{4-57}$$

which then establishes them as

$$\kappa_n \approx \frac{n\pi}{r_t - r_h} \tag{4-58}$$

The accuracy with which these characteristic values check the true roots of Eq. 4-11 is a reasonable measure of the accuracy that can be expected from the asymptotic solution. Consequently the values of the characteristic numbers $\kappa_n r_h$ are tabulated below for the first 10 roots calculated numerically and estimated from the asymptotic result for the radius ratio $r_h/r_t = 0.6$. For all practical purposes the asymptotic values are identical

n	$\kappa_n r_h$	$\frac{n\pi}{r_t/r_h - 1}$
1	4.758051	4.712389
2	9.448369	9.424778
3	14.182998	14.137167
4	18.861456	18.849556
5	23.571475	23.561945
6	28.282281	28.274334
7	32.993535	32.986723
8	37.705076	37.699113
9	42.416800	42.411502
10	47.128604	47.123891

with the exact characteristic numbers for $n > 4$. Furthermore the error involved in employing the asymptotic values entirely is not very large and usually well within the error made in applying the throughflow analysis in the first place. In this case the functions for the radial velocity distribution may be written

$$J_1(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_1(\kappa_n r) \approx \frac{-2\left(\frac{r_t}{r_h} - 1\right)}{\sqrt{\frac{r}{r_t}}} \sin n\pi \left(\frac{r - r_h}{r_t - r_h}\right) \tag{4-59}$$

Using an argument that is exactly similar, the functions appropriate to the axial velocity perturbation may be written asymptotically as

$$J_0(\kappa_n r)Y_1(\kappa_n r_h) - J_1(\kappa_n r_h)Y_0(\kappa_n r) = \frac{-2\left(\frac{r_t}{r_h} - 1\right)}{\sqrt{\frac{r}{r_h}}} \cos n\pi \left(\frac{r - r_h}{r_t - r_h}\right) \tag{4-60}$$

The use of these asymptotic functions simplifies the numerical calculation of a given problem in two ways. First, the actual determination of the characteristic values is made simpler and the evaluation of the functions is trivial. Second, the infinite series involved may often be summed directly to give a closed result. For example, the function $G(r, z; \alpha, \beta)$ appearing within the integral giving the radial velocity for a blade row of finite chord, Eq. 4-23 and 4-24, may be written asymptotically as

$$G(r, z; \alpha, \beta) \approx \frac{1}{8\pi r}\sqrt{\frac{\alpha r}{r_h}} \ln \left[\frac{\cosh \pi\left(\frac{z - \beta}{r_t - r_h}\right) + \cos \pi\left(\frac{\alpha - r}{r_t - r_h}\right)}{\cosh \pi\left(\frac{z - \beta}{r_t - r_h}\right) + \cos \pi\left(\frac{\alpha - r_h}{r_t - r_h} + \frac{r - r_h}{r_t - r_h}\right)}\right] \tag{4-61}$$

This demonstrates the fact, which was intuitively clear before, that the function $G(r, z; \alpha, \beta)$ has a logarithmic singularity at $\alpha = r, z = \beta$. This is equivalent to the statement that an annular vortex ring at the point α, β induces infinite radial velocities in the immediate neighborhood of the ring. In a similar manner the function $K(r, z; \alpha, \beta)$ involved in calculation of the axial velocity perturbation, Eq. 4-29 and 4-30, yields the asymptotic expression

$$K(r, z; \alpha, \beta) \approx \frac{1}{8\pi r}\sqrt{\frac{\alpha r}{r_h}} \ln \left\{\left[\cosh \left(\frac{z - \beta}{r_t - r_h}\right) + \cos \pi\left(\frac{\alpha - r}{r_t - r_h}\right)\right]\left[\cosh \pi\left(\frac{z - \beta}{r_t - r_h}\right) + \cos \pi\left(\frac{\alpha - r_h}{r_t - r_h} + \frac{r - r_h}{r_t - r_h}\right)\right]\right\} \tag{4-62}$$

In exactly the same fashion as the radial velocity, the axial velocity perturbation induced by an annular vortex element is logarithmically infinite in the neighborhood of the element. Eq. 4-61 and 4-62 are extremely useful in problems that require consideration of a blade row of finite chord. Using these results the integrands of Eq. 4-23 and 4-27 may be evaluated algebraically and the integration carried out by conventional numerical means, insuring the appropriate treatment of the logarithmic singularity.

For many examples the series involved in a calculation cannot be summed directly but they can be placed in a convenient closed form for calculation. Consider, for instance, the actuator disk problem which was solved previously; the radial velocity is given by Eq. 4-9 and 4-10, and the coefficients are given explicitly by Eq. 4-21 et seq. Formal substitution of the coefficient into Eq. 4-9 and 4-10 yields a series which converges as $1/n^2$ but which cannot be summed directly into a closed form. The first derivative of this series can be summed easily, however, so that the radial velocity component appears as a definite integral. After carrying out these processes and some simplification, the radial velocity becomes

$$u^{(1)} = -\frac{a\omega}{\pi}\sqrt{\frac{r_h}{r}}\left(\frac{r_h}{r_t} + \sqrt{\frac{r_t}{r_h}}\right)\int_{r_h}^{r_t} \ln\left[1 + e^{\frac{-2\pi|z|}{r_t - r_h}} + 2e^{\frac{-\pi|z|}{r_t - r_h}}\cos\pi\left(\frac{\xi - r_h}{r_t - r_h}\right)\right] d\xi \quad (4\text{-}63)$$

The integrand is regular within the domain of interest and elementary numerical integration may be used to determine the radial velocity at any point of the field. The asymptotic representations for the Bessel functions permit convenient computational procedures of this sort in many instances.

C,5. Solutions for Variable Hub and Tip Radii.

It is nearly always the case in actual turbomachines that hub radius, tip radius or both vary along the direction of flow. In many instances, for example in the early stages of a multistage compressor, the hub and tip radii vary so much that the change of radius through a given blade row must be taken into account in determining the throughflow. It is true that a significant portion of this change in radius may be to compensate for density changes in the fluid which, in the present section, are being neglected. However, the general flow pattern is not so greatly changed by this compressibility effect but that the incompressible flow patterns give most of the necessary information.

In order that the conditions of linearization be satisfied it is necessary that the distortion of the hub and tip contours induce only first order radial velocities. Since the axial velocity is of zeroeth order this restriction requires that the slopes of the surfaces be of order ϵ at the most. Then it is clear that the problem of throughflow with variable hub and tip radii may be stated

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}\left[1 - \frac{1}{rw^{(0)2}}\frac{\partial (rv^{(0)})^2}{\partial r}\right] + \frac{\partial^2 u^{(1)}}{\partial z^2} = \phi(r, z) \quad (5\text{-}1)$$

with the boundary conditions

$$u^{(1)}(r, -\infty) = u^{(1)}(r, \infty) = 0 \quad (5\text{-}2)$$

and

$$\frac{u^{(1)}}{w^{(0)}} = \frac{dr_t}{dz} \quad \text{on} \quad r = r_t \quad (5\text{-}3)$$

$$\frac{u^{(1)}}{w^{(0)}} = \frac{dr_h}{dz} \quad \text{on} \quad r = r_h \quad (5\text{-}4)$$

where the quantities $r_t(z)$ and $r_h(z)$ refer to the variable hub and tip radii given by Eq. 3-34 and 3-35. Since the problem is linear it can be solved in two parts. The solution of the inhomogeneous differential equation with homogeneous boundary conditions, $u^{(1)}(r_h, z) = u^{(1)}(r_t, z) = 0$, is the first part and this was discussed in the section covering constant hub and tip radii. The second part is the solution of the homogeneous differential equation:

$$\frac{\partial^2 u^{(1)}}{\partial^2 r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} - \frac{1}{r^2}\left[1 - \frac{1}{rw^{(0)2}}\frac{\partial (rv^{(0)})^2}{\partial r}\right]u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = 0 \quad (5\text{-}5)$$

with inhomogeneous boundary conditions given in Eq. 5-2, 5-3, and 5-4. The sum of these two solutions satisfies both the inhomogeneous right-hand side of Eq. 5-1 and the inhomogeneous boundary conditions.

Throughflow with variable hub radius. Consider the particular instance where $dr_t/dz = 0$ and $dr_h/dz = f_h(z)$ in Eq. 5-3, 5-4 above. Assume furthermore that the zeroeth order tangential velocity $v^{(0)}$ is either zero or of the vortex type. The most convenient technique of solution is the Fourier transform with respect to z, the axial direction. Denote

$$U(r, k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u^{(1)}(r, z)e^{-ikz}dz \quad (5\text{-}6)$$

the Fourier transform of the radial velocity component. Then the homogeneous differential equation (Eq. 5-5) becomes

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \left(k^2 + \frac{1}{r^2}\right)U = 0 \quad (5\text{-}7)$$

while the boundary conditions are

$$U(r_t, k) = 0 \quad (5\text{-}8)$$

$$U(r_h, k) = \frac{w^{(0)}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f_h(z)e^{-ikz}dz \equiv w^{(0)}F_h(k) \quad (5\text{-}9)$$

Now the differential equation (Eq. 5-7) has solutions $J_1(ikr)$ and $Y_1(ikr)$. By employing the Fourier inversion theorem it is not difficult to show that the radial velocity induced by the hub radius variation is

$$\frac{u^{(1)}(r, z)}{w^{(0)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_h(k) \frac{J_1(ikr)Y_1(ikr_t) - J_1(ikr_t)Y_1(ikr)}{J_1(ikr_h)Y(ikr_t) - J_1(ikr_t)Y_1(ikr_h)} e^{ikz} dk \tag{5-10}$$

Explicit solutions for particular forms of the hub shape are obtained by evaluation of the integral in Eq. 5-10, most generally this involves contour integration.

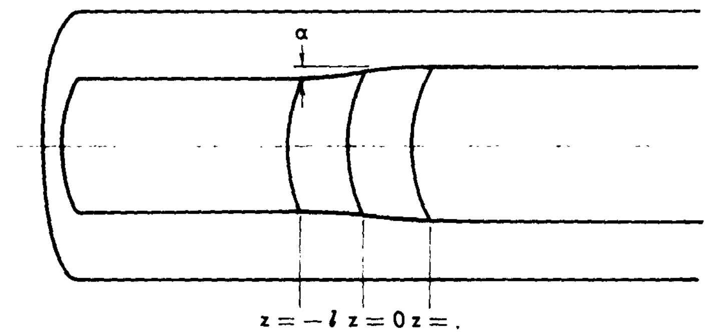


Fig. C,5a. Axial turbomachine with variable hub radius.

Consider a particular example where the hub slope is given by

$$\begin{aligned} f_h(z) &= \frac{\pi r_h}{2l} \cos \frac{\pi z}{2l}; \qquad |z| \leqq l \\ f_h(z) &= 0; \qquad\qquad\qquad |z| \geqq l \end{aligned} \tag{5-11}$$

as shown in Fig. C,5a. Then the Fourier transform of the boundary shape yields

$$F_h(k) = \left(\frac{r_h}{l}\right) \frac{1}{l} \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\cos kl}{(\pi/2l)^2 - k^2} \tag{5-12}$$

In anticipation of employing the technique of contour integration to evaluate the radial velocity component, it is worthwhile to note that the integral in Eq. 5-10 may be rewritten in terms of the Hankel functions $H_1^{(1)}(ikr)$ and $H_1^{(2)}(ikr)$ of the first and second kind to give

$$\frac{u^{(1)}(r, z)}{w^{(0)}} = \frac{r_h}{l} \frac{\pi}{4l} \int_{-\infty}^{\infty} \frac{\cos kl}{(\pi/2l)^2 - k^2} \left[\frac{H_1^{(1)}(ikr)H_1^{(2)}(ikr_t) - H_1^{(1)}(ikr_t)H_1^{(2)}(ikr)}{H_1^{(1)}(ikr_h)H_1^{(2)}(ikr_t) - H_1^{(1)}(ikr_t)H_1^{(2)}(ikr_h)} \right] e^{ikz} dk \tag{5-13}$$

But if the complex variable σ is denoted $\sigma = \zeta + ik$, then the complex integral to consider is

$$\frac{-r_h}{l}\frac{\pi i}{4l}\int \frac{\cosh \sigma l}{(\pi/2l)^2 + \sigma^2}\left[\frac{H_1^{(1)}(\sigma r)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r)}{H_1^{(1)}(\sigma r_h)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r_h)}\right] e^{\sigma z} d\sigma \tag{5-14}$$

and it is necessary to evaluate this over the imaginary axis. Now the integral has poles at $\sigma = \pm i\pi/2l$ and at the roots of the denominator

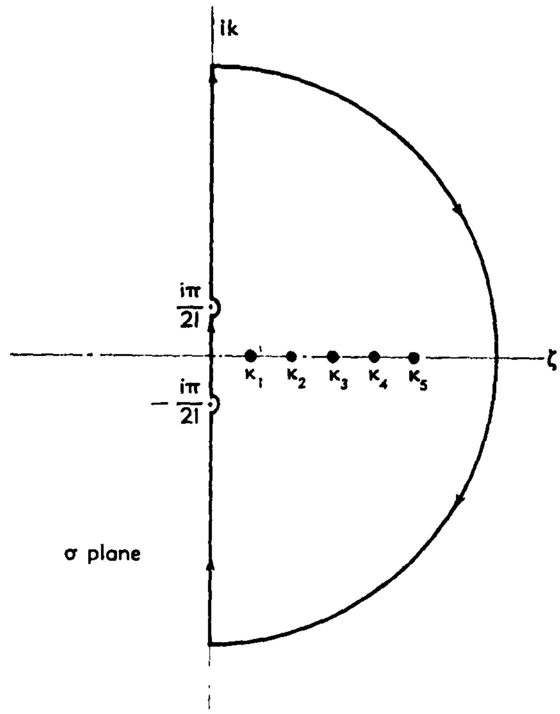


Fig. C,5b. Contour for evaluation of radial velocity integral, $z \leq -l$.

$H_1^{(1)}(\sigma r_h)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r_h) = 0$. There are an infinite number of these roots on the positive real axis; they are in fact just the roots κ_n of

$$J_1(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_1(\kappa_n r_h) = 0$$

which occurred previously in the problem for constant hub and tip radii. Using the contour of Fig. C,5b indented about the two singularities on the imaginary axis, the integral along the imaginary axis may be evaluated in terms of the residues on the real axis provided the integral along the arc vanishes. This condition is assured when $z < -l$, that is, upstream of the wall curvature. Evaluation of the residues gives the value of the definite integral in Eq. 5-13 to be

$$\frac{u^{(1)}}{w^{(0)}} = -\frac{r_h}{l}\frac{\pi^2}{2l}$$
$$\sum_1^\infty \frac{\cosh \kappa_n l}{(\pi/2l)^2 + \kappa_n^2} e^{\kappa_n z} \left\{ \frac{J_1(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_1(\kappa_n r)}{\begin{array}{l} r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] \\ + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)] \end{array}} \right\} \tag{5-15}$$

In exactly the same manner, with the exception that the arc of the contour is drawn in the left half plane, the solution for $z > l$ may be written

$$\frac{u^{(1)}}{w^{(0)}} = -\frac{r_h}{l}\frac{\pi^2}{2l}$$
$$\sum_1^\infty \frac{\cosh \kappa_n l}{(\pi/2l)^2 + \kappa_n^2} e^{-\kappa_n z} \left\{ \frac{J_1(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_1(\kappa_n r)}{\begin{array}{l} r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] \\ + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)] \end{array}} \right\} \tag{5-16}$$

To find an appropriate solution in the region $-l \leqq z \leqq l$ requires somewhat more consideration. The complex integral

$$I_1 = -\frac{r_h}{l}\frac{\pi i}{4l}$$
$$\int \frac{(\cosh \sigma l + \sinh \sigma l)}{(\pi/2l)^2 + \sigma^2} \left[\frac{H_1^{(1)}(\sigma r)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r)}{H_1^{(1)}(\sigma r_h)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r_h)} \right] e^{\sigma z} \tag{5-17}$$

converges when evaluated over a contour consisting of the entire imaginary axis and large semicircle in the right half plane, provided that $z > -l$. Similarly the integral

$$I_2 = -\frac{r_h}{l}\frac{\pi i}{4l}$$
$$\int \frac{(\cosh \sigma l - \sinh \sigma l)}{(\pi/2l)^2 + \sigma^2} \left[\frac{H_1^{(1)}(\sigma r)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r)}{H_1^{(1)}(\sigma r_h)H_1^{(2)}(\sigma r_t) - H_1^{(1)}(\sigma r_t)H_1^{(2)}(\sigma r_h)} \right] e^{\sigma z} \tag{5-18}$$

converges when evaluated over a contour with its large semicircle in the left half plane provided $z < l$. These two integrals have the common region of convergence $-l \leqq z \leqq l$ and half of their sum is equal to the integral required, that is $u^{(1)}/w^{(0)} = \frac{1}{2}(I_1 + I_2)$. Carrying out the evaluation in detail gives the radial velocity distribution

$$\frac{u^{(1)}(r, z)}{w^{(0)}} = \frac{\pi r_h}{2l} \left[\frac{I_1\left(\frac{\pi r}{2l}\right) K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right) K_1\left(\frac{\pi r}{2l}\right)}{I_1\left(\frac{\pi r_h}{2l}\right) K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right) K_1\left(\frac{\pi r_h}{2l}\right)} \right] \cos\frac{\pi z}{2l}$$
$$- \frac{r_h}{l}\frac{\pi^2}{2l} \sum_1^\infty \frac{\cosh \kappa_n z}{(\pi/2l)^2 + \kappa_n^2} e^{-\kappa_n l} \left\{ \frac{J_1(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_1(\kappa_n r)}{\begin{array}{l} r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] \\ + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)] \end{array}} \right\} \tag{5-19}$$

It is an elementary matter to develop the corresponding relationship for the axial velocity perturbations. From the continuity equation it follows that

$$\frac{w^{(1)}}{w^{(0)}} = -\int_{-\infty}^{z} \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{ru^{(1)}}{w^{(0)}}\right) dz \tag{5-20}$$

where the appropriate representation of $u^{(1)}/w^{(0)}$ must be employed in each of the three regions. Assuming the axial velocity undistorted far upstream of the hub curvature, the axial velocity perturbation in the region $-\infty \leqq z \leqq -l$ may be written, using Eq. 5-15, as

$$\frac{w^{(1)}}{w^{(0)}} = \frac{r_h}{l}\frac{\pi^2}{2l} \sum_{1}^{\infty} \frac{\cosh \kappa_n l}{(\pi/2l)^2 + \kappa_n^2} e^{\kappa_n z} \left\{ \frac{J_0(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r)}{r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)]} \right\}$$

$$-\infty \leqq z \leqq -l \tag{5-21}$$

Similarly

$$\frac{w^{(1)}}{w^{(0)}} = \frac{r_h}{l}\frac{\pi}{2}\left[\frac{I_0\left(\frac{\pi r}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_0\left(\frac{\pi r}{2l}\right)}{I_1\left(\frac{\pi r_h}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_1\left(\frac{\pi r_h}{2l}\right)}\right]\left(1 + \sin\frac{\pi z}{2l}\right)$$

$$+ \frac{r_h}{l}\frac{\pi^2}{2l}\sum_{1}^{\infty}\frac{1 + e^{-\kappa_n l}\sinh(\kappa_n z)}{(\pi/2l)^2 + \kappa_n^2}\left\{\frac{J_0(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r)}{r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)]}\right\}$$

$$-l \leqq z \leqq l \tag{5-22}$$

holds in the region where the actual wall curvature exists. Finally, downstream of the wall distortion,

$$\frac{w^{(1)}}{w^{(0)}} = \frac{r_h}{l}\pi\left[\frac{I_0\left(\frac{\pi r}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_0\left(\frac{\pi r}{2l}\right)}{I_1\left(\frac{\pi r_h}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_1\left(\frac{\pi r}{2l}\right)}\right]$$

$$+ \frac{r_h}{l}\frac{\pi^2}{2l}\sum_{1}^{\infty}\frac{2 - e^{-\kappa_n z}\cosh \kappa_n l}{(\pi/2l)^2 + \kappa_n^2}\left\{\frac{J_0(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r)}{r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] + r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)]}\right\}$$

$$l \leqq z \tag{5-23}$$

Up to the present time, exact calculations utilizing these results have not been carried out, partly for the reason that an adequate approximation to this solution may be made. This will be indicated in a later article where an example will be presented.

Throughflow with variable tip radius. When it is the tip radius rather than the root radius that varies along the direction of flow, the procedure for calculating the flow is changed only slightly. If now $dr_t/dz = f_t(z)$ and $dr_h/dz = 0$, Eq. 5-9 is replaced by

$$U(r, k) = \frac{w^{(0)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f_t(z) e^{-ikz} dz = w^{(0)} F_t(k) \tag{5-24}$$

Similarly the Fourier inversion gives, corresponding to Eq. 5-10,

$$\frac{u^{(1)}(r, z)}{w^{(0)}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_t(k) \left[\frac{J_1(ikr_h) Y_1(ikr) - J_1(ikr) Y_1(ikr_h)}{J_1(ikr_h) Y_1(ikr_t) - J_1(ikr_t) Y_1(ikr_h)} \right] e^{ikz} dk \tag{5-25}$$

Note here that only the numerator of the bracketed term is changed from its value in Eq. 5-10 where the hub radius is varying. Hence, in the ensuing contour integration, the only differences that appear from the previous case are modifications of the numerator. If, for example, the slope of the tip contour is given as

$$\begin{aligned} f_t(z) &= \frac{\pi r_t}{2l} \cos \frac{\pi z}{2l}; \qquad |z| \leqq l \\ f_t(z) &= 0; \qquad |z| \geqq l \end{aligned} \tag{5-26}$$

while the hub diameter remains constant, the appropriate solutions for the radial and axial velocities can be obtained by substituting the expressions,

$$\begin{gathered} J_1(\kappa_n r_h) Y_1(\kappa_n r) - J_1(\kappa_n r) Y_1(\kappa_n r_h) \\ J_1(\kappa_n r_h) Y_0(\kappa_n r) - J_0(\kappa_n r) Y_1(\kappa_n r_h) \\ I_1\left(\frac{\pi r_h}{2l}\right) K_0\left(\frac{\pi r}{2l}\right) - I_0\left(\frac{\pi r}{2l}\right) K_1\left(\frac{\pi r_h}{2l}\right) \end{gathered} \tag{5-27}$$

respectively, in place of the expressions,

$$\begin{gathered} J_1(\kappa_n r) Y_1(\kappa_n r_t) - J_1(\kappa_n r_t) Y_1(\kappa_n r) \\ J_0(\kappa_n r) Y_1(\kappa_n r_t) - J_1(\kappa_n r_t) Y_0(\kappa_n r) \\ I_0\left(\frac{\pi r}{2l}\right) K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right) K_0\left(\frac{\pi r}{2l}\right) \end{gathered} \tag{5-28}$$

at the appropriate positions in the numerators of Eq. 5-15, 5-16, 5-19, 5-21, 5-22, and 5-23.

It is clear also that, since the problem is a linear one, the situation where both hub and tip radii vary can be treated by superposition of the perturbations caused by hub variation only and tip variation only.

C,6. Effects of Upstream Conditions and Compressibility. The throughflow with strong tangential velocities imposed far upstream was treated earlier only when the tangential velocity distribution was that corresponding to either a vortex ($v^{(0)} \sim 1/r$) or a solid body rotation ($v^{(0)} \sim r$). In practical cases the tangential velocity is often of a more complex nature and cannot be treated easily within the framework of the theory developed so far, since the variable coefficients that occur in the partial differential equation for radial velocity give rise to special functions whose type depends upon the velocity distribution. This is quite unsatisfactory for the development of a useful theory. Furthermore it is not unusual that a strong variation of the axial velocity $w^{(0)}(r)$ should be imposed far upstream of the blade row. This introduces a similar situation to the general tangential velocity distribution. Both of these upstream velocity distributions have in common the property that, in the absence of compressibility effects, they influence only the coefficient $q(r)$ in Eq. 3-28 and leave the coefficient $p(r) = 1/r$.

On the other hand, when the tangential and axial Mach number values are of significant size to merit consideration, the coefficient $p(r)$ and the coefficient of $\partial^2 u^{(1)}/\partial z^2$ are involved so that the problem becomes a good bit more complicated. Since it is almost never possible to consider the Mach number values M_θ and M_z as independent of the radius, it is necessary to deal with the situation where all of the coefficients in Eq. 3-28 are, to some extent, functions of the radius.

It is clear that anything to be done must, moreover, be of an approximate nature since an exact treatment would prove intractable. The method to be developed will utilize an asymptotic solution of the ordinary differential equation

$$\frac{d^2U}{dr^2} + p(r)\frac{dU}{dr} + [\kappa_n^2(1 - M_z^2) + q(r)]U = 0$$

that arises from Eq. 3-28. The method, first applied to this problem by Rannie [*22*], rests on the fact that the first characteristic value κ_1 is large in some sense. Physically this requires that the hub ratio r_h/r_t should not be too small, since $\kappa_1 \cong \pi/(r_t - r_h)$. Under these conditions the problem may be treated by the well-developed procedures described by Erdelyi [*23*].

Variation of tangential and axial velocities. Suppose that flow may be considered incompressible but that large and arbitrary variations with radius occur in the tangential velocity $v^{(0)}(r)$ and axial velocity $w^{(0)}(r)$ prescribed far upstream of the blade row. Then, writing the coefficients in detail, Eq. 3-28 becomes

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + \frac{1}{r}\frac{\partial u^{(1)}}{\partial r} + \left[-\frac{1}{r^2} + \frac{2v^{(0)}}{r^2 w^{(0)2}}\frac{d(rv^{(0)})}{dr} - \frac{r}{w^{(0)}}\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right)\right]u^{(1)} + \frac{\partial^2 u^{(1)}}{\partial z^2} = \varphi(r, z) \quad (6\text{-}1)$$

If solutions are of the form $e^{\pm\kappa_n x}U_n(r)$, as was appropriate for the blade row in a channel of constant hub and tip radius, the function $U_n(r)$ satisfies the equation

$$\frac{d^2U_n}{dr^2} + \frac{1}{r}\frac{dU_n}{dr} + \left[\kappa_n^2 - \frac{1}{r^2} - \frac{r}{w^{(0)}}\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right) + \frac{2v^{(0)}}{r^2w^{(0)2}}\frac{d(rv^{(0)})}{dr}\right]U_n = 0 \tag{6-2}$$

where the characteristic value κ_n^2 is to be considered large in comparison with other quantities in the problem. For convenience, introduce the new dependent variable $V_n = \sqrt{r}\,U_n$; the differential equation transforms into

$$\frac{d^2V_n}{dr^2} + \left[\kappa_n^2 - \frac{3}{4r^2} - \frac{r}{w^{(0)}}\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right) + \frac{2v^{(0)}}{r^2w^{(0)2}}\frac{d(rv^{(0)})}{dr}\right]V_n = 0 \tag{6-3}$$

so that the first order differential is suppressed. For any distributions $v^{(0)}(r)$ and $w^{(0)}(r)$ the coefficient of $V_n(r)$ is a certain known function $F(r)$ which may be written explicitly as

$$F(r) = -\frac{3}{4r^2} - \frac{r}{w^{(0)}}\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right) + \frac{2v^{(0)}}{r^2w^{(0)2}}\frac{d(rv^{(0)})}{dr} \tag{6-4}$$

The differential equation to be solved is then

$$\frac{d^2V_n}{dr^2} + [\kappa_n^2 + F(r)]V_n = 0 \tag{6-5}$$

which in the case of constant hub and tip radii must satisfy the conditions

$$V_n(r_t) = V_n(r_h) = 0 \tag{6-6}$$

Now this may be considered as a differential equation with large parameter κ_n and solved by an asymptotic method. Choose the solution to be of the form,

$$V_n(r) = e^{\alpha\kappa_n r + \sum_0^\infty \varphi_j(r)\kappa_n^{-j}} \tag{6-7}$$

that is, the argument of the exponential function is expanded in inverse powers of the large parameter κ_n. Substitution into the differential equation (Eq. 6-5) gives the formal result

$$\sum_0^\infty \kappa_n^{-j}\varphi_j'' + (\alpha\kappa_n)^2 + 2\alpha\kappa_n\sum_0^\infty \kappa_n^{-j}\varphi_j' + \left(\sum_0^\infty \kappa_n^{-j}\varphi_j'\right)^2 + \kappa_n^2 + F(r) = 0 \tag{6-8}$$

and since κ_n sets the order of magnitude, the coefficients of each power of κ_n must vanish identically. The coefficient of κ_n^2, the greatest power of κ_n, gives $\alpha^2 + 1 = 0$, or

$$\alpha = \pm i \tag{6-9}$$

Similarly the coefficient of κ_n may be written as $2\alpha\varphi_0' = 0$ so that the coefficient φ_0 is a constant. Proceeding to the power κ_n^0, the coefficient reads

$$\varphi_0'' + 2\alpha\varphi_1' + (\varphi_0')^2 + F(r) = 0 \tag{6-10}$$

Taking account of the constancy of φ_0, this gives the function φ_1 as

$$\varphi_1 = -\frac{1}{2\alpha}\int_a^r F(r)dr \tag{6-11}$$

where the lower limit of the integral is left arbitrary for the moment. Further coefficients may readily be obtained in a similar manner, but those obtained so far are adequate to demonstrate the technique. To the order of $1/\kappa_n$ then, the functions V_n become

$$V_n(r) = e^{\pm i\left[\kappa_n + \varphi_0 + \frac{1}{2\kappa_n}\int_a^r F(r)dr\right]}$$

Since it is required that the V_n vanish at hub and tip radii, choose the linear combination

$$V_n(r) = \sin\left[\kappa_n(r - r_h) + \frac{1}{2\kappa_n}\int_{r_h}^r F(r)dr\right] \tag{6-12}$$

where values of φ_0 and a have been selected to make the V_n vanish at $r = r_h$. The large parameter κ_n must be chosen so that

$$\kappa_n(r_t - r_h) + \frac{1}{2\kappa_n}\int_{r_h}^{r_t} F(r)dr = n\pi$$

To the same order of accuracy, then, the value of κ_n may be calculated as

$$\kappa_n = \frac{n\pi}{r_t - r_h}\left[1 - \frac{r_t - r_h}{2n^2\pi^2}\int_{r_h}^{r_t} F(r)dr\right] \tag{6-13}$$

Transforming back now to the original dependent variable and denoting $\frac{1}{2}\int_{r_h}^{r} F(r)dr \equiv g(r)$ the appropriate characteristic functions of Eq. 6-1 may be written

$$\frac{1}{\sqrt{r}}\sin\left\{n\pi\frac{r - r_h}{r_t - r_h} + \frac{r_t - r_h}{n\pi}\left[g(r) - \frac{r - r_h}{r_t - r_h}g(r_t)\right]\right\} e^{\pm\frac{n\pi x}{r_t - r_h}\left[1 - \frac{r_t - r_h}{n^2\pi^2}g(r_t)\right]} \tag{6-14}$$

where terms of higher order than $[(r_t - r_h)/n\pi]^2$ have been deleted. It is now a simple matter to build solutions from these in precisely the manner that the Bessel function combination was used in the development following Eq. 4-9. The functions possess the appropriate orthogonality property since they satisfy a problem of the Sturm-Liouville type.

Before discussing particular solutions, two features of this asymptotic solution should be observed. In the first place, suppose either that the function $F(r) = 0$ or that the expansion is cut off at terms of the order $1/\kappa_n$. Then the functions U_n, asymptotic solutions to Eq. 6-2, are of the form

$$U_n \sim \frac{1}{\sqrt{r}} \sin n\pi \left(\frac{r - r_h}{r_t - r_h}\right) \tag{6-15}$$

and hence are identical, with the exception of an irrelevant constant factor, to the asymptotic form of the Bessel function combination introduced in Eq. 4-59. In fact exactly the same approximation is being made in each case.

In the second place consider the upstream flow where $w^{(0)} = \text{const}$ and $v^{(0)} = bw^{(0)}r/r_t$. Then

$$F(r) = -\frac{3}{4r^2} + \frac{4b^2}{r_t^2}$$

and the integral

$$g(r) \equiv \frac{1}{2}\int_{r_h}^{r}\left(-\frac{3}{4r^2} + \frac{4b^2}{r_t^2}\right)dr = \frac{2b^2(r - r_h)}{r_t^2} - \frac{3}{8}\left(\frac{1}{r_h} - \frac{1}{r}\right) \tag{6-16}$$

The characteristic values are then

$$\kappa_n = \frac{n\pi}{r_t - r_h}\left[1 - \frac{2b^2}{n^2\pi^2}\left(1 - \frac{r_h}{r_t}\right)^2 + \frac{3}{8n^2\pi^2}\frac{r_t}{r_h}\left(1 - \frac{r_h}{r_t}\right)^2\right] \tag{6-17}$$

and the characteristic functions may be written down explicitly as

$$\frac{1}{\sqrt{r}} \sin n\pi \left\{\frac{r - r_h}{r_t - r_h}\left[1 - \frac{3}{8n^2\pi^2}\left(\frac{r_t}{r_h} - 1\right)^2\left(\frac{r_h}{r} - \frac{r_h}{r_t}\right)\right]\right\}$$
$$e^{\pm \frac{n\pi z}{r_t - r_h}\left[1 - \frac{2b^2}{n^2\pi^2}\left(1 - \frac{r_h}{r_t}\right)^2 + \frac{3}{8n^2\pi^2}\frac{r_t}{r_h}\left(1 - \frac{r_h}{r_t}\right)^2\right]} \tag{6-18}$$

Note that the radial dependence of the characteristic functions does not contain any influence of rotation parameter b; the term having coefficient $3/8n^2\pi^2$ arises only in the approximation to the Bessel functions. The upstream solid body rotation only affects the axial dependence (exponential decay) of the solutions. Speaking generally, the influence of an imposed upstream solid body rotation reduces the rate of decay upstream and downstream of a disturbance from

$$\exp\left(\pm\frac{n\pi z}{r_t - r_h}\right) \text{ to } \exp\left\{\pm\frac{n\pi z}{r_t - r_h}\left[1 - \frac{2b^2}{n^2\pi^2}\left(1 - \frac{r_h}{r_t}\right)^2\right]\right\}$$

This is precisely the effect observed when this problem was treated exactly and may be compared with the results following Eq. 4-34. The solutions obtained here are, in fact, the correct asymptotic representations of those obtained previously when b is not large.

Now, having been satisfied that the asymptotic solutions to the differential equation (Eq. 6-1) agree with exact solutions under circumstances where they can be compared, the asymptotic method may be applied to problems where exact solutions can be obtained only numerically. As an example consider the case of a blade row operating downstream of a guide vane that imparts a solid body rotation given again by

$$v^{(0)} = w_0 b \frac{r}{r_t} \tag{6-19}$$

where w_0 is the mean axial velocity. In general, when a homogeneous fluid of constant stagnation enthalpy flows through a guide vane of this type, the axial velocity distribution is disturbed although its mean value remains constant. It will be shown later that far downstream of the guide vane the axial velocity profile is given with reasonable accuracy as

$$w^{(0)} = w_0 \left\{1 + k^2 \left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)\right]^2\right\} \tag{6-20}$$

where actually $k^2 = b^2/2$. It is now straightforward to calculate the functions $F(r)$ and $g(r)$. Substitution into Eq. 6-4 gives, after a little manipulation,

$$F(r) = -\frac{3}{4r^2} + \frac{4b^2}{r_t^2} \frac{1}{[a^2 - k^2(r/r_h)^2]^2} \tag{6-21}$$

where the quantity a^2 has been used to denote

$$a^2 = 1 + \frac{1}{2} k^2 \left[1 + \left(\frac{r_t}{r_h}\right)^2\right] \tag{6-22}$$

The function $g(r) \equiv \frac{1}{2}\int_{r_h}^{r} F(r)dr$ may also be evaluated explicitly as

$$g(r) = \frac{3}{8}\left(\frac{1}{r} - \frac{1}{r_h}\right) + \frac{b^2}{a^2}\frac{r_h}{r_t^2}\left[\frac{\frac{r}{r_h}}{a^2 - \left(k\frac{r}{r_h}\right)^2} - \frac{1}{a^2 - k^2}\right]$$

$$+ \frac{b^2}{2a^3}\frac{r_h/r_t^2}{k} \ln\left[\left(\frac{a + k\frac{r}{r_h}}{a + k}\right)\left(\frac{a - k}{a - k\frac{r}{r_h}}\right)\right] \tag{6-23}$$

Substitution of this result into Eq. 6-14 gives the characteristic function in terms of which may be expanded the asymptotic solution for any blade row far downstream of an entrance guide vane.

The effect on the radial velocity distribution is probably of less interest than is the effect on rate of decay of the disturbance upstream and downstream of the blade row. Referring again to Eq. 6-14, the modification to the decay rate depends only upon the value of $g(r_t)$. From Eq. 6-23 this becomes directly

$$g(r_t) = -\frac{3}{8}\left(\frac{1}{r_h} - \frac{1}{r_t}\right) + \frac{b^2}{a^2}\frac{r_h}{r_t^2}\left[\frac{\frac{r_t}{r_h}}{a^2 - \left(k\frac{r_t}{r_h}\right)^2} - \frac{1}{a^2 - k^2}\right]$$

$$+ \frac{b^2}{2a^3k}\frac{r_h}{r_t^2}\ln\left[\left(\frac{a + k\frac{r_t}{r_h}}{a + k}\right) \Big/ \left(\frac{a - k}{a - k\frac{r_t}{r_h}}\right)\right] \quad (6\text{-}24)$$

As was observed before, the term $-\frac{3}{8}[(1/r_h) - (1/r_t)]$ arises from asymptotic approximation to the original Bessel functions and is generally unimportant. All of the other terms in Eq. 6-24 are positive since $r_t/r_h > 1$. As a result the argument of the exponential function in Eq. 6-14 is decreased; therefore the range of the disturbance upstream and downstream of the blade row is extended over that without initial rotation or axial velocity variation. When the axial velocity is constrained to remain uniform, that is $k = 0$ and $a^2 = 1$, the result reduces to that given in the exponent of Eq. 6-18. On the other hand, when the axial velocity alone is present (i.e. $b = 0$), the resulting function reduces to the first two terms of Eq. 6-24.

This particular result given by Eq. 6-23 and 6-24 for the upstream conditions distorted by a zeroeth order rotational velocity and axial velocity variation will prove very useful in the construction of approximate solutions applicable in compressor design procedures.

Effect of compressibility. Compressibility influences that affect the throughflow directly are usually not of major concern in compressor design. The reason for this is that, although the Mach number relative to the rotating blades may be in the transonic regime, the meridional Mach number, upon which the change from elliptic to hyperbolic equation depends, almost invariably will be in the subsonic region. Thus although compressibility influences in the throughflow do exist and exert their influence upon the flow field, it must be kept in mind that some of the most significant compressibility effects are those associated with the detailed blade geometry.

To illustrate the compressibility influences without becoming unnecessarily enmeshed in detailed calculations, consider a rather over-simplified example. Suppose that both the zeroeth order axial velocity $w^{(0)}$ and sonic velocity $a^{(0)}$ are constant and that the tangential velocity is a solid body rotation, $v^{(0)} = bw^{(0)}r/r_t$. This may be realized physically with no difficulty but, it must be admitted, the example differs somewhat from the circumstances usually encountered downstream of a guide vane. Solutions must be obtained to the homogeneous partial differential equation,

$$\frac{\partial^2 u^{(1)}}{\partial r^2} + p(r)\frac{\partial u^{(1)}}{\partial r} + q(r)u^{(1)} + (1 - M_z^2)\frac{\partial^2 u^{(1)}}{\partial z^2} = 0 \qquad (6\text{-}25)$$

where the functions $p(r)$ and $q(r)$ are given by Eq. 3-29 and 3-30. These functions, as well as the coefficient of $\partial^2 u^{(1)}/\partial z^2$, involve the Mach numbers $M_z = w^{(0)}/a^{(0)}$ and $M_\theta = v^{(0)}/a^{(0)}$. Clearly the axial Mach number M_z is a constant and the tangential Mach number may be expressed

$$M_\theta^2 = \frac{b^2 w^{(0)2}}{a^{(0)2}}\frac{r^2}{r_t^2} = M_z^2 b^2\left(\frac{r}{r_t}\right)^2 \qquad (6\text{-}26)$$

in terms of the axial Mach number and the radius. With these observations, the coefficients in Eq. 6-25 may be written explicitly as

$$p(r) = \frac{1}{r}\left[1 + \gamma M_z^2 b^2\left(\frac{r}{r_t}\right)^2\right] \qquad (6\text{-}27)$$

$$q(r) = -\frac{1}{r^2} + \frac{b^2}{r_t^2}[4 - (4 - \gamma)M_z^2] + (\gamma - 1)\frac{b^4}{r_t^4}M_z^2 r^2 \qquad (6\text{-}28)$$

If solutions of the differential equation (Eq. 6-25) are sought in the form $U_n(r)e^{\pm\kappa_n z}$, the ordinary differential equation for $U_n(r)$ is

$$\frac{d^2U_n}{dr^2} + \frac{1}{r}\left[1 + \gamma M_z^2 b^2\left(\frac{r}{r_t}\right)^2\right]\frac{dU_n}{dr} + \left\{(1 - M_z^2)\kappa_n^2 + 4\frac{b^2}{r_t^2}\left[1 - \left(1 - \frac{\gamma}{4}\right)M_z^2\right] - \frac{1}{r^2} + (\gamma - 1)M_z^2\frac{b^4}{r_t^4}r^2\right\}U_n = 0 \qquad (6\text{-}29)$$

The term involving the first derivatives may be eliminated in the conventional manner; the transformation is simply

$$U_n(r) = \frac{1}{\sqrt{r}}V_n(r)e^{-\frac{\gamma}{4}M_z^2 b^2\left(\frac{r}{r_t}\right)^2} \qquad (6\text{-}30)$$

which reduces, of course, to the transformation used previously when the axial Mach number M_z, and hence the tangential Mach number M_θ, vanish. After substitution of this relation into the differential equation (Eq. 6-29) and carrying out some simplification, the differential equation for $V_n(r)$ becomes

$$\frac{d^2V_n}{dr^2} + \left\{(1 - M_z^2)\kappa_n^2 - \frac{3}{4r^2} + \frac{b^2}{r_t^2}\left[4 - \left(4 + \frac{3\gamma}{2}\right)M_z^2\right] + \frac{b^4}{r_t^4}M_z^2\left[(\gamma - 1) - \frac{\gamma^2 M_z^2}{2}\right]r^2\right\}V_n = 0 \qquad (6\text{-}31)$$

Now calling

$$\bar{\kappa}_n^2 = (1 - M_s^2)\kappa_n^2 \tag{6-32}$$

and

$$F(r) = -\frac{3}{4r^2} + \frac{b^2}{r_t^2}\left[4 - \left(1 + \frac{\gamma}{2}\right)M_s^2\right] + \frac{b^4}{r_t^4}M_s^2\left[(\gamma - 1) - \frac{\gamma^2}{2}M_s^2\right]r^2 \tag{6-33}$$

The above differential equation assumes the form

$$\frac{d^2V_n}{dr^2} + [\bar{\kappa}_n^2 + F(r)]V_n = 0 \tag{6-34}$$

which is exactly the equation treated previously in the paragraphs following Eq. 6-5. The results of that analysis may be carried over and applied directly to the present problem. It is a simple calculation to show then that

$$g(r) = \frac{3}{8}\left(\frac{1}{r} - \frac{1}{r_h}\right) + \frac{2b^2}{r_t^2}\left[1 - \left(1 + \frac{3\gamma}{8}\right)M_s^2\right](r - r_h) + \frac{b^4}{6r_t^4}M_s^2\left[(\gamma - 1) - \frac{\gamma^2 M_s^2}{2}\right](r^3 - r_h^3) \tag{6-35}$$

Treating the problem as before, it is clear that the appropriate solutions of Eq. 6-34 are

$$V_n = \sin\left[\bar{\kappa}_n(r - r_h) + \frac{1}{\bar{\kappa}_n}g(r)\right]$$

so that, to the appropriate order of magnitude,

$$\bar{\kappa}_n = \frac{n\pi}{r_t - r_h}\left[1 - \frac{r_t - r_h}{n^2\pi^2}g(r_t)\right] \tag{6-36}$$

which gives

$$V_n = \sin\left\{n\pi\left(\frac{r - r_h}{r_t - r_h}\right) + \frac{r_t - r_h}{n\pi}\left[g(r) - \frac{r - r_h}{r_t - r_h}g(r_t)\right]\right\}$$

very much as before. The characteristic functions of the problem may then be written, taking account of the transformation given by Eq. 6-30,

$$\frac{1}{\sqrt{r}}\, e^{-\frac{\gamma}{4}M_s^2 b^2\left(\frac{r}{r_t}\right)^2} \sin\left\{n\pi\left(\frac{r - r_h}{r_t - r_h}\right) + \frac{r_t - r_h}{n\pi}\left[g(r) - \frac{r - r_h}{r_t - r_h}g(r_t)\right]\right\} e^{\pm\frac{n\pi}{\sqrt{1-M_s^2}}\frac{z}{r_t - r_h}\left[1 - \frac{r_t - r_h}{n^2\pi^2}g(r_t)\right]} \tag{6-37}$$

The compressibility correction thus enters into the problem in a fairly complicated manner, affecting both the radial variation and the axial decay of the disturbance. The axial decay term is worth examining in some detail. The decay factor is explicitly

$$\exp\left(\pm\frac{n\pi}{\sqrt{1-M_z^2}}\frac{z}{r_t-r_h}\left\{1-\frac{2b^2}{n^2\pi^2}\left(1-\frac{r_h}{r_t}\right)^2\left[1-\left(1+\frac{3\gamma}{8}\right)M_z^2\right]\right.\right.$$
$$-\frac{b^4}{6n^2\pi^2}M_z^2\left(\gamma-1-\frac{\gamma^2M_z^2}{2}\right)\left[1-\left(\frac{r_h}{r_t}\right)^3\right]\left(1-\frac{r_h}{r_t}\right)$$
$$\left.\left.+\frac{3}{8n^2\pi^2}\frac{r_t}{r_h}\left(1-\frac{r_h}{r_t}\right)^2\right\}\right) \tag{6-38}$$

In the absence of any swirl velocity component, that is where $b = 0$, the only correction is the obvious scale contraction in the z direction given by $z/\sqrt{1 - M_z^2}$. This effect tends to crowd all disturbances caused by the blade into a region close to the blade row. It is simply the Prandtl-Glauert contraction and is associated with any compressible flow predominantly in the direction of z and has no peculiar association with the turbomachine problem. This scale contraction was first applied to the axial turbomachine problem by Horlock [*13*]. When a tangential velocity of significant magnitude is present, which is invariably the case in any practical example, other Mach number corrections enter. For example the extension of the disturbed region, as controlled by the term

$$\frac{-2b^2}{n^2\pi^2}\left(1-\frac{r_h}{r_t}\right)^2\left[1-\left(1+\frac{3\gamma}{8}\right)M_z^2\right]$$

may be modified by 15 per cent or more by the compressibility influence $[1 + (3\gamma/8)]M_z^2$. Here again the compressibility effect tends to decrease the extent of the disturbance. Except for rather large values of b (ratio of tangential to axial velocities) corresponding to severe off-design conditions, the effect of axial scale contraction $z/\sqrt{1 - M_z^2}$ dominates the other compressibility influences.

It is perhaps appropriate to reiterate the fact that the compressibility effect discussed here is often not the one of primary interest to compressor or turbine design. Local blade channel choking and the accompanying losses depend strongly upon the local blade geometry and consequently are properly treated under cascade theory. However, it is of extreme importance to have a good calculation of the axially symmetric throughflow in order to know gas velocities and flow angles in the neighborhood of the blades. Thus there exists one compressibility effect in calculating the throughflow and another in determining the flow about a blade cascade placed in that throughflow.

There appears to exist one class of problems in which the effects of compressibility may not be segregated into the two familiar categories of throughflow and blade characteristics. This example is the transonic compressor where the blade tips operate at a relative Mach number larger than unity while the blade roots operate at a subsonic Mach number.

A recent investigation by McCune [17] for a finite number of blades with thickness, but without lift, has shown that the detailed flow about blades at the rotor tip influence in a major way detailed flow about the blades at the root. This result, one which is not an uncommon one in transonic flow fields, may be interpreted as meaning that the roots and tips of the blades may not be considered as independent cascades operating in their particular flow fields. Since this flow may not be divided, even approximately, in the manner of which the flow has been treated in the foregoing work, the transonic compressor represents a strictly three-dimensional problem that must be treated in complete detail. It should be mentioned that although these results were obtained for nonturning blades it is probable that similar, though less drastic, results would be obtained for the turning blade row.

C,7. Approximations to the Throughflow. It is amply clear that while the results of the preceding sections are ideal for investigating the throughflow in detail for one given blade row, the general complexity of the calculations makes it desirable to search out simple and approximate techniques for routine use in turbomachine work, particularly for consideration of multiple blade rows. One approximation [2,3] which has been used to considerable extent is the so-called radial equilibrium theory. Here one considers only the flow far upstream and far downstream of a blade row. At these sections all curvature of the meridional flow has ceased and the flow is determined by equilibrium between the radial pressure gradient and the centripetal acceleration caused by the motion of gas elements about the axis of symmetry. The shortcoming of this analysis is that no description is given of how near to the blade row the modification of axial velocity takes place. Information of this sort is particularly necessary when blade rows are spaced closely enough so that the flow fields of adjacent blade rows overlap. In such cases of mutual interference an approximation to the flow field is required also. It will prove possible to approximate the detailed development of the axial velocity profile with considerable accuracy and with sufficient simplicity for use in the many complex situations. Finally it will prove possible to approximate the effects of variation in hub and tip radii as well as the modification due to compressibility.

Radial equilibrium theory. Of the many possible ways to approach radial equilibrium theory, one will be chosen here [10] that parallels the linearized development given in the previous section. Excluding the blade force normal to the stream surfaces, which need only be considered in very special cases, the tangential vorticity is given by Eq. 2-43 as

$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} = \frac{\rho}{\rho_0} r \left\{ \frac{\partial h}{\partial \psi} - \frac{v}{r} \frac{\partial (rv)}{\partial \psi} - T \frac{\partial s}{\partial \psi} \right\}$$

For the present considerations, neglect the entropy variation. Far downstream of a single blade row, say at $z = \infty$, the radial velocities vanish and the variation in stream function $\delta\psi = -r \frac{\rho}{\rho_0} w\delta r$. Then, since all other quantities become functions of the radius only, it follows that

$$w \frac{dw}{dr} = \frac{dh^0}{dr} - \frac{v}{r} \frac{d}{dr} (rv) \tag{7-1}$$

Now if the total enthalpy h^0 and the angular momentum rv were known functions of the radius at $z = \infty$, it would be a simple matter to integrate this relation. Both the enthalpy and the angular momentum are transported along the stream surfaces and change only upon passage through the blade row. Each of these quantities consists then of two parts, that transported from far upstream ($z = -\infty$) and that imparted by the blade row. Consider for the moment the transport of angular momentum, given by Eq. 2-25. To the order of ϵ, the order of magnitude of the blade forces, Eq. 2-25 possesses a zeroeth order part,

$$w^{(0)} \frac{\partial(rv^{(0)})}{\partial z} = 0 \tag{7-2}$$

and a first order part,

$$u^{(1)} \frac{\partial(rv^{(0)})}{\partial r} + w^{(0)} \frac{\partial(rv^{(1)})}{\partial z} = rf_\theta \tag{7-3}$$

From Eq. 7-2 the initial angular momentum is unchanged along the axial direction so that the quantity $rv^{(0)}$ is known far downstream. The first order contribution $rv^{(1)}$ follows from Eq. 7-3. Integration with respect to z gives simply

$$rv^{(1)} = \frac{1}{w^{(0)}} \int_{-\infty}^{z} rf_\theta dz - \frac{1}{w^{(0)}} \frac{d(rv^{(0)})}{dr} \int_{-\infty}^{z} u^{(1)} dz \tag{7-4}$$

The first integral is just the angular momentum imparted by the blade row. Since the tangential force f_θ vanishes outside of the blade row, it is sufficient to extend the integration only across the blade chord and

$$\frac{1}{w^{(0)}} \int_{-\infty}^{z} rf_\theta dz = \frac{1}{w^{(0)}} \int_{-c/2}^{c/2} rf_\theta dz \equiv r\Delta v^{(1)} \tag{7-5}$$

for any point downstream of the blade row. This change in angular momentum across the blade row, $r\Delta v^{(1)}$, will be considered prescribed. The second integral in Eq. 7-4 represents the perturbation of angular momentum due to the radial transport of the angular momentum from far upstream. It is clear, in fact, that $\int_{-\infty}^{z} (u^{(1)}/w^{(0)})dz$ is equal to the radial distance that a stream surface moves in passing from far upstream to a point z. The angular momentum is transported along these stream surfaces so that the angular momentum which exists at a distance z downstream and radius r is not $rv^{(0)}$ but rather

$$v^{(0)}\left(r - \epsilon \int_{-\infty}^{z} \frac{u^{(1)}}{w^{(0)}} dz\right)$$

To first order, then, an angular momentum perturbation,

$$-\frac{d(rv^{(1)})}{dr} \epsilon \int_{-\infty}^{z} \frac{u^{(1)}}{w^{(0)}} dz$$

exists due to the radial transport of initial angular momentum. This is the origin of the last term of Eq. 7-4.

It is not really convenient to express the radial transport in terms of the radial perturbation velocity $u^{(1)}$ inasmuch as the axial perturbation velocity is the quantity of main interest. The radial and axial velocity components are related through the continuity equation, but the integral $\int_{-\infty}^{z} (u^{(1)}/w^{(0)})dz$ which is required can be given in terms of $w^{(1)}$ by some direct physical reasoning. Since the stream surface bounds a constant mass flow of fluid between the hub and local radius of the stream surface, the surface must be displaced to accommodate a variation in mass flow $\epsilon \int_{r_h}^{r} \rho w^{(1)} 2\pi r dr$ where $w^{(1)}$ is the axial velocity perturbation from the flow when the stream surface was cylindrical. This mass flow variation is compensated by decreasing the radius of the stream surface by an amount $-\Delta r$ through which the mass flow is, to the first order, $-\rho w^{(0)} 2\pi r \Delta r$. The mass flow integral and this last expression must be equal for the stream surface to bound a constant mass flow. Consequently

$$\Delta r \equiv \int_{-\infty}^{z} \frac{u^{(1)}}{w^{(0)}} dz = -\frac{1}{rw^{(0)}} \int_{r_h}^{r} w^{(1)} r dr \tag{7-6}$$

With the results of Eq. 7-5 and 7-6 it is possible to express the perturbation angular momentum, far downstream of the blade row, as

$$rv^{(1)} = r\Delta v^{(1)} + \frac{1}{rw^{(0)}} \frac{d(rv^{(0)})}{dr} \int_{r_h}^{r} w^{(1)} r dr \tag{7-7}$$

The same consideration as given in Eq. 7-2 et seq. gives a similar expression for the stagnation enthalpy perturbation far downstream:

$$h^{(1)} = \omega r \Delta v^{(1)} + \frac{1}{rw^{(0)}} \frac{dh^{(0)}}{dr} \int_{r_h}^{r} w^{(1)} r dr \tag{7-8}$$

where the fact has been used that the enthalpy perturbation $\Delta h^{(1)}$ across the blade row is equal to $\omega r \Delta v^{(1)}$, as may be deduced from Eq. 2-36.

Now Eq. 7-7 and 7-8 give the information necessary to express Eq. 7-1, for the axial velocity, in terms of known quantities. It is appropriate to split this equation into its zeroeth and first order parts. The zeroeth order part is a trivial statement of the conservation of zeroeth order enthalpy, angular momentum, and axial velocity profile along cylindrical surfaces. The first order part is the one of interest in computing the perturbation in the axial velocity profile. It may be written

$$w^{(0)}\frac{dw^{(1)}}{dr}+\left[\frac{1}{r^4w^{(0)}}\frac{d}{dr}(rv^{(0)})^2-\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right)\right]\int_{r_h}^{r}w^{(1)}rdr \qquad (7\text{-}9)$$

where now the expressions for $rv^{(1)}$ and $h^{(1)}$ may be entered from Eq. 7-7 and 7-8. After some simplification this gives

$$w^{(0)}\frac{dw^{(1)}}{dr}+\left[\frac{1}{r^4w^{(0)}}\frac{d}{dr}(rv^{(0)})^2-\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right)\right]\int_{r_h}^{r}w^{(1)}rdr$$
$$=-\left(\frac{v^{(0)}}{r}-\omega\right)\frac{d}{dr}(r\Delta v^{(1)})-\frac{1}{r^2}\frac{d}{dr}(rv^{(0)})(r\Delta v^{(1)}) \qquad (7\text{-}10)$$

where the zeroeth order part of Eq. 7-1 has been employed in the reduction.

In the particular case that the upstream axial velocity is uniform and the upstream tangential velocity vanishes, then

$$\frac{dw^{(2)}}{dr}=\frac{\omega}{w^{(0)}}\frac{d}{dr}(r\Delta v^{(1)}) \qquad (7\text{-}11)$$

which may be integrated directly. As before, it appears that a stationary blade row produces no distortion, to the first order, when there is no initial distortion far upstream. Thus the guide vane requires special treatment, and a second order analysis of this particular problem gives

$$\frac{dw^{(1)}}{dr}=-\frac{r\Delta v^{(1)}}{r^2w^{(0)}}\frac{d}{dr}(r\Delta v^{(1)}) \qquad (7\text{-}12)$$

It is justified to utilize this second order analysis in the case of the entrance vane because the angular momentum increment $r\Delta v^{(1)}$ across it is generally much larger than that across other blade rows. Hence the axial velocity disturbance $w^{(2)}$ calculated in this particular case may be sufficiently large that it becomes the upstream distortion $w^{(0)}$ for succeeding blade rows.

Returning to the general case of Eq. 7-10, this integro-differential equation may be converted to a second order differential equation by differentiation and subsequent elimination of the integral. To do this it is convenient to denote

$$\frac{1}{w^{(0)}}\left[\frac{1}{r^4w^{(0)}}\frac{d}{dr}(rv^{(0)})^2-\frac{d}{dr}\left(\frac{1}{r}\frac{dw^{(0)}}{dr}\right)\right]=h(r) \qquad (7\text{-}13)$$

$$\frac{1}{w^{(0)}}\left[\left(\frac{v^{(0)}}{r}-\omega\right)\frac{d}{dr}(r\Delta v^{(1)})+\frac{1}{r^2}\frac{d}{dr}(rv^{(0)})r\Delta v^{(1)}\right]=k(r) \qquad (7\text{-}14)$$

so that Eq. 7-10 becomes

$$\frac{dw^{(1)}}{dr} + h(r)\int_{r_h}^{r} w^{(1)}rdr = -k(r) \tag{7-15}$$

The corresponding second order differential equation is

$$\frac{d^2w^{(1)}}{dr^2} - \frac{d}{dr}[\ln h(r)]\frac{dw^{(1)}}{dr} + rh(r)w^{(1)} = k(r)\frac{d}{dr}\ln\left[\frac{h(r)}{k(r)}\right] \tag{7-16}$$

The solution to this must satisfy two conditions. One is simply that the perturbation axial velocity does not modify the mass throughflow; thus

$$\int_{r_h}^{r_t} w^{(1)}rdr = 0 \tag{7-17}$$

The second condition is that the solution satisfy Eq. 7-15, which it did before differentiation. Since the solution satisfies Eq. 7-16 by definition, it need in addition satisfy Eq. 7-15 at only one point. It is convenient to choose this point as $r = r_t$ inasmuch as here the integral vanishes. Therefore it is sufficient that

$$\frac{dw^{(1)}}{dr}(r_t) = -k(r_t) \tag{7-18}$$

Eq. 7-16 and the conditions given by Eq. 7-17 and 7-18 complete the mathematical problem for the throughflow far downstream of an arbitrary blade row. It should be noted here that there is no difficulty in accounting for compressibility in this calculation. The density then enters through the continuity relation into Eq. 7-6 et seq. and in the condition of Eq. 7-7.

In general Eq. 7-16 is difficult to integrate and numerical analysis is required. In fact, for numerical analysis, it is often more convenient to work with Eq. 7-15. An iterative solution is applicable, for example, where the first approximation is obtained by integrating $-k(r)$ and neglecting the integral $\int_{r_h}^{r} w^{(1)}rdr$. This first order solution is then used in the integral to obtain a more accurate second approximation. Some examples will now help to illustrate the calculation of the throughflow far downstream from a blade row.

Flow downstream of an entrance vane. Consider an entrance vane system that imparts a solid body rotation

$$\Delta v^{(1)} = bw_0\frac{r}{r_t} \tag{7-19}$$

Upstream of the guide vane the flow is axial with a uniform velocity $w^{(0)} \equiv w_0$. Let the station far downstream of the guide vane be denoted by subscript 1. Then according to Eq. 7-12

$$\frac{dw^{(2)}}{dr} = -b^2w_0\left(\frac{2r}{r_t^2}\right) \tag{7-20}$$

Integrating this and satisfying the integral condition given by Eq. 7-17 yields, after a very elementary calculation,

$$w_1^{(2)} = \frac{b^2 w_0}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right] \tag{7-21}$$

A typical profile corresponding to this calculation is shown in Fig. C,7a.

It should be noted here that the small quantity, denoted ϵ in the formal perturbation analysis, is the constant b which gives the ratio of

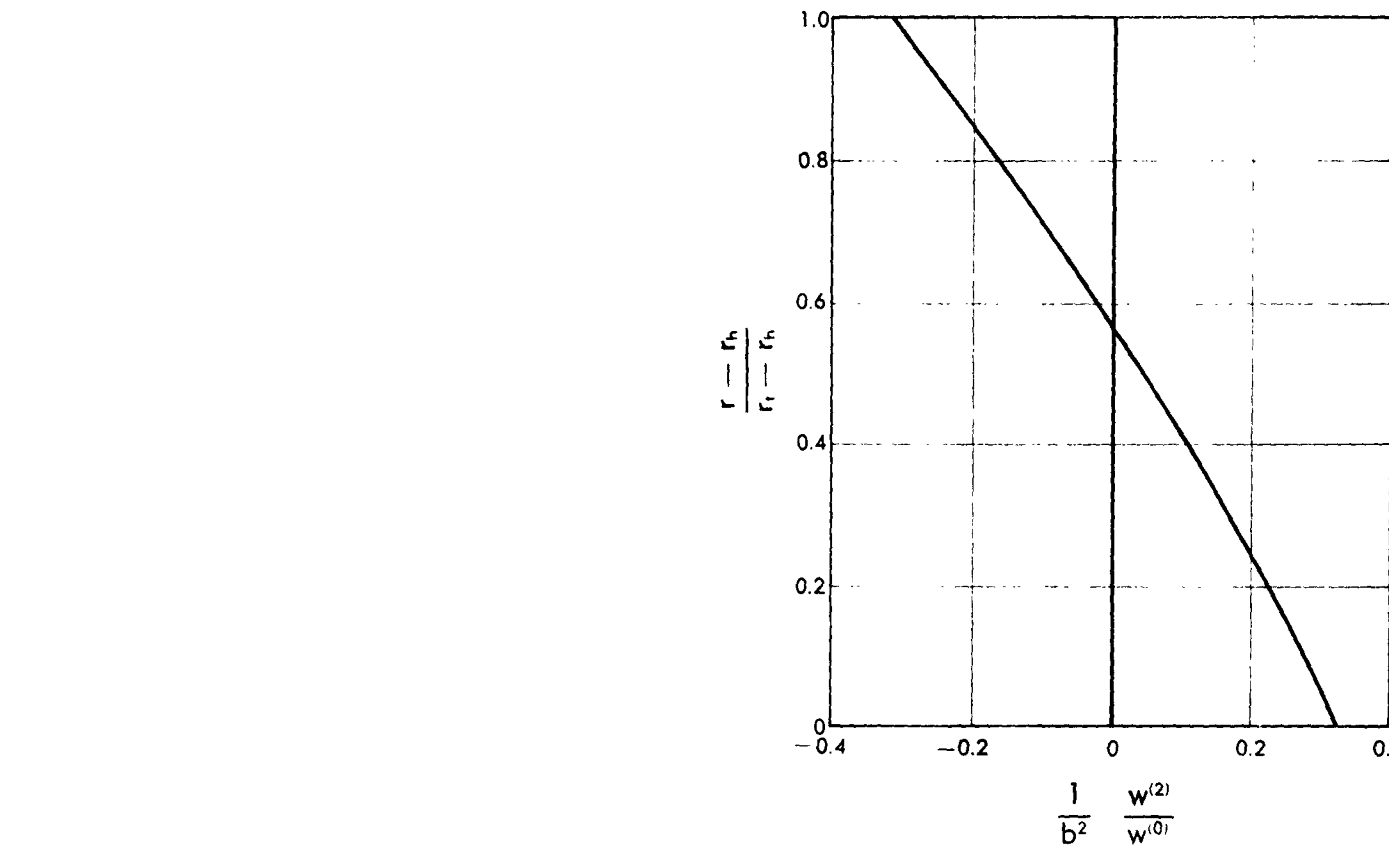


Fig. C,7a. Axial velocity profile far downstream of entrance guide vane, $v^{(1)} = bw^{(0)}r/r_t$, $r_h/r_t = 0.6$.

the tangential velocity at the blade tip to the undisturbed axial velocity. In guide vanes this need not be small and may, in fact, be about unity. In spite of this, the procedure gives a very satisfactory approximation to the flow far downstream.

Rotor far downstream from entrance vane. With the results that have been developed it is possible to continue discussion of throughflow in an axial compressor, considering the blade rows to be spaced sufficiently far apart that the development of each axial velocity profile is complete before the next blade row is encountered. Consider the rotor, then, which imparts a vortex rotation

$$\Delta v_2^{(1)} = aw_0 \frac{r_t}{r} \tag{7-22}$$

and hence imparts the same enthalpy rise at all radii. So far as the rotor is concerned the upstream conditions are not uniform but have a zeroeth order axial and tangential velocity distortion,

$$v_1^{(0)} = bw_0 \frac{r}{r_t} \tag{7-23}$$

$$w_1^{(0)} = w_0 \left\{1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]\right\}$$

A word of explanation should be added here inasmuch as these were considered perturbations in the guide vane calculation and now appear as zeroeth order quantities. The fact is that the tangential velocity disturbance is not strictly a perturbation, although it was treated as such to the second order, and may not be omitted from calculations of the ensuing stages. The axial velocity disturbance is generally of much smaller magnitude, depending upon the hub ratio, and it may be neglected at appropriate steps of the work. Then according to Eq. 7-13 and 7-14 the functions $h(r)$ and $k(r)$ become

$$h(r) = \frac{4b^2}{r}\left\{1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]\right\}^{-2} \tag{7-24}$$

$$k(r) = \frac{2abw_0}{r}\left\{1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]\right\}^{-1} \tag{7-25}$$

In these expressions the axial velocity disturbance far upstream of the rotor is characterized by the presence of the term

$$\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]$$

Its influence is usually a small one, becoming less significant as the hub ratio r_h/r_t approaches unity. Neglecting these terms, Eq. 7-16 for the axial velocity perturbation far downstream of the rotor blade row may be written

$$\frac{d^2w_2^{(1)}}{dr^2} + \frac{1}{r}\frac{dw_2^{(1)}}{dr} + \left(\frac{2b}{r_t}\right)^2 w_2^{(1)} = 0 \tag{7-26}$$

where the subscript 2 has been employed to denote conditions downstream of the rotor. The solution consists of Bessel functions of order zero, and may be written as the linear combination

$$w_2^{(1)} = AJ_0\left(2b\frac{r}{r_t}\right) + BY_0\left(2b\frac{r}{r_t}\right) \tag{7-27}$$

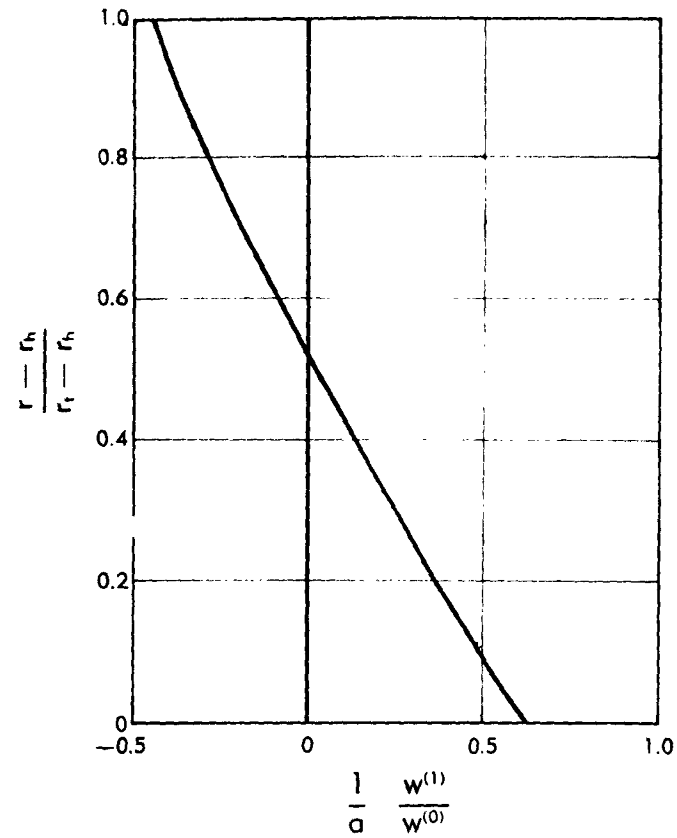


Fig. C,7b. Axial velocity far downstream of rotor blade row, $v^{(1)} = aw^{(0)}r_t/r$, $r_h/r_t = 0.6$.

where the constants are to be determined from the conditions given by Eq. 7-17 and 7-18. Direct substitution gives, for Eq. 7-18,

$$AJ_1(2b) + BY_1(2b) = w_0 a \tag{7-28}$$

On the other hand, the integral condition (Eq. 7-17) gives

$$A\left[J_1(2b) - \frac{r_h}{r_t} J_1\left(2b\frac{r_h}{r_t}\right)\right] + B\left[Y_1(2b) - \frac{r_h}{r_t} Y_1\left(2b\frac{r_h}{r_t}\right)\right] \tag{7-29}$$

Some calculation then gives the solution for the axial velocity perturbation downstream of the rotor, where the quantity $2b/r_t$ is denoted κ

$$w_2^{(1)} = aw_0 \left\{ \frac{\left[Y_1(\kappa r_h) - \frac{r_t}{r_h} Y_1(\kappa r_t)\right] J_0(\kappa r) + \left[J_1(\kappa r_h) - \frac{r_t}{r_h} J_1(\kappa r_t)\right] Y_0(\kappa r)}{J_1(\kappa r_t)Y_1(\kappa r_h) - J_1(\kappa r_h)Y_1(\kappa r_t)} \right\} \tag{7-30}$$

This particular distortion is shown in Fig. C,7b. The complete axial velocity profile downstream of the rotor is made up of the uniform axial velocity w_0, the disturbance $w_1^{(2)}$ (Eq. 7-21) caused by the inlet guide vane, and the additional distortion (Eq. 7-30) caused by the rotor.

Additional rotors and stators. In the normal multistage turbomachine the succeeding rotor and stator blade rows, respectively, add and subtract rotational velocity components of about the same magnitude. It will be assumed in the present example that tangential velocity changes across rotor and stator blade rows are equal and opposite and that the tangential velocity change induced always has the vortex distribution. Thus across

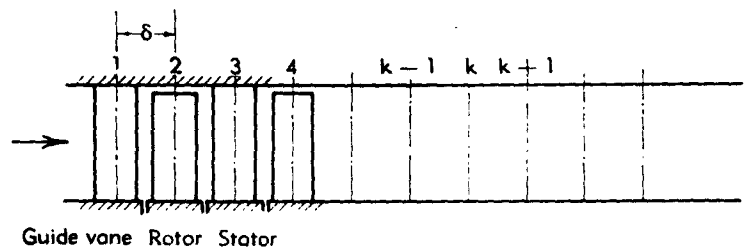


Fig. C,7c. Schematic diagram of multistage axial compressor showing blade row numbering.

the stator following the rotor discussed above the change in tangential velocity is

$$\Delta v_3^{(1)} = -a w_0 \frac{r_t}{r} \tag{7-31}$$

and since the changes in axial and tangential velocity across the first rotor are definitely of first order, the axial and tangential velocities "far upstream" of the first stator are identical with those "far upstream" of the first rotor. Consequently the axial velocity distortion across the first stator is exactly the negative of that across the first rotor, that is

$$w_3^{(1)} = -w_2^{(1)} \tag{7-32}$$

It is simple to see, then, what the complete axial velocity profile becomes downstream of each blade row of the axial compressor, a schematic diagram of which is shown in Fig. C,7c. Upstream of the guide vane the axial velocity is uniform and equal to w_0. Between the guide vane and rotor, which are supposed here to be far apart as compared with the distance required for the change in axial velocity to take place, the axial velocity is just

$$w_1(r) = w_0 \left\{1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]\right\} \tag{7-33}$$

Between the following rotor and guide vane, the total axial velocity is

$$\frac{w_2(r)}{w_0} = 1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]$$
$$+ a\left\{\frac{\left[Y_1(\kappa r_h) - \frac{r_t}{r_h}Y_1(\kappa r_t)\right]J_0(\kappa r) + \left[J_1(\kappa r_h) - \frac{r_t}{r_h}J_1(\kappa r_t)\right]Y_0(\kappa r)}{J_1(\kappa r_t)Y_1(\kappa r_h) - J_1(\kappa r_h)Y_1(\kappa r_t)}\right\} \tag{7-34}$$

where the rotor has introduced the additional vortex motion indicated by Eq. 7-22. Downstream of the ensuing stator row the vortex motion is removed and the axial velocity profile reverts to that downstream of the guide vane,

$$w_3(r) = w_1(r) \tag{7-35}$$

Similarly, downstream of the second rotor, the axial velocity profile becomes exactly what it was downstream of the first rotor, that is

$$w_4(r) = w_2(r) \tag{7-36}$$

The velocity profile continues to oscillate in this manner, attaining one definite axial velocity profile downstream of each rotor, a different one downstream of each stator.

Development of axial velocity profiles. The basic deficiency of the radial equilibrium analysis given in the preceding subarticle is that, although it gives the equilibrium axial velocity profiles far upstream and downstream of the blade row, it fails to describe the manner in which this change takes place. There is no clue in radial equilibrium theory as to whether the transition takes place very close to the blade row in question or whether it is spread out over many blade chord lengths. This question becomes of extreme importance in a multistage compressor where the change in axial velocity caused by one blade row may not be complete before the next blade row is reached. Hence the radial equilibrium velocity profiles are never actually attained and the entrance and exit angles to the blade row must be designed to axial velocity profiles that depend upon the flow fields of the neighboring blade rows. This phenomenon is known as the mutual interference between blade rows.

The methods developed for detailed calculation of the flow field about a blade row of definite chord or an actuator disk may be employed to study the development of the axial velocity profile. The general complexity of the method prohibits its application to multistage compressor problems and actually makes an undesirable amount of labor out of the routine design of a single blade row. Therefore some method for approximating the development of the axial velocity profile is needed, particularly for use in multistage turbomachines [*9,10,11,12*].

Toward this end it should be remembered that, so far as the actuator disk is concerned, the axial velocity profile is known at three stations. Far upstream of the blade row the perturbations associated with the blade row in question vanish. Far downstream of the blade row, the axial velocity perturbation is calculated from the radial equilibrium theory according to Eq. 7-11, 7-12, or 7-16. From detailed calculations carried out previously, in fact from the observation that the radial velocity is symmetrical about the actuator disk, it follows that the change in axial velocity is half completed at the plane of the actuator disk. If $w^{(0)}(r)$ is the axial velocity far upstream of the blade row and $w^{(1)}(r, z)$ is the perturbation axial velocity, it is appropriate to denote $w^{(1)}(r, \infty)$ the perturbation axial velocity far downstream of the blade row calculated from radial equilibrium theory. Hence, making the origin $z = 0$ in the plane of the actuator disk and taking $w^{(1)}(r, \infty)$ as being known from radial equilibrium theory, the axial velocity at $z = -\infty$ is

$$w^{(0)}(r) \tag{7-37}$$

at $z = 0$

$$w^{(0)}(r) + \tfrac{1}{2}w^{(1)}(r, \infty) \tag{7-38}$$

and at $z = \infty$

$$w^{(0)}(r) + w^{(1)}(r, \infty) \tag{7-39}$$

Now it is clear upon inspection of the complete solution for axial velocity profile, such as that given in Eq. 4-27 and 4-28, that each Fourier-Bessel component of the axial velocity disturbance decays upstream of the actuator disk by an exponential factor $e^{-\kappa_n|z|}$ where κ_n is the appropriate characteristic value for the Fourier-Bessel component considered. For an approximation it is sufficiently accurate to assume that all components decay at the same rate or, in other words, that the entire axial velocity perturbation decays as some exponential function $e^{\lambda\frac{z}{r_t - r_h}}$ when $z < 0$. Referring again to Eq. 7-37 and 7-38 it is appropriate to write

$$w = w^{(0)}(r) + \tfrac{1}{2}w^{(1)}(r, \infty)e^{\lambda\frac{z}{r_t - r_h}} \tag{7-40}$$

which satisfies the requirement at both the actuator disk and far upstream. For the remainder of the axial velocity profile to develop in a manner symmetrical with the development upstream, it is necessary that

$$w = w^{(0)} + w^{(1)}(r, \infty)\left(1 - \tfrac{1}{2}e^{-\lambda\frac{z}{r_t - r_h}}\right) \tag{7-41}$$

This approximation clearly satisfies both conditions given by Eq. 7-38 and 7-39.

The problem yet remains to choose the constant λ in such a manner as to make this approximation a good one. Clearly, if the perturbation axial velocity $w^{(1)}(r, \infty)$ consisted of only the first Fourier-Bessel term in the expansion (cf. Eq. 4-27 and 4-28) the value of λ would be very nearly π since the asymptotic value of $\kappa_1 = \pi/(r_t - r_h)$. Although axial velocity disturbances encountered in practice invariably include many more than the first harmonic component, the first harmonic does constitute a large portion of the disturbance. Consequently, for all but the region very near the actuator disk, the first harmonic component dominates the flow pattern and $\lambda = \pi$ is a very reasonable value to use. Calculations were carried out by Marble [10] to determine what value of λ provided the least mean square error from an exact solution. For a blade row imparting a solid body rotation it was found that this optimum λ had the value 3.25 for a hub ratio $r_h/r_t = 0.5$ and rapidly approached the value π as r_h/r_t increased toward unity. Hence in consideration of all other approximations connected with the throughflow theory, it seems completely adequate to choose $\lambda = \pi$, or in general the first characteristic value.

Using this exponential approximation to the detailed development of the throughflow it is a simple matter to write down the solution for flow through an entrance vane imparting a solid body rotation. Referring to the example following Eq. 7-22 for which radial equilibrium flow was computed,

$$w^{(0)}(r) = w^{(0)} \qquad \text{and} \qquad w^{(1)}(r, \infty) = \frac{w_0 b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]$$

If again this entrance vane is represented by an actuator disk located at $z = 0$, then the exponential approximation to the axial velocity field becomes

$$w(r, z) = w_0 \left\{1 + \frac{b^2}{4}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r_h}{r_t}\right)^2\right] e^{\pi\frac{z}{r_t - r_h}}\right\} \qquad z \to 0 \quad (7\text{-}42)$$

and

$$w(r, z) = w_0 \left\{1 + \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right]\left(1 - \frac{1}{2}e^{-\pi\frac{z}{r_t - r_h}}\right)\right\} \quad (7\text{-}43)$$

The development of this velocity profile is shown in Fig. C,7d for various distances upstream and downstream of the actuator disk. Comparison with the exact (linearized) solution indicates negligible error of this approximation except in the immediate vicinity of the actuator disk, that is for $|z/(r_t - r_h)| < 0.1$. The whole actuator disk concept breaks down in this region anyway because this distance is usually well within the blade chord. Consequently the exponential approximation, Eq. 7-42 and 7-43, may be used wherever the actuator disk itself is valid.

The general utility of the exponential approximation may be extended by superposition of actuator disk solutions, using exponential approximation, to make up a blade of finite chord. Actually this is an unnecessarily complex procedure because a quite adequate approximation to a blade row of finite chord may be obtained with two, or at most three, actuator disks situated at appropriate axial positions along the blade chord.

Using the exponential approximation it is possible to discuss the problems of mutual interference between blade rows in a multistage axial

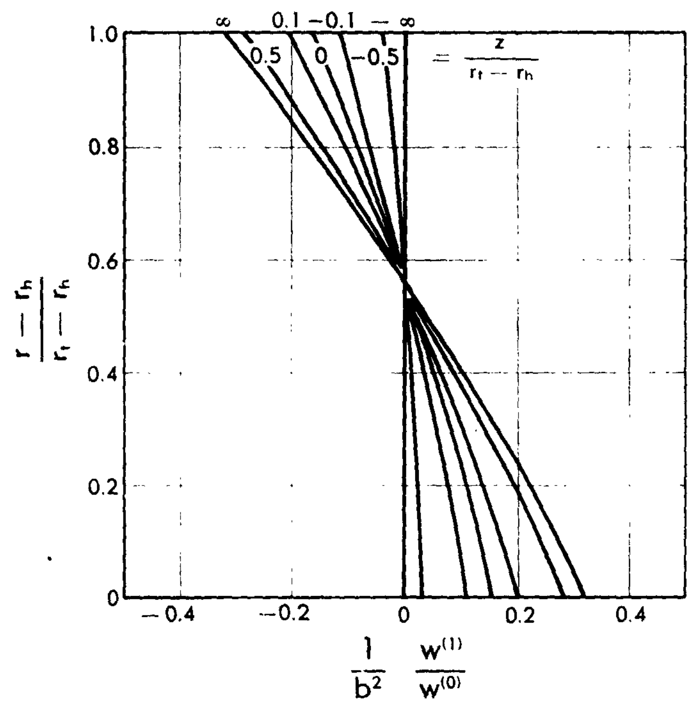


Fig. C,7d. Development of axial velocity profile upstream and downstream of entrance guide vane, calculated using exponential approximation, $v^{(1)} = bw^{(0)}r/r_t$, $r_h/r_t = 0.60$.

compressor. Consider, for example, the multistage compressor for which the radial equilibrium axial velocity profiles were determined in the work following Eq. 7-19. Let the individual blade rows be separated by a distance δ so that the entrance guide vane is represented by an actuator disk at $z = 0$, the first rotor by an actuator disk at $z = \delta$, the first stator at $z = 2\delta$, and so on, as indicated in Fig. C,7c. The change in axial velocity imposed by each of these blade rows is known and, hence, to determine the complete flow field it is merely necessary to superpose the effects of individual blade rows in the appropriate manner.

It is convenient to denote the radial equilibrium axial velocity perturbation across the guide vane by

$$\Delta w_z^{(2)} \equiv w_0 \frac{b^2}{2}\left[1 + \left(\frac{r_h}{r_t}\right)^2 - 2\left(\frac{r}{r_t}\right)^2\right] \tag{7-44}$$

and the magnitude of radial equilibrium axial velocity perturbation across each of the other rows by

$$\Delta w^{(1)} \equiv$$

$$w_0 a \left\{\frac{\left[Y_1(\kappa r_h) - \frac{r_h}{r_t} Y_1(\kappa r_t)\right] J_0(\kappa r) + \left[J_1(\kappa r_h) - \frac{r_t}{r_h} J_1(\kappa r_t)\right] Y_0(\kappa r)}{J_1(\kappa r_t) Y_1(\kappa r_h) - J_1(\kappa r_h) Y_1(\kappa r_t)}\right\} \tag{7-45}$$

The perturbation given by Eq. 7-45 will appear with a positive sign for a rotor blade row and with a negative sign for a stator. The perturbation axial velocity flow field for the guide vane is then

$$\Delta_1 w(r, z) = \begin{Bmatrix} \Delta w_z^{(2)} \cdot \frac{1}{2} e^{\pi \frac{z}{r_t - r_h}} & z \leqq 0 \\ \Delta w_z^{(2)}(1 - \frac{1}{2} e^{-\frac{z}{r_t - r_h}}) & z \geqq 0 \end{Bmatrix} \tag{7-46}$$

while that for the first rotor is just

$$\Delta_2 w(r, z) = \begin{Bmatrix} \Delta w^{(1)} \cdot \frac{1}{2} e^{\pi\left(\frac{z-\delta}{r_t - r_h}\right)} & z \leqq \delta \\ \Delta w^{(1)}(1 - \frac{1}{2} e^{-\pi\left(\frac{z-\delta}{r_t - r_h}\right)}) & z \geqq \delta \end{Bmatrix} \tag{7-47}$$

The first stator imparts a disturbance which is the negative of that due to the first rotor and is displaced a distance δ farther downstream. A similar procedure will give the flow field for any of the subsequent rotors and stators. In general the perturbation associated with any of the blade rows downstream of the stator may be written

$$\Delta_n w(r, z) = \begin{Bmatrix} (-1)^n \Delta w^{(1)} \cdot \frac{1}{2} e^{\pi\left[\frac{z-(n-1)\delta}{r_t - r_h}\right]} & z \leqq (n-1)\delta \\ (-1)^n \Delta w^{(1)}(1 - \frac{1}{2} e^{-\pi\left[\frac{z-(n-1)\delta}{r_t - r_h}\right]}) & z \geqq (n-1)\delta \end{Bmatrix} \tag{7-48}$$

It is now simply a matter of summation over the perturbations given by Eq. 7-46 and 7-48 to find the axial velocity field at any point of the multistage axial turbomachine. Suppose that there are a total of N stages and hence N rotors and N stators following an entrance guide vane. The various blade rows are then numbered from 1 through $2N + 1$. If the axial velocity distribution is desired in the section between the kth and $(k + 1)$th blade rows, it may be written down using the appropriate solutions upstream and downstream of this section. When k is unity or larger, the complete axial velocity may be written

$$w(r, z) = w_0 + \Delta w_s^{(2)}(1 - \tfrac{1}{2}e^{-\pi\frac{z}{r_t - r_h}})$$
$$+ \sum_{n=2}^{k} (-1)^n \Delta w^{(1)}(1 - \tfrac{1}{2}e^{-\pi\left[\frac{z-(n-1)\delta}{r_t - r_h}\right]}) + \sum_{k+1}^{2N+1} (-1)^n \Delta w^{(1)} \cdot \tfrac{1}{2}e^{\pi\left[\frac{z-(n-1)\delta}{r_t - r_h}\right]}$$
$$k\delta \leqq z \leqq (k+1)\delta \quad (7\text{-}49)$$

For the velocity distribution upstream of the guide vane, i.e. the particular circumstance where $k = 0$, the solution may be written

$$w(r, z) = w_0 + \Delta w_s^{(2)} \cdot \tfrac{1}{2}e^{\pi\frac{z}{r_t - r_h}}$$
$$+ \sum_{n=2}^{2N+1} (-1)^n \Delta w^{(1)} \cdot \tfrac{1}{2}e^{\pi\left(\frac{z-(n-1)\delta}{r_t - r_h}\right)} \qquad z \leqq 0 \quad (7\text{-}50)$$

Using Eq. 7-49 and 7-50 the flow field through the entire turbomachine may be found by computing the individual sections corresponding to particular values of k. It will be of particular interest to determine the behavior of the flow field in the vicinity of the first few blade rows to observe just how many stages are involved in the transition region until a steady repeating flow pattern is achieved. Then it will be of interest to calculate what this steady repeating pattern is for a stage deeply embedded in the compressor. From this latter it will be possible to observe the effects of mutual interference of blade rows.

The transition from undisturbed flow far upstream to the periodic pattern developed through the stages far aft of the guide vane is observed most easily by studying the shape of a stream surface which lies at the middle of the annulus far upstream of the guide vane. The radial velocity distribution is required to compute streamline shape, for the local slope is just $u^{(1)}(r, z)/w_0$ where $u^{(1)}(r, z)$ is the perturbation radial velocity. Since the continuity equation relates the radial and axial velocities as

$$\frac{1}{r}\frac{\partial ru}{\partial r} = -\frac{\partial w}{\partial z}$$

it is an elementary integration to obtain the radial velocity distribution from Eq. 7-49 and 7-50. In this manner it follows that, to the accuracy of the exponential approximation,

$$u(r, z) = \sum_{1}^{2N+1} \Delta_n u_{max} e^{-\pi\left|\frac{z-(n-1)\delta}{r_t - r_h}\right|} \quad (7\text{-}51)$$

where $\Delta_n u_{max}$ is calculated from the $\Delta_n w^{(1)}$ as

$$\Delta_n u_{max} = \frac{-\pi}{2\frac{r}{r_t}\left(1 - \frac{r_h}{r_t}\right)} \int_{r_h}^{r} \Delta_n w^{(1)}(r) r\,dr \quad (7\text{-}52)$$

Now since $u(r, z)/w_0$ is the flow angle, the shape of the desired stream surface is easily constructed from Eq. 7-51 either graphically or by direct integration with respect to z. The values of $\Delta_n w^{(1)}(r)$ are known and the result is shown in Fig. C,7e. The scale of the vertical motion of the streamline is grossly magnified to show the effects. It is seen that the periodic flow is established surprisingly quickly, essentially by the time the second rotor is reached. The transient state caused by the inlet vanes and the first rotor is of very short duration, partly because the distortion due to the rotor is of the same sense as that due to the guide vanes and assists in completing rapidly the distortion due to the guide vanes.

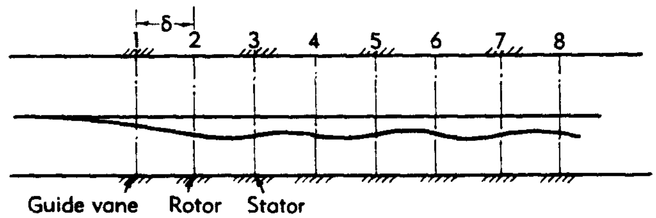


Fig. C,7e. Deflection of middle stream surface near entrance of multistage axial compressor, vertical scale magnified.

It may be deduced from these results that most of the blade rows in a multistage turbomachine are deeply imbedded, in the sense that they do not feel the influence of the compressor ends. Consequently the second point mentioned above, the steady repeating pattern for a deeply embedded stage, takes on considerable significance. To investigate this flow pattern it is convenient to transform the axis to the plane of the kth blade row, that is to introduce a variable ζ such that $z = \zeta + (k - 1)\delta$. Eq. 7-49 then becomes

$$
\begin{aligned}
w(r, \zeta) = w_0 &+ \Delta w_z^{(2)}\left(1 - e^{-\pi\left[\frac{\zeta + (k-1)\delta}{r_t - r_h}\right]}\right) \\
&+ \sum_{n=2}^{k} (-1)^n w^{(1)}\left(1 - \tfrac{1}{2}e^{-\pi\left[\frac{\zeta + (k-n)\delta}{r_t - r_h}\right]}\right) \\
&+ \sum_{k+1}^{2N+1} (-1)^n \Delta w^{(1)} \tfrac{1}{2} e^{\pi\left[\frac{\zeta + (k-n)\delta}{r_t - r_h}\right]} \qquad \text{(7-53)}
\end{aligned}
$$

In anticipation of making an infinite number of blade rows both upstream and downstream of the case in question, it is appropriate to introduce new indices for each of the two sums in Eq. 7-53. In the sum $\sum_{n=2}^{k}$ call $j = k - n$ and in the sum $\sum_{k+1}^{2N+1}$ call $j = n - k - 1$, then with a little rearrangement,

$$w(r, \zeta) = w_0 + \Delta w_s^{(2)}\left(1 - \tfrac{1}{2}e^{-\pi\left[\frac{\zeta+(k+1)\delta}{r_t-r_h}\right]}\right) + \tfrac{1}{2}[1 + (-1)^k]\Delta w^{(1)}$$
$$+ (-1)^{k-1}\frac{\Delta w^{(1)}}{2}\left(e^{-\pi\zeta}\sum_0^{k-2}(-1)^j e^{-j\pi\delta} + e^{\pi\zeta}e^{-\pi\delta}\sum_0^{2N-k}(-1)^j e^{-j\pi\delta}\right) \quad (7\text{-}54)$$

where the factor $\frac{1}{2}[1 + (-1)^k]$ is unity when k is even (the section under consideration is just downstream of a rotor) or zero when k is odd (the section is just downstream of a stator). Now if the stage is deeply embedded, one may consider that there are an infinite number of similar stages upstream and downstream of the section being investigated. Thus both k and $2N - k$ approach ∞ so that the coefficient of $\Delta w_s^{(2)}$ becomes unity and both series may be summed simply. After a little manipulation the result may be written

$$w(r, \zeta) = w_0 + \Delta w_s^{(2)} + \frac{\Delta w^{(1)}}{2} + (-1)^k\frac{\Delta w^{(1)}}{2}\left[1 - \frac{\cosh\pi\left(\frac{\zeta - \frac{1}{2}\delta}{r_t - r_h}\right)}{\cosh\pi\left(\frac{\frac{1}{2}\delta}{r_t - r_h}\right)}\right] \quad (7\text{-}55)$$

The flow thus consists of a mean velocity given by $w^{(0)} + \Delta w_s^{(2)} + (\Delta w^{(1)}/2)$ and a component,

$$(-1)^k\frac{\Delta w^{(1)}}{2}\left[1 - \frac{\cosh\pi\left(\frac{\zeta - \frac{1}{2}\delta}{r_t - r_h}\right)}{\cosh\pi\left(\frac{\frac{1}{2}\delta}{r_t - r_h}\right)}\right] \quad (7\text{-}56)$$

which fluctuates along the z axis. The mean velocity profile is given by the sum of the average velocity w_0, the distortion $\Delta w_s^{(2)}$ caused by the guide vanes and half of the distortion produced by the next rotor. The fluctuating part reaches its maximum value halfway between the two blade rows, that is where $\zeta = \frac{1}{2}\delta$. Here it is equal to

$$(-1)^k\frac{\Delta w^{(1)}}{2}\left[1 - \frac{1}{\cosh\pi\left(\frac{\frac{1}{2}\delta}{r_t - r_h}\right)}\right] \quad (7\text{-}57)$$

Clearly then, if the blade spacing becomes very large,

$$\cosh\frac{\pi}{2}\frac{\delta}{r_t - r_h} \to \infty$$

and the fluctuation in axial velocity profile is just

$$(-1)^k\frac{\Delta w^{(1)}}{2} \quad (7\text{-}58)$$

This is, of course, just the result of simple radial equilibrium theory which, as already pointed out, does hold exactly if the blade rows are spaced infinitely far apart along the z axis.

When the blade rows are not spaced a large distance apart, the flow fields of adjacent blade rows tend to cancel each other to a certain extent, that is they interfere. The amplitude of the fluctuation in axial velocity profile from one section to the next is reduced below that given by radial equilibrium theory, Eq. 7-58, by just the factor

$$1 - \frac{1}{\cosh \frac{\pi}{2}\left(\frac{\delta}{r_t - r_h}\right)} \tag{7-59}$$

appearing in Eq. 7-57. The second term of Eq. 7-59 is designated the

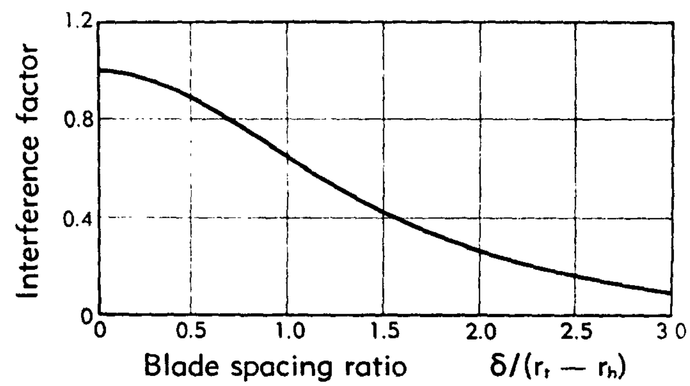


Fig. C,7f. Blade interference factor as a function of blade spacing ratio, $\delta/(r_t - r_h)$.

mutual interference factor and gives the fraction of the radial equilibrium change of axial velocity profile which is prohibited by interference of the blade rows in a section of a multistage compressor made up of identical stages. The mutual interference factor depends only upon the blade spacing ratio $\delta/(r_t - r_h)$, the ratio of blade spacing to blade length. A plot of this factor is shown in Fig. C,7f. It appears then that when the blade spacing is large, of the order of twice as large as the blade length, practically the whole radial equilibrium shift takes place after one blade row, before the next one is encountered. If the blade spacing is small, say about half of the blade length or less as in the early stages of axial compressors, then only about one third of the radial equilibrium may take place before the field of the next blade row takes effect. In view of the rapidity with which this oscillating pattern develops, as discussed in connection with Fig. C,7e, this mutual interference factor may be used with considerable assurance beyond the fourth blade row.

Effects of upstream conditions and compressibility. A few words may well be added here concerning some factors which were found to affect the exponential coefficients in the asymptotic expansions used to investigate large upstream tangential and axial velocity distortions and also compressible throughflow. In the first place it may be recalled that investigation following Eq. 4-31 showed that when the flow upstream of a blade row possessed a strong solid body rotation given by $v^{(0)} = bw_0r/r_t$, the exponential decay of this perturbation was given by $e^{-\lambda_n|z|}$, cf. Eq. 4-34 and 4-35, where

$$\lambda_n \cong \sqrt{\left(\frac{n\pi}{r_t - r_h}\right)^2 - \left(\frac{2b}{r_t}\right)^2} \tag{7-60}$$

For use in the exponential approximation under these circumstances, the ppropriate factor in the exponent should not be simply π, but rather

$$\pi\sqrt{1 - \left(\frac{2b}{\pi}\right)^2\left(1 - \frac{r_h}{r_t}\right)^2} \tag{7-61}$$

In cases of practical importance, such as mutual interference in the axial compressor discussed in the previous section, the value of the radical may be as low as 0.80. Reference to Eq. 7-59 shows that the influence of upstream solid body rotation tends to reduce the effective blade spacing δ, in the present example to 0.80 of its geometrical value. The physical effect is to increase the interference factor, as may be seen from Fig. C,7f, and consequently to reduce the axial velocity distortion from one blade row to another.

In general the effects of upstream tangential velocity and axial velocity distortion were treated by the method of asymptotic solution of the differential equation. The appropriate exponential factors were determined explicitly in Eq. 6-13. Thus when the upstream axial velocity and tangential velocity are given as $w^{(0)}(r)$ and $v^{(0)}(r)$ respectively, the appropriate factor in the exponent is

$$\pi\left\{1 \quad \frac{r_t - r_h}{2\pi^2}\int_{r_h}^{r_t}\left[\frac{2v^{(0)}}{r^2w^{(0)2}}\frac{d(rv^{(0)})}{dr} - \frac{r}{w^{(0)}}\frac{d}{dr}\left(\frac{dw^{(0)}}{rdr}\right) - \frac{3}{4r^2}\right]dr\right\} \tag{7-62}$$

instead of simply π. Since this expression is only a constant to be employed in the exponential approximation, the integral may be evaluated numerically without difficulty.

The effect of compressibility has been treated also by means of asymptotic methods in a previous section but not in a completely general manner. For the case of a solid body rotation $v^{(0)} = bw_0r/r_t$ and a uniform axial velocity $w^{(0)} = w_0 = \text{const}$, the asymptotic solution for the characteristic functions was carried out in the development following Eq. 6-25. In particular it was found that the decay factor, the exponential part of the characteristic functions, was given in Eq. 6-38. From this then it is clear that the appropriate constant to employ in the exponential approximation to the throughflow is not exactly π, but

$$\frac{\pi}{\sqrt{1 - M_z^2}} \left\{ 1 - \frac{2b^2}{\pi^2}\left(1 - \frac{r_h}{r_t}\right)^2 \left(1 - \frac{\gamma + 2}{8} M_z^2\right) \right.$$
$$\left. - \frac{b^4}{6\pi^2} M_z^2 \left[(\gamma - 1) - \frac{\gamma M_z^2}{2}\right] \left(1 - \frac{r_h}{r_t}\right)^3 \left(1 - \frac{r_h}{r_t}\right) - \frac{3}{8\pi^2}\frac{r_t}{r_h}\left(1 - \frac{r_h}{r_t}\right)^2 \right\} \tag{7-63}$$

This is simply a number for any set of operating conditions and reduces to the former value for incompressible flow with upstream solid body rotation when the Mach numbers approach zero. So far as compressibility is concerned, the strongest effect is usually the factor $\sqrt{1 - M_z^2}$ in the denominator. As pointed out previously this is simply the Prandtl-Glauert correction associated with the axial velocity. Effects of the rotational velocities appear only in the other terms of the numerator. The predominant effect of compressibility is to increase the constant in the exponential approxima. on, opposite the influence of the solid body rotation far upstream. With reference to the multistage compressor discussed previously, the compressibility correction tends to increase the effective blade row spacing along the axis. For ordinarily encountered axial velocities the compressibility correction may increase the value of the appropriate exponential constant from π to 1.15π. It is also worth noting that in a conventional compressor the two main corrections to the exponential constant, the solid body rotation imparted by the guide vane and the compressibility correction, tend very nearly to compensate each other and make the appropriate constant π.

Variable hub and tip radii. It has recently been pointed out to the author by G. Oates that it is possible to develop an exponential approximation for the effect of variable hub and tip radii, the exact solution for which was discussed previously in detail. It has not as yet been developed to the extent of exponential approximation for perturbations due to blade rows and has not at all found its way into practice. Consequently the discussion here will be limited to the example of hub radius variations which was worked out in detail following Eq. 5-11. Take the amplitude of the sine wave to be αr_h so that the hub radius is r_h upstream of the contraction and $r_h(1 + 2\alpha)$ downstream. If now the axial velocity profile is uniform and the tangential velocity vanishes (or has the distribution of a vortex) far upstream of the contraction, then the axial velocity profile is certainly uniform far downstream of the profile. By continuity, the uniform perturbation on the axial velocity far downstream is

$$w^{(1)}(r, \infty) = \frac{2r_h^2\alpha}{r_t^2 - r_h^2} \tag{7-64}$$

and according to Eq. 5-23, it is clear then that

$$\frac{2r_h^2}{r_t^2 - r_h^2} = \frac{\pi r_h}{l}\left[\frac{I_0\left(\frac{\pi r}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_0\left(\frac{\pi r}{2l}\right)}{I_1\left(\frac{\pi r_h}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_1\left(\frac{\pi r_h}{2l}\right)}\right]$$

$$+ \frac{r_h}{l}\frac{\pi^2}{2l}\sum_1^\infty \frac{2 - e^{-\kappa_n z}\cosh \kappa_n\zeta}{(\pi/2l)^2 + \kappa_n^2}\left\{\frac{J_0(\kappa_n r)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r)}{r_h[J_0(\kappa_n r_h)Y_1(\kappa_n r_t) - J_1(\kappa_n r_t)Y_0(\kappa_n r_h)] - r_t[J_1(\kappa_n r_h)Y_0(\kappa_n r_t) - J_0(\kappa_n r_t)Y_1(\kappa_n r_h)]}\right\} \tag{7-65}$$

In particular the radial dependence of the two functions on the right-hand side of Eq. 7-65 cancels to give a constant result. Employing this relation, it appears upon writing Eq. 5-22 at the point $z = 0$ in the center of the converging section, that

$$\frac{w^{(1)}(r, 0)}{w^{(0)}} = \frac{r_h^2\alpha}{r_t^2 - r_h^2} \tag{7-66}$$

and consequently the axial velocity is uniform here also. This result could also have been obtained directly by a symmetry argument with respect to $z = 0$.

The exponential approximation in this case consists in noting that, in addition to the above observations, the terms of principal importance in the summations are those corresponding to the first characteristic value, $\kappa_1 \approx \pi/(r_t - r_h)$. Now denoting

$$T(r) = -\frac{\pi r_h}{l}\left[\frac{I_0\left(\frac{\pi r}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_0\left(\frac{\pi r}{2l}\right)}{I_1\left(\frac{\pi r_h}{2l}\right)K_1\left(\frac{\pi r_t}{2l}\right) - I_1\left(\frac{\pi r_t}{2l}\right)K_1\left(\frac{\pi r_h}{2l}\right)}\right] \tag{7-67}$$

The axial velocity distribution may be denoted in the following approximate form:

$$\frac{w^{(1)}(r, z)}{w^{(0)}} = \alpha\left[\frac{1}{(r_t/r_h)^2 - 1} + \frac{1}{2}T(r)\right]\frac{1}{2}\cosh \pi\frac{l}{r_t - r_h}\, e^{\pi\frac{z}{r_t - r_h}} \qquad z \leqq l \tag{7-68}$$

$$\frac{w^{(1)}(r, z)}{w^{(0)}} = \alpha\left[\frac{1}{(r_t/r_h)^2 - 1} + \frac{1}{2}T(r)\right]\left(e^{-\pi\frac{l}{r_t - r_h}}\sinh \pi\frac{z}{r_t - r_h} - \sin \pi\frac{z}{2l}\right) + \frac{\alpha}{(r_t/r_h)^2 - 1}\left(1 + \sin \pi\frac{z}{2l}\right) \qquad -l \leqq z \leqq l \tag{7-69}$$

$$\frac{w^{(1)}(r, z)}{w^{(0)}} = \alpha\left[\frac{2}{(r_t/r_h)^2 - 1} + T(r)\right]\left(1 - \frac{1}{2}\cosh \pi\frac{l}{r_t - r_h}\, e^{-\pi\frac{z}{r_t - r_h}}\right) + \frac{2\alpha}{(r_t/r_h)^2 - 1} \qquad l \leqq z \tag{7-70}$$

Since $T(r)$ can be evaluated directly, this approximation for the axial velocity perturbation is very simple to use. Its accuracy is also adequate for any but extremely unusual needs.

The axial velocity profiles have been calculated at several axial positions using the exponential approximation given above, and the results

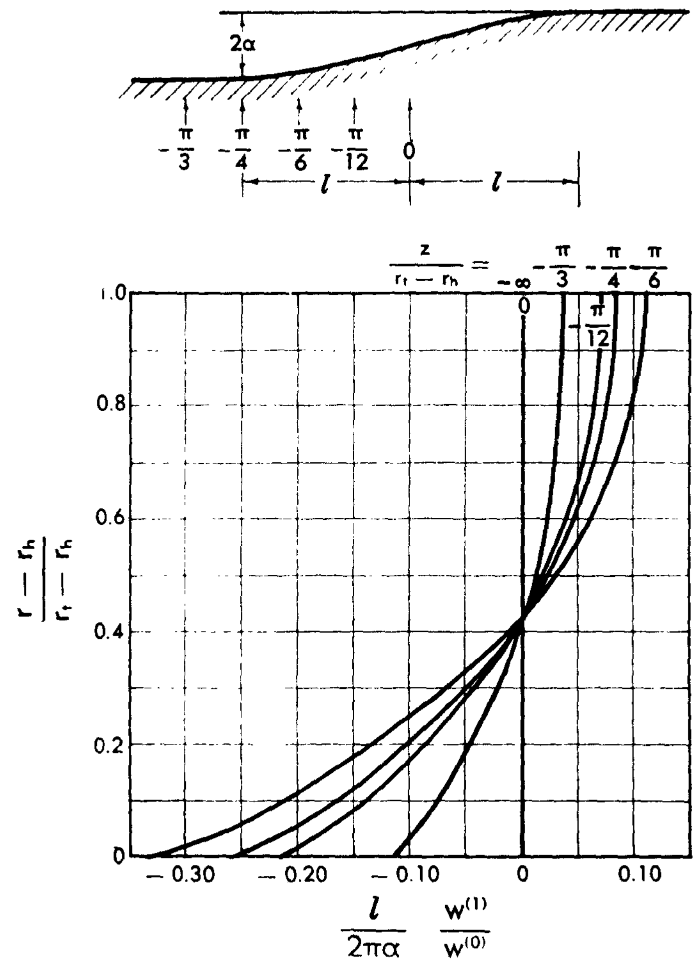


Fig. C,7g. Approximate axial velocity profiles caused by variation in hub radius, $r_h/r_t = 0.6$.

are shown in Fig. C,7g. Comparison with the exact solution suggests that they are more reliable than a significant number of terms in the Fourier-Bessel expansion and, of course, incomparably easier to calculate. Combined with the axial velocity distortion associated with an actuator disk, the results give a more adequate description of the throughflow for a blade row in channel of varying cross section. It is obvious that a similar approximation may be developed for a variable tip radius.

In summary it may be said that the exponential approximation combined with a radial equilibrium theory provides all of the throughflow information and the required accuracy for turbomachine technology. While some specific problems may require the detailed exact treatment, it is probable that, when these problems have been investigated thoroughly, they too will permit approximate treatment similar to the exponential approximation.

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