A Contextual Range-Dependent Model for Choice under Risk

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Abstract

We introduce a context-dependent theory for choice under risk, called range utility theory. It builds on Parducci's range principle from psychophysics and modifies expected utility by positing that risky prospects are evaluated relative to the range of consequences of all prospects in the decision context. When the context is fixed, choices typically exhibit the four-fold pattern of risk preferences, yet are fully consistent with expected utility (linear in probabilities) without invoking rank-principles. We illustrate this advantage in game theory contexts. As the same time, when the context varies, the relative value of an alternative also does, yielding different forms or preference reversals, some of which have been robustly documented.

Keywords: Expected utility, range effects, context effects, preference reversal, reference dependence.

JEL codes: D81, D91, D03

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1 Introduction

Von Neumann & Morgenstern (1944) defined rational preferences in the presence of risk by a series of simple postulates supporting the Expected Utility (EU) representation. They also beautifully demonstrated that EU guarantees the existence of equilibrium in zero-sum games (the minmax theorem), a result famously generalized by Nash (1950) to all finite games. Its normative appeal, elegance and simplicity made EU theory a golden standard in applications involving decisions under risk.

Since then, EU theory applied to monetary risks with wealth levels being the consequence has been shown to be systematically violated in numerous experiments. Many attempts have been made to propose better descriptive theories, with cumulative prospect theory (CPT) standing out as the most popular contender (Kahneman & Tversky, 1979; Tversky & Kahneman, 1992). However, as Barberis (2013, p. 173) notes "It is curious, then, that so many years after the publication of the 1979 paper, there are relatively few well-known and broadly accepted applications of prospect theory in economics."

Prospect theory builds on two ideas: reference dependence (Markowitz, 1952) and nonlinear probability weighting (Edwards, 1954). While the former reinterprets the domain over which the EU axioms are defined—with outcomes representing changes relative to a reference point rather than levels of wealth—the latter departs from EU by introducing rank-ordering of outcomes and probability weighting.

We claim that by denying the essence of the EU this latter feature makes prospect theory less attractive in applications. For example nonlinear probability weighting implies violations of the reduction of compounded lotteries. To illustrate consider the implementation of a mixed strategy equilibrium, assuming one exists (under CPT, equilibrium may fail to exist). The ex-ante optimal mixed strategy is not optimal after I flip a coin to determine my own strategy. Moreover, CPT requires rank-ordering the outcomes, which introduces complexity and hinders tractability (Baucells et al., 2023).

Instead of entirely replacing EU as prospect theory does, what we propose in this paper is a model that extends EU by allowing for context dependence. While EU is preserved within each decision context, its violations are allowed when a given alternative, or a comparison between two alternatives, is set in a different context. Thus our model retains the elegance and tractability of EU implied by linear probability weighting, and at the same time is capable of accommodating systematic violations of EU. This makes it suitable for applications.

In our model, probability weighting is replaced by a range distortion function that deforms the relative utility of the outcomes compared to the range of outcomes. When eliciting certainty equivalents of binary prospects, and only then, CPT and our model become mathematically identical, with the range distortion function being the inverse of the probability weighting function. To elicit certainty equivalents of prospect with three or more outcomes, or compare any two prospects each having two or more outcomes, the two models are different. The notion of an S-shaped range distortion function has a strong support in psychophysics and neuroscience.

In psychophysical judgment, Parducci (1965) demonstrated that a particular stimulus is judged relative to the range of the stimuli in the context. Such range adaptation effects are well documented in neuroeconomics (Rustichini et al., 2017; Stauffer et al., 2014; Zimmermann et al., 2018) and multi-attribute choice problems (Soltani et al., 2012; Kőszegi & Szeidl, 2013; Bushong et al., 2021; Somerville, 2022). We think of range distortion as 'continuous' way to model the process that the brain follows to encode inputs of different width into a fixed neuronal output range. This is in contrast with 'discrete' process models such as the priority heuristic Brandstätter et al. (2006); Katsikopoulos & Gigerenzer (2008), where the minimum gain and the maximum gain across the prospects being compared also feature prominently, and that can successfully account for some well-documented EU paradoxes. In the same way that (Drechsler et al., 2014) axiomatizes the priority heuristic, our main contribution is to axiomatize range utility theory.

Range effects also appear in large empirical studies. In the context of human reinforcement learning, Bavard et al. (2018) find that the algorithm that best matches subject's behavior is based on range-adaptation and reference dependence (see also Palminteri & Lebreton, 2021). Peterson et al. (2021) conducted largest experiment on risky choice to date and used the resulting dataset to power machine-learning algorithms that were constrained to produce psychological theories of risky choice. They tested a long list of well-known models and heuristics. They found that context-dependence is a key feature of a new, more accurate model, and identified minimum outcome and maximum outcome in the context – i.e. range – as well as outcome variability as the most important context effects.

In line with these findings, we introduce a formal context-dependent decision model that accounts for range adaptation and reference dependence, called range utility theory (RUT). As a key theoretical advantage, RUT retains a vNM linear structure when the range is fixed, and reproduces the utility shapes advanced by Markowitz (1952). On the descriptive side, RUT successfully explains the robust but elusive preference reversal phenomenon (Lichtenstein & Slovic, 1971; Grether & Plott, 1979); and the model is quite compatible with observed risk attitudes (e.g., Gonzalez & Wu (1999)), without the need of invoking nonlinear probability weighting.

Finally, we discuss the theoretical and practical advantages of RUT over rank-dependent models when it comes to elementary game theory applications. Indeed, range utility theory is linear in probabilities, hence allows a standard treatment of mixed strategies, provided the range is set by the payoffs of the game. If a player faces a new game with a different range, then her vNM utility for such game needs to be recalculated.

Our model builds on range dependent utility of Kontek & Lewandowski (2017). In this earlier model, each outcome in a prospect is normalized relative to the highest and lowest outcome of *that* prospect. Here, such outcome would be normalized relative to the highest and lowest outcome in the context, which may be wider. Thus, range dependent utility cannot in principle address preference reversals. Moreover, range dependent utility leans on scale and shift invariance, a rather simplistic axiom (if all the prizes of a risky prospect are shifted by some amount, then its certainty equivalents must also shift by this same amount, even if the shift turns some gains into losses or losses into gains, or substantially changes the decision maker's wealth).

2 Model overview

2.1 Certainty equivalents of binary gambles

We begin with the biseparable model, which has repeatedly been shown to improve significantly over EU at predicting certainty equivalents of binary prospects (Gonzalez & Wu, 1999; Tversky & Kahneman, 1992). Given outcomes $G > L$ and probability $\pi \in [0,1]$ of getting G (instead of L), let c be the sure outcome that meets the indifference $c \sim \pi G + (1-\pi)L$. To predict this relation for all binary lotteries, we employ (Ghirardato & Marinacci, 2001; Quiggin, 1982; Luce, 1991):

$$v(c) = v(L) + w(\pi)[v(G) - v(L)]. \tag{1}$$

where $v : \mathbf{R} \to \mathbf{R}$ and $w : [0,1] \to [0,1]$ are strictly increasing functions with $w(0) = 0$ and $w(1) = 1$. Wakker (2010, Section 6.2) notes that many choice theories reduce to (1) for binary prospects.[1]

The standard interpretation of w is that of a inverse-S distortion applied to rank-ordered probabilities. We now show, however, that it can also be interpreted as the inverse of a function that distorts normalized utilities on a range. To see this, rewrite (1) as

$$\frac{v(c) - v(L)}{v(G) - v(L)} = w(\pi).$$

Next, let $D = w^{-1}$, and apply this transformation to both sides to obtain:

$$D\left(\frac{v(c) - v(L)}{v(G) - v(L)}\right) = \pi. \tag{2}$$

Noting that $D(0) = 0$ and $D(1) = 1$, the right hand side can be seen as the expectation of the gamble using the modified utility $D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$. Specifically,

$$\pi = \pi D(1) + (1-\pi)D(0) = \pi D\left(\frac{v(G)-v(L)}{v(G)-v(L)}\right) + (1-\pi)D\left(\frac{v(L)-v(L)}{v(G)-v(L)}\right). \tag{3}$$

This suggests a model based on normalizing utilities on a range, $\frac{v(x)-v(L)}{v(G)-v(L)}$, then applying a range distortion function, $D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$, and finally take expectations.

2.2 The Range

When having to choose one out of several prospects, how is the range determined? One could argue that the range is perhaps given by the range of stimuli the individual has received recently (e.g., in an experimental setting, the range could be influenced by the payoffs of the current task as well as payoffs observed in preceding tasks). The range could also be determined by the memory of outcomes in situations similar to the one described in the task (e.g., the stakes one may gamble in a casino can set the range of a gambling question). At the same time, a case could be made that individuals need to focus their attention to the task at hand, and that the range is given by the set of payoffs involved in the current decision task. While it is an empirical question to tell which of these is the case, the goal of this paper is to set up the theory in the simplest of setups. To this end, we will assume that the range is determined by the payoffs involved in the current decision task.

[1]E.g. disappointment theory (Gul, 1991), prospective reference theory (Viscusi, 1989), or the RAM and TAX models (Birnbaum, 2008).

Thus, when eliciting the certainty equivalent of a prospect, the range is given by the the minimum and maximum payoff of the prospect; and when having to choose one out of two or more prospects, the range $[L, G]$ is given by the minimum and maximum payoffs in the support of all prospects.

2.3 Transforming the Probability Equivalents

Moving passed certainty equivalets of binary lotteries, there is the question on what it is that shall be weighted. Some theories maintain the original idea of weighting individuals probabilities (Edwards, 1954; Kahneman & Tversky, 1979; Birnbaum, 2008). Nowadays, most models including CPT apply (differential) weighting of cumulative probabilities (Tversky & Kahneman, 1992). Range Utility Theory proposes a third alternative: weighting of the probability equivalents.

Suppose the individual is to choose one out of two or more prospects, and let $[L, G]$ be the range of this decision tasks. Let $x \in [L, G]$ be a possible payoff. Its probability equivalent is the $\pi(x)$ that meets $x \sim \pi(x)G + (1 - \pi(x))L$. Of course, if $x = G$, then $\pi(G) = 1$; and if $x = L$, then $\pi(L) = 0$. We now show that RUT can be heuristically derived using Luce & Raiffa (1957, p. 27) procedure of replacing outcomes by their probability equivalents, but with weighting applied to these.

Under EU, the probability equivalent solves $\upsilon(x) = \upsilon(L) + \pi(x)[\upsilon(G) - \upsilon(L)]$, resulting in $\pi(x) = \frac{\upsilon(x)-\upsilon(L)}{\upsilon(G)-\upsilon(L)}$. After obtaining the probability equivalent of each and every outcome, we proceed by replacing each outcome x by the equally preferred binary lottery of getting G with probability $\pi(x)$ and L otherwise. As a result, each prospect reduces to an equivalent lottery having probability $\pi = \sum_{x \in X} p(x)\pi(x)$ of getting G and L otherwise. It is then obvious that the best prospect is the one having highest

$$\pi = \sum_{x \in X} p(x) \frac{\upsilon(x) - \upsilon(L)}{\upsilon(G) - \upsilon(L)}.$$

By linearity, this rule is equivalent to choosing the prospect with the highest expected utility, $\sum_{x \in X} p(x)\upsilon(x)$, hence independent of G and L.

When redoing this exercise using the biseparable model, what changes is that the probability equivalent now solves $\upsilon(x) = \upsilon(L) + w(\pi(x))[\upsilon(G) - \upsilon(L)]$. Letting $D = w^{-1}$, we obtain $\pi(x) = D\left(\frac{\upsilon(x)-\upsilon(L)}{\upsilon(G)-\upsilon(L)}\right)$. After replacing each outcome x by the equally preferred binary lottery of getting G with probability $\pi(x)$ and L otherwise, we again conclude that the best prospect is the one having the highest overall probability $\pi = \sum_{x \in X} p(x)\pi(x)$ of getting G, now given by:

$$\pi = \sum_{x \in X} p(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right).$$

Unless D is linear, this decision rule cannot be simplified, and generally depend on G and L. This is, in essence, range utility theory (RUT). Our main contribution is to provide preference foundations for RUT.

2.4 The Shape of the Range Distortion Function

The range distortion function, at least in the axiomatization we propose, is slightly more constrained than the (inverse of a) probability weighting function. The restriction is necessary to disentangle v and D.

Definition 1. *A function D is admissible if it is either linear or otherwise possesses a single interior fixed point* $\alpha \in (0, 1)$*; is continuous and strictly increasing with* $D(0) = 0$ *and* $D(1) = 1$*, and satisfies the reflection property:*

$$D(z) = 1 - \tfrac{1-\alpha}{\alpha} D\left(\tfrac{\alpha}{1-\alpha}(1 - z)\right),\ z \in [\alpha, 1]. \tag{4}$$

For $\alpha = 0.5$, the reflection property simplifies to $D(z) = 1 - D(1 - z)$. By (4), a function D defined on $[0, \alpha]$ and having $D(\alpha) = \alpha$ can be automatically extended to $[\alpha, 1]$.

In our examples, we will employ the inverse of the Goldstein-Einhorn pwf on $[0, \alpha]$, and its reflection on $[\alpha, 1]$, given by:[2]

$$D(z) = \begin{cases} \left[1 + \left(\frac{\alpha}{1-\alpha}\right)^{(1-\gamma)/\gamma} \left(\frac{1-z}{z}\right)^{1/\gamma}\right]^{-1}, & \text{if } z \in (0, \alpha], \\ 1 - \left[\frac{\alpha}{1-\alpha} + \left(\frac{1}{1-z} - \frac{\alpha}{1-\alpha}\right)^{1/\gamma}\right]^{-1}, & \text{if } z \in [\alpha, 1). \end{cases} \tag{5}$$

For $\alpha = 0.5$, the function fully coincides with the inverse of the symmetric Goldstein-Einhorn pwf (Goldstein & Einhorn, 1987), and given by:[3]

$$D(z) = \frac{z^{1/\gamma}}{z^{1/\gamma} + (1 - z)^{1/\gamma}},\ \gamma > 0. \tag{6}$$

[2]The original pwf is $w(p) = \delta p^\gamma / (\delta p^\gamma + (1-p)^\gamma)$, whose inverse is $D(z) = z^{1/\gamma} / (z^{1/\gamma} + \delta^{1/\gamma}(1-z)^{1/\gamma})$. The first expression in (5) follows from dividing both the numerator and denominator by $z^{1/\gamma}$ and observing that the fixed point satisfies $\delta = \left(\frac{\alpha}{1-\alpha}\right)^{1-\gamma}$.

[3]The open-ended modification of this function, given by $D(z) = \frac{z^{1/\gamma}}{z^{1/\gamma}+M}$, $z \geq 0$, is routinely used to fit neuroeconomic data (Tymula & Glimcher, 2022).

A second example of a D would be the two-sided power function, given by $D(z) = \alpha(z/\alpha)^{1/\gamma}$ on $z \in [0, \alpha]$, and by (4) on $[\alpha, 1]$. Its inverse has been proposed as a probability weighting function by Diecidue et al. (2009).

A third very tractable example, while technically not fully admissible, is the inverse of the neo-additive weighting function (Chateauneuf et al., 2007). Let $0 < \alpha < \gamma$. If $\gamma \leq 1$, then the function is continuous (but not strictly increasing), taking value 0 on $z \in [0, \frac{\alpha}{1-\alpha}(1-\gamma)]$, value 1 on $z \in [\gamma, 1]$, and value

$$D(z) = \frac{1-\alpha}{\gamma-\alpha} z - \alpha \frac{1-\gamma}{\gamma-\alpha} \text{ on } z \in [\tfrac{\alpha}{1-\alpha}(1-\gamma), \gamma].$$

And if $\gamma > 1$, then the function is equal to the above expression on $z \in (0, 1)$, but discontinuous at 0 and 1.

In all three cases, if $0 < \gamma < 1$, then the proposed function is S-shaped; and if $\gamma > 1$, the it is inverse S-shaped.

In Appendix B, we provide a method to elicit v and D under RUT's framework.

3 Model Predictions

For predictive purposes, and in agreement with both neurobiological evidence (Stauffer et al., 2014) as well as empirical analysis (see Figure 4 in Appendix C based on Gonzalez & Wu (1999) data), we assume D is S-shaped. For prediction, we also assume outcomes are changes of wealth relative to the reference point.

In all numerical examples, we employ the specification,

$$\begin{aligned} &v(x) = x^{0.8}, x \geq 0, \;\; v(x) = -2|x|^{0.8}, x < 0; \;\text{ and} \\ &D(z) = z^2/(z^2 + (1-z)^2), \;\text{ or } w(p) = p^{0.5}/(p^{0.5} + (1-p)^{0.5}). \end{aligned} \tag{7}$$

3.1 The Four-fold Pattern of Risk Preferences

Intuitively, and S-shaped D magnifies what is in the center of the range, very much like a lens. This increases the sensitivity to outcomes in the center, and reduces the sensitivity to outcomes falling near the edge of the range. This results in elevated certainty equivalents for small probabilities (risk seeking) and reduced certainty equivalents for high probabilities (risk aversion). In other words, the certainty equivalents that fall in the periphery of the range $[L, G]$ are brought towards its center. Hence, the model naturally explains the four-fold pattern of risk preferences.

For example, the certainty equivalent of $(100, 0; 0.05, 0.95)$ under EU using $v(x) = x^{0.8}$ would be 2.4 (risk averse). This number results from $v(2.4)/v(100) = 0.05$. On the range $[0, 100]$, a value of 2.4 seems very small, and after applying the distortion function, the certainty equivalent becomes 12.3, well above the expected value of the prospect (risk seeking). In view of (2), this number results from $D\left(v(12.3)/v(100)\right) = 0.05$.

3.2 Certainty Equivalent of a Ternary Prospect

We now illustrate that RUT neither contains nor is contained in the class of rank-dependent models. To make this point suffices to consider the certainty equivalent of a ternary gains prospect. Let c be indifferent to $(L, M, G; 1 - \pi - \pi', \pi', \pi)$. Below are the expressions for $\frac{v(c)-v(L)}{v(G)-v(L)}$ according to expected utility (EU), cumulative prospect theory (CPT), and RUT, respectively.

$$\frac{v(c) - v(L)}{v(G) - v(L)} = \begin{cases} \pi + \pi' \frac{v(M)-v(L)}{v(G)-v(L)}. & \text{EU} \\ w(\pi) + [w(\pi + \pi') - w(\pi)] \frac{v(M)-v(L)}{v(G)-v(L)}. & \text{CPT} \\ D^{-1}\left(\pi + \pi' D\left(\frac{v(M)-v(L)}{v(G)-v(L)}\right)\right). & \text{RUT} \end{cases}$$

Note that if w and D are linear, then both CPT and RUT collapse into EU; and if $\pi' = 0$ (binary case), then both CPT and RUT collapse into the biseparable model. If $\pi' > 0$ (ternary case), however, then CPT and RUT are distinctly different.

Table 1: Certainty Equivalents of some binary and ternary gains prospects.

Prospect	EU	CPT	RUT
$(0, 100; 0.67, 0.33)$	25.3	33.2	33.2
$(0, 100; 0.33, 0.67)$	60.2	51.2	51.2
$(0, 20, 100; 0.33, 0.23, 0.44)$	42.2	42.2	40.3
$(0, 80, 100; 0.45, 0.33, 0.22)$	41.6	41.6	44.0
$(0, 20, 100; 0.33, 0.29, 0.38)$	37.5	39.7	37.5
$(0, 80, 100; 0.33, .56, 0.11)$	50.0	45.4	50.0

In Table 1, we compute the certainty equivalents of several prospects under EU, CPT, and RUT, respectively, using the parametric specification in (7). The first two lines illustrate that for binary gain prospects, RUT and CPT are equivalent, but potentially very different from EU. For ternary prospects, the next two lines set prospects so that EU and CPT coincide, and show that RUT can produce predictions on opposite sides (higher or lower). Finally, the last two lines set EU equivalent to RUT, and show that CPT can also produce predictions on opposite sides.

In a game situation, Baucells et al. (2023) consider an investment game with six pure strategies. All three models (EU, CPT, RUT) call for a randomization over these six strategies, but with different mixing probability predictions. Of these, RUT's prediction fits the observed frequency of play better than CPT or EU, and RUT can uniquely produce some qualitative features seen in the data.

3.3 The Relevance of Irrelevant Alternatives

How does the range affect the comparison between two prospects? We now examine how the comparison between prospects A and B changes as we add an irrelevant alternative that modifies the range. Specifically, we argue that increasing G, the upper end of the range, makes individuals less risk averse or even risk seeking. In contrast, decreasing L, the lower end of the range, makes individuals more risk averse. As before, we stick to the v and D given in (7). Let $A = (100, 0; 0.5, 0.5)$ be a risky prospect, and B the sure outcome indifferent to A, 42 in this case.

First, we increase G by merely adding $C = (200, 0; 0.1, 0.9)$ into the choice set. In the context of these three alternatives, C is the least attractive (hence irrelevant). By changing the range from $[0, 100]$ to $[0, 200]$, however, the irrelevant alternative C alters the original indifference between A and B. The key is to see how the normalized utility of 100 and of 42 change as we increase the range by Δ. Note that $v(100)/v(100 + \Delta)$ lives near the upper part of the unit interval, hence $D\left(v(100)/v(100 + \Delta)\right)$ is quite insensitive to Δ. In contrast, $v(42)/v(100+\Delta)$ falls in the intermediate portion of the unit interval, hence $D\left(v(42)/v(100 + \Delta)\right)$ drops quite rapidly as Δ increases. As a result, increasing the range by $\Delta > 0$ makes A strictly preferred to B. Intuitively, in the presence of a larger gain, the safe gain of 42 gets relativized, hence the individual becomes less risk averse.

Next, we return to A vs. B, and decrease L by adding $C' = (100, -100; 0.6, 0.4)$ into the choice set. As before, C' is irrelevant, but changes the range to $[-100, 100]$ and alters the original indifference between A and B. In this case, $D\left(\frac{v(100)-v(-\Delta)}{v(100)-v(-\Delta)}\right)$ for B is quite insensitive to Δ, whereas $D\left(\frac{v(42)-v(-\Delta)}{v(100)-v(-\Delta)}\right)$ for A increases quite rapidly as Δ increases. As a result, decreasing the range by $\Delta > 0$ makes B strictly preferred to A. Intuitively, in the presence of a low payoff (a loss), the safe gain of 42 becomes more salient, hence the individual becomes more risk averse.

Clearly, this argument applies regardless of the shape of v, and holds for any S-shaped range distortion function. Formally, let $A = (y, 0; \alpha, 1-\alpha)$ and $B = (x, 1)$ be such that $A \sim B$ in the context of $\{A, B\}$. Let $C = (y+\Delta, 0; \epsilon, 1-\epsilon)$ and $C' = (y, -\Delta; \epsilon', 1-\epsilon')$. Under RUT, it is easy to find values of $\Delta, \epsilon, \epsilon' > 0$ such that when choosing from $\{A, B, C\}$, we have that $A \succ B \succ C$; and when choosing from $\{A, B, C'\}$, we have that $B \succ A \succ C'$. Note also that by lowering x a bit, we can have a strict preference reversal when adding C; and by increasing x a bit we can produce a strict preference reversal when adding C'.

3.4 The Classical Preference Reversal

The previous discussion sheds light into the classical preference reversal phenomenon (Lichtenstein & Slovic, 1971). Consider a P-bet of moderate to high probability, say $(x, 0; \pi, 1-\pi) = (40, 0; 0.6, 0.4)$, and a \$-bet with a wide range and smaller probability, say $(y, 0; \beta, 1-\beta) = (70, 0; 0.3, 0.7)$. In line with the fourfold pattern, it has been repeatedly shown that individuals express a higher certainty equivalent for the \$-bet. In direct choice, however, most individuals reverse their preference and express $P \succ \$$ (Grether & Plott, 1979). This striking pattern cannot be reconciled with choice models (such as CPT or range dependent utility) where the preference ordering must agree with the ordering of the certainty equivalents.

We now argue that RUT can naturally explain the reversal, as the individual becomes less risk averse with respect to the P-bet as the range varies from $[0, 40]$ (certainty equivalent task) to $[0, 70]$ (choice task). Indeed, according to RUT, the values of $c_\$$ and c_P solve $D\left(v(c_\$)/v(y)\right) = \beta$, and $D\left(v(c_P)/v(x)\right) = \pi$, respectively. Hence, the relation $c_P < c_\$$ occurs whenever $v(x)/v(y) < D^{-1}(\beta)/D^{-1}(\pi)$. Indeed, using the parameters in (7), we predict $c_P = 19 < c_\$ = 22$.

In direct comparison, the pattern $P \succ \$$ occurs whenever $\pi D\left(v(x)/v(y)\right) > \beta$. By rearranging these two inequalities, we conclude that RUT predicts the preference reversal pattern whenever

$$D^{-1}\left(\frac{\beta}{\pi}\right) < \frac{v(x)}{v(y)} < \frac{D^{-1}(\beta)}{D^{-1}(\pi)}. \tag{8}$$

If D is S-shaped (and D^{-1} inverse-S shaped), then it is not difficult to design bets meeting this condition. Indeed, this inequality also holds using (7). Specifically, we get $D^{-1}(\beta/\pi) = 0.5$, $v(x)/v(y) = 0.64$ and $D^{-1}(\beta)/D^{-1}(\pi) = 0.72$, consistent with (8) and the two reversal conditions.

More generally, if we set $\pi > \alpha$ and $\beta = \alpha\pi$, then we are guaranteed that $D^{-1}(\beta/\pi) = D^{-1}(\alpha) = \alpha = \beta/\pi$. Because D is S-shaped (crosses the identity line from below once) and $\beta < \alpha < \pi$, we have that $D^{-1}(\beta) > \beta$ and $D^{-1}(\pi) < \pi$, hence, $D^{-1}(\beta)/D^{-1}(\pi) > \beta/\pi$. To comply with (8), it suffices to find $x < y$ so that $v(x)/v(y)$ falls in between β/π and $D^{-1}(\beta)/D^{-1}(\pi)$.

Thus, in light of RUT, the preference reversal phenomenon is a natural context effect that arises when the same P-bet is considered under two different ranges. Specifically, under a wider range, the individual becomes less risk averse towards this high probability prospect.

4 Preference Foundations

In this Section, we provide preference foundations for RUT having D either linear, or with a fixed point at $\alpha = 0.5$. In Appendix A, we extend the formulation to any admissible range distortion function with arbitrary $\alpha \in (0, 1)$.

Let $X = \mathbb{R}$ be the set of consequences. They can be interpreted as wealth positions, or changes in wealth relative to some reference point. A *prospect* is a probability distribution over X with finite support. The set of prospects on X is denoted by Δ, and contains all functions $p : X \to [0, 1]$ with $\sum_{x \in X} p(x) = 1$. Here, the sum is over the elements of $\text{supp}(p) = \{x \in X : p(x) > 0\}$. We embed X in Δ so that $x \in X$ denotes either a consequence or a degenerate prospect with $p(x) = 1$. The range of a prospect is the convex hull of its support, denoted by $R(p) = \text{Co}(\text{supp}(p))$. The range $[L, G]$ associated with a choice between two prospects $p, q \in \Delta$ is given by $R(p, q) = \text{Co}(R(p) \cup R(q))$. More generally, for choice out of a menu of prospects, the range is given by $R(p_1, p_2, ..., p_n) = \text{Co}[\cup_{i \in \{1,2,...,n\}} \text{supp}(p_i)]$. Finally, let $\mathcal{R}$ be the set of ranges $[L, G]$ with $L < G$.

4.1 Axioms

Preferences are given by a binary relation $\succcurlyeq \subset \Delta \times \Delta$ satisfying the following:

A1 Weak order: $\succcurlyeq$ is complete. Moreover, $\forall p, q, r \in \Delta$, if $R(p, q) = R(q, r) = R(p, r)$, $p \succcurlyeq q$ and $q \succcurlyeq r$, then $p \succcurlyeq r$.

A2 Continuity in probabilities: $\forall p, q, r \in \Delta$, if $R(p, q) = R(q, r) = R(p, r)$ and $p \succ q \succ r$, then $\exists \alpha, \beta \in (0, 1)$ such that $\alpha p + (1 - \alpha)r \succ q \succ \beta p + (1 - \beta)r$.

A3 **Independence:** $\forall p, q, r \in \Delta$, if $p \succ q$ and $R(p,q) = R(q,r) = R(p,r)$, then $\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$ for all $\alpha \in (0,1]$.

A4 **Monotonicity:** $\forall x, y \in X$, if $x > y$, then $\alpha x + (1-\alpha)p \succ \alpha y + (1-\alpha)p$ for all $\alpha \in (0,1]$ and $p \in \Delta$.

A5 **Continuity in outcomes:** For all $p \in \Delta$, the sets $\{x \in R(p) : p \succ x\}$ and $\{x \in R(p) : p \prec x\}$ are open in $R(p)$.

A1–A3 are standard vNM axioms, but restricted to triples where each pair comparison possesses the same range. This is consistent with Harless & Camerer (1994), who conduct a meta-study of experimental studies and conclude that linear probability weighting performs remarkably well if individuals are choosing from gambles with the same outcome range, and quite poorly otherwise. These axioms are also weaker than the corresponding axioms of Kontek & Lewandowski (2017). The latter require prospects p, q, r to either have identical range or be degenerate with support in that range. Here we only require that $R(p,q) = R(q,r) = R(p,r)$. For example, having $R(p) = [0,1]$, $R(q) = [2,3]$, and $R(r) = [0,3]$ would satisfy A1–A3, even though the ranges differ.

Note that we require continuity in probabilities (A3), but only when the range does not change. Preferences may be discontinuous when a probability of a given outcome turns into or away from 0, and such change induces a change in the range. In fact, these security and potential effects were well recognized by Cohen (1992) and Lopes (1987). They are also consistent with experimental results. Abdellaoui & Munier (1998) found that EU performs reasonably well near the hypotenuse of the Marschak-Machina triangle (where the range is the same as in the interior), but the indifference curves change abruptly near the legs (where the range discontinuously changes).[4]

Axiom A5 ensures continuity in outcomes, i.e. small changes in prospect prizes do not cause preference jumps. Together with the other axioms it implies

[4] Kontek (2018) fixes three outcomes and analyzes how the certainty equivalent (CE) of different lotteries change if one shifts 1% probability mass from the middle outcome towards the highest outcome. When such change makes the middle outcome disappear, the CE increases by just 1.5%, whereas when the change makes the higher outcome possible, the CE increases by 13.4%. Similarly, shifting 1% probability from the middle outcome towards the lowest outcome produces a change of 0.7% when the middle outcome disappears, and of 7.4% when the lowest outcome becomes possible. Kontek (2018) concludes that models with discontinuities at the legs such as range dependent utility, RUT, or Prospective Reference (Viscusi, 1989) perform much better than continuous models such as EU or CPT.

existence of certainty equivalents, which by A4 are also unique.

The following lemma is a preliminary step towards our main representation result. It uses A1–A5 to produce an uncountable number of EU representations, each with a vNM utility $u_R : R \to \mathbb{R}$ specific to a given range. All proofs are relegated to Appendix D.

Lemma 1. *$\succcurlyeq$ satisfies A1–A5 iff for each $R \in \mathcal{R}$ there exists a continuous and strictly increasing utility function $u_R : R \to \mathbb{R}$, such that for $p, q \in \Delta$ with $R(p, q) = R$,*

$$p \succcurlyeq q \iff \sum_{x \in X} p(x) u_R(x) \geq \sum_{x \in X} q(x) u_R(x). \tag{9}$$

Moreover, each u_R is unique up to positive affine transformation.

Requiring a different vNM utility u_R for each range seems excessive and impractical. We do not expect people to completely change their mind if – other things being the same – the range changes just slightly. This calls for additional axioms that, by introducing regularity between different payoff ranges, will reduce the number of degrees of freedom and increase predictive power.

In strengthening the axioms we want to make sure that while we eliminate some erratic or random behavior we can still capture robust empirical patterns. One such pattern is the preference reversal phenomenon. In order to allow for such reversals, our model will not require choices between binary prospects to agree with how their certainty equivalents compare. Our sixth axiom postulates that consistency between the two must hold only for equal chance binary prospects.

A6 Midpoint Consistency: For $L \neq G$ and $l \neq g$, let $x \sim 0.5G + 0.5L$ and $y \sim 0.5g + 0.5l$. Then, $0.5G + 0.5L \succcurlyeq 0.5g + 0.5l$ iff $x \succcurlyeq y$.

Using probability 0.5 might sound arbitrary. We could replace consistency for equal chance binary prospects with a more general consistency for binary prospects with some fixed probability $\alpha \in (0, 1)$ of the greater outcome. Such a possibility and a more general representation result is introduced in Appendix A. Here, we stay with $\alpha = 1/2$, focal in decision theory since the beginning (Ramsey, 1931, p. 168). Edwards (1954, p. 389), who pioneered the notion that individuals subjectively distort probability values, suggested 0.5 as the most natural undistorted value. Also note that 0.5 is the only probability for which the ordering of the payoffs doesn't matter because $(x, y; 0.5) = (y, x; 0.5)$ for all x, y. This makes the model independent of rank and therefore only focuses on range adaptation.

Our empirical analysis of the Gonzalez & Wu (1999) data is supportive of an admissible D with α near 0.5 (see Appendix C). More generally, there is some collinearity between D and v in the biseparable model, which makes the location of the D fixed point, the value of α, either less important or even irrelevant for prediction purposes. For example, consider simple prospects with only one non-zero payoff (88 out of 165 prospects in the Gonzalez & Wu (1999) data are simple prospects) and assume that their CEs are represented by a biseparable model (2) for some S-shaped D function with a unique non-trivial fixed point at $\alpha \in (0, 1)$ and some utility function v. It is easy to check that replacing D with $\tilde{D}(x) = D(x^\gamma)$ and v with $\tilde{v}(x) = v(x)^{1/\gamma}$ for some $\gamma > 0$ yields an equivalent model for this set of prospects. Note that regardless of the value of α, $\tilde{D}$ can have its only non-trivial fixed point anywhere in the range $(0, 1)$, depending on the value of γ . For example, setting $\gamma = \log_{0.5} D^{-1}(0.5)$ sets the fixed point $\tilde{D}$ to 0.5: $\tilde{D}(0.5) = D(0.5^{\log_{0.5} D^{-1}(0.5)}) = D(D^{-1}(0.5)) = 0.5$. As shown by our analysis of the GW99 data, for more complex prospects this argument does not hold exactly, but it still holds approximately.

Our last axiom is a special case of the midpoint independence axiom from Quiggin & Wakker (1994), applied to certainty equivalents of binary prospects instead of a general preference between any two prospects. This axiom is also a special case of bisymmetry, an axiom known at least since Aczél (1948), who used it to characterize the quasilinear mean. As seen in (1), our midpoint independence provides a biseparable representation of the certainty equivalents of binary prospects.

A7 Midpoint Independence: Let $L_1 < G_1$, $L_3 < G_3$, $\alpha \in (0, 1]$, and

$$x_1 \sim \alpha G_1 + (1-\alpha)L_1,\ L_2 \sim 0.5L_1 + 0.5L_3,$$
$$x_3 \sim \alpha G_3 + (1-\alpha)L_3,\ \text{and}\ G_2 \sim 0.5G_1 + 0.5G_3.$$

Then, $x_2 \sim 0.5x_1 + 0.5x_3$ iff $x_2 \sim \alpha G_2 + (1-\alpha)L_2$.

4.2 Main Representation Result

Theorem 1. *A preference relation $\succcurlyeq$ satisfies A1–A7 if and only if there exist an admissible range distortion function D satisfying $D(z) = 1 - D(1 - z)$, $z \in [0, 1]$, and a continuous and strictly increasing utility function $v : X \to \mathbb{R}$, such that for any pair of prospects p, q with $R(p, q) = [L, G]$,*

$$p \succcurlyeq q \iff \sum_{x\in X} p(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right) \geq \sum_{x\in X} q(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right). \quad (10)$$

Moreover, D is unique, and v is unique up to positive affine transformation.

We sketch how the axioms imply the representation. The construction of the range distortion function requires to apply A6 and A7 in tandem. We begin by constructing a nondecreasing v such that $v(x) = 0.5v(G) + 0.5v(L)$ holds whenever $x \sim 0.5G+0.5L$ does, for *some* specific values L, x, G in the grid used to construct v. Then we use A7 to show that the latter equivalence extends to *all* values satisfying the indifference condition. By A6, not only the CE's but also the comparison between any two equal chance binary prospects is EU-represented by v. The following remark, which (10) satisfies, is what A6-A7 accomplishes.

Remark 1. *For any $L < G$ and $l < g$, we have that $0.5G + 0.5L \succcurlyeq 0.5g + 0.5l$ if and only if $0.5v(G) + 0.5v(L) \geq 0.5v(g) + 0.5v(l)$.*

For a fixed $\alpha \in [0, 1]$, denote by x the Certainty Equivalent of $\alpha g + (1 - \alpha)l$, for all $l \leq g$. By Remark 1 and A6, $v(x)$ is a function of $v(l)$ and $v(g)$. A7 is used to show that this function (denoted by W), whose domain is the set of all $\mathbf{y} := (y_1, y_2) \in \mathbf{R}^2$, with $y_1 \leq y_2$, satisfies the Jensen's equation, i.e. $W(0.5\mathbf{y} + 0.5\mathbf{y}') = 0.5W(\mathbf{y}) + 0.5W(\mathbf{y}')$. Since W is non-negative, hence bounded from below, the only solution to Jensen's equation is a linear W from which it follows that $v(x) = (1 - W(0, 1))v(l) + W(0, 1)v(g)$. After defining $D^{-1}(\alpha) = W(0, 1)$, and observing that $u_{[l,g]}(x) = \alpha$ by representation (9) we obtain that $u_{[l,g]}(x) = D\left(\frac{v(x)-v(l)}{v(g)-v(l)}\right)$. This can be repeated for any $\alpha \in [0, 1]$ and hence (10) follows.

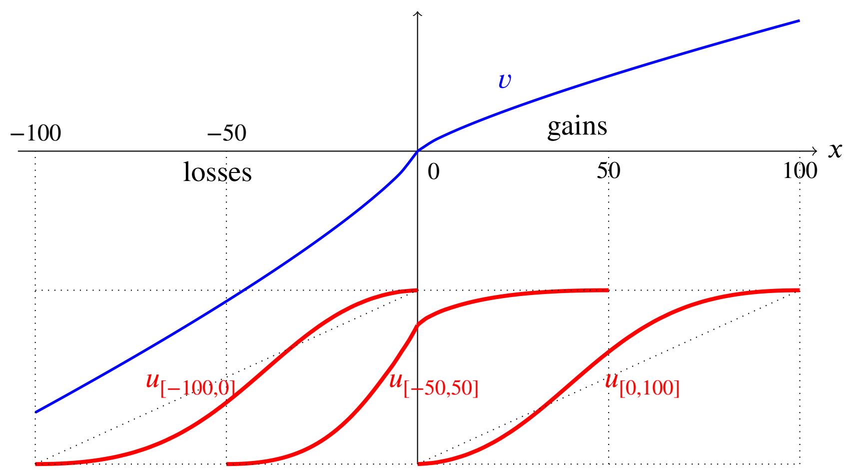
Finally, A6 is used to prove that $D(z) = 1 - D(1 - z)$. This condition implies that D is either linear, or has a unique fixed point at 0.5. Thus, for binary prospects with probability 0.5, the effect of D disappears, and preferences are represented by a range-independent v.

5 Discussion

5.1 Reinterpreting Markowitz's Utility

In range utility theory, the vNM utility $u_{[L,G]} : [L, G] \to [0, 1]$ adapts to the range by adding an S-shaped deformation of some underlying v. The top panel of Figure 1 sets v as a value function from prospect theory (Kahneman & Tversky, 1979). The bottom panel shows the corresponding $u_{[L,G]}$ as we apply a symmetric D and vary $[L, G]$.

Figure 1: Top: The value function, $v(x) = x^{0.8}$, $x \geq 0$, $v(x) = -2v(-x)$, $x < 0$. Bottom: The vNM utility $u_{[L,G]}(x) = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$, $x \in [L, G]$, using $D(z) = z^2/(z^2 + (1-z)^2)$, for three exemplary ranges $[L, G]$.



For gains and losses separately, $u_{[0,100]}$ and $u_{[-100,0]}$ resemble the double-S utility proposed by Markowitz (1952) in order to accommodate commonly observed risk preference patterns (see Scholten & Read (2014) for a comparison between Prospect Theory's and Markowitz's fourfold patterns). Note that even though D is symmetric around $(0.5, 0.5)$, $u_{[L,G]}$ cross the linear u below the mid-point of the range for gains and (symmetrically) above it for losses to capture reflection effect. This is caused by the curvature of v being concave for gains and concave for losses. When the reference point falls in the interior of the range, e.g., $u_{[-50,50]}$, the utility function is an S with a kink at the reference point, as advocated by Kahneman & Tversky (1979). Note that even if v is piece-wise linear, $u_{[-50,50]}$ would become S-shaped.

If ranges and the reference point remain fixed then the utility function could be complex, but no violations of EU are predicted. In order to explain the true departure from EU, such as the Allais paradoxes one can invoke certainty effects (i.e. adjustment of the reference point), as proposed by Schneider & Day (2016).

5.2 Application to Game Theory

In game theory, the natural choice space is that of profiles of mixed strategies; with players having preferences over lotteries generated by mixed strategies. Existence of equilibrium is guaranteed if preferences are quasi-concave (Crawford, 1990, p. 140). In particular, it covers the linear case of expected utility as well as rank-dependent utility with concave probability weighting. If preferences are not quasi-concave (e.g., rank-dependent utility with inverse S-shaped probability weighting function), then equilibrium may fail to exist (Crawford, 1990, pp. 137–138, Dekel et al., 1991, p. 234).[5] Our range utility theory, with the range specified by the game itself and arbitrary range distortion function, exhibits linear preferences, hence guarantees existence.[6]

Table 2: Payoff Matrix of the Inspection Game

	Inspect	Not inspect
Disobey	$-f$, 0	s , $-\ell$
Obey	0 , $1-c$	0 , 1

To illustrate the use of RUT to game theory, and showcase some of the insights, consider a standard 2x2 inspection game. Here, the inspectee can choose to Disobey or Obey a particular law or regulation, and the inspector can either Inspect him or Not. Payoffs in units of v are given in Table 2.[7] The EU benchmark exhibits a unique mixed strategy equilibrium given by

$$(p^{\text{EU}}_{\text{Disobey}}, p^{\text{EU}}_{\text{Inspect}}) = \left(\frac{c}{c+\ell}, \frac{s}{s+f}\right).$$

Assume the inspectee follows RUT. Note that the range of this game for the inspectee is $[-f, s]$, in units of v. For simplicity, assume $\alpha = 0.5$ so that $D(z) = 1 - D(1-z)$. As usual, the goal of the inspector is to make the inspectee indifferent between obeying or not, which occurs if $p_{\text{Inspect}} \cdot D(0) + (1 - p_{\text{Inspect}}) \cdot D(1)$

[5]The literature has proposed two ways to convexify the best response function. One is Crawford's equilibrium in beliefs. The other have the players take a compound lottery view (one's strategic choice comes first, the mixing of the rest of players comes second, non-linear probability weighting only applies to mixtures by the rest of players, and one's probabilities are treated linearly) (Dekel et al., 1991, Theorem 1, Metzger & Rieger, 2019, Proposition 1). The two approaches yield different predictions, and Crawford's is particularly complex to compute (e.g., a closed form solution for the equilibrium probability in a simple 2x2 game may fail to exist).

[6]For more on comparing the CPT model with RUT in the context of game theory, see Baucells et al. (2023).

[7]E.g., if the inspectee's reference point is 0, and v is piecewise linear and loss averse, then f is equal to the monetary penalty times the coefficient of loss aversion.

is equal to $D\left(\frac{0+f}{s+f}\right)$, or

$$p^{\text{RUT}}_{\text{Inspect}} = 1 - D\left(\frac{f}{s+f}\right) = D\left(\frac{s}{s+f}\right).$$

If $f = s$, then there is no distortion and $p^{\text{RUT}}_{\text{Inspect}} = p^{*}_{\text{Inspect}} = 0.5$. If $s > f$, then the concavity of D above 0.5 implies that $p^{\text{RUT}}_{\text{Inspect}} > p^{\text{EU}}_{\text{Inspect}} > 0.5$; and if $f > s$, then the convexity of D below 0.5 implies that $p^{\text{RUT}}_{\text{Inspect}} < p^{\text{EU}}_{\text{Inspect}} < 0.5$. Hence, we can use range effects to reproduce the narrative that, compared to $p^{\text{EU}}_{\text{Inspect}}$, the inspector may lower the inspection rate because the inspectee over-weights the small probability of being caught. This prediction allows us to import insights from behavioral economics into game theory in a conceptually harmonious way.

Next, if the inspector also follows RUT, then for the inspectee to make the inspector indifferent requires

$$p^{\text{RUT}}_{\text{Disobey}} = \frac{D\left(\frac{c}{1+\ell}\right)}{D\left(\frac{c}{1+\ell}\right) + D\left(\frac{\ell}{1+\ell}\right)}.$$

Again, if $\ell = c$, then $p^{\text{RUT}}_{\text{Disobey}} = p^{*}_{\text{Disobey}} = 0.5$; if $\ell < c \leq 0.5$, then $p^{\text{RUT}}_{\text{Disobey}} > p^{\text{EU}}_{\text{Disobey}} > 0.5$; and if $c < \ell \leq 0.5$, then $p^{\text{RUT}}_{\text{Disobey}} < p^{\text{EU}}_{\text{Disobey}} < 0.5$. Thus, an S-shaped range distortion function pushes equilibrium probabilities away from 0.5 and towards 0 or 1.

6 Contexts With More Than Two Prospects

Until now we analyzed contexts with at most two prospects. In many settings, however, the individual faces a menu of alternatives, C, and the menu determines the range $R(C) = \text{Co}[\cup_{p \in C}\text{supp}(p)]$. We wish to use our $\succcurlyeq$ for pairwise choice to define a contextual preference $\succcurlyeq_R$, for comparing alternatives within a broad menu range $R = R(C)$.

To deal with choices from menus, we import the notion of trembling hand. That is, the individual has a small probability ϵ of choosing any alternative in C at the moment of execution, so that the range of possible outcomes always stays at $R(C)$. Let $\Delta(R)$ be the set of prospects with support contained in R. We define $\succcurlyeq_R \subset \Delta^2(R)$ from $\succcurlyeq$ as follows:

$$p \succcurlyeq_R q \iff (1-\epsilon)p + \epsilon r \succcurlyeq (1-\epsilon)q + \epsilon r$$

for some $\epsilon > 0$ and some prospect r with $R(r) = R$. To ensure that $\succcurlyeq_R$ does not depend on the specific choice of $\epsilon > 0$ and r, we strengthen A3 to:

A3′ Strong Independence: For all $p, q, r \in \Delta$ with $R(p, q) = R(q, r) = R(p, r)$ and $\alpha \in (0, 1)$, $p \succcurlyeq q \iff \alpha p + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)r$.

Lemma 2. *Let $\succcurlyeq$ satisfy A3′ and $p, q \in R([L, G])$. If $\alpha p + (1 - \alpha)r \succcurlyeq \alpha q + (1 - \alpha)r$ holds for* **some** *$\alpha \in (0, 1)$ and some $r \in \Delta$ with $R(r) = [L, G]$, then the same holds for* **all** *$\alpha \in (0, 1)$ and all $r \in \Delta$ with $R(r) = [L, G]$.*

From choice data one can only infer a weak preference of the chosen alternative vs. each of the remaining alternatives from the menu. Accordingly, we introduce the observable part. A choice function is a mapping $\Gamma : \mathcal{C} \to \mathcal{C}$ such that $\emptyset \neq \Gamma(C) \subset C$ for any $C \in \mathcal{C}$.

We will say that Γ satisfies *the within-range axiom of revealed preference*, if for any two contexts C, C' having the same range and any two prospects p, q contained in $C \cap C'$, if $p \in \Gamma(C)$ and $q \in \Gamma(C')$, then $p \in \Gamma(C')$.

For a context $C \in \mathcal{C}$, prospect r, and $\alpha \in [0, 1]$, we denote by $\alpha C + (1 - \alpha)r$ a context in which every prospect p in C is replaced by $\alpha p + (1 - \alpha)r$. We will say that Γ satisfies *within-range independence*, if for any context C and prospect $p \in C$, we have that $p \in \Gamma(C)$ is equivalent to $\alpha p + (1 - \alpha)r \in \Gamma(\alpha C + (1 - \alpha)r)$ for any $\alpha \in (0, 1]$ and $r \in \Delta$ such that $R(C \cup \{r\}) = R(C)$.

Let $\succcurlyeq$ satisfy A1 and A3′. We say that Γ is *consistent with* $\succcurlyeq$ if, for any context C with $R(C) = R$,

$$p \in \Gamma(C) \iff p \succcurlyeq_R q \text{ for all } q \in C. \tag{11}$$

Our main result is to connect the observable choice data from menus with a binary preference.

Theorem 2. *A choice function Γ satisfies the within-range axiom of revealed preference and within-range independence if and only if there exist a a preference relation $\succcurlyeq$ satisfying A1 and A3′ that is consistent with Γ.*

7 Conclusions

Mainstream microeconomic theory has had mixed reactions to developments in behavioral economics. Some observed behavior such as present bias and violations of dynamic consistency have been relatively well accepted. These can be welcomed in the conceptual edifice by simply splitting the decision maker into a collection of different selves indexed by time, each being a (more or less sophisticated) player in a game. Similarly, that individuals encode monetary consequences as changes of wealth relative to a reference point requires to think more carefully about the payoff function, but does not create fundamental incompatibilities.

Non-linear probability weighting, however, is an entirely different proposition because it casts doubts on the existence of equilibrium in games, and renders even simple 2x2 games difficult to solve. Such behavioral theories cannot peacefully coexist with mainstream micro-economics.

In this paper, we have introduced range utility theory. It acknowledges the neurobiological reality that individuals normalize outcomes in a range, and that such normalization is accompanied by a systematic distortion of the peripheral outcomes. Range effects manifest in multiple forms. For example, the certainty equivalents of binary prospects will be pulled towards the center of the range, and result in the fourfold pattern of risk preferences (without invoking non-linear probability weighting). Preferences when comparing two prospects having different ranges may not match their certainty equivalents obtained in isolation, as seen in the robust preference reversal phenomenon. At the same time, when a number of alternatives is simultaneously considered in a fixed broad range, range utility theory predicts choices consistent with expected utility and hence more rational.

Thus, range utility theory offers a framework that is more familiar to expected utility and mainstream economic theory, while at the same time accounts for some highly irrational but robust patterns in observed behavior.

A Range Utility Theory with a general α

Instead of committing to equal chance binary prospects, one could weaken axiom A6 to require consistency for some $\alpha \in (0, 1)$. In what follows we present the representation of this more general case. As a useful side-effect we present an entirely different replacement for A7. We first strengthen A6:

$\overline{\text{A6}}$ **α-consistency**. For some $\alpha \in (0, 1)$ and all $L \le l < g \le G$, the certainty equivalents $x \sim \alpha G + (1 - \alpha)L$ and $y \sim \alpha g + (1 - \alpha)l$ satisfy

$$x \succcurlyeq y \iff \alpha G + (1 - \alpha)L \succcurlyeq \alpha g + (1 - \alpha)l.$$

Note that consistency is required to hold for one level of α, but without specifying which. Our final axiom seeks to capture the range principle of Parducci (1965), whereby outcomes that hold the same relative position within a range are treated equally. Thus, 0.7 in the range $[0, 1]$ would be treated the same as 7 in the range $[0, 10]$. These numerical values, however, are expressed in utils. More specifically, according to (Wedell et al., 1990, p.320) the value of a stimulus is determined by the proportion of the subjective range of stimuli lying below it. The range value of stimulus i in some context can be expressed algebraically as: $\frac{S_i - S_{min}}{S_{max} - S_{min}}$ where S_i is the subjective value of the stimulus independent of the range, and S_{max} and S_{min} are the maximum and minimum subjective values of the context.

We adapt this principle for decisions under risk as follows. Because α-consistency asserts that no preference reversals occur for α-binary prospects, we can use those prospects to elicit a function υ that reflects tastes over outcomes independent of range effects. It is thus our equivalent of the scale S. The context is given by all outcomes in the choice problem and hence the relative perception of x in the range $[L, G]$, i.e. our incarnation of $\frac{S_i - S_{min}}{S_{max} - S_{min}}$, is given by $\beta' = \frac{\upsilon(x) - \upsilon(L)}{\upsilon(G) - \upsilon(L)}$. We then postulate that all outcomes that hold the same relative position in their respective ranges are treated equally, i.e. possess the same probability equivalent. Such probability equivalent, however, may be some β different from β', due to range effects. Before formally stating the postulate, we show how to construct υ from preferences, in a recursive way. The method modifies the standard construction for $\alpha = 1/2$ (Quiggin & Wakker, 1994).

For some fixed $\alpha \in (0, 1)$, and two outcomes $x(1) > x(0)$ we set $\upsilon(x(1)) = 1$ and $\upsilon(x(0)) = 0$. For $n = 0$, find $x(1/2) \sim \alpha x(1) + (1 - \alpha)x(0)$ and set $\upsilon(x(1/2)) = \alpha$.

For $n \geq 1$, we proceed recursively as follows. For $k \in \{0, 1, ..., 2^n - 1\}$, find

$$x((2k+1)/2^{n+1}) \sim \alpha x((k+1)/2^n) + (1-\alpha)x(k/2^n) \text{ and set}$$
$$v(x((2k+1)/2^{n+1})) = \alpha v(x((k+1)/2^n)) + (1-\alpha)v(x(k/2^n)).$$

The domain and range of this recursively defined function are $X_\alpha \subset [x(0), x(1)]$ and $A_\alpha \subset [0, 1]$, respectively. Note that for $\alpha = 1/2$, A_α is the set of dyadic rational numbers and X_α is the midpoint indifference grid. Finally we extend the function to $v : [x(0), x(1)] \to [0, 1]$ by letting $v(x) = \sup\{v(x(k/2^n)) : x(k/2^n) \preccurlyeq x\}$.

$\overline{\text{A7}}$ **Range-principle:** Let α satisfy α-consistency and v be constructed using α-binary prospects. If $\beta' = \frac{v(x)-v(L)}{v(G)-v(L)}$ and $x \sim \beta G + (1-\beta)L$ hold for some $L < x < G$, then, for all $L' < x' < G'$,

$$\beta' = \frac{v(x') - v(L')}{v(G') - v(L')} \iff x' \sim \beta G' + (1-\beta)L'.$$

Theorem 3. *A preference relation $\succcurlyeq$ satisfies A1–A5, $\overline{A6}$–$\overline{A7}$ if and only if for some $\alpha \in (0, 1)$ there exist an admissible range distortion function $D : [0, 1] \to [0, 1]$, and a continuous and strictly increasing utility function $v : X \to \mathbb{R}$ such that for any pair of prospects p, q with $R(p, q) = [L, G]$,*

$$p \succcurlyeq q \iff \sum_{x \in X} p(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right) \geq \sum_{x \in X} q(x) D\left(\frac{v(x) - v(L)}{v(G) - v(L)}\right). \quad (12)$$

Moreover, D is unique and v is unique up to positive linear transformation.

The proof is given in Appendix D. We now highlight the differences to the proof of Theorem 1. From construction, we only know that v is nondecreasing. We also know that $v(x) = \alpha v(G) + (1-\alpha)v(L)$ whenever $x \sim \alpha G + (1-\alpha)L$ holds for *some* specific values of x in (L, G). We first use $\overline{A7}$ to show the latter equivalence extends to *any* values satisfying the indifference condition. Then, we use α-consistency to show that v is strictly increasing and thus represents choices between α-binary prospects. This latter fact is summarized in the following remark:

Remark 2. *By (12), for any $L < G$, $L' < G'$, we have that $\alpha G + (1-\alpha)L \succcurlyeq \alpha G' + (1-\alpha)L'$ if and only if $\alpha v(G) + (1-\alpha)v(L) \geq \alpha v(G') + (1-\alpha)v(L')$.*

The systematic relationship in $\overline{A7}$ between the probability equivalent β and the relative position in the utility scale, $\beta' = \frac{v(x)-v(L)}{v(G)-v(L)}$, defines a range-independent function $D(\beta') = \beta$. By Lemma 1, we have that $u_{[L,G]} = \beta$. Connecting the two pieces yields $u_{[L,G]} = \beta = D(\beta') = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$, and (10) follows.

Finally, α-consistency is used to prove that D satisfies (4). This condition implies that D is either linear, or has a unique fixed point in $(0,1)$. Thus, for binary prospects with probability α, the effect of D disappears, and preferences are represented by a range-independent v. By (4), a function D defined on $[0,\alpha]$ can be automatically extended to $[\alpha,1]$.

B Eliciting v and D

To elicit v and D in the RUT model, and obtain D's fixed point α, we propose adapting the method by Abdellaoui (2000) in such a way that it always employs certainty equivalents of binary prospects. The procedure has two stages (one for v, another for D), and $n \geq 3$ steps in each stage, identifying n points of these two functions.

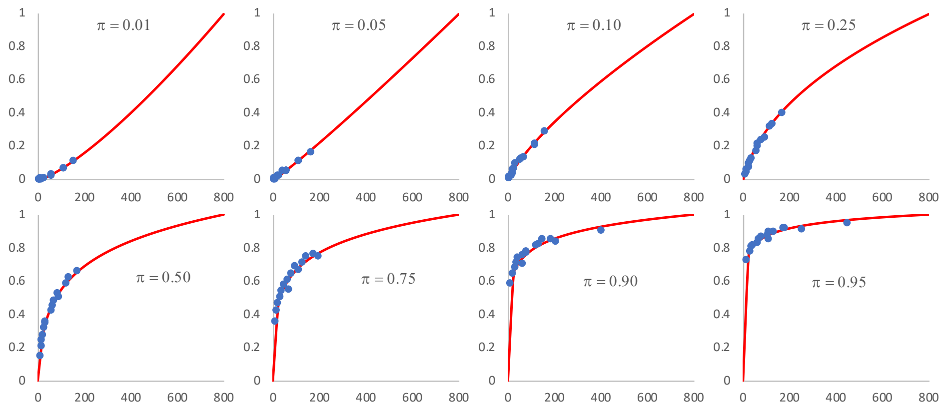
In stage I, fix a range $L < a_0$, and choose a probability (say 0.5) and an outcome $l \in (L, a_0)$. In step 1, find the certainty equivalent of the prospect $(a_0, l; 0.5)$. Then, find $a_1 > a_0$ such that the prospect $(a_1, L; 0.5)$ has the same certainty as the prospect $(a_0, l; 0.5)$.[8] In step 2, find the certainty equivalent of $(a_1, l; 0.5)$; and then we find $a_2 > a_1$ such that the prospect $(a_2, L; 0.5)$ has the same certainty as the prospect $(a_1, l; 0.5)$. Repeat these steps until obtaining a_n. Under the biseparable model (which agrees with RUT in all these elicitations of certainty equivalents), the utility differences $v(a_i) - v(a_{i-1})$, $i = 1, ..., n$, are all equal to $[v(l) - v(L)](1/D^{-1}(0.5) - 1)$; hence we can set $v(a_i) = i/n$, $i = 0, ..., n$.

In stage II, and for $i = 1, ..., n$, find the probability b_i such that the certainty equivalent of $(a_n, a_0; b_i)$ is equal to a_i. According to RUT, we have that $D^{-1}(b_i) = \frac{v(a_i)-v(a_0)}{v(a_n)-v(a_0)}$, yielding $b_i = D(i/n)$.

If D is S-shaped, then we expect to find one i^* such that $b_i \leq i/n$ for $i < i^*$ and $b_i > i/n$ for $i \geq i^*$. Then, α can be found using linear interpolation in the interval $[b_{i^*}, b_{i^*+1})$. If no such i^* exists, then adjust the step size or the number of step. In order to meet the reflection property in (4), we recommend a parametric fit of an admissible D, from which α also obtains.

[8] In the original method meant for CPT, one would find the a_1 that makes $(a_1, L; 0.5)$ and $(a_0, l; 0.5)$ indifferent. Under RUT, we insist on finding the a_1 that equates the certainty equivalents, so that the biseparable evaluation applies.

Figure 2: Fit of selected v_π's.



C Reinterpreting Gonzalez and Wu

While it is well known that the EU desciptively fails on the set of binary prospects, it is instructive to see when and how it fails. We use the data of Gonzalez & Wu (1999) who report certainty equivalents of 165 two-outcome prospects $(L, G; 1-\pi, \pi)$ with 15 different pairs (L, G), and 11 different probability levels π.[9]

We begin with predicting the certainty equivalents using the expected utility model (EU). To so so, we employ the flexible two-parameter utility

$$v(x) = (W + x)^\theta / \theta, \;\; \theta \in \mathbb{R} \text{ and } W \geq 0. \tag{13}$$

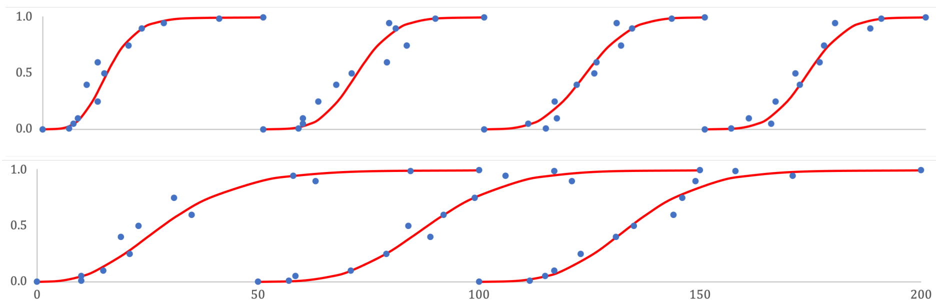
Then, we find the parameters that minimize the mean absolute percentage error (MAPE) between observed and predicted certainty equivalents. The result is a moderately concave utility with $(W, \theta) = (52, -0.95)$ and a MAPE of 0.318.

Under the assumptions of EU, fitting (13) to a subset of the data should produce the same utility (up to estimation error). To test this prediction, we estimate 11 separate versions of (13), one for each probability level π, over the subset of prospects involving π-binary prospects. As a result, we obtain 11 pair parameters (W_π, θ_π), yielding 11 functions v_π.

Figure 2 shows v_π for 8 representative probability levels. Note that as probability π increases, the shape of v_π changes in a systematic way. If EU were true

[9]The 15 outcome pairs are $(0, 25)$, $(0, 50)$, $(0, 75)$, $(0, 100)$, $(0, 150)$, $(0, 200)$, $(0, 400)$, $(0, 800)$, $(25, 50)$, $(50, 75)$, $(50, 100)$, $(50, 150)$, $(100, 150)$, $(100, 200)$, and $(150, 200)$; and the 11 probability levels are $\pi \in \{0.01, 0.05, 0.1, 0.25, 0.4, 0.5, 0.6, 0.75, 0.9, 0.95, 0.99\}$. For each of the prospects they report median CE values for a group of 10 subjects.

Figure 3: Fit of the expected utility model (14) for selected ranges $[L, G]$;



on the whole data, we would expect different v_π's to look roughly the same. Here, however, v_π gradually changes from slightly convex for small π, to very concave for large π. The average MAPE of these fits is half the previous one, MAPE $= 0.155$. We conclude that each v_π is not a noisy incarnation of v.

Similarly, we can fit the EU model to prospects with a fixed payoff range $[L, G]$ (and varying probability π):

$$u_{[L,G]}(c) = \pi. \tag{14}$$

Figure 3 shows $u_{[L,G]}$ for several representative ranges.[10] Here we observe a remarkable regularity across ranges: in particular, all utilities are S-shaped, and the curvature and the interior fixed point change gradually as a result of moving or stretching the range.

Of course, we don't want a theory consisting of a series of EU functions: v_π, one for each π, or $u_{[L,G]}$, one for each $[L, G]$. Intuitively, you can use the regularities of different v_π's or $u_{[L,G]}$'s to get a more parsimonious model. In particular, RUT combines these models into one, reducing the number of distinct functions from uncountably many to just two, setting each of the above two 'local' utilities v_π and $u_{[L,G]}$ equal to:

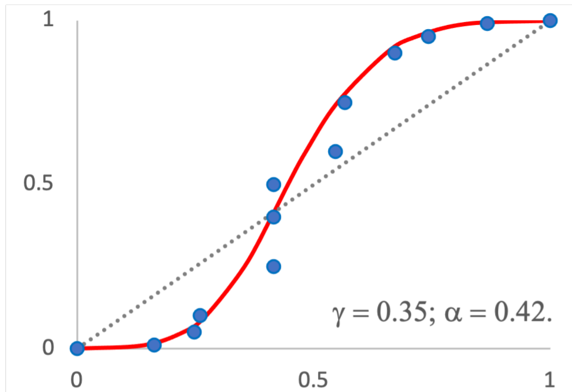
$$D\left(\frac{v(c) - v(L)}{v(G) - v(L)}\right) = \pi. \tag{15}$$

[10]We use the following functional specification:

$$u_{[L,G]}(x) = \left(1 + \left(\frac{(9.45+G)^\theta - (9.45+x)^\theta}{(9.45+x)^\theta - (9.45+L)^\theta}\right)^{1/\gamma}\right)^{-1}.$$

As a result of fitting (14) for each $[L, G]$, we obtain 15 pairs of parameters $(\theta_{[L,G]}, \gamma_{[L,G]})$, yielding 15 functions $u_{[L,G]}$. The average MAPE of these fits is 0.098.

Figure 4: Fit of the D given in (5) (red line) to the data from Gonzalez & Wu (1999) (blue dots).



for each range $[L, G]$ and probability value π.

We conduct a parametric fit of the data using v as in (13) and D as in (5). We obtain $(W, \theta, \gamma, \alpha) = (10.89, 0.20, 0.35, 0.42)$ and a MAPE of 0.119, while greatly reducing the number of parameters. Note that the estimated value of α equals 0.42. If we eliminate a parameter by setting $\alpha = 0.5$, then we obtain $(W, \theta, \gamma, \alpha) = (8.13, 0.14, 0.35, 0.5)$ and a MAPE of 0.122

Finally, we conduct a non-parametric estimation of the range distortion function. To do this, we estimate a biseparable model (15) where v is given by (13) and D^{-1} is nonparametric, i.e.

$$c = v^{-1}(v(L) + q_\pi[v(G) - v(L)]).$$

We only require q_π to be weakly monotonic in π, i.e. $q_\pi \leq q'_\pi$ whenever $\pi < \pi'$. The MAPE of this model is 0.1103. The pairs (q_π, π) are shown in Figure 4 with the previous parametric version of D superimposed on them. Note that q_π is greater than π for $\pi \in \{0.01, 0.05, 0.1, 0.25, 0.4\}$ and the opposite is true for $\{0.5, 0.6, 0.75, 0.9, 0.95, 0.99\}$. This confirms that D^{-1} is inverse S-shaped and D is S-shaped. Its fixed point is somewhere in the interval $(0.4, 0.5)$.

D Proofs

Proof of Lemma 1. It is standard to verify the necessity of A1–A5. For sufficiency, we follow Kreps (1988, Theorem 5.4 and 5.11), while keeping in mind that our A1, A2, and A3 are the relaxed versions of his (5.1), (5.3) and (5.2).

Lemma 3. *A1–A4 imply the following (we omit the "for all" quantifier for brevity):*

(i) *Let $p \succ q$. Then, $\beta > \alpha \iff \beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$.*

(ii) *Let $R(p,q) = R(p,r) = R(q,r)$ and $\alpha \in (0,1)$. Then, $p \succcurlyeq q \iff \alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r$.*

(iii) *Let $R(p,q) = R(q,r) = R(p,r)$, $p \succcurlyeq q \succcurlyeq r$ and $p \succ r$. Then there exists a unique $\alpha^* \in [0,1]$ such that $q \sim \alpha^* p + (1-\alpha^*)r$.*

(iv) *Let $\alpha \in [0,1]$. Then, $x \geq y \iff \alpha x + (1-\alpha)p \succcurlyeq \alpha y + (1-\alpha)p$.*

(v) *Let $R(p) \subset [L,G]$. Then $G \succcurlyeq_{[L,G]} p \succcurlyeq_{[L,G]} L$.*

Claims (i), (ii), and (iii) follow from Kreps (1988, Lemma 5.6) while keeping in mind that (ii) and (iii) apply to the subset of prospects satisfying the range restrictions. Claim (iv) follows from A4 and reflexivity.

Regarding (v), recall that $p \succcurlyeq_R q$ whenever $\alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r$ for some $\alpha \in (0,1)$ and $r \in \Delta$ with $R(r) = R$. Note that $\succcurlyeq_R$ is well defined (see Lemma 2) and that (ii) is identical to A3$'$. Consider any prospects p, r with $R(p) \subseteq R(r) = [L,G]$, and let $\alpha \in (0,1)$. Choose one of p's outcome, $x_1 \in \text{supp}(p)$, and remove $\alpha p(x_1)$ probability from x_1 in $\alpha p + (1-\alpha)r$ and normalize, to obtain $r' := \frac{(1-\alpha)p(x_1)}{1-\alpha p(x_1)} x_1 + \sum_{x \notin \{x_1\}} \frac{\alpha p(x) + (1-\alpha)r(x)}{1-\alpha p(x_1)} x$. By (iv) and $G \geq x_1$,

$$\alpha p(x_1)G + [1-\alpha p(x_1)]r' \succcurlyeq \alpha p(x_1)x_1 + [1-\alpha p(x_1)]r'.$$

For another outcome x_2 in the support of p, we again have that $G \geq x_2$ implies

$$\alpha p(x_2)G + [1-\alpha p(x_2)]r'' \succcurlyeq \alpha p(x_2)x_2 + [1-\alpha p(x_2)]r''$$

where $r'' = \frac{\alpha p(x_1)}{1-\alpha p(x_2)} G + \sum_{x \notin \{x_1,x_2\}} \frac{\alpha p(x)}{1-\alpha p(x_2)} x + \sum_x \frac{(1-\alpha)r(x)}{1-\alpha p(x_2)} x$. Because the range $[L,G]$ remains fixed, we can use transitivity (A1) to obtain

$$\alpha[p(x_1) + p(x_2)]G + \sum_{x \notin \{x_1,x_2\}} \alpha p(x)x + \sum_x (1-\alpha)r(x)x \succcurlyeq \sum_x [\alpha p(x) + (1-\alpha)r(x)]x.$$

Running this argument over all finite outcomes of p produces $\alpha G + (1-\alpha)r \succcurlyeq \alpha p + (1-\alpha)r$. Because the logic does not depend on the choice of r and α, we conclude that $G \succcurlyeq_{[L,G]} p$. Adapting the same construction for L yields $G \succcurlyeq_{[L,G]} p \succcurlyeq_{[L,G]} L$.

Construction of $U(p)$. Continuing with the same prospects, we know by A4 that $\alpha G + (1-\alpha)r \succ \alpha L + (1-\alpha)r$. By (iii), there is a unique $\alpha^* \in [0,1]$ satisfying $\alpha p + (1-\alpha)r \sim \alpha^*(\alpha G + (1-\alpha)r) + (1-\alpha^*)(\alpha L + (1-\alpha)r)$, which we write as $p \sim_{[L,G]} \alpha^* G + (1-\alpha^*)L$. We define the utility of a prospect as

$$U_{[L,G]}(p) = \alpha^*.$$

That such $U_{[L,G]}$ is affine is a standard proof, hence omitted.

$U_{[L,G]}$ represents $\succcurlyeq$ restricted to pairs with $R(p,q) = [L,G]$. Take a pair $p \succcurlyeq q$ with $R(p,q) = [L,G]$, and let r be any prospect with $R(r) = [L,G]$. In this case, by (ii), $p \succcurlyeq q$ if and only if $p \succcurlyeq_{[L,G]} q$. By the definition of $U_{[L,G]}$, we have that $U_{[L,G]}(p)G + (1 - U_{[L,G]}(p))L \sim_{[L,G]} p$. By transitivity, $U_{[L,G]}(p)G + (1 - U_{[L,G]}(p))L \succcurlyeq_{[L,G]} q$. Repeating the same for q yields $U_{[L,G]}(p)G + (1 - U_{[L,G]}(p))L \succcurlyeq_{[L,G]} U_{[L,G]}(q)G + (1 - U_{[L,G]}(q))L$, which by (ii) is equivalent to $U_{[L,G]}(p)G + (1 - U_{[L,G]}(p))L \succcurlyeq U_{[L,G]}(q)G + (1 - U_{[L,G]}(q))L$. By (i), this is equivalent to $U_{[L,G]}(p) \geq U_{[L,G]}(q)$.

Conversely, assume $U_{[L,G]}(p) \geq U_{[L,G]}(q)$. By (i), it follows that $U_{[L,G]}(p)G + (1 - U_{[L,G]}(p))L \succcurlyeq U_{[L,G]}(q)G + (1 - U_{[L,G]}(q))L$. Use the definition of $U_{[L,G]}(p)$ and $U_{[L,G]}(q)$, transitivity twice, and then (ii), to conclude that $p \succcurlyeq q$.

Construction of $u(x)$. Define the utility of outcomes as $u_{[L,G]}(x) := U_{[L,G]}(x)$, $x \in [L,G]$, and use induction on the size of the support of the prospect to obtain the expected utility form of $U_{[L,G]}$ from the affinity of $U_{[L,G]}$. That $u_{[L,G]}$ is strictly increasing uses the same argument as in proving that $p \succcurlyeq q$ implies $U_{[L,G]}(p) \geq U_{[L,G]}(q)$ with x, y replacing p, q and (iv) instead of (ii).

Finally, we show that $u_{[L,G]}$ is continuous. Given $[L,G] \in \mathcal{R}$, let $u = u_{[L,G]}$. Suppose that u is discontinuous at $x \in [L,G]$. Then there a sequence $\{x_n\}_{n\in\mathbb{N}}$: $\lim_{n\to+\infty} x_n = x$ all belonging to $[L,G]$ such that $\lim_{n\to+\infty} u(x_n)$ exists (including limits ∞ and $-\infty$) and $\theta := \lim_{n\to+\infty} u(x_n) \neq u(x)$. Suppose $\theta > u(x)$. If $\theta = +\infty$, then $x_n \succ x_N \succ x$ for some fixed N and all large n. Take $p = 0.5x + 0.5x_N$. Then $p \succ x$ and yet for sufficiently large n, $x_n \succ p$. This in turn implies that the set $\{x \in [L,G] : p \succ x\}$ is not open. If $\theta \neq \infty$, let $\theta - u(x) = \beta$, and let x_N be such that $|\theta - u(x_N)| < \beta/2$. Then the gamble $p = 0.5x_N + 0.5x$ has expected utility satisfying $\theta - \beta/4 = \theta/2 + u(x)/2 + \beta/4 > U(p) > \theta/2 + u(x)/2 - \beta/4 = u(x) + \beta/4$. Thus $p \succ x$, but for all sufficiently large n, $x_n \succ p$, which implies that the set $\{x \in [L, G] : p \succ x\}$ is not open, thus resulting in a contradiction of A5. A similar argument works for $\theta < u(x)$. This can be repeated for any $p \in \Delta$.

The uniqueness part of the Lemma is standard and hence omitted. $\square$

Proof of Theorem 1. We first show **the axioms are necessary**. It is immediate to see that (10) satisfies A1 to A5 (v and D continuous implies $u_{[L,G]}$ is continuous). Axioms A6 and A7 require more detail.

A6) Take D and v satisfying the representation. We will show that Midpoint consistency holds. Let $L \leq l < g \leq G$ and $x \sim 0.5G + 0.5L$ and $y \sim 0.5g + 0.5l$. Using the biseparable representation as in (1) yields

$$\begin{aligned} x \succcurlyeq y &\iff D^{-1}(0.5)v(G) + (1 - D^{-1}(0.5))v(L) \geq D^{-1}(0.5)v(g) + (1 - D^{-1}(0.5))v(l) \\ &\iff D^{-1}(0.5) \geq \frac{v(l) - v(L)}{v(G) - v(g) + v(l) - v(L)}. \end{aligned}$$

At the same time, we exploit $D(z) = 1 - D(1 - z)$ to conclude that:

$$\begin{aligned} & 0.5G + 0.5L \succcurlyeq 0.5g + 0.5l \\ \iff & 0.5D(1) + 0.5D(0) \geq 0.5D\left(\tfrac{v(g)-v(L)}{v(G)-v(L)}\right) + 0.5D\left(\tfrac{v(l)-v(L)}{v(G)-v(L)}\right) \\ \iff & \left[1 - D\left(\tfrac{v(g)-v(L)}{v(G)-v(L)}\right)\right] \geq D\left(\tfrac{v(l)-v(L)}{v(G)-v(L)}\right) \\ \iff & D\left(\tfrac{v(G)-v(g)}{v(G)-v(L)}\right) \geq D\left(\tfrac{v(l)-v(L)}{v(G)-v(L)}\right) \\ \iff & 0.5 \geq \frac{v(l) - v(L)}{v(G) - v(g) + v(l) - v(L)}. \end{aligned} \tag{16}$$

Clearly, if $D(0.5) = 0.5$, then $x \succcurlyeq y \iff 0.5G + 0.5L \succcurlyeq 0.5g + 0.5l$, so that Midpoint consistency holds.

A7) Using the representation we can express the antecedent of A7 as: $v(x_i) = D^{-1}(\alpha)v(G_i) + (1 - D^{-1}(\alpha))v(L_i)$, $i = 1, 3$, and $v(G_2) = 0.5v(G_3) + 0.5v(G_1)$, $v(L_2) = 0.5v(L_3) + 0.5v(L_1)$. Using these and the representation we have that $x_2 \sim \alpha G_2 + (1 - \alpha)L_2$ if and only:

$$\begin{aligned} v(x_2) &= D^{-1}(\alpha)v(G_2) + (1 - D^{-1}(\alpha))v(L_2) \\ &= D^{-1}(\alpha)[0.5v(G_3) + 0.5v(G_1)] + (1 - D^{-1}(\alpha))[0.5v(L_3) + 0.5v(L_1)] \\ &= 0.5[D^{-1}(\alpha)v(G_3) + (1 - D^{-1}(\alpha))v(L_3)] + 0.5[D^{-1}(\alpha)v(G_1) + (1 - D^{-1}(\alpha))v(L_1)] \\ &= 0.5v(x_3) + 0.5v(x_1) \end{aligned}$$

and the latter is equivalent to $x_2 \sim 0.5x_3 + 0.5x_1$ as desired.

Given Lemma 1 what is left to prove **sufficiency** is to establish the following biseparable representation: $x \sim \alpha G + (1-\alpha)L \iff v(x) = D^{-1}(\alpha)v(G)) + (1 - D^{-1}(\alpha))v(L)$ for all $G > L$. Indeed note that the RHS of the above equivalence can be rewritten as $D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right) = \alpha$. On the other hand, Lemma 1 implies that the LHS is represented by $u_{[L,G]}(x) = \alpha$. Combining the two gives: $u_{[L,G]}(x) = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$. This can be repeated for any $\alpha \in [0,1]$ and $x \in [L,G]$ and hence the specific form of $u_{[L,G]}$ is established.

We now establish the biseparable representation. For this we adapt Quiggin & Wakker (1994)'s proof, hereafter referred to as QW94. We first show that their Axioms $R.2$, $2'a$, $2'b$, and 3 are implied by ours. Indeed, $R.2$ states that certainty equivalents exist for each prospect. This is implied by representation (9) as applied to lotteries using that u_R is continuous. Because u_R are strictly increasing, certainty equivalents are also unique. Their Axiom $2'a$ states that $\alpha \geq \beta$ implies $\alpha g + (1-\alpha)l \succcurlyeq \beta g + (1-\beta)l$ and is implied by (i) in the proof of The Their Axiom $2'b$ states that $x_1' \succcurlyeq x_1$ and $x_2' \succcurlyeq x_2$ implies $0.5x_1' + 0.5x_2' \succcurlyeq 0.5x_1 + (1-0.5)x_2$. To see it is implied by ours, let $R = \mathrm{Co}(x_1, x_1', x_2, x_2')$ and $\alpha \in (0,1)$. Applying Lemma 3.iv twice we obtain that $x_1' \succcurlyeq x_1$ iff $\alpha x_1' + (1-\alpha)x_2' \succcurlyeq_R \alpha x_1 + (1-\alpha)x_2'$ and that $x_2' \succcurlyeq x_2$ iff $\alpha x_1 + (1-\alpha)x_2' \succcurlyeq_R \alpha x_1 + (1-\alpha)x_2$. By transitivity of $\succcurlyeq_R$ in $A1$ we get $\alpha x_1' + (1-\alpha)x_2' \succcurlyeq_R \alpha x_1 + (1-\alpha)x_2$. By Lemma 3.ii, the latter is equivalent to $\alpha x_1' + (1-\alpha)x_2' \succcurlyeq \alpha x_1 + (1-\alpha)x_2$. Setting $\alpha = 0.5$ yields the desired conclusion.

The above argument also shows that $x_1 \sim x_1', x_2 \sim x_2' \to \alpha x_1 + (1-\alpha)x_2 \sim \alpha x_1' + (1-\alpha)x_2', \alpha \in [0,1]$ (Lemma 3.1 in QW94 restricted to binary prospects). Finally, Lemma 3.iii-iv implies that $x \preccurlyeq y \preccurlyeq z \to y \sim \alpha z + (1-\alpha)x$ for some $\alpha \in [0,1]$ (Axiom 3 of QW94). Indeed, take $x \preccurlyeq y \preccurlyeq z$. By Lemma 3.iv the trivial case of $x \sim y \sim z$ implies $x = y = z$. Assume that $x \succ z$. By Lemma 3.iv, $x \preccurlyeq_{[x,z]} y \preccurlyeq_{[x,z]} z$. Then by Lemma 3.iii, there is a unique $\alpha \in [0,1]$ such that $\alpha z + (1-\alpha)x \sim_{[x,z]} y$, which by Lemma 3.iv implies $\alpha z + (1-\alpha)x \sim y$.

The remaining Axioms 1 and 4 in QW94 are stronger than ours. In particular, our $A1$ is weaker than their standard transitivity. Our Midpoint Independence applies only to certainty equivalents of binary prospects, hence is a special case of their key Axiom 4. We will show that they are sufficient for our representation. Following QW94, we first assume that there are worst and best outcomes $x_0 < x_1$ and derive the representation in this case. This assumption will then be relaxed in the final step of the proof. We divide the proof of sufficiency into 10 steps.

Step 1. *Dyadic construction of* v. By Lemma 3.v, for any $p \in \Delta$, $x_0 \preccurlyeq_{[x_0,x_1]} p \preccurlyeq_{[x_0,x_1]} x_1$. Set $v(x_0) = 0$ and $v(x_1) = 1$. By (9) there exist a unique $x_{1/2} \sim 0.5x_1 + 0.5x_0$. By Lemma 3.ii, Lemma 3.i, $A1$, and Lemma 3.iv, respectively, we have that $x_{1/2} \sim_{[x_0,x_1]} 0.5x_1 + 0.5x_0$, $x_1 \succ_{[x_0,x_1]} 0.5x_1 + 0.5x_0 \succ_{[x_0,x_1]} x_0$, $x_1 \succ_{[x_0,x_1]} x_{1/2} \succ_{[x_0,x_1]} x_0$, and $x_1 \succ x_{1/2} \succ x_0$. Similarly, find $x_{1/4} \sim 0.5x_{1/2} + 0.5x_0$, $x_{3/4} \sim 0.5x_1 + 0.5x_{1/2}$, $x_{1/8}, x_{3/8}, ...,$, and inductively every $x_{k/2^n}$. Then, set $v(x_{k/2^n}) = k/2^n$. By construction, $v(x_{k/2^n}) \geq v(x_{k'/2^{n'}})$ iff $x_{k/2^n} \succcurlyeq x_{k'/2^{n'}}$.

Step 2. $0.5x_{k/2^n} + 0.5x_{k'/2^{n'}} \sim x_{(k/2^n + k'/2^{n'})/2}$. When restricted to equal chance binary mixtures, Midpoint Consistency and Midpoint Independence jointly imply Axiom 4 in QW94, which is all that they invoke to establish this indifference.

Step 3. *Extending the domain of* v. Define $v : x \mapsto \sup\{v(x_{k/2^n}) : x_{k/2^n} \preccurlyeq x\}$. For $p \in \Delta$, define $U(p) := v(x)$ where $x \sim p$ is the certainty equivalent. Following QW94 we then show that:

a) $x' \succcurlyeq x \to v(x') \geq v(x)$: follows from the definition of the extended v.

b) $p \succcurlyeq q \to U(p) \geq U(q)$: in our case, it holds in two special cases:

   b.1) Both p and q are binary equal chance prospects,

   b.2) Either p and q have the same range, or one of the prospects is degenerate with support contained in the range of the other prospect.

c) $U(0.5l + 0.5g) = 0.5v(g) + 0.5v(l)$: this holds in our case as the proof invokes Axiom $2'b$ (implied by our axioms), a), b.1), and Step 2.

Step 4. *Construction of* $w = D^{-1} : [0,1] \to [0,1]$. For $\alpha \in [0,1]$, define $w(\alpha) = U(\alpha x_1 + (1-\alpha)x_0)$. Obviously, $w(0) = 0$, $w(1) = 1$, and, by definition of $x_{1/2}$, $w(0.5) = 0.5$. Since $u_{[x_0,x_1]}$ in Lemma 1 is strictly increasing and continuous, then so is w. Hence, its inverse D exists and is also strictly increasing and continuous.

Step 5. v *is surjective*. It follows from their axiom $R.2$ and b.2), both implied by our axioms.

Step 6. v *represents preferences over outcomes*. In view of a), the missing part is the implication $x' \succ x \to v(x') > v(x)$. The proof invokes their Axioms 3, $2'a$, $2'b$, Step 5, and c), all implied by our axioms. Unlike QW94, that v represents preferences over outcomes does not imply U represents preferences over (binary) prospects. Yet the two special cases in which it does are $b.1$) and $b.2$). In particular, for all outcomes l, l', g, g':

$$\tfrac{1}{2}l + \tfrac{1}{2}g \succcurlyeq \tfrac{1}{2}l' + \tfrac{1}{2}g' \iff \tfrac{1}{2}v(l) + \tfrac{1}{2}v(g) \geq \tfrac{1}{2}v(l') + \tfrac{1}{2}v(g'). \tag{17}$$

Step 7. *Jensen's (functional) Equation*. Let α be fixed. Since v represents $\succcurlyeq$ over outcomes, and because of Lemma 3.1 of QW94 (implied by our axioms) one can write for $l \leq g$, $U((1-\alpha)l + \alpha g) = W(v(l), v(g))$ for a function W. For simplicity of notation, from now on we will identify outcomes with their v values in this step. The domain of W is the set of all $(l, g) \in [0,1]^2$ with $l \leq g$. We show that W satisfies Jensen's equation, i.e. $W\left(\frac{(l,g)+(l',g')}{2}\right) = \frac{W(l,g)+W(l',g')}{2}$ for all $(l, g), (l', g')$ in the domain of W. For simplicity of notation, we write x_i', G_i', L_i' for $v(x_i), v(G_i), v(L_i)$, respectively. Assume the antecedent of Midpoint Independence and $x_2 \sim \alpha G_2 + (1-\alpha)L_2$. Hence $x_2' = U(\alpha G_2 + (1-\alpha)L_2) = W(L_2', G_2') = W\left(\frac{(L_1',G_1')+(L_3',G_3')}{2}\right)$, where the last equality holds because $L_2' = 0.5L_3' + 0.5L_1'$ and $G_2' = 0.5G_3' + 0.5G_1'$ as implied by c). At the same time, by Midpoint Independence, $x_2 \sim 0.5x_3 + 0.5x_1$, hence $x_2' = 0.5x_3' + 0.5x_1'$ by c). By the antecedent of Midpoint Consistency, $x_3' = U(\alpha G_3 + (1-\alpha)L_3)$ and $x_1' = U(\alpha G_1 + (1-\alpha)L_1)$, so that $x_2' = \frac{W(L_1',G_1')+W(L_3',G_3')}{2}$. Combining gives the Jensen's equation.

Step 8. *The representation*. Let α be fixed. By definition of w, $w(1) = U(x_1) = W(1,1)$ and $w(\alpha) = U((1-\alpha)x_0 + \alpha x_1) = W(0,1)$. As argued by QW94 even though the solutions to Jensen's equation exist that are nonlinear, these are excluded in our case since W is bounded from below (nonnegative). It follows that W is linear and hence $x \sim \alpha g + (1-\alpha)l$ implies $v(x) = U((1-\alpha)l + \alpha g) = W(v(l), v(g)) = W[v(l), v(l) + (v(g) - v(l))] = v(l)W(1,1) + [v(g) - v(l)]W(0,1) = v(l) + [v(g) - v(l)]w(\alpha)$, which is the representation we seek. Because D and u_R are continuous and strictly increasing, so is v. Cardinal uniqueness of v and uniqueness of $w = D^{-1}$ is straightforward give the uniqueness part of Lemma 1 and the definitions of v and w.

Step 9. *Extending the representation beyond* $[x_0, x_1]$. For each $y \succcurlyeq x_1 \succ x_0 \succcurlyeq z$, we can construct a representation for prospects with outcomes $\{x \in \mathbb{R} : y \succcurlyeq x \succcurlyeq z\}$ as already done. By the cardinal uniqueness of v and uniqueness of D^{-1}, we can ensure that the two representations coincide on the overlapping domain and this uniquely determines the extended representation over the remaining domain.

Because prospects have finite supports, outcomes involved in any prospect are bounded, and the representation is uniquely determined for all prospects.

Step 10. *D satisfies $D(z) = 1-D(1-z)$, $z \in [0,1]$.* Suppose $L \leq l \leq g \leq G$, $L < G$ and $\frac{1}{2}g+\frac{1}{2}l \sim \frac{1}{2}G+\frac{1}{2}L$. By the representation, $\frac{1}{2}D\left(\frac{v(g)-v(L)}{v(G)-v(L)}\right)+\frac{1}{2}D\left(\frac{v(l)-v(L)}{v(G)-v(L)}\right) = \frac{1}{2}$, or $D(z) = 1 - D\left(\frac{v(l)-v(L)}{v(G)-v(L)}\right)$, where $z = \frac{v(g)-v(L)}{v(G)-v(L)}$. At the same time, by (17), $\frac{1}{2}v(g) + \frac{1}{2}v(l) = \frac{1}{2}v(G) + \frac{1}{2}v(L)$ or $\frac{v(l)-v(L)}{v(G)-v(L)} = 1 - z$, and the result follows. Finally, note that by continuity and strict monotonicity of v, z can take any value in between $[0, 1]$. This concludes the proof of existence. □

Proof of Lemma 2. Consider $[L, G] \in \mathcal{R}$. Take two prospects p, q with $R(p, q) \subset [L, G]$ and fix them. Suppose that for some $\alpha \in (0, 1)$ and some prospect r with $R(r) = [L, G]$,

$$\alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r. \tag{18}$$

We want to prove that $\beta p+(1-\beta)s \succcurlyeq \beta q+(1-\beta)s$ for all $\beta \in (0, 1)$ and all prospects s with $R(s) = [L, G]$. Define $p' := \alpha p+(1-\alpha)r$ and $q' := \alpha q+(1-\alpha)r$. It is clear that $R(p', q') = [L, G]$. We first show that r in (18) can be replaced by any prospect s with $R(s) = [L, G]$. Note that $\alpha \in (0, 1)$ can be written as $\alpha = \gamma\rho$ for some $\gamma, \rho \in (0, 1)$. Note that $p' = \gamma(\rho p+(1-\rho)r)+(1-\gamma)r$ and $q' = \gamma(\rho q+(1-\rho)r)+(1-\gamma)r$. So by A3$'$, $p' \succcurlyeq q'$ implies $\rho p + (1-\rho)r \succcurlyeq \rho q + (1-\rho)r$, which – again by A3$'$ – implies that $\gamma(\rho p+(1-\rho)r)+(1-\gamma)s \succcurlyeq \gamma(\rho q+(1-\rho)r)+(1-\gamma)s$ for any prospect s with $R(s) = [L, G]$. Rewriting, rearranging and substituting $1-\beta = \gamma(1-\rho)$ we obtain $\beta(\gamma\rho/\beta p+(1-\gamma)/\beta s)+(1-\beta)r \succcurlyeq \beta(\gamma\rho/\beta q+(1-\gamma)/\beta s)+(1-\beta)r$ which by applying A3$'$ twice implies $\beta(\gamma\rho/\beta p+(1-\gamma)/\beta s)+(1-\beta)s \succcurlyeq \beta(\gamma\rho/\beta q+(1-\gamma)/\beta s)+(1-\beta)s$. That in turn after simplifying and substituting $\alpha = \gamma\rho$ gives $\alpha p + (1-\alpha)s \succcurlyeq \alpha q + (1-\alpha)s$ for any prospect s with $R(s) = [L, G]$.

We now show that α in (18) can be replaced by any $\beta \in (0, \alpha]$. Any such β can be written as $\beta = \theta\alpha$ for $\theta \in (0, 1]$. By A3$'$ (with p', q' defined as before) $p' \succcurlyeq q'$ implies $\theta p' + (1-\theta)r \succcurlyeq \theta q' + (1-\theta)r$ for all $\theta \in (0, 1]$. Rewriting gives $\beta p + (1-\beta)r \succcurlyeq \beta q + (1-\beta)r$.

Finally we show that α in (18) can be replaced by any $\beta \in (\alpha, 1)$. Any such β can be written as α/θ, for $\theta \in (\alpha, 1)$. Note that $p' \succcurlyeq q'$ can then be rewritten as $\theta(\beta p + (1-\beta)r) + (1-\theta)r \succcurlyeq \theta(\beta q + (1-\beta)r) + (1-\theta)r$, which by A3$'$ implies $\beta p + (1-\beta)r \succcurlyeq \beta q + (1-\beta)r$. Since this is true for any $\theta \in (\alpha, 1)$, the latter is true for any $\beta \in (\alpha, 1)$. Summing up, α in (18) can be replaced by any $\beta \in (0, 1)$.

For any $\beta \in (0,1)$ we can repeat the first step of the proof and show that r can be replaced by any prospect s with $R(s) = [L, G]$. $\square$

Proof of Theorem 2. ($\Leftarrow$) Given a choice function Γ, let $\succcurlyeq$ be a preference relation satisfying A1 and A3$'$ that is consistent with Γ. We prove:

Γ *satisfies the within-range axiom of revealed-preference*: Consider two contexts C, C' having the same range $[L, G]$ and two prospects p, q, both contained in $C \cap C'$, such that $p \in \Gamma(C)$ and $q \in \Gamma(C')$. We need to show that $p \in \Gamma(C')$. Because Γ is consistent with $\succcurlyeq$, p is in $\Gamma(C)$ and q is in C, hence $p \succcurlyeq_{[L,G]} q$. By the definition of $\succcurlyeq_{[L,G]}$, the latter is equivalent to $\alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r$, for some $\alpha \in (0,1)$ and $r \in \Delta$ such that $R(r) = [L, G]$. Similarly, because Γ is consistent with $\succcurlyeq$ and $q \in \Gamma(C')$, so $q \succcurlyeq_{[L,G]} s$ for all $s \in C'$, which by the definition of $\succcurlyeq_{[L,G]}$ implies $\alpha' q + (1-\alpha')r' \succcurlyeq \alpha' s + (1-\alpha')r'$, for some $\alpha' \in (0,1)$ and $r' \in \Delta$ such that $R(r') = [L, G]$. By Lemma 2 we can choose $\alpha' = \alpha$ and $r' = r$. Hence, by A1, $\alpha p + (1-\alpha)r \succcurlyeq \alpha s + (1-\alpha)r$. Finally by definition of $\succcurlyeq_{[L,G]}$, we obtain $p \succcurlyeq_{[L,G]} s$. This is true for all $s \in C'$, which, by consistency with Γ is equivalent to $s \in \Gamma(C')$.

Γ *satisfies within-range independence*: Consider a context C. We need to show that $p \in \Gamma(C)$ if and only if $\alpha p + (1-\alpha)r \in \Gamma(\alpha C + (1-\alpha)r)$ for any $\alpha \in (0,1]$ and prospect r such that $R(C \cup \{r\}) = R(C)$. Consider a context C such that $R(C) = [L, G]$ and take any $p \in C$. Since $\succcurlyeq$ is consistent with Γ, $p \in \Gamma(C) \iff p \succcurlyeq_{[L,G]} q$ for all $q \in C$. By definition, $p \succcurlyeq_{[L,G} q$ is equivalent to $\alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r$ for some $\alpha \in (0,1)$ and prospect r such that $R(r) = [L, G]$. Setting $\alpha p + (1-\alpha)r$, $\alpha q + (1-\alpha)r$, r and $\gamma \in (0,1)$ for p, q, r, α, respectively, in A3$'$, we obtain that the latter is equivalent to $\gamma(\alpha p + (1-\alpha)r) + (1-\gamma)r \succcurlyeq \gamma(\alpha q + (1-\alpha)r) + (1-\gamma)r$. By definition this is equivalent to $\alpha p + (1-\alpha)r \succcurlyeq_{[L,G]} \alpha q + (1-\alpha)r$. Because $\succcurlyeq$ is consistent with Γ, we conclude that $\alpha p + (1-\alpha)r \in \Gamma(\alpha C + (1-\alpha)r)$.

($\Rightarrow$) Let Γ be a choice function satisfying the within-range axiom of revealed-preference and within-range independence. Consider $\succcurlyeq$ defined as $p \succcurlyeq q$ if $p \in \Gamma(\{p, q\})$. We first show that $\succcurlyeq$ satisfies A1 and A3$'$ and then, that it is consistent with Γ. Since $\Gamma(C)$ is nonempty for any context C, $\succcurlyeq$ is complete. We now prove transitivity. Consider prospects p, q, r such that $R(p, q) = R(q, r) = R(p, r)$ and suppose that $p \succcurlyeq q$ and $q \succcurlyeq r$. By definition of Γ, $p \in \Gamma(\{p, q\})$ and $q \in \Gamma(\{q, r\})$. We want to show that $p \in \Gamma(\{p, r\})$ and hence $p \succcurlyeq r$. We first show that $p \in \Gamma(\{p, q, r\})$. Since Γ is nonempty, there are three possibilities. If $p \in \Gamma(\{p, q, r\})$, then we are done. If $q \in \Gamma(\{p, q, r\})$, then since $p \in \Gamma(\{p, q\})$ and $R(p, q, r) = R(p, q)$ (this is implied by $R(p, q) = R(q, r) = R(p, r)$), so by the within-range axiom of revealed preference we have that $p \in \Gamma(\{p, q, r\})$. The third case is when $r \in \Gamma(\{p, q, r\})$. Since $q \in \Gamma(\{q, r\})$ and $R(p, q, r) = R(q, r)$, then by the within-range axiom of revealed preference $q \in \Gamma(\{p, q, r\})$ and by the previous argument $p \in \Gamma(\{p, q, r\})$. We now prove that $p \in \Gamma(p, r)$. Since $\Gamma(p, r)$ is nonempty, it either contains p and we are done, or r. In the latter case, since $p \in \Gamma(\{p, q, r\})$ and $R(p, q, r) = R(p, r)$, by the within-range axiom of revealed preference, we get the desired conclusion, and hence $p \succcurlyeq r$. This finishes the proof of transitivity.

Let p, q, r be three prospects such that $R(p, q) = R(p, r) = R(q, r)$. Then obviously $R(p, q, r) = R(p, q)$ and A3′ follows directly from the definition of $\succcurlyeq$ and within-range independence of Γ. We now prove (11). Take $C \in \mathcal{C}$. If $p \in \Gamma(C)$, then take prospect r such that $R(r) = R(C) = R$ and some $\alpha \in (0, 1)$. By within-range independence $\alpha p + (1-\alpha)r \in \Gamma(\alpha C + (1-\alpha)r)$. The within-range axiom of revealed preference then implies that $\alpha p + (1-\alpha)r \in \Gamma(\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r)$ for any $q \in C$, which by definition of $\succcurlyeq$, is equivalent to $\alpha p + (1-\alpha)r \succcurlyeq \alpha q + (1-\alpha)r$ for any $q \in C$. This in turn implies $p \succcurlyeq_R q$, for all $q \in C$. We now prove the reverse direction of (11). For a contradiction, suppose that $p \prec_R q$, for some $q \in C$. This is equivalent to $\alpha p + (1-\alpha)r \prec \alpha q + (1-\alpha)r$ for some $\alpha \in (0, 1)$ and $r \in \Delta$ s.t. $R(r) = R(C) = R$. By definition of $\succcurlyeq$, it is in turn equivalent to $\alpha p + (1-\alpha)r \notin \Gamma(\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r)$ and since Γ is nonempty it implies that $\alpha q + (1-\alpha)r = \Gamma(\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r)$. By the within-range axiom of revealed preference, $\alpha p + (1-\alpha)r \notin \Gamma(\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r)$ implies that either $\alpha q + (1-\alpha)r \notin \Gamma(\alpha p + (1-\alpha)r, \alpha q + (1-\alpha)r)$ or $\alpha p + (1-\alpha)r \notin \Gamma(\alpha C + (1-\alpha)r)$ for any C containing both p and q. Since the first possibility is not true, then the latter must be true. But by within-range independence the latter implies $p \notin \Gamma(C)$, which proves the reverse direction of (11) and completes the proof. $\square$

Proof of Theorem 3. Necessity is analogous as in the proof of Theorem 1. The only difference is $\overline{A}7$, which we now prove:

$\overline{A}7$) Take α and the corresponding D and v satisfying the representation. Take some $L < x < G$ such that $x \sim \beta G + (1-\beta)L$ and $v(x) = \beta' v(G) + (1-\beta')v(L)$. By (12), the former is equivalent to $D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right) = \beta$ while the latter can be rewritten as $\beta' = \frac{v(x)-v(L)}{v(G)-v(L)}$. Take any $L' < x' < G'$ such that $\beta' = \frac{v(x')-v(L')}{v(G')-v(L')}$. Since D does not depend on $[L, G]$, $D\left(\frac{v(x')-v(L')}{v(G')-v(L')}\right) = \beta$, which is equivalent to $x' \sim \beta G' + (1-\beta)L'$.

Next, we prove **sufficiency**. For $\alpha \in (0, 1)$ define $\Delta_\alpha := \{p \in \Delta : p = \alpha G + (1-\alpha)L, L \leq G\}$, i.e. the set of all α-binary prospects, possibly degenerate ones. In steps 1–5 we will show that the preferences over Δ_α, where α satisfies α-consistency, possess an EU representation with vNM utility function υ, as constructed using the α-outcome grid and then extended to $[x(0), x(1)]$. In step 3 and 6 we construct a continuous and strictly increasing range distortion function D and show that the $u_{[L,G]}$ from Lemma 1 conforms with (12). Step 7 verifies (4) and step 8 shows that α is either unique or universal. Step 9 proves the relative uniqueness of D and υ which combined with the uniqueness of $u_{[L,G]}$ from Lemma 1 gives uniqueness of D and υ. In steps 1-9 we assume that for some outcomes $x(0) < x(1)$, $x(0) \succcurlyeq x \succcurlyeq x(1)$ for all $x \in X$ and construct the utility on the interval $[x(0), x(1)]$. In step 10 we relax this assumption and extend the representation beyond $[x(0), x(1)]$.

Step 1. *υ EU-represents certainty equivalents on Δ_α.*

Take $\alpha \in (0, 1)$ satisfying α-consistency (exist by $\overline{A6}$) and fix it for Steps 1–7. Take υ as constructed using the α-binary prospects. Given the representation in Lemma 1, certainty equivalents exist due to continuity of u_R, and by strict monotonicity of u_R, $k < k'$ implies $x(k/2^n) < x(k'/2^n)$. Hence the α grid on $[x(0), x(1)]$ is well defined and $\upsilon(x((2k+1)/2^{n+1})) = \alpha\upsilon(x((k+1)/2^n)) + (1-\alpha)\upsilon(x(k/2^n))$ holds.

We now apply $\overline{A7}$ to show that υ EU-represents CEs on Δ_α. For $n \geq 1$ and $k \in \{0, ..., 2^n - 1\}$, let $x = x((2k+1)/2^{n+1})$, $g = x((k+1)/2^n)$ and $l = x(k/2^n)$. By construction $x \sim \alpha g + (1-\alpha)l$ and $\upsilon(x) = \alpha\upsilon(g) + (1-\alpha)\upsilon(l)$. By $\overline{A7}$, for any $l' < x' < g'$, $x' \sim \alpha g' + (1-\alpha)l' \iff \upsilon(x') = \alpha\upsilon(g') + (1-\alpha)\upsilon(l')$.

Recall that $\upsilon(x) = \sup\{\upsilon(x(k/2^n)) : x(k/2^n) \preccurlyeq x\}$, $x \in [x(0), x(1)]$, and hence it follows straightforwardly that υ is nondecreasing. We must prove it is strictly increasing, but for that we need the following three intermediate steps:

Step 2. *A_α is dense in* $[0, 1]$.

Let $A_{0.5}$ be the set of dyadic rationals on $[0, 1]$, and $a : A_{0.5} \to A_\alpha$ map each dyadic number $k/2^n$ to its counterpart $a(k/2^n)$. Because a is monotonic and $A_{0.5}$ is dense, it can extend to a continuous function $\bar{a}$. Because $\bar{a} : [0, 1] \to [0, 1]$ is continuous and onto, and $A_{0.5}$ is dense in $[0, 1]$, we have that $A_\alpha = \bar{a}(A_{0.5})$ is dense in $[0, 1]$.

Step 3. *Construction of a continuous and nondecreasing D.*

For $\beta \in [0, 1]$, define $D^{-1} : [0, 1] \to [0, 1]$ as $D^{-1}(\beta) = \upsilon(x)$, where $x \sim \beta x(1) + (1-\beta)x(0)$. Note that $D^{-1}(0) = 0$, $D^{-1}(1) = 1$ and $D^{-1}(\alpha) = \alpha$, for

α-consistent. To see that D^{-1} is nondecreasing, let $\beta > \beta'$. By (i), $\beta x(1) + (1 - \beta)x(0) \succ \beta' x(1) + (1 - \beta')x(0)$. Let x, x' be such that: $x \sim \beta x(1) + (1 - \beta)x(0)$, $x' \sim \beta' x(1) + (1 - \beta')x(0)$. By A1 applied to the range $[x(0), x(1)]$ $x \succ x'$ and by A4 $x > x'$. Since v is nondecreasing (Step 1), so $v(x) \geq v(x')$ and hence $D^{-1}(\beta) \geq D^{-1}(\beta')$. Also D^{-1} is continuous: for every $x(k/2^n)$ there exists a β such that $\beta x(1) + (1 - \beta)x(0) \sim x(k/2^n)$, i.e. $D^{-1}(\beta) = a(k/2^n)$. By Step 2, A_α is dense in $[0, 1]$ so the range of D^{-1} is dense in $[0, 1]$. The nondecreasing D^{-1} cannot have jumps and hence must be continuous.

Step 4. *v is surjective.*

To show that $v(X) = [0, 1]$, let $\beta' \in [0, 1]$. By the properties of D, we can find β such that $\beta' = D^{-1}(\beta)$. By Lemma 1, there exists $x \sim \beta x(1) + (1 - \beta)x(0)$, which by construction satisfies $\beta' = v(x)$.

Step 5. *v is strictly increasing and* (2) *holds.*

We know v is nondecreasing, hence remains to establish that v is strictly increasing. Suppose not, e.g., there are $x > y$ with $v(x) = v(y)$. We define $\beta = v(x) = v(y)$. Either $\beta \neq 0$ or $\beta \neq 1$. We assume the latter. Take any $\beta < \theta < 1$. By Step 4 there exists x_θ such that $v(x_\theta) = \theta$. Let $x_{\alpha\theta+(1-\alpha)\beta} \sim \alpha x_\theta + (1 - \alpha)x$ and $y_{\alpha\theta+(1-\alpha)\beta} \sim \alpha x_\theta + (1 - \alpha)y$. By Step 1, $v(x_{\alpha\theta+(1-\alpha)\beta}) = v(y_{\alpha\theta+(1-\alpha)\beta}) = \alpha\theta + (1 - \alpha)\beta$ whereas A4 and $x > y$ imply $\alpha x_\theta + (1 - \alpha)x \succ \alpha x_\theta + (1-\alpha)y$, so by $\overline{A6}$, $x_{\alpha\theta+(1-\alpha)\beta} > y_{\alpha\theta+(1-\alpha)\beta}$. Such outcomes can be constructed for each $\theta \in (\beta, 1)$ (uncountable number) and $\{[y_{\alpha\theta+(1-\alpha)\beta}, x_{\alpha\theta+(1-\alpha)\beta}]\}_{\theta\in(\beta,1)}$ gives an uncountable number of mutually disjoint real intervals, which is impossible because it would lead to uncountably many distinct rational numbers, one for each interval $(y_{\alpha\theta+(1-\alpha)\beta}, x_{\alpha\theta+(1-\alpha)\beta})$, a contradiction. So v must be strictly increasing.

We now prove that v EU-represents preferences over Δ_α. Take two triples $L < x < G$ and $L' < x' < G'$ such that $x \sim \alpha G + (1-\alpha)L$ and $x' \sim \alpha G' + (1-\alpha)L'$. By Step 1, $v(x) = \alpha v(G) + (1 - \alpha)v(L)$ and $v(x') = \alpha v(G') + (1 - \alpha)v(L')$. Since v is representing on X, so $v(x) \geq v(y) \iff x \geq y$ and by $\overline{A6}$ the latter is equivalent to $\alpha G + (1 - \alpha)L \succeq \alpha G' + (1 - \alpha)L'$. So (2) follows.

Step 6. *$u_{[L,G]}$ conforms with* (10). By Step 5 v is strictly increasing. So D^{-1} must be strictly increasing as well by repeating the same argument as in Step 3 with strictly increasing instead of nondecreasing v. So the inverse function D exists which is also strictly increasing and continuous. By Step 3, $x' \sim \beta x(1) + (1 - \beta)x(0)$

implies that $v(x') = D^{-1}(\beta)$ for any $\beta \in [0,1]$. By Lemma 1, $u_{[x(0),x(1)]}(x') = \beta$. So $u_{[x(0),x(1)]}(x) = D(v(x')) = \beta$. Define $v(x') = \beta' = \beta' v(x(1)) + (1-\beta')vx((0))$. Consider $x \in [L,G]$ such that $x \sim \beta G + (1-\beta)L$. By Lemma 1, $u_{[L,G]}(x) = \beta$. On the other hand, by $\overline{A7}$, it must be that $v(x) = \beta' v(G) + (1-\beta')v(L)$ or $\frac{v(x)-v(L)}{v(G)-v(L)} = \beta' = D^{-1}(\beta)$ or $\beta = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right)$. So $u_{[L,G]}(x) = D\left(\frac{v(x)-v(L)}{v(G)-v(L)}\right) = \beta$. Because this holds for any β, and $u_{[L,G]} : [L,G] \to [0,1]$ is surjective, such $u_{[L,G]}(x)$ is defined for any $x \in [L,G]$. Finally, we substitute this utility form in Lemma 1 to obtain (12). By Lemma 1, for $p,q \in \Delta$ such that $R(p,q) = [L,G]$, $p \succcurlyeq q \iff \beta \geq \beta' \iff \beta G + (1-\beta)L \succcurlyeq \beta' G + (1-\beta)L$ where $\beta = \sum_{x\in X} p(x)u_{[L,G]}(x)$, $\beta' = \sum_{x\in X} q(x)u_{[L,G]}(x)$. Applying the definition of $u_{[L,G]}(x)$ for each x yields (12).

Step 7. *D satisfies condition* (4). Suppose $L \leq l \leq g \leq G$, $L < G$ and $\alpha g + (1-\alpha)l \sim \alpha G + (1-\alpha)L$. Using the representation, we have that $\alpha D\left(\frac{v(g)-v(L)}{v(G)-v(L)}\right) + (1-\alpha)D\left(\frac{v(l)-v(L)}{v(G)-v(L)}\right) = \alpha$, or

$$D\left(\frac{v(g)-v(L)}{v(G)-v(L)}\right) = 1 - \frac{1-\alpha}{\alpha} D\left(\frac{v(l)-v(L)}{v(G)-v(L)}\right).$$

At the same time, and because of the particular use of α mixtures, v also represents this indifference. Hence, $\alpha v(g) + (1-\alpha)v(l) = \alpha v(G) + (1-\alpha)v(L)$ or

$$\frac{v(l)-v(L)}{v(G)-v(L)} = \frac{\alpha}{1-\alpha}\left(1 - \frac{v(g)-v(L)}{v(G)-v(L)}\right).$$

Replacing this expression into the right hand side of the former yields the result. This can be done for different values of l ranging from $[L,x]$ and g ranging from $[x,G]$ such that $\alpha g + (1-\alpha)l \sim \alpha G + (1-\alpha)L \sim x$. Because $\alpha = \frac{v(x)-v(L)}{v(G)-v(L)}$, it follows that $\frac{v(g)-v(L)}{v(G)-v(L)} \geq \alpha$. Hence, D satisfies (4) for $z \in [\alpha, 1]$.

Step 8. *Uniqueness of* α. Clearly, (4) is compatible with $D(z) = z$ for all $z \in [0,1]$, or with D having a unique interior fixed point. We now show these are the only possibilities. In what follows we will 'pivot' points around α, i.e., obtain $D(z)$ from $D\left(\frac{\alpha}{1-\alpha}(1-z)\right)$ using (4). First, assume the set of fixed points is dense somewhere. By continuity of D, the set of fixed points is closed. Hence, it contains an interval, say $[a,b]$. Use $\alpha = b$ to pivot each point in the interval and conclude that the larger interval $[a, a+\frac{b-a}{b}]$ must also be a fixed point. Applying this repeatedly on both sides results in $[0,1]$ being a fixed point. Next, assume the set of fixed points is nowhere dense, and that $0 < \alpha < \beta < 1$ are two fixed points, with no fixed points on (α, β). Pivoting α around β yields $\beta_1 = \beta + \frac{1-\beta}{\beta}(\beta - \alpha)$, with no fixed points on (β, β_1). Pivoting β around α yields $\alpha_1 = \alpha - \frac{\alpha}{1-\alpha}(\beta - \alpha)$ with no fixed points on (α_1, α); and pivoting β_1 around α yields $\alpha_2 = \alpha - \frac{\alpha}{1-\alpha}(\beta_1 - \alpha)$. Next, we reverse the direction and pivot α_2 around α_1. The result should yield α, as otherwise we are not recovering the original points. Hence, $\alpha_1 + \frac{1-\alpha_1}{\alpha_1}(\alpha_1 - \alpha_2) = \alpha$, which reduces to $(\beta - \alpha)(1 - 2\alpha) = 0$. Thus, either $\beta = \alpha$ (a contradiction), or $\alpha = 1/2$. Next, we pivot α_1 around β to obtain $\beta_2 = \beta + \frac{1-\beta}{\beta}(\beta - \alpha_1)$. Again, pivoting β_2 around β_1 should return β. This produces the condition $\beta_1 - \frac{\beta_1}{1-\beta_1}(\beta_2 - \beta_1) = \beta$, which reduces to $(\beta - \alpha)(\alpha + \beta - 3\alpha\beta) = 0$. Thus, either $\beta = \alpha$ (a contradiction), or $\beta = \frac{\alpha}{3\alpha - 1}$. But if $\alpha = 1/2$, then $\beta = 1$, a contradiction. We conclude that either $D(z) = z$, $z \in [0, 1]$, or there is only one interior fixed point.

Step 9. *Relative uniqueness of D and v.*

Suppose there is (v_1, D_1) and (v_2, D_2) satisfying, for all $[L, G]$ and $x \in [L, G]$, $D_1\left(\frac{v_1(x)-v_1(L)}{v_1(G)-v_1(L)}\right) = D_2\left(\frac{v_2(x)-v(L)}{v_2(G)-v_2(L)}\right)$. Let $f = D_1^{-1} \circ D_2$, so that $\frac{v_1(x)-v_1(L)}{v_1(G)-v_1(L)} = f\left(\frac{v_2(x)-v(L)}{v_2(G)-v_2(L)}\right)$. We shall now use the standard Cauchy functional equations to show that f must be simultaneously power and inverse power, hence linear.

Fix L. For any $G' \in (L, G)$ and $x \in [L, G']$, we have that

$$\frac{v_1(x)-v_1(L)}{v_1(G)-v_1(L)} = \frac{v_1(x)-v_1(L)}{v_1(G')-v_1(L)} \frac{v_1(G')-v_1(L)}{v_1(G)-v_1(L)},$$

$$f\left(\frac{v_2(x)-v(L)}{v_2(G)-v_2(L)}\right) = f\left(\frac{v_2(x)-v(L)}{v_2(G')-v_2(L)}\right) f\left(\frac{v_2(G')-v_2(L)}{v_2(G)-v_2(L)}\right),$$

$$f\left(\frac{v_2(x)-v(L)}{v_2(G')-v_2(L)} \frac{v_2(G')-v_2(L)}{v_2(G)-v_2(L)}\right) = f\left(\frac{v_2(x)-v(L)}{v_2(G')-v_2(L)}\right) f\left(\frac{v_2(G')-v_2(L)}{v_2(G)-v_2(L)}\right).$$

Letting $y = \frac{v_2(x)-v(L)}{v_2(G')-v_2(L)}$ and $z = \frac{v_2(G')-v_2(L)}{v_2(G)-v_2(L)}$, we obtain $f(yz) = f(y)f(z)$. By Aczél (1966, p. 145), the function f must be power. Thus, $D_2(p) = D_1(p^\beta)$, $\beta > 0$.

Next, fix G. For any $L' \in (L, G)$ and $x \in [L', G]$, we have that

$$\begin{aligned}\frac{v_1(x)-v_1(L)}{v_1(G)-v_1(L)} &= \frac{v_1(x)-v_1(L')}{v_1(G)-v_1(L)} + \frac{v_1(L')-v_1(L)}{v_1(G)-v_1(L)} \\ &= 1 - \left(1 - \frac{v_1(L')-v_1(L)}{v_1(G)-v_1(L)}\right)\left(1 - \frac{v_1(x)-v_1(L')}{v_1(G)-v_1(L')}\right),\end{aligned}$$

$$f\left(\frac{v_2(x)-v_2(L)}{v_2(G)-v_2(L)}\right) = 1 - \left(1 - f\left(\frac{v_2(x)-v_2(L')}{v_2(G)-v_2(L')}\right)\right)\left(1 - f\left(\frac{v_2(L')-v_2(L)}{v_2(G)-v_2(L)}\right)\right).$$

Replacing $\frac{v_2(x)-v_2(L)}{v_2(G)-v_2(L)} = 1 - \left(1 - \frac{v_2(L')-v_2(L)}{v_2(G)-v_2(L)}\right)\left(1 - \frac{v_2(x)-v_1(L')}{v_2(G)-v_2(L')}\right)$ and letting $y = \frac{v_2(L')-v_2(L)}{v_2(G)-v_2(L)}$ and $z = \frac{v_2(x)-v_2(L')}{v_2(G)-v_2(L')}$, we obtain $1 - f(1 - (1 - y)(1 - z)) = (1 - f(1 -$

$(1-y)))(1-f(1-(1-z)))$. By Aczél (1966, p. 145), the function $1-f(1-p)$ must be power, which implies $D_2(p)=D_1(1-(1-p)^{\beta'})$, $\beta'>0$. But if $D_1(p^{\beta})=D_1(1-(1-p)^{\beta'})$, then $p^{\beta}+(1-p)^{\beta'}=1$, which holds iff $\beta=\beta'=1$. Hence, $f(p)=p$, $D_1=D_2$, and $v_1=v_2$.

Step 10. *Extend the construction beyond* $[x(0), x(1)]$.

If all outcomes are indifferent, then all prospects are indifferent and the result is trivial. Otherwise, fix some $x(1)\succ x(0)$. For each $x'\succcurlyeq x(1)\succ x(0)\succcurlyeq x''$ we can repeat the process for $x(1), x(0)$ and construct a representation for prospects with outcomes $\{x\in X : x'\succcurlyeq x\succcurlyeq x''\}$. By uniqueness of D and v, the new representation can be made to coincide with the old one for prospects over the overlapping range $[x(0), x(1)]$. The support of any prospect is finite and hence bounded, so the new representation is uniquely identified for all prospects. This completes the proof of Theorem 3. □

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