Homework 1

Due January 22, 2018 PHY 204B

Problem 9.5.2. If Ψ is a solution of Laplace's equation, $\nabla^2\Psi = 0$, show that $\partial\Psi/\partial z$ is also a solution.

Problem 9.6.2. Solve the wave equation, Eq. (9.89), subject to the indicated conditions. Determine $\psi(x,t)$ given that at $t = 0$, $\psi_0(x) = \delta(x)$ (Dirac delta function) and the initial time derivative of ψ is zero.

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} \tag{9.89}$$

Problem 9.7.3. Solve the PDE

$$\frac{\partial\psi}{\partial t} = a^2\frac{\partial^2\psi}{\partial x^2},$$

to obtain $\psi(x,t)$ for a rod of infinite extent (in both the $+x$ and $-x$ directions), with a heat pulse at time $t = 0$ that corresponds to $\psi_0(x) = A\delta(x)$.

Problem 19.1.2. In the analysis of a complex waveform (ocean tides, earthquakes, musical tones, etc.), it might be more convenient to have the Fourier series written as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\alpha_n\cos(nx - \theta_n).$$

Show that this is equivalent to Eq. (19.1) with

$$\begin{aligned} a_n &= \alpha_n\cos\theta_n, & \alpha_n^2 &= a_n^2 + b_n^2, \\ b_n &= \alpha_n\sin\theta_n, & \tan\theta_n &= b_n/a_n. \end{aligned}$$

Note. The coefficients α_n^2 as a function of n define what is called the **power spectrum**. The importance of α_n^2 lies in their invariance under a shift in the phase θ_n.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos nx + \sum_{n=1}^{\infty}b_n\sin nx. \tag{19.1}$$

Problem 19.1.11. Verify that $\delta(\varphi_1 - \varphi_2) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\varphi_1-\varphi_2)}$ is a Dirac delta function by showing that it satisfies the definition,

$$\int_{-\pi}^{\pi}f(\varphi_1)\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}e^{im(\varphi_1-\varphi_2)}\mathrm{d}\varphi_1 = f(\varphi_2)$$

Hint. Represent $f(\varphi_1)$ by an exponential Fourier series.

Problem 19.1.14. Given

$$\varphi_1(x) \equiv \sum_{n=1}^{\infty} \frac{\sin nx}{n} = \begin{cases} -\dfrac{1}{2}(\pi + x), & -\pi \le x < 0, \\ \dfrac{1}{2}(\pi - x), & 0 < x \le \pi, \end{cases}$$

show by integrating that

$$\varphi_2(x) \equiv \sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \begin{cases} \dfrac{1}{4}(\pi + x)^2 - \dfrac{\pi^2}{12}, & -\pi \le x \le 0, \\ \dfrac{1}{4}(\pi - x)^2 - \dfrac{\pi^2}{12}, & 0 \le x \le \pi. \end{cases}$$

Problem 19.2.6. Develop the Fourier series representation of

$$f(t) = \begin{cases} 0, & -\pi \le \omega t \le 0, \\ \sin \omega t, & 0 \le \omega t \le \pi. \end{cases}$$

This is the output of a simple half-wave rectifier. It is also an approximation of the solar thermal effect that produces "tides" in the atmosphere.

Problem 19.2.9. A triangular wave (Fig. 19.4) is represented by

$$f(x) = \begin{cases} x, & 0 < x < \pi \\ -x, & -\pi < x < 0. \end{cases}$$

Represent $f(x)$ by a Fourier series.

Problem 19.2.13.

(a) Find the Fourier series representation of

$$f(x) = \begin{cases} 0, & -\pi < x \le 0 \\ x, & 0 \le x < \pi. \end{cases}$$

(b) From the Fourier expansion show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots.$$

Problem 19.3.2. Determine the partial sum, s_n, of the series in Eq. (19.33) by using

$$\text{(a)} \quad \frac{\sin mx}{m} = \int_0^x \cos my \mathrm{d}y, \qquad \text{(b)} \quad \sum_{p=1}^{n} \cos(2p-1)y = \frac{\sin 2ny}{2\sin y}.$$

Do you agree with the result given in Eq. (19.40)?

$$f(x) = \frac{2h}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right) \tag{19.33}$$

$$\int_{\pi}^{\infty} \frac{\sin \xi}{\xi} \mathrm{d}\xi = -\,\mathrm{si}\,(\pi) \tag{19.40}$$