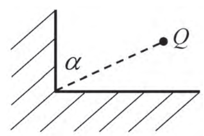
Problem Set 5

Due February 16, 2018 PHY 200B

8.2 Point Charge near a Corner Two semi-infinite and grounded conducting planes meet at a right angle as seen edge-on in the diagram. Find the charge induced on each plane when a point charge Q is introduced as shown.

Rena: Also state the range for α and explain why your answer makes sense when α approaches the extremes of its range.



8.4 A Dielectric Slab Intervenes An infinite slab with dielectric constant $\kappa = \epsilon/\epsilon_0$ lies between $z = a$ and $z = b = a+c$. A point charge q sits at the origin of coordinates. Let $\beta = (\kappa - 1)/(\kappa + 1)$ and use solutions of Laplace's equation in cylindrical coordinates to show that

$$\varphi(z > b) = \frac{q(1-\beta^2)}{4\pi\epsilon_0} \int_0^\infty \mathrm{d}k \, \frac{J_0(\kappa\rho)\exp(-kz)}{1 - \beta^2 \exp(-2kc)} = \frac{q(1-\beta^2)}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{\beta^{2n}}{\sqrt{(z+2nc)^2 + \rho^2}}.$$

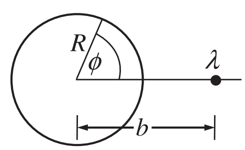
Note: The rightmost formula is a sum over image potentials, but it is much more tedious to use images from the start.

8.7 Images in Spheres I A point charge q is placed at a distance $2R$ from the center of an isolated, conducting sphere of radius R. The force on q is observed to be zero at this position. Now move the charge to a distance $3R$ from the center of the sphere. Show that the force on q at its new position is repulsive with magnitude

$$F = \frac{1}{4\pi\epsilon_0} \frac{173}{5184} \frac{q^2}{R^2}.$$

Hint: A spherical equipotential surface remains an equipotential surface if an image point charge is placed at its center.

8.10 Force between a Line Charge and a Conducting Cylinder Let b the perpendicular distance between an infinite line with uniform charge per unit length λ and the center of an infinite conducting cylinder with radius $R = b/2$.



(a) Show that the charge density induced on the surface of the cylinder is

$$\sigma(\phi) = -\frac{\lambda}{2\pi R}\left(\frac{3}{5 - 4\cos\phi}\right).$$

(b) Find the force per unit length on the cylinder by an appropriate integration over $\sigma(\phi)$.

(c) Confirm your answer to (b) by computing the force per unit length on the cylinder by another method.

Hint: Let the single image line inside the sphere fix the potential of the cylinder.

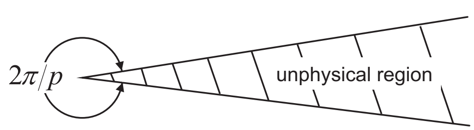
Rena: Feel free to use integral tables, Mathematica, or other integration aids on this one.

8.16 The Charge Induced by Induced Charge Maintain the plane $z = 0$ at potential V and introduce a grounded conductor somewhere into the space $z > 0$. Use the "magic rule" for the Dirichlet Green function to find the charge density $\sigma(x, y)$ induced on the $z = 0$ plane by the charge $\sigma_0(\mathbf{r})$ induced on the surface S_0 of the grounded conductor.

8.18 Free-Space Green Function in Polar Coordinates The free-space Green function in two dimensions (potential of a line charge) is $G_0^{(2)}(\mathbf{r}, \mathbf{r}') = -\ln|\mathbf{r} - \mathbf{r}'|/2\pi\epsilon_0$. Use the method of direct integration to reduce the two-dimensional equation $\epsilon_0\nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$ to a one-dimensional equation and establish the alternative representation

$$G_0^{(2)}(\mathbf{r}, \mathbf{r}') = -\frac{1}{2\pi\epsilon_0}\ln\rho_> + \frac{1}{2\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\frac{\rho_<^m}{\rho_>^m}\cos m(\phi - \phi').$$

8.24 Electrostatics of a Cosmic String A *cosmic string* is a one-dimensional object with an extraordinarily large linear mass density ($\mu \sim 10^{22}$ kg/m) which (in some theories) formed during the initial cool-down of the Universe after the Big Bang. In two-dimensional (2D) general relativity, such an object distorts flat space-time into an extremely shallow cone with the cosmic string at its apex. Alternatively, one can regard flat 2D space as shown below: undistorted but with a tiny wedge-shaped region removed from the physical domain. The usual angular range $0 \le \phi < 2\pi$ is thus reduced to $0 \le \phi < 2\pi/p$ where $p^{-1} = 1 - 4G\mu/c^2$, G is Newton's gravitational constant, and c is the speed of light. The two edges of the wedge are indistinguishable so any physical quantity $f(\phi)$ satisfies $f(0) = f(2\pi/p)$.



(a) Begin with no string. Show that the free-space Green function in 2D is

$$G_0(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\frac{1}{2\pi\epsilon_0}\ln|\boldsymbol{\rho} - \boldsymbol{\rho}'|.$$

(b) Now add the string so $p \neq 1$. To find the modified free-space Green function $G_0^p(\boldsymbol{\rho}, \boldsymbol{\rho'})$, a representation of the delta function is required which exhibits the proper angular behavior. Show that a suitable form is

$$\delta(\phi - \phi') = \frac{p}{2\pi} \sum_{m=-\infty}^{\infty} e^{imp(\phi-\phi')}.$$

(c) Exploit the ansatz

$$G_0^p(\rho, \phi, \rho', \phi') = \frac{p}{2\pi} \sum_{m=-\infty}^{\infty} e^{imp(\phi-\phi')} G_m(\rho, \rho')$$

to show that

$$G_0^p(\rho, \phi, \rho', \phi') = \frac{1}{2\pi} \sum_{m=1}^{\infty} \cos\left[mp(\phi - \phi')\right] \frac{1}{m} \left(\frac{\rho_<}{\rho_>}\right)^{mp} - \frac{p}{2\pi} \ln \rho_>.$$

(d) Perform the indicated sum and find a closed-form expression for G_0^p. Check that $G_0^1(\boldsymbol{\rho}, \boldsymbol{\rho'})$ correctly reproduces your answer in part (a).

(e) Show that a cosmic string at the origin and a line charge q at $\boldsymbol{\rho}$ are attracted with a force

$$\mathbf{F} = (p-1) \frac{q^2 \hat{\boldsymbol{\rho}}}{4\pi\epsilon_0 \rho}.$$