Homework 3

Due February 9, 2018 PHY 204B

Problem 11.3.7. Show that

$$\oint_C \frac{\mathrm{d}z}{z^2+z} = 0,$$

in which the contour C is a circle defined by $|z| = R > 1$.

Hint. Direct use of the Cauchy integral theorem is illegal. The integral may be evaluated by expanding into partial fractions and then treating the two terms individually. This yields 0 for $R > 1$ and $2\pi i$ for $R < 1$.

Problem 11.4.1. Show that

$$\frac{1}{2\pi i}\oint z^{m-n-1}\,\mathrm{d}z\,, \quad m \text{ and } n \text{ integers}$$

(with the contour encircling the origin once), is a representation of the Kronecker δ_{mn}.

Problem 11.4.2. Evaluate

$$\oint_C \frac{\mathrm{d}z}{z^2-1},$$

where C is the circle $|z-1| = 1$.

Problem 11.4.3. Assuming that $f(z)$ is analytic on and within a closed contour C and that the point z_0 is within C, show that

$$\oint_C \frac{f'(z)}{z-z_0}\,\mathrm{d}z = \oint_C \frac{f(z)}{(z-z_0)^2}\,\mathrm{d}z\,.$$

Problem 11.4.6. Evaluate

$$\oint_C \frac{e^{iz}}{z^3}\,\mathrm{d}z\,,$$

for the contour a square with sides of length $a > 1$, centered at $z = 0$.

Problem 11.4.9. Evaluate

$$\oint_C \frac{f(z)}{z(2z+1)^2}\,\mathrm{d}z\,,$$

for the contour the unit circle.

Hint. Make a partial fraction expansion.

Problem 11.5.2. Derive the binomial expansion

$$(1+z)^m = 1 + mz + \frac{m(m-1)}{1\cdot 2}z^2 + \ldots = \sum_{n=0}^{\infty} \binom{m}{n} z^n$$

for m, any real number. The expansion is convergent for $|z| < 1$. Why?

Problem 11.5.3. A function $f(z)$ is analytic on and within the unit circle. Also, $|f(z)| < 1$ for $|z| \leq 1$ and $f(0) = 0$. Show that $|f(z)| < |z|$ for $|z| \leq 1$.

Hint. One approach is to show that $f(z)/z$ is analytic and then to express $[f(z_0)/z_0]^n$ by the Cauchy integral formula. Finally, consider absolute magnitudes and take the nth root. This exercise is sometimes called Schwarz's theorem.

Problem 11.5.5. Prove that the Laurent expansion of a given function about a given point is unique; that is, if

$$f(z) = \sum_{n=-N}^{\infty} a_n (z-z_0)^n = \sum_{n=-N}^{\infty} b_n (z-z_0)^n,$$

show that $a_n = b_n$ for all n.

Hint. Use the Cauchy integral formula.

Problem 11.5.7. Obtain the Laurent expansion of $ze^z/(z-1)$ about $z = 1$.

Problem 11.5.8. Obtain the Laurent expansion of $(z-1)e^{1/z}$ about $z = 0$.