Homework 2

Due February 2, 2018 PHY 204B

Problem 19.2.16. Confirm the delta function nature of your Fourier series of exercise 19.2.15 by showing that for any $f(x)$ that is finite in the interval $[-\pi, \pi]$ and continuous at $x = 0$,

$$\int_{-\pi}^{\pi} f(x)\,[\text{Fourier expansion of } \delta_\infty(x)]\,\mathrm{d}x = f(0).$$

Solution for exercise 19.2.15 from solution manual:

$$\delta_n(x) = \frac{1}{2\pi} + \frac{2n}{\pi}\sum_{m=1}^{\infty}\frac{\sin(m/2n)}{m}\cos mx$$

Problem 19.2.17.

(a) Show that the Dirac delta function $\delta(x-a)$, expanded in a Fourier sine series in the half-interval $(0, L)$ $(0 < a < L)$ is given by

$$\delta(x-a) = \frac{2}{L}\sum_{n=1}^{\infty}\sin\left(\frac{n\pi a}{L}\right)\sin\left(\frac{n\pi x}{L}\right).$$

Note that this series actually describes $-\delta(x+a) + \delta(x-a)$ in the interval $(-L, L)$.

(b) By integrating both sides of the preceding equation from 0 to x, show that the cosine expansion of the square wave

$$f(x) = \begin{cases} 0, & 0 \le x < a \\ 1, & a < x < L, \end{cases}$$

is

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)\cos\left(\frac{n\pi x}{L}\right)$$

for $0 \le x < L$.

(c) Show that the term $\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi a}{L}\right)$ is the average of $f(x)$ on $(0, L)$.

Problem 11.2.3. Find the analytic function

$$w(z) = u(x,y) + iv(x,y)$$

(a) if $u(x,y) = x^3 - 3xy^2$

(b) if $v(x,y) = e^{-y}\sin x$

Problem 11.2.6. Show that given the Cauchy–Riemann equations, the derivative $f'(z)$ has the same value for $\mathrm{d}z = a\mathrm{d}x + ib\mathrm{d}y$ (with neither a nor b zero) as it has for $\mathrm{d}z = \mathrm{d}x$.

Problem 11.2.7. Using $f\left(re^{i\theta}\right) = R(r,\theta)e^{i\Theta(r,\theta)}$, in which $R(r,\theta)$ and $\Theta(r,\theta)$ are differentiable real functions of r and θ, show that the Cauchy–Riemann conditions in polar coordinates become

(a) $\dfrac{\partial R}{\partial r} = \dfrac{R}{r}\dfrac{\partial \Theta}{\partial \theta}$

(b) $\dfrac{1}{r}\dfrac{\partial R}{\partial \theta} = -R\dfrac{\partial \Theta}{\partial r}$

Hint. Set up the derivative first with δz radial and then with δz tangential.

Problem 11.2.11. Two–dimensional irrotational fluid flow is conveniently described by a complex potential $f(z) = u(x,v) + iv(x,y)$. We label the real part, $u(x,y)$, the velocity potential, and the imaginary part, $v(x,y)$, the stream function. The fluid velocity $\mathbf{V}$ is given by $\mathbf{V} = \nabla u$. If $f(z)$ is analytic:

(a) Show that $\mathrm{d}f/\mathrm{d}z = V_x - iV_y$.

(b) Show that $\nabla \cdot \mathbf{V} = 0$ (no sources or sinks).

(c) Show that $\nabla \times \mathbf{V} = 0$ (irrotational, nonturbulent flow).

Problem 11.3.1. Show that $\displaystyle\int_{z_1}^{z_2} f(z)\mathrm{d}z = -\int_{z_2}^{z_1} f(z)\mathrm{d}z$.

Problem 11.3.3. Show that the integral

$$\int_{3+4i}^{4-3i} \left(4z^2 - 3iz\right)\mathrm{d}z$$

has the same value on the two paths:

(a) the straight line connecting the integration limits

(b) an arc on the circle $|z| = 5$

Problem 11.3.6. Verify that

$$\int_0^{1+i} z^* \mathrm{d}z$$

depends on the path by evaluating the integral for the two paths shown in Fig. 11.7. Recall that $f(z) = z^*$ is not an analytic function of z and that Cauchy's integral theorem therefore does not apply.

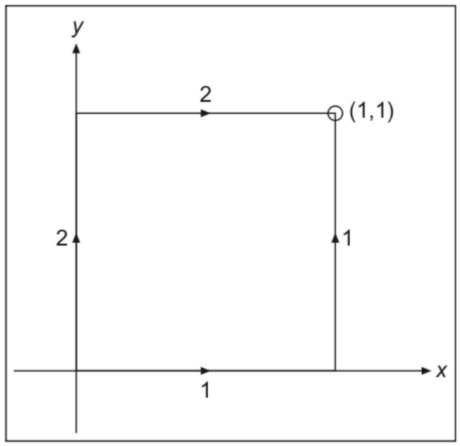

Figure 1: Fig. 11.7 from book.