Problem Set 4

Due February 7, 2018 PHY 200B

Problem 7.1. Use the orthogonality properties of the spherical harmonics to prove the following identities for a function $\varphi(\boldsymbol{r})$ which satisfies Laplace's equation in and on an origin-centered spherical surface S of radius R:

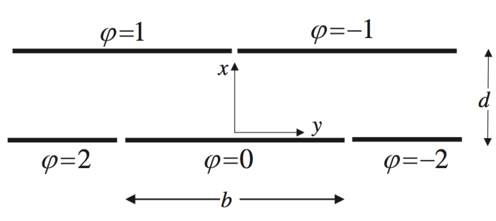
(a) $\displaystyle\int_S \mathrm{d}S\varphi(\boldsymbol{r}) = 4\pi R^2\varphi(0)$.

(b) $\displaystyle\int_S \mathrm{d}Sz\varphi(\boldsymbol{r}) = \frac{4\pi}{3}R^4\frac{\partial\varphi}{\partial z}\bigg|_{\boldsymbol{r}=0}$.

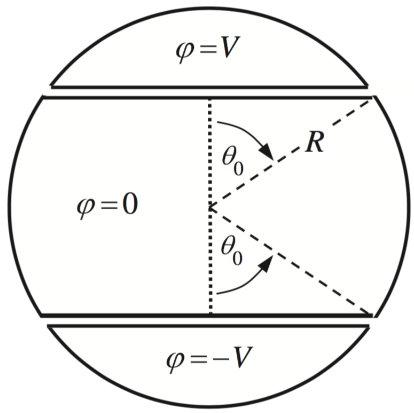
Problem 7.4. The z–axis runs down the center of an infinitely long heating duct with a square cross section. For a real metal duct (not a perfect conductor), the electrostatic potential $\varphi(x, y)$ varies *linearly* along the side walls of the duct. Suppose that the duct corners at $(\pm a, 0)$ are held at potential $+V$ and the duct corners at $(0, \pm a)$ are held at potential $-V$. Find the potential inside the duct beginning with the trial solution

$$\varphi(x, y) = A + Bx + Cy + Dx^2 + Ey^2 + Fxy.$$

Problem 7.6. The parallel plates of a *microchannel plate* electron multiplier are segmented into conducting strips of width b so the potential can be fixed on the strips at staggered values. We model this using infinite-area plates, a finite portion of which is shown below. Find the potential $\varphi(x, y)$ between the plates and sketch representative field lines and equipotentials. Note the orientation of the x– and y–axes.



Problem 7.10. A spherical shell of radius R is divided into three conducting segments by two very thin air gaps located at latitudes θ_0 and $\pi - \theta_0$. The center segment is grounded. The upper and lower segments are maintained at potentials V and $-V$, respectively. Find the angle θ_0 such that the electric field inside the shell will be as nearly constant as possible near the center of the sphere.



Problem 7.12. A spherical conducting shell centered at the origin has radius R_1 and is maintained at potential V_1. A second spherical conducting shell maintained at potential V_2 has radius $R_2 > R_1$ but is centered at the point $s\hat{\mathbf{z}}$ where $s \ll R_1$.

(a) To lowest order in s, show that the charge density induced on the surface of the inner shell is

$$\sigma(\theta) = \epsilon_0 \frac{R_1 R_2 (V_2 - V_1)}{R_2 - R_1} \left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3} \cos\theta \right].$$

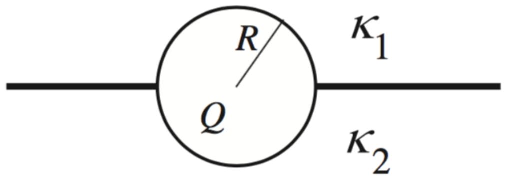
Hint: Show first that the boundary of the outer shell is $r_2 \approx R_2 + s\cos\theta$.

(b) To lowest order in s, show that the force exerted on the inner shell is

$$\boldsymbol{F} = \int \mathrm{d}S \frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}} 2\pi R_1^2 \int_0^\pi \mathrm{d}\theta \sin\theta \frac{\sigma^2(\theta)}{2\epsilon_0} \cos\theta = -\frac{Q^2}{4\pi\epsilon_0} \frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3}.$$

(c) Integrate the force in (b) to find the capacitance of this structure to second order in s.

Problem 7.14. A conducting sphere with radius R and charge Q sits at the origin of coordinates. The space outside the sphere above the $z = 0$ plane has dielectric constant κ_1. The space outside the sphere below the $z = 0$ plane has dielectric constant κ_2.

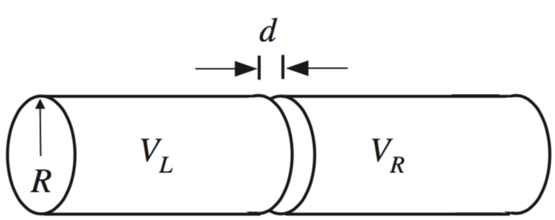


(a) Find the potential everywhere outside the conductor.
(b) Find the distributions of free charge and polarization charge wherever they may be.

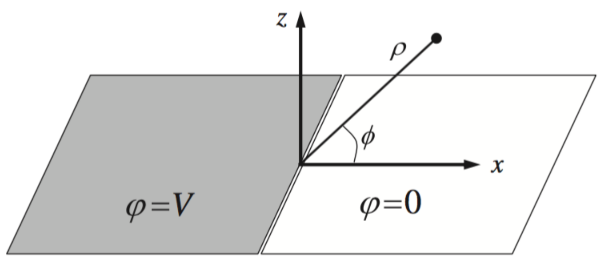
Problem 7.18. Two semi-infinite, hollow cylinders of radius R are coaxial with the z–axis. Apart from an insulating ring of thickness $d \to 0$, the two cylinders abut one another at $z = 0$ and are held at potentials V_L and V_R. Find the potential everywhere inside both cylinders. You will need the integrals

$$\lambda \int_0^1 \mathrm{d}s\, s J_0(\lambda s) = J_1(\lambda) \qquad \text{and} \qquad 2\int_0^1 \mathrm{d}s\, s J_0(x_n s) J_0(x_m s) = J_1^2(x_n)\delta_{nm}.$$

The real numbers x_m satisfy $J_0(x_m) = 0$.



Problem 7.23. The $x > 0$ half of a conducting plane at $z = 0$ is held at zero potential. The $x < 0$ half of the plane is held at potential V. A tiny gap at $x = 0$ prevents electrical contact between the two halves.



(a) Use a change-of-scale argument to conclude that the $z > 0$ potential $\varphi(\rho, \phi)$ in plane polar coordinates cannot depend on the radial variable ρ.
(b) Find the electrostatic potential in the $z > 0$ half-space.
(c) Make a semi-quantitative sketch of the electric field lines and use words to describe the most important features.