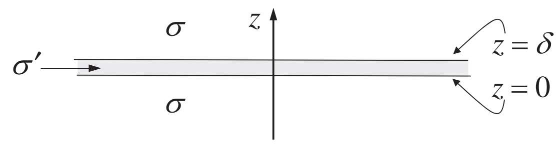
Problem Set 6

Due March 5, 2018 PHY 200B

9.5 **Membrane Boundary Conditions** A thin membrane with conductivity σ' and thickness δ separates two regions with conductivity σ.

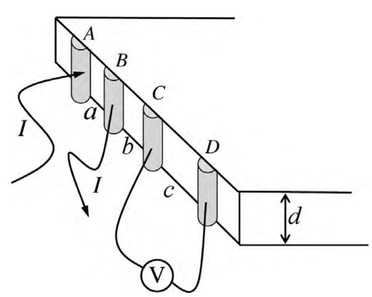


Assume uniform current flow in the z-direction in the figure above. When δ is small, it makes sense to seek "across-the-membrane" matching conditions for the electrostatic potential $\varphi(z)$ defined entirely in terms of quantities defined outside the membrane. Find the potential in all three regions of the figure and prove that suitable matching conditions are

$$\varphi(z=\delta^+) - \varphi(z=0^-) = \delta\frac{\sigma}{\sigma'}\left.\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right|_{z=0^-}$$

$$\left.\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right|_{z=\delta^+} - \left.\frac{\mathrm{d}\varphi}{\mathrm{d}z}\right|_{z=0^-} = 0.$$

9.17 **van der Pauw's Formula** The diagram below shows an ohmic film with conductivity σ, thickness d, infinite length, and semi-infinite width. A total current I enters the film at the point A through a line contact (modeled as a half-cylinder with negligible radius) and exits the film similarly at the point B. The potential difference $V_D - V_C$ between the contact at C and the contact at D determines the resistance $R_{AB,CD}$. The contact separations are a, b, and c, as indicated.



(a) Show that the electrostatic potential produced at point C by the current injected at point A is

$$\varphi_{AC} = -\frac{I}{\pi d\sigma}\ln(a+b).$$

(b) Prove that

$$\exp(-\pi d\sigma R_{AB,\,CD}) + \exp(-\pi d\sigma R_{BC,\,DA}) = 1.$$

9.23 **The Resistance of a Shell** A spherical shell with radius a has conductivity σ in the angular range $\alpha_1 < \theta < \pi - \alpha_2$. Otherwise, the shell is perfectly conducting and a potential difference V is maintained between $\theta = 0$ and $\theta = \pi$.

(a) Solve Laplace's equation to find the potential, surface current density, and resistance of the shell between $\theta = 0$ and $\theta = \pi$.

(b) Divide the shell into many thin rings. Find the resistance of each and combine them to find the resistance and confirm the answer derived in part (a).

Hint: The substitution $y = \ln[\tan(\theta/2)]$ will be useful.

9.24 **The Resistance of the Atmosphere** The conductivity of the Earth's atmosphere increases with height due to ionization by solar radiation. At a height of about $H = 50\,\text{km}$, the atmosphere can be considered practically an ideal conductor. Experiment shows that height dependence of the conductivity of the atmosphere can be approximated by

$$\sigma(r) = \sigma_0 + A(r - r_0)^2,$$

where $r_0 = 6.4 \times 10^6$ m is the radius of the Earth and r is the distance from the center of the Earth to the observation point. The conductivity at the surface of the Earth is $\sigma_0 = 3 \times 10^{-14}\,\text{S/m}$ and the constant $A = 0.5 \times 10^{-20}\,\text{S/m}^3$. Experiment also shows that an electric field $E_0 \approx -100\,\text{V/m}$ exists near the Earth's surface and is directed downward. Estimate the resistance of the atmosphere.

10.6 **Two Approaches to the Field of a Current Sheet**

(a) Use the Biot-Savart law to find $\mathbf{B}(\mathbf{r})$ everywhere for a current sheet at $x = 0$ with $\mathbf{K} = K\hat{\mathbf{z}}$.

(b) Check your answer to part (a) by superposing the magnetic field from an infinite number of straight current-carrying wires.

10.8 **The Magnetic Field of Planar Circuits**

(a) Let I be the current carried by a wire bent into a planar loop. Place the origin of coordinates at an observation point P in the plane of the loop. Show that the magnitude of the magnetic field at the point P is

$$B(P) = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{\mathrm{d}\phi}{r(\phi)},$$

where $r(\phi)$ is the distance from the origin of coordinates at P to the point on the loop located at an angle ϕ from the positive x-axis.

(b) Show that the magnetic field at the center of a current-carrying wire bent into an ellipse with major and minor axes $2a$ and $2b$ is proportional to a complete elliptic integral of the second kind. Show that you get easily understandable answers when $a = b$ and when $a \to \infty$ with b fixed.

(c) An infinitesimally thin wire is wound in the form of a planar coil which can be modeled using an effective surface current density $\mathbf{K} = K\hat{\boldsymbol{\phi}}$. Find the magnetic field at a point P on the symmetry axis of the coil. Express your answer in terms of the angle α subtended by the coil at P.

10.20 **Magnetic Potentials** The magnetic scalar potential in a volume V is $\psi(x, y, z) = (C/2)\ln\left(x^2 + y^2\right)$. Find a vector potential $\mathbf{A} = A_x\hat{\mathbf{x}} + A_y\hat{\mathbf{y}}$ which produces the same magnetic field.

10.24 **Lamb's Formula** A quantum particle with charge q, mass m, and momentum $\mathbf{p}$ in a magnetic field $\mathbf{B}(\mathbf{r}) = \boldsymbol{\nabla} \times \mathbf{A}(\mathbf{r})$ has velocity $\boldsymbol{v}(\mathbf{r}) = \mathbf{p}/m - (q/m)\mathbf{A}(\mathbf{r})$. This means that a charge distribution $\rho(\mathbf{r})$ generates a "diamagnetic current" $\mathbf{j}(\mathbf{r}) = -(q/m)\rho(\mathbf{r})\mathbf{A}(\mathbf{r})$ when it is placed in a magnetic field.

(a) Show that $\mathbf{A}(\mathbf{r}) = \frac{1}{2}\mathcal{B} \times \mathbf{r}$ is a legitimate vector potential for a uniform magnetic field $\mathcal{B}$.

(b) Let $\rho(\mathbf{r}) = \rho(r)$ be the spherically symmetric charge distribution associated with the electrons of an atom. Expose the atom to a uniform magnetic field $\mathcal{B}$ and show that the ensuing diamagnetic current induces a vector potential

$$\mathbf{A}_{\text{ind}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\mathcal{B} \times \mathbf{r}}{6m}\left[\frac{1}{r^3}\int\limits_{r'<r} \mathrm{d}^3r'\,\rho(r')r'^2 + \int\limits_{r'>r} \mathrm{d}^3r'\,\frac{\rho(r')}{r'}\right].$$

(c) Expand $\mathbf{A}_{\text{ind}}$ for small values of r and show that the diamagnetic field at the atomic nucleus can be written in terms of $\varphi(0)$, the electrosstatic potential at the nucleus produced by $\rho(r)$:

$$\mathbf{B}_{\text{ind}}(0) = \frac{e\varphi(0)}{3mc^2}\mathcal{B}.$$

This formula was obtained by Willis Lamb in 1941. He had been asked by I.I. Rabi to determine whether $\mathbf{B}_{\text{ind}}(0)$ could be ignored when nuclear magnetic moments were extracted from molecular beam data.