Hyperbolic PDEs

The Wave Equation

- The wave equation is the canonical second-order hyperbolic PDE of computational science and engineering. We will consider the one-dimensional example of a wave propagating along a string

$$\frac{\partial^2 u}{\partial t^2} = \frac{Tg}{w}\frac{\partial^2 u}{\partial x^2}$$

- T=tension, w is the weight per unit length, and g is gravitational acceleration.

Finite-Differencing the Wave Equation

- Our first impulse might be to use FTCS as for the diffusion equation. BUT
  - For hyperbolic equations, FTCS is unconditionally unstable.
- Therefore we will try CTCS (centered time, centered space)

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\Delta t^2} = \frac{Tg}{w}\left(\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}\right)$$

CTCS

Collecting terms gives us

$$u_i^{n+1} = \frac{Tg\Delta t^2}{w\Delta x^2}\left(u_{i+1}^n + u_{i-1}^n\right) - u_i^{n-1} + 2\left(1 - \frac{Tg\Delta t^2}{w\Delta x^2}\right)u_i^n$$

We must have u^{-1} in order to start the method.

Frequently we obtain this from an estimate of the initial velocity, using a centered-time difference

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = g(x)$$

Or we can take one initial step with a one-step method.

Conditions on CTCS

- We will not derive it, but the condition for stability and accuracy on the CTCS explicit method is

$$\Delta t = \frac{\Delta x}{\sqrt{Tg/w}}$$

- If Δt is greater than this value, the method is not guaranteed to converge; but it also turns out that if Δt is less than this value, accuracy is lost.
- Note that √(Tg/w) is the wave speed. So this amounts to the condition that cΔt=Δx for the pure wave equation.

Dispersive and Dissipative Errors

- The solutions to hyperbolic equations are *waves* and due to this they are prone to two types of error we have not encountered in our study of PDEs: dispersive and dissipative errors.
- Dispersive errors occur when the numerical wave velocity does not equal the exact (analytic) wave velocity.
- Dissipative errors occur when the amplitude decays (as if diffusing away).
- An ideal method has no dissipation or dispersion. In practice, there are tradeoffs to control one or the other.

The Advection Equation

- The advection equation is

$$\partial_t u + \nabla \cdot \left(\vec{v} u \right) = 0$$

- It is hyperbolic in nature. Like other hyperbolic equations, it *transmits discontinuities*, making its numerical simulation delicate.

FTCS/CTCS

- We might first think of forward-time centered space, but as for the wave equation this method is *unconditionally unstable* for advection.
- Therefore we fall back on centered-time centered-space as for the wave equation. But since the advection equation is first order it has some interesting consequences.

Donor Cell

- We can rescue the FTCS method (somewhat) if we make a few changes to the algorithm. This is to use *upwind differencing* to compute the approximation to the time derivative.

$$\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta t} = v_i^n \frac{u_i^n - u_{i-1}^n}{\Delta x}, v_i^n > 0$$

$$\frac{u_{i+1}^{n+1} - u_i^{n+1}}{\Delta t} = v_i^n \frac{u_{i+1}^n - u_i^n}{\Delta x}, v_i^n \leq 0$$

- The donor cell method is first order in time and space.

Lax-Wendroff

- The Lax-Wendroff method uses quadratic interpolation to derive a more accurate approximation. In implementation, we form fluxes at the right and left boundaries (half steps) and difference the fluxes. This imitates the physical situation, in which the net change in the quantity within a grid zone is due to the flux in minus the flux out.
- This method is second order in space and time.

1-D Lax-Wendroff Scheme

- Half-step quantity

$$u_l = \frac{1}{2}\left(u_i + u_{i-1}\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[v_i u_i - v_{i-1} u_{i-1}\right]$$

$$u_r = \frac{1}{2}\left(u_{i+1} + ui\right) - \frac{1}{2}\frac{\Delta t}{\Delta x}\left[v_{i+1} u_{i+1} - v_i u_i\right]$$

- Half-step velocities

$$v_l = \frac{1}{2}\left(v_i + v_{i-1}\right)$$

$$v_r = \frac{1}{2}\left(v_{i+1} + v_i\right)$$

1-D Lax-Wendroff (Continued)

- The final step is to form *fluxes*
- $F_l = u_l v_l$
- $F_r = u_r v_r$
- Then
- $u_i = u_i - (\Delta t/\Delta x)(F_r - F_l)$

Leapfrog

- Applying centered differences to the advection equation yields

$$\frac{u_i^{n+1} - u_i^{n-1}}{2\Delta t} = -\frac{v_{i+1}^n u_{i+1}^n - v_{i-1}^n u_{i-1}^n}{2\Delta x}$$

- We can rewrite this as

$$u_i^{n+1} - u_i^{n-1} = -\frac{\Delta t}{\Delta x}\left(F_{i+1}^n - F_{i-1}^n\right)$$

- Where F is the flux vu. In our formulation here it is a *conserved* quantity.
- This method is second order in time and space.

Leapfrog Decoupling

- As was the case for the wave equation, the leapfrog method must have some means of determining u^{-1}. However, unlike the wave equation, the terms in u^n_i are absent. This means that the leapfrog solution will decouple into two independent solutions! To avoid this we must either take a FTCS step between every N leapfrog steps, or we must add some explicit coupling such as introducing a second-order centered difference with a small coefficient.

Artificial Viscosity

- If we add a small term

$$\eta\left(u_{i+1}^{n} - 2u_{i}^{n} + u_{i-1}^{n}\right), \eta \ll 1$$

to the leapfrog equations, we have added a small diffusivity (recall that $u_t+u_{xx}=0$ is the diffusion equation). Artificial (or numerical) viscosity is present in all finite-difference methods for advection due to truncation error, but in this case we have explicitly introduced it in order to transfer information from one cell to adjacent cells, which does not happen in the unadorned leapfrog scheme.

Stability of Explicit Methods

- The explicit methods we have seen so far are subject to the Courant-Friedrichs-Lewy condition

$$\frac{|v|\Delta t}{\Delta x} \leq 1$$

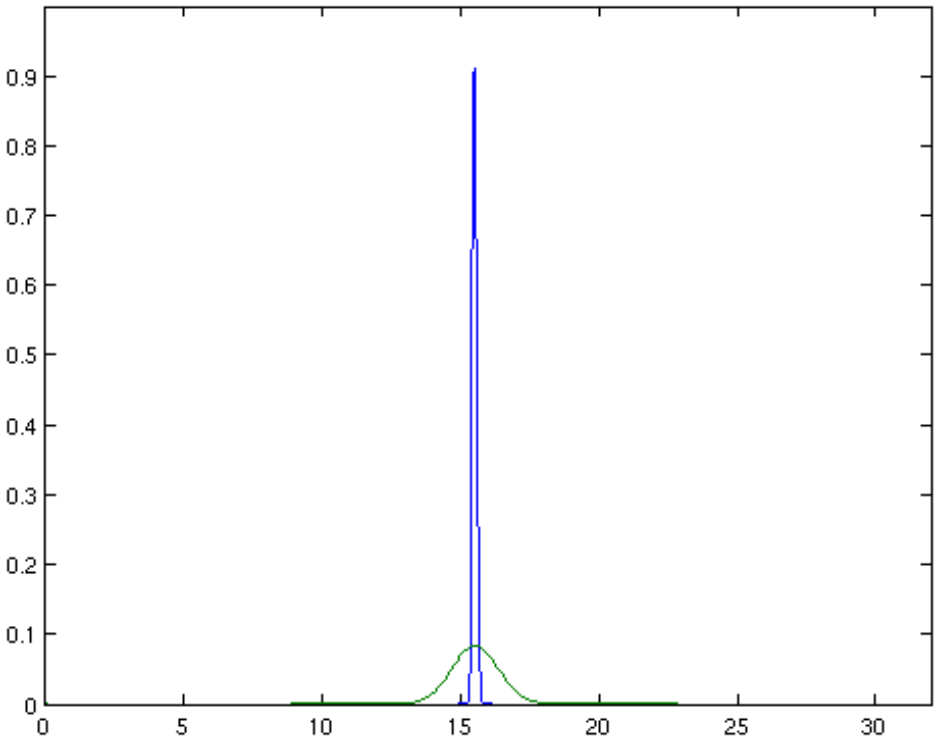
- If this condition is not satisfied, the method is unstable. This is a sometimes severe restriction on the step sizes we can utilize.

Demonstrations

- Results for an extremely simple case: one (spatial) dimension, Gaussian pulse advection.
- Donor cell and Lax-Wendroff are coded as normal. Leapfrog is coded without any filtering or additional numerical viscosity.

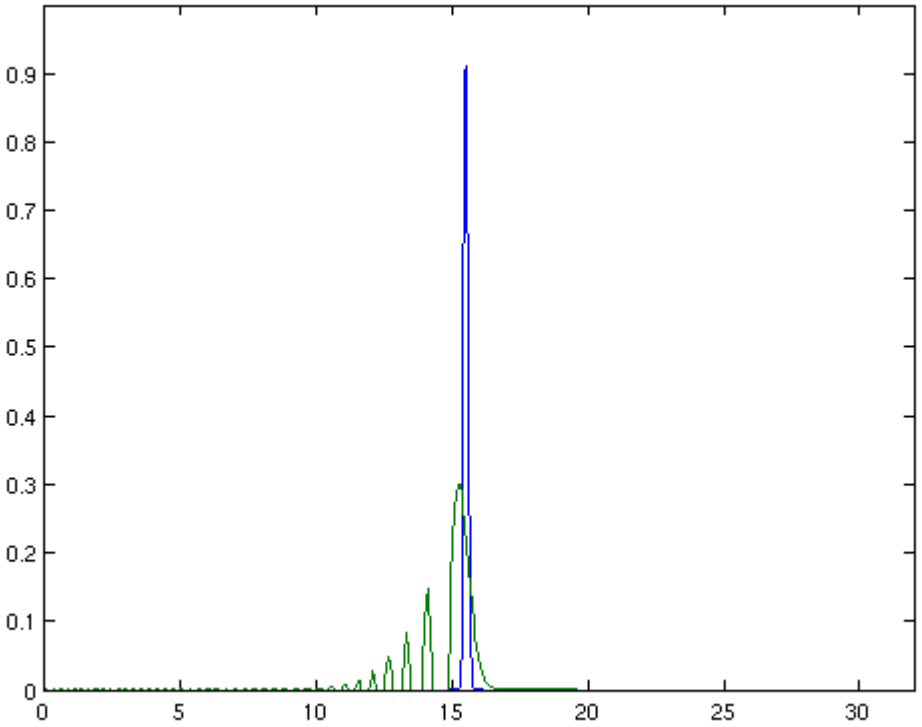
Donor Cell Demo

- The donor-cell method is not dispersive, but it is highly dissipative.



Lax-Wendroff Demo

- This method is not as dissipative as donor cell, but it has substantial dispersion.



Leapfrog (Basic) Demo

- The method is both dissipative and dispersive.