Partial Differential Equations

Definition

- A partial differential equation is a relationship among the partial derivatives of a function, the function itself, and a source function. That is,

$$F(x_1,\ldots,x_n,u,\frac{\partial u}{\partial x_1},\frac{\partial u}{\partial x_2},\ldots,\frac{\partial u}{\partial x_n},\frac{\partial^2 u}{\partial x_1},\frac{\partial^2 u}{\partial x_2^2},\ldots,\frac{\partial^2 u}{\partial x_n})=S$$

- Is the general second-order partial differential equation for a function of n independent variables.

Existence and Uniqueness of Solution

- Like for boundary value problems of ordinary differential equations, only more so, an arbitrary PDE with arbitrary boundary and initial values is not guaranteed to have a solution, or if it has a solution a unique one.
- The Cauchy-Kovalevsky (Kovalesvkaya) theorem says that a certain class of problems (Cauchy problems) have unique solutions that depend continuously on the conditions.
- Many PDEs are ill posed but those we encounter to describe physical systems are nearly always well posed.

A Famous Ill-Posed Example

- The Laplace Equation

$$\nabla^2 u = 0$$

- is an equation that occurs in many fields of science and engineering. Nearly all the time it is well posed. However, with the boundary conditions u(x,0)=0, $\partial_y u(x,0) = \frac{\sin(nx)}{n}$
- The solution goes to infinity if nx is not an integer multiple of π.

Classification of PDEs

- First- and Second-order partial differential equations broadly fall into three categories based on their *characteristics.*
- The general 2^{nd}-order linear PDE may be written

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + e\frac{\partial u}{\partial x} + f\frac{\partial u}{\partial y} + ug(x,y) = 0$$

where a, b, c, e, and f depend only on x and y or are constant.

Principle Part

- Only the highest-order derivatives are important for the behavior. They make up the *principle part* of the equation. In our general linear second-order equation, the principle part can be written as

$$P \equiv a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2}$$

Characteristic Directions

- A vector ξ defines a direction in space. For the PDE the direction defined by some vector is said to be *characteristic* if

  $$P(\mathbf{x}, \xi)=0$$

  where x denotes the vector of independent variables

  The characteristic directions are the normals to the characteristic surface.

Discriminant

- For our linear two-dimensional equation we define the discriminant as

  $D=b^2-ac$

  If $D<0$, then $\sqrt{(b^2-ac)}$ is not real.

  If $D=0$, $b^2=ac$

  If $D>0$, $\sqrt{(b^2-ac)}$ is real.

Characteristics

- The *characteristics* of a PDE are the curves that satisfy the equation

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

- In two dimensions the characteristic surfaces reduce to characteristic curves.

Elliptic Equations

- If the discriminant has only imaginary values there are no real characteristic curves and the equation is *elliptic*.
- Example: the Poisson equation

$$u_{xx} + u_{yy} = S(x,y)$$

- Here we have used the standard notation of indicating a partial derivative by a subscript.

Example

- The principle part of the Poisson equation is

  $u_{xx}+u_{yy}$

The characteristic equation is

$dy/dx=\sqrt{-1} =i$

Thus there are no real characteristics.

Parabolic Equations

- If the discriminant is zero then there is exactly one characteristic curve and the equation is *parabolic*. The characteristic equation is

$$\frac{dy}{dx} = \frac{b}{a}$$

Example

- The basic heat equation is

$$u_t + cu_{xx} = 0$$

- The principle part is cu_{xx}. The characteristic equation in this case is dy/dx=0 or y=α; that is, the characteristic curves are straight lines defined by constant values of y.

Hyperbolic Equations

- If the discriminant has two real values the equation is *hyperbolic*. The characteristic equations are

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - ac}}{a}$$

Example

- The basic wave equation is

  $$u_{xx}-c^2u_{tt}$$

  The principle part is $u_{xx}-c^2u_{tt}$. The characteristic equation is

  $$\frac{dy}{dt}=\frac{\pm\sqrt{-(-c^2)}}{1}=\pm c$$

The characteristic curves are straight lines

$$y=\pm ct+\gamma,$$

which are straight lines with a slope of c.

Mixed Types and Nonlinear Equations

- If the coefficients are not constant, then the type of the equation can vary from one location to another.
- Our results so far are true for linear equations. Nonlinear equations do not have simple characteristic curves/surfaces, but often behave in many respects like one of the three fundamental types.