Derivation on Derivatives of Curvature in Hybrid A* Algorithm

1 Notation and Definitions

For a vector $\boldsymbol{\vec{v}}=\{\boldsymbol{x, y, z, ...}\}$, the derivative of this vector on a variable $\boldsymbol{a}$ is defined as:

$$\frac{\partial \vec{v}}{\partial a} = \left\{\frac{\partial x}{\partial a}, \frac{\partial y}{\partial a}, \frac{\partial z}{\partial a}, ...\right\} \tag{1}$$

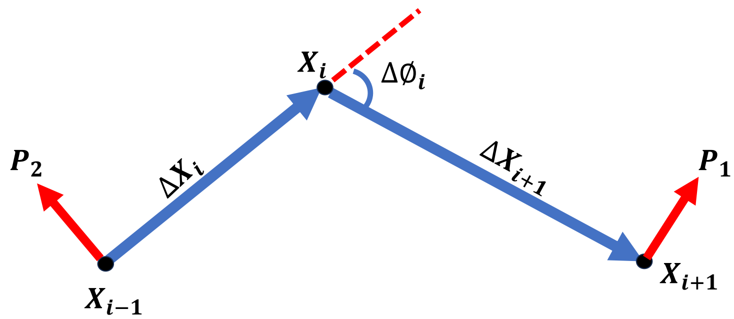

Figure 1: Illustration of points and vectors.

Suppose we get arbitrary 3 successive points in Hybrid A* path, the 3 points are defined as $\boldsymbol{X_{i-1}} = (\boldsymbol{x_{i-1}, y_{i-1}})$,$\boldsymbol{X_i} = (\boldsymbol{x_i, y_i})$ and $\boldsymbol{X_{i+1}} = (\boldsymbol{x_{i+1}, y_{i+1}})$, and we further define 2 vectors $\boldsymbol{\Delta X_i = X_i - X_{i-1}}$ and $\boldsymbol{\Delta X_{i+1} = X_{i+1} - X_i}$, Denote $\boldsymbol{\Delta\phi_i}$ as the angle between $\boldsymbol{\Delta X_i}$ and $\boldsymbol{\Delta X_{i+1}}$, thus

$$\cos(\Delta\phi_i) = \frac{\Delta X_i^T \cdot \Delta X_{i+1}}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \tag{2}$$

We define the the curvature at point $\boldsymbol{X_i}$ as:

$$\kappa_i = \frac{\Delta\phi_i}{|\Delta X_i|} \tag{3}$$

Define $\boldsymbol{\vec{a} \perp \vec{b} = \vec{a} - \frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2} \cdot \vec{b}}$, and we denote:

$$\begin{aligned} p_1 &= \frac{\Delta X_i \perp (-\Delta X_{i+1})}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \\ &= \frac{1}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \cdot (\Delta X_i - \frac{\Delta X_i^T \cdot (-\Delta X_{i+1})}{|\Delta X_{i+1}|^2} \cdot (-\Delta X_{i+1})) \\ &= \frac{1}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \cdot (\Delta X_i - \frac{\Delta X_i^T \cdot \Delta X_{i+1}}{|\Delta X_{i+1}|^2} \cdot \Delta X_{i+1}) \end{aligned} \tag{4}$$

$$\begin{aligned} p_2 &= \frac{(-\Delta X_{i+1}) \perp (\Delta X_i)}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \\ &= \frac{1}{|\Delta X_i| \cdot |\Delta X_{i+1}|} \cdot (-\Delta X_{i+1} + \frac{\Delta X_{i+1}^T \cdot \Delta X_i}{|\Delta X_i|^2} \cdot \Delta X_i) \end{aligned} \tag{5}$$

2 Preliminaries

Suppose $\boldsymbol{L_1 = |\Delta X_i| = |X_i - X_{i-1}| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}}$

$$\begin{aligned}
\frac{\partial L_1}{\partial x_{i-1}} &= \frac{-(x_i - x_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} \\
\frac{\partial L_1}{\partial y_{i-1}} &= \frac{-(y_i - y_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} \\
\frac{\partial L_1}{\partial X_{i-1}} &= \left\{\frac{\partial L_1}{\partial x_{i-1}}, \frac{\partial L_1}{\partial y_{i-1}}\right\} = -\frac{\Delta X_i}{|\Delta X_i|} \qquad (6)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{\partial L_1}{\partial x_i} &= \frac{(x_i - x_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} \\
\frac{\partial L_1}{\partial y_i} &= \frac{(y_i - y_{i-1})}{\sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}} \\
\frac{\partial L_1}{\partial X_i} &= \left\{\frac{\partial L_1}{\partial x_i}, \frac{\partial L_1}{\partial y_i}\right\} = \frac{\Delta X_i}{|\Delta X_i|} \qquad (7)
\end{aligned}$$

Suppose: $\boldsymbol{L_2 = \Delta X_i^T \cdot \Delta X_{i+1} = (x_i - x_{i-1})(x_{i+1} - x_i) + (y_i - y_{i-1})(y_{i+1} - y_i)}$

$$\begin{aligned}
\frac{\varphi L_2}{\varphi X_{i-1}} &= \left\{\frac{\varphi L_2}{\varphi x_{i-1}}, \frac{\varphi L_2}{\varphi y_{i-1}}\right\} \\
&= \{-(x_{i+1} - x_i), -(y_{i+1} - y_i)\} \\
&= -\Delta X_{i+1} \qquad (8) \\
\frac{\varphi L_2}{\varphi X_i} &= \left\{\frac{\varphi L_2}{\varphi x_i}, \frac{\varphi L_2}{\varphi y_i}\right\} \\
&= \{(x_{i+1} - x_i) - (x_i - x_{i-1}), (y_{i+1} - y_i) - (y_i - y_{i-1})\} \\
&= \Delta X_{i+1} - \Delta X_i \qquad (9) \\
\frac{\varphi L_2}{\varphi X_{i+1}} &= \left\{\frac{\varphi L_2}{\varphi x_{i+1}}, \frac{\varphi L_2}{\varphi y_{i+1}}\right\} \\
&= \{(x_i - x_{i-1}), (y_i - y_{i-1})\} \\
&= \Delta X_i \qquad (10)
\end{aligned}$$

We also define:

$$\begin{aligned}
p_0 &= \frac{\varphi \Delta\phi_i}{\varphi \cos(\Delta\phi_i)} \\
&= \frac{\varphi \cos^{-1}(\cos(\Delta\phi_i))}{\cos(\Delta\phi_i)} \\
&= \frac{-1}{(1 - \cos^2(\Delta\phi_i))^{\frac{1}{2}}} \qquad (11)
\end{aligned}$$

3 Derivatives of $\cos(\Delta\phi_i)$

$$
\begin{aligned}
\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_{i-1}} &= \frac{1}{|\Delta X_{i+1}|}\cdot\frac{\varphi\frac{\Delta X_i^T\cdot\Delta X_{i+1}}{|\Delta X_i|}}{\varphi X_{i-1}} \\
&= \frac{1}{|\Delta X_{i+1}|}\cdot\frac{-\Delta X_{i+1}\cdot|\Delta X_i| - \Delta X_i^T\cdot\Delta X_{i+1}\cdot\frac{-\Delta X_i}{|\Delta X_i|}}{|\Delta X_i|^2} \\
&= \frac{1}{|\Delta X_{i+1}|}\cdot(-\frac{\Delta X_{i+1}}{|\Delta X_i|} + \frac{\Delta X_i^T\cdot\Delta X_{i+1}}{|\Delta X_i|^3}\cdot\Delta X_i) \\
&= \frac{1}{|\Delta X_{i+1}|\cdot|\Delta X_i|}\cdot(-\Delta X_{i+1} + \frac{\Delta X_{i+1}^T\cdot\Delta X_i}{|\Delta X_i|^2}\cdot\Delta X_i) \\
&= p_2
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_{i+1}} &= \frac{1}{|\Delta X_i|}\cdot\frac{\varphi\frac{\Delta X_i^T\cdot\Delta X_{i+1}}{|\Delta X_{i+1}|}}{\varphi X_{i+1}} \\
&= \frac{1}{|\Delta X_i|}\cdot\frac{\Delta X_i\cdot|\Delta X_{i+1}| - \Delta X_i^T\cdot\Delta X_{i+1}\cdot\frac{\Delta X_{i+1}}{|\Delta X_{i+1}|}}{|\Delta X_{i+1}|^2} \\
&= \frac{1}{|\Delta X_i|}\cdot(\frac{\Delta X_i}{|\Delta X_{i+1}|} - \frac{\Delta X_i^T\cdot\Delta X_{i+1}}{|\Delta X_{i+1}|^3}\cdot\Delta X_{i+1}) \\
&= \frac{1}{|\Delta X_{i+1}|\cdot|\Delta X_i|}\cdot(\Delta X_i - \frac{\Delta X_{i+1}^T\cdot\Delta X_i}{|\Delta X_{i+1}|^2}\cdot\Delta X_{i+1}) \\
&= p_1
\end{aligned}
\tag{13}
$$

Similarly,we can derive:

$$
\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_i} = -p1 - p2 \tag{14}
$$

4 Derivatives of κ_i

$$
\begin{aligned}
\frac{\varphi\kappa_i}{\varphi X_{i-1}} &= \frac{\frac{\varphi\Delta\Phi_i}{\varphi X_{i-1}}\cdot|\Delta X_i| - \Delta\phi_i\cdot\frac{|\varphi\Delta X_i|}{\varphi X_{i-1}}}{|\Delta X_i|^2} \\
&= \frac{1}{|\Delta X_i|}\cdot\frac{\varphi\Delta\phi_i}{\varphi\cos(\Delta\phi_i)}\cdot\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_{i-1}} - \frac{\Delta\phi_i}{|\Delta X_i|^2}\cdot\frac{\varphi|\Delta X_i|}{\varphi X_{i-1}}
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\frac{\varphi\kappa_i}{\varphi X_i} &= \frac{\frac{\varphi\Delta\Phi_i}{\varphi X_i}\cdot|\Delta X_i| - \Delta\phi_i\cdot\frac{|\varphi\Delta X_i|}{\varphi X_i}}{|\Delta X_i|^2} \\
&= \frac{1}{|\Delta X_i|}\cdot\frac{\varphi\Delta\phi_i}{\varphi\cos(\Delta\phi_i)}\cdot\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_i} - \frac{\Delta\phi_i}{|\Delta X_i|^2}\cdot\frac{\varphi|\Delta X_i|}{\varphi X_i}
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
\frac{\varphi\kappa_i}{\varphi X_{i+1}} &= \frac{\frac{\varphi\Delta\Phi_i}{\varphi X_{i+1}}\cdot|\Delta X_i| - \Delta\phi_i\cdot\frac{|\varphi\Delta X_i|}{\varphi X_{i+1}}}{|\Delta X_i|^2} \\
&= \frac{1}{|\Delta X_i|}\cdot\frac{\varphi\Delta\phi_i}{\varphi\cos(\Delta\phi_i)}\cdot\frac{\varphi\cos(\Delta\phi_i)}{\varphi X_{i+1}} \quad (\frac{\varphi|\Delta X_i|}{\varphi X_{i+1}} = 0)
\end{aligned}
\tag{17}
$$