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The Flow of Power-Law Non-Newtonian Fluids in Concentric Annuli

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A simple algebraic solution is presented for the volume rate of flow of a power-law non-Newtonian fluid through a concentric annulus in laminar flow. This expression is valid for all values of the flow behavior index n (not just reciprocal integers as in previous solutions) and all values of the annulus aspect ratio $\sigma = R_1/R$. The present solution allows calculation of either volume flow or pressure loss directly using only a pocket calculator. Previous authors obtained their results in terms of a definite integral which they could not evaluate for arbitrary n values. It is shown how this integral may be evaluated analytically to obtain a simple algebraic result valid for all n, σ in the range.

Introduction

The problem of axial laminar flow of power-law non-Newtonian fluids in concentric annuli was first studied by Fredrickson and Bird (1958), who presented results obtained by power series expansions for limited values of n applied to the arguments of certain integrals which could not be analytically integrated. Vaughn and Bergman (1966) objected to the results of Fredrickson and Bird as not being an accurate representation of experimental data, as did Bird (1965). However, the data which were presented in objection to the theory were not well represented by a power-law model. This was shown by Russell and Christiansen (1974), who used a three-constant equation to fit the data of Bergman (1962) and by Ashare et al. (1965), who reanalyzed Fredrickson's (1959) data using a three-constant model. Tiu and Bhattacharyya (1973, 1974) have presented data which substantiated the theoretical results of Fredrickson and Bird (1958), showing that they do give an accurate representation of experimental data, when the viscometric data are truly power-law. Therefore, the theoretical results of Fredrickson and Bird (1958) are useful and valuable.

Since the concentric annulus is an industrially important flow geometry and the power-law rheological model is so convenient to use, it would be highly desirable to have a simple analytical expression for the relation between the axial pressure loss and the volume rate of flow. Fredrickson and Bird (1958) were unable to obtain such an analytical expression and were forced to express their results in terms of an integral which required numerical quadrature for all but integer values of the parameter $s = 1/n$, where n is the flow behavior index of the power-law model. For this special case they obtained a cumbersome power series solution. This is rather awkward since most practical values of s are non-integer, necessitating graphical interpolation from Fredrickson and Bird's small-scale graphs or direct numerical quadrature of the integral for the case at hand.

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The purpose of this paper is to present a simple analytical solution to the volume flow relation which is valid for arbitrary, non-integer values of $s = 1/n$. This simple expression eliminates the need for either graphical interpolation or direct numerical quadrature and renders the computation of the pressure loss–volume flow relation for power-law fluids in concentric annuli a simple matter requiring at most a pocket calculator.

Theory

The stress distribution for axial laminar flow of an arbitrary inelastic non-Newtonian fluid through a concentric annulus is known to be (Fredrickson and Bird, 1958)

$$\tau_{rz} = \frac{PR}{2}\left(\xi - \frac{\lambda^2}{\xi}\right) \tag{1}$$

where

$$P = \frac{p_0 - p_L}{L} + \rho g \tag{2}$$

is the total pressure loss per unit length L, $\xi = r/R$, and R is the radius of the outer pipe of the annulus as illustrated in Figure 1. The constant of integration is chosen so that τ_{rz} $(r = \lambda R) = 0$. In laminar flow this corresponds to the location of maximum velocity.

The power-law rheological model may be represented for this geometry as (Fredrickson and Bird, 1958)

$$\tau_{rz} = -m\left|\frac{dv_z}{dr}\right|^{n-1}\left(\frac{dv_z}{dr}\right) \tag{3}$$

When eq 3 is combined with eq 1 and integrated formally, one obtains (Fredrickson and Bird, 1958)

$$v_{zi} = R\left(\frac{PR}{2m}\right)^s \int_\sigma^\xi \left(\frac{\lambda^2}{x} - x\right)^s dx \qquad \sigma \le \xi \le \lambda \tag{4}$$

$$v_{z0} = R\left(\frac{PR}{2m}\right)^s \int_\xi^1 \left(x - \frac{\lambda^2}{x}\right)^s dx \qquad \lambda \le \xi \le 1 \tag{5}$$

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Table I. Values of $\lambda(n, \sigma)$ Computed from Eq 6

	σ										
n	0.05	0.08	0.10	0.20	0.30	0.40	0.50	0.60	0.70	0.80	0.90
0.10	0.2534	0.3118	0.3442	0.4687	0.5632	0.6431	0.7140	0.7788	0.8389	0.8954	0.9489
0.15	0.2675	0.3251	0.3567	0.4776	0.5694	0.6473	0.7167	0.7804	0.8397	0.8957	0.9490
0.20	0.2811	0.3374	0.3682	0.4856	0.5749	0.6509	0.7191	0.7818	0.8404	0.8960	0.9491
0.25	0.2939	0.3488	0.3787	0.4927	0.5797	0.6541	0.7211	0.7830	0.8411	0.8963	0.9491
0.30	0.3060	0.3593	0.3884	0.4991	0.5840	0.6570	0.7229	0.7840	0.8416	0.8965	0.9492
0.35	0.3173	0.3690	0.3972	0.5048	0.5878	0.6595	0.7245	0.7850	0.8421	0.8967	0.9492
0.40	0.3278	0.3779	0.4052	0.5100	0.5912	0.6617	0.7259	0.7858	0.8426	0.8969	0.9493
0.45	0.3376	0.3861	0.4126	0.5146	0.5943	0.6637	0.7271	0.7865	0.8430	0.8971	0.9493
0.50	0.3466	0.3936	0.4193	0.5189	0.5970	0.6655	0.7283	0.7872	0.8433	0.8972	0.9493
0.55	0.3550	0.4005	0.4255	0.5227	0.5995	0.6671	0.7293	0.7878	0.8436	0.8973	0.9494
0.60	0.3628	0.4069	0.4312	0.5262	0.6018	0.6686	0.7303	0.7884	0.8439	0.8975	0.9494
0.65	0.3700	0.4128	0.4364	0.5294	0.6039	0.6700	0.7311	0.7889	0.8442	0.8976	0.9494
0.70	0.3767	0.4182	0.4412	0.5324	0.6059	0.6713	0.7319	0.7893	0.8444	0.8977	0.9495
0.75	0.3829	0.4232	0.4457	0.5351	0.6076	0.6724	0.7326	0.7898	0.8446	0.8978	0.9495
0.80	0.3887	0.4279	0.4498	0.5377	0.6093	0.6735	0.7333	0.7902	0.8449	0.8979	0.9495
0.85	0.3940	0.4322	0.4537	0.5400	0.6108	0.6745	0.7339	0.7905	0.8450	0.8979	0.9495
0.90	0.3990	0.4362	0.4572	0.5422	0.6122	0.6754	0.7345	0.7909	0.8452	0.8980	0.9495
0.95	0.4037	0.4400	0.4605	0.5442	0.6135	0.6762	0.7350	0.7912	0.8454	0.8981	0.9495
1.00	0.4080	0.4435	0.4637	0.5461	0.6147	0.6770	0.7355	0.7915	0.8455	0.8981	0.9496

where $s = 1/n$, and the boundary conditions $v_z(\xi = \sigma) = 0$, $v_z(\xi = 1) = 0$ have been used.

The value of λ is determined by the condition $v_{zi}(\lambda) = v_{z0}(\lambda)$

$$\int_{\sigma}^{\lambda}\left(\frac{\lambda^2}{x} - x\right)^s dx - \int_{\lambda}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx = 0 \quad (6)$$

The volume rate of flow is

$$Q = 2\pi R^2 \int_{\sigma}^{1} \xi v_z \, d\xi \quad (7)$$

If one introduces eq 4 and 5 into eq 7 one has

$$\frac{Q}{\pi R^3} = 2\left(\frac{PR}{2m}\right)^s \left\{\int_{\sigma}^{\lambda} \xi d\xi \int_{\sigma}^{\xi}\left(\frac{\lambda^2}{x} - x\right)^s dx + \int_{\lambda}^{1} \xi d\xi \int_{\xi}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx\right\} \quad (8)$$

It can easily be shown that by interchanging the order of integration in the two iterated integrals in eq 8 one may obtain

$$I \equiv \int_{\sigma}^{\lambda} \xi d\xi \int_{\sigma}^{\xi}\left(\frac{\lambda^2}{x} - x\right)^s dx + \int_{\lambda}^{1} \xi d\xi \int_{\xi}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx = \int_{\sigma}^{\lambda}\left(\frac{\lambda^2}{x} - x\right)^s dx \int_{x}^{\lambda} \xi \, d\xi + \int_{\lambda}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx \int_{\lambda}^{x} \xi d\xi = \frac{1}{2}\int_{\sigma}^{\lambda}(\lambda^2 - x^2)(\lambda^2 - x^2)^s x^{-s} \, dx + \frac{1}{2}\int_{\lambda}^{1}(x^2 - \lambda^2)(x^2 - \lambda^2)^s x^{-s} \, dx = \tfrac{1}{2}\int_{\sigma}^{1} |\lambda^2 - x^2|^{s+1} \, x^{-s} \, dx \quad (9)$$

Thus, one obtains Fredrickson and Bird's (1958) result

$$\frac{Q}{\pi R^3} = \left(\frac{PR}{2m}\right)^s \int_{\sigma}^{1} |\lambda^2 - x^2|^{s+1} x^{-s} \, dx \quad (10)$$

These authors were unable to reduce the integral in eq 10 further and presented curves and numerical values of the integral obtained by power series expansion for values of $s = p$ where $p = 1, 2, 3, \ldots$, are integers. These analytical expressions, which are rather cumbersome series expansions, are not valid if $s \neq p$. They also break down when $p >> 1$ and/or $1 - \sigma << 1$.

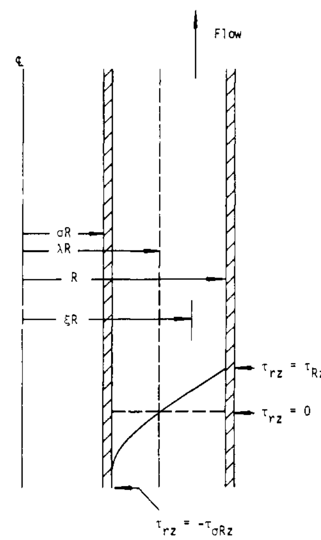

Figure 1. Schematic representation of coordinates describing axial flow in a concentric annulus.

Analytical Evaluation of Flow Integral

The integral in eq 10 may be evaluated analytically for arbitrary values of s by the following procedure. Consider the first iterated integral in eq 8. If one integrates by parts, one finds

$$\int_{\sigma}^{\lambda} \xi d\xi \int_{\sigma}^{\xi}\left(\frac{\lambda^2}{x} - x\right)^s dx = \tfrac{1}{2}\lambda^2 \int_{\sigma}^{\lambda}\left(\frac{\lambda^2}{x} - x\right)^s dx - \tfrac{1}{2}\int_{\sigma}^{\lambda} \xi^{2-s}(\lambda^2 - \xi^2)^s \, d\xi \quad (11)$$

Similarly, for the second iterated integral in eq 8 by integrating by parts, one finds

$$\int_{\lambda}^{1} \xi d\xi \int_{\xi}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx = -\frac{1}{2}\lambda^2 \int_{\lambda}^{1}\left(x - \frac{\lambda^2}{x}\right)^s dx + \frac{1}{2}\int_{\lambda}^{1} \xi^{2-s}(\xi^2 - \lambda^2)^s \, d\xi \quad (12)$$

When eq 11 and 12 are added together

$$I = \frac{1}{2}\lambda^2\left\{\int_\sigma^\lambda\left(\frac{\lambda^2}{x} - x\right)^s dx - \int_\lambda^1\left(x - \frac{\lambda^2}{x}\right)^s dx\right\} - \frac{1}{2}\int_\sigma^\lambda \xi^{2-s}(\lambda^2 - \xi^2)^s\, d\xi + \frac{1}{2}\int_\lambda^1 \xi^{2-s}(\xi^2 - \lambda^2)^s\, d\xi \quad (13)$$

The term enclosed in braces in eq 13 is seen to be just eq 6, the determining condition for λ, and clearly vanishes. Thus

$$I = -\frac{1}{2}\int_\sigma^\lambda (\lambda^2 - \xi^2)^s\xi^{2-s}\, d\xi + \frac{1}{2}\int_\lambda^1 (\xi^2 - \lambda^2)^s\xi^{2-s}\, d\xi \quad (14)$$

Now, in the first integral of eq 14 if one chooses $u = \xi^{1-s}$ and $dv = (\lambda^2 - \xi^2)\xi d\xi$, integrating once by parts gives

$$-\frac{1}{2}\int_\sigma^\lambda (\lambda^2 - \xi^2)^s\xi^{2-s}d\xi = -\frac{1}{4}\left(\frac{1}{s+1}\right)\left\{\sigma^{1-s}(\lambda^2 - \sigma^2)^{1+s} + (1-s)\int_\sigma^\lambda (\lambda^2 - \xi^2)^{s+1}\xi^{-s}\, d\xi\right\} \quad (15)$$

Similarly, in the second integral of eq 14, by choosing $u = \xi^{1-s}$ and $dv = (\xi^2 - \lambda^2)^s\xi d\xi$, integrating by parts gives

$$\frac{1}{2}\int_\lambda^1 (\xi^2 - \lambda^2)^s\xi^{2-s}d\xi = \frac{1}{4}\left(\frac{1}{s+1}\right)\left\{(1 - \lambda^2)^{s+1} - (1-s)\int_\lambda^1 (\xi^2 - \lambda^2)^{s+1}\xi^{-s}\, d\xi\right\} \quad (16)$$

Combining eq 14–16 with eq 8 gives the result

$$\frac{Q}{\pi R^3} = \left(\frac{PR}{2m}\right)^s\left\{\frac{1}{2}\left(\frac{1}{1+s}\right)\left[(1 - \lambda^2)^{1+s} - \sigma^{1-s}(\lambda^2 - \sigma^2)^{1+s} - (1-s)\int_\sigma^1 |\lambda^2 - \xi^2|^{s+1}\xi^{-s}\, d\xi\right]\right\} \quad (17)$$

Clearly eq 10 and 17 must be equivalent results. If one eliminates $Q/\pi R^3$ between these two equations, one finds

$$\int_\sigma^1 |\lambda^2 - \xi^2|^{s+1}\xi^{-s}d\xi = \frac{1}{3+s}[(1 - \lambda^2)^{1+s} - \sigma^{1-s}(\lambda^2 - \sigma^2)^{1+s}] \quad (18)$$

which is valid for arbitrary values of s for all values of σ in the range $0 \le \sigma \le 1$. Thus, the analytical expression for the volume rate of flow of a power-law non-Newtonian fluid through a concentric annulus is

$$Q = \frac{n\pi R^3}{1+3n}\left(\frac{PR}{2m}\right)^s[(1 - \lambda^2)^{1+s} - \sigma^{1-s}(\lambda^2 - \sigma^2)^{1+s}] \quad (19)$$

where $s = 1/n$. The value of λ to be used in eq 19 is given by solution of eq 6. Table I contains values of λ computed for a number of values of n, σ. These values of λ can be used with eq 19 to compute Q if $PR/2$ is given or to compute $PR/2$ if Q is given. We emphasize again that eq 19 and Table I are valid for arbitrary values of n and σ. For values of n or σ different from those given in the table, simple interpolation will give the λ values to be used.

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Theoretical and Experimental Studies of the Fluid Dynamics of a Two-Roll Coater

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Two cylindrical rolls are partially immersed in a Newtonian fluid so that the roll axes are parallel to each other and lie in the plane of the free surface of the fluid. The rolls nearly touch each other, and when they counterrotate, fluid is drawn into the region between them. Classical lubrication theory is used to predict the thickness of the fluid film withdrawn from the bath by the rolls. The effect of gravity is accounted for. For large capillary numbers the theory agrees well with data obtained with various Newtonian fluids. Gravity effects become measurable at small capillary numbers and the data show the trends predicted by the model. Some discussion is offered regarding the nature of boundary conditions imposed at the film-splitting stagnation point. It is shown that the use of the more accurate Coyne–Elrod condition does not significantly alter the model based on the approximate but simple Prandtl–Hopkins condition, at least for the range of conditions normally met in the laboratory or in coating practice.

Introduction

The application of rollers for the deposition of uniform liquid films onto moving or stationary webs is a common industrial operation. It is especially widespread in the photographic, printing, and paper–pulp industries where coating-thickness specifications are often very strict. Schematic diagrams of some typical roll coating processes are shown in Figure 1. Such "rolling geometries" are quite commonplace in lubrication engineering and the fundamental hydrodynamic analyses for these systems are available in texts on hydrodynamic lubrication (e.g., Pinkus and Sternlicht, 1961; Cameron, 1966).

Calendering is also an important operation that resembles roll coating. It differs from roll coating principally in the way by which the fluid separates downstream from the nip (the position where the rigid surfaces are at closest

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